Quantum Information Processing (2024) 23:100
https://doi.org/10.1007/s11128-024-04309-w

# Quantifying coherence of quantum channels based on the generalized $\alpha$-$z$-relative Rényi entropy

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Received: 18 April 2023 / Accepted: 7 February 2024 / Published online: 10 March 2024


**Abstract**

By using the Choi–Jamiołkowski isomorphism, we propose a well-defined coherence measure of quantum channels based on the generalized $\alpha$-$z$-relative Rényi entropy. In addition, we present an alternative coherence measure of quantum channels by quantifying the commutativity between the channels and the completely dephasing channels with the generalized $\alpha$-$z$-relative Rényi entropy. Some elegant properties of the measures are illustrated in detail. Explicit formulas of these coherence measures are derived for some detailed typical quantum channels.

**Keywords** Quantum coherence · Generalized $\alpha$-$z$-relative Rényi entropy · Quantum channel · Choi–Jamiołkowski isomorphism

## 1 Introduction

As a fundamental feature of quantum physics, coherence plays an essential role in quantum information processing. Based on the framework of quantifying the coherence of quantum states [1], quantifications of quantum coherence have been extensively studied in terms of the $l_1$-norm [1], relative entropy [1], skew information [2, 3], fidelity [4, 5] and generalized $\alpha$-$z$-relative Rényi entropy [6], with various applications in quantum entanglement, quantum algorithm, quantum meteorology and quantum biology [7–24]. Yu et al. [25] have presented an alternative framework for quantifying coherence.

Quantum channels characterize the general evolutions of quantum systems [26]. In recent years, fruitful results have been obtained on studies of quantum channels

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[27–43]. Datta et al. [44] investigated the coherence of quantum channels by using the Choi–Jamiołkowski isomorphism. Xu [45] proposed a framework to quantify the coherence of quantum channels by using the Choi–Jamiołkowski isomorphism, and defined the $l_1$-norm coherence measure of quantum channels. Based on this framework, some quantifiers of coherence for quantum channels have been given successively, such as maximum relative entropy [46], robustness [46], fidelity [47], skew information and Hellinger distance [48]. Luo et al. [49] introduced the coherence weight of quantum channels to investigate the quantum resource theory of dynamical coherence. Kong et al. [50] presented an alternative framework to quantify the coherence of quantum channels.

On the other hand, Meznaric et al. [51] formulated a measure of nonclassicality of a quantum operation, which is defined by quantifying the commutativity between a quantum operation and a completely dephasing operation based on the relative entropy. Fan et al. [52] studied the commutativity between a channel and a completely dephasing channel based on the trace distance, and quantified the coherence of quantum channels via commutativity.

The paper is organized as follows. In Sect. 2, we present the definition of a coherence measure for quantum channels based on the generalized $\alpha$-$z$-relative Rényi entropy via Choi–Jamiołkowski isomorphism, and verify that it is a well-defined coherence measure. In Sect. 3, we study the commutativity between the channels and the completely dephasing channels based on the generalized $\alpha$-$z$-relative Rényi entropy, and derive several elegant properties. In Sect. 4, we obtain explicit formulas of coherence measures with respect to some typical channels for above two newly defined measures. Finally, we conclude with a summary in Sect. 5.

## 2 Coherence of quantum channels by using Choi–Jamiołkowski isomorphism based on the generalized $\alpha$-$z$-relative Rényi entropy

For two arbitrary quantum states $\rho, \sigma$ and $\alpha, z \in \mathbb{R}$, the generalized $\alpha$-$z$-relative Rényi entropy is defined by [6],

$$D_{\alpha,z}(\rho,\sigma) = \frac{f_{\alpha,z}^{\frac{1}{\alpha}}(\rho,\sigma) - 1}{\alpha - 1}, \tag{1}$$

where

$$f_{\alpha,z}(\rho,\sigma) = \mathrm{Tr}\left(\sigma^{\frac{1-\alpha}{2z}}\rho^{\frac{\alpha}{z}}\sigma^{\frac{1-\alpha}{2z}}\right)^{z}. \tag{2}$$

Let $\{|i\rangle\}_{i=1}^{d}$ be a set of orthonormal basis of a $d$-dimensional Hilbert space $H$. The set $\mathcal{I}$ of quantum states is said to be incoherent if all the density matrices are diagonal in this basis. The quantum coherence $C_{\alpha,z}(\rho)$ of a quantum state $\rho$ induced by the generalized $\alpha$-$z$-relative Rényi entropy,

$$C_{\alpha,z}(\rho) = \min_{\sigma\in\mathcal{I}} D_{\alpha,z}(\rho,\sigma), \tag{3}$$

is a well-defined coherence measure in each of the following cases [6]:

(1) $\alpha \in (0, 1)$ and $z \geq \max\{\alpha, 1-\alpha\}$;
(2) $\alpha \in (1, 2]$ and $z = \{1, \frac{\alpha}{2}\}$;
(3) $\alpha > 1$ and $z = \alpha$.

It can be found that $C_{\alpha,z}(\rho)$ reduces to $\ln 2 \cdot C_r(\rho)$ and $2 \cdot C_s(\rho)$ when $z = 1, \alpha \to 1$ and $z = 1, \alpha = \frac{1}{2}$, respectively, where $C_r(\rho)$ denotes the relative entropy of coherence [1] and $C_s(\rho)$ denotes the skew information of coherence [3].

Let $H_A$ and $H_B$ be two Hilbert spaces with dimensions $|A|$ and $|B|$, orthonormal bases $\{|i\rangle\}_i$ and $\{|\beta\rangle\}_\beta$, respectively. We assume that $\{|i\rangle\}_i$ and $\{|\beta\rangle\}_\beta$ are fixed and adopt the tensor basis $\{|i\beta\rangle\}_{i\beta}$ as the fixed basis when considering the multipartite system $H_{\mathrm{AB}} = H_A \otimes H_B$. Denote by $\mathcal{D}(H_A)$ and $\mathcal{D}(H_B)$ the set of all density operators on $H_A$ and $H_B$, respectively. Denote by $\mathcal{C}_{\mathrm{AB}}$ the set of all channels from $\mathcal{D}(H_A)$ to $\mathcal{D}(H_B)$, $\mathcal{SC}_{ABA'B'}$ the set of all superchannels from $\mathcal{C}_{\mathrm{AB}}$ to $\mathcal{C}_{A'B'}$, $\mathcal{IC}_{\mathrm{AB}}$ the set of incoherent channels in $\mathcal{C}_{\mathrm{AB}}$, and $\mathcal{ISC}_{ABA'B'}$ the set of incoherent superchannels in $\mathcal{SC}_{ABA'B'}$. A quantum channel $\phi \in \mathcal{C}_{\mathrm{AB}}$ is a completely positive trace-preserving (CPTP) map. A coherence measure $C$ of quantum channels should satisfy the following conditions [45]:

(a) Faithfulness: $C(\phi) \geq 0$ for any $\phi \in \mathcal{C}_{\mathrm{AB}}$, and $C(\phi) = 0$ if and only if $\phi \in \mathcal{IC}_{\mathrm{AB}}$;
(b) Nonincreasing under $\mathcal{ISC}s$: $C(\phi) \geq C[\Theta(\phi)]$ for any $\Theta \in \mathcal{ISC}_{ABA'B'}$;
(c) Nonincreasing under $\mathcal{ISC}s$ on average: $C(\phi) \geq \sum_m p_m C(\phi_m)$ for any $\Theta \in \mathcal{ISC}_{ABA'B'}$, with $\{K_m\}_m$ an incoherent expression of $\Theta$, $p_m = \frac{\mathrm{Tr}(K_m J_\phi K_m^\dagger)}{|A'|}$ and $J_{\phi_m} = |A'|\frac{K_m J_\phi K_m^\dagger}{\mathrm{Tr}(K_m J_\phi K_m^\dagger)}$;
(d) Convexity: $C\left(\sum_m p_m \phi_m\right) \leq \sum_m p_m C(\phi_m)$ for any $\{\phi_m\}_m \subset \mathcal{C}_{\mathrm{AB}}$ and probability $\{p_m\}_m$.

Following the idea in [25], the authors in [50] proposed an alternative framework for quantifying the coherence of quantum channels which substitutes (c) and (d) with the following additivity,

$$C(\phi) = p_1 C(\phi_1) + p_2 C(\phi_2), \tag{4}$$

where $p_1 + p_2 = 1$, $\phi_1 \in \mathcal{C}_{AB_1}$, $\phi_2 \in \mathcal{C}_{AB_2}$, $\phi \in \mathcal{C}_{\mathrm{AB}}$, $|B| = |B_1| + |B_2|$, and $\phi(|i\rangle\langle\beta|) = p_1\phi_1(|i\rangle\langle\beta|) \oplus p_2\phi_2(|i\rangle\langle\beta|)$.

According to Theorem 3 in [45], if $C$ is a coherence measure for quantum states which satisfies (a)-(d), then the coherence measure of quantum channels is defined as

$$C(\phi) = C\left(\frac{J_\phi}{|A|}\right), \tag{5}$$

where $J_\phi$ is the Choi matrix corresponding to $\phi$. For convenience, we denote $\frac{J_\phi}{|A|}$ by $M_\phi$.

Suppose that the Kraus representation of a quantum channel $\phi$ is $\phi(\rho) = \sum_n K_n \rho K_n^\dagger$. According to Eq. (2) in [47], we have

$$M_\phi = (\mathbf{Id} \otimes \phi)|\varphi\rangle\langle\varphi| = \sum_n (\mathbb{I} \otimes K_n)|\varphi\rangle\langle\varphi|(\mathbb{I} \otimes K_n)^\dagger.$$

Here $|\varphi\rangle = \frac{1}{\sqrt{|A|}} \sum_{i=0}^{|A|-1} |ii\rangle$ is a maximally entangled state in Hilbert space $H_A \otimes H_A$, $\mathbf{Id}$ is the identity channel, and $\mathbb{I}$ is the identity operator.

**Definition 1** The generalized $\alpha$-$z$-relative Rényi entropy of two arbitrary quantum channels $\phi, \widetilde{\phi} \in \mathcal{C}_{\text{AB}}$ is defined as

$$D_{\alpha,z}(\phi, \widetilde{\phi}) = \frac{f_{\alpha,z}^{\frac{1}{\alpha}}(M_\phi, M_{\widetilde{\phi}}) - 1}{\alpha - 1}. \tag{6}$$

**Definition 2** The coherence measure of a channel $\phi$ induced by the generalized $\alpha$-$z$-relative Rényi entropy is defined by

$$C_{\alpha,z}(\phi) = \min_{\widetilde{\phi} \in \mathcal{IC}_{\text{AB}}} D_{\alpha,z}(\phi, \widetilde{\phi}) = \min_{M_{\widetilde{\phi}} \in \mathcal{I}} \frac{f_{\alpha,z}^{\frac{1}{\alpha}}(M_\phi, M_{\widetilde{\phi}}) - 1}{\alpha - 1}. \tag{7}$$

In particular, when $z = 1, \alpha \in (0, 1) \cup (1, 2]$, by using the Corollary 2 in [6], we have

$$C_{\alpha,1}(\phi) = \frac{\sum_{i,\beta} \langle i\beta | M_\phi^\alpha | i\beta \rangle^{\frac{1}{\alpha}} - 1}{\alpha - 1}. \tag{8}$$

$C_{\alpha,1}(\phi)$ reduces to $\ln 2 \cdot C_r(\phi)$ and $2 \cdot C_s(\phi)$ when $\alpha \to 1$ and $\alpha = \frac{1}{2}$, where $C_r(\phi)$ denotes the relative entropy of coherence of quantum channels and $C_s(\phi)$ denotes the skew information of coherence of quantum channels [48].

**Theorem 1** *$C_{\alpha,z}(\phi)$ defined in Eq. (7) is a well-defined coherence measure.*

***Proof*** According to Eqs. (2), (6) and (7), $C_{\alpha,z}(\phi)$ can be further rewritten as

$$C_{\alpha,z}(\phi) = \begin{cases} \frac{1 - \max_{M_{\widetilde{\phi}} \in \mathcal{I}} f_{\alpha,z}^{\frac{1}{\alpha}}(M_\phi, M_{\widetilde{\phi}})}{1-\alpha} & 0 < \alpha < 1, \\ \frac{\min_{M_{\widetilde{\phi}} \in \mathcal{I}} f_{\alpha,z}^{\frac{1}{\alpha}}(M_\phi, M_{\widetilde{\phi}}) - 1}{\alpha - 1} & \alpha > 1. \end{cases}$$

From the Lemma 1 in [6], it is easy to see that $C_{\alpha,z}(\phi) \geq 0$, and $C_{\alpha,z}(\phi) = 0$ if and only if $\phi = \widetilde{\phi}$. Thus, $C_{\alpha,z}(\phi)$ satisfies the condition (a).

When $\alpha > 1$, denote $\Theta^{'} = \frac{|A|}{|A^{'}|}\Theta$ with $\Theta \in \mathcal{ISC}_{\mathcal{ABA'B'}}$. Thus, $J_{\Theta'}$ is a CPTP map. Direct calculation shows that

$$\begin{aligned} f_{\alpha,z}(J_{\Theta'(\phi)}, J_{\Theta'(\widetilde{\phi})}) =& f_{\alpha,z}\left(\frac{|A|}{|A^{'}|}J_{\Theta(\phi)}, \frac{|A|}{|A^{'}|}J_{\Theta(\widetilde{\phi})}\right) \\ =& \frac{|A|}{|A^{'}|} f_{\alpha,z}(J_{\Theta(\phi)}, J_{\Theta(\widetilde{\phi})}) \\ =& |A| f_{\alpha,z}\left(\frac{J_{\Theta(\phi)}}{|A^{'}|}, \frac{J_{\Theta(\widetilde{\phi})}}{|A^{'}|}\right). \end{aligned}$$

Utilizing the Lemma 2 in [6], we have $f_{\alpha,z}(J_{\Theta'(\phi)}, J_{\Theta'(\widetilde{\phi})}) \leq f_{\alpha,z}(J_{\phi}, J_{\widetilde{\phi}})$. Then $D_{\alpha,z}(\Theta(\phi), \Theta(\widetilde{\phi})) \leq D_{\alpha,z}(\phi, \widetilde{\phi})$. Therefore,

$$\begin{aligned} C_{\alpha,z}(\Theta(\phi)) &= \min_{\widetilde{\phi}\in\mathcal{IC}_{\text{AB}}} D_{\alpha,z}(\Theta(\phi), \widetilde{\phi}) \\ &\leq \min_{\widetilde{\phi}\in\mathcal{IC}_{\text{AB}}} D_{\alpha,z}(\Theta(\phi), \Theta(\widetilde{\phi})) \\ &\leq \min_{\widetilde{\phi}\in\mathcal{IC}_{\text{AB}}} D_{\alpha,z}(\phi, \widetilde{\phi}) \\ &= C_{\alpha,z}(\phi). \end{aligned}$$

It can be seen that $C_{\alpha,z}(\Theta(\phi)) \leq C_{\alpha,z}(\phi)$ when $\alpha > 1$. The case of $0 < \alpha < 1$ can be easily proved in the same way. Hence, the condition (b) follows immediately.

Next we prove that $C_{\alpha,z}(\phi)$ satisfies Eq. (4). Suppose that $M_{\phi}$ is block-diagonal in the reference $\{|i\beta\rangle\}_{i\beta}$,

$$M_{\phi} = p_1 M_{\phi_1} \oplus p_2 M_{\phi_2},$$

where $p_1, p_2 > 0$ with $p_1 + p_2 = 1$, and $M_{\phi_1}$ and $M_{\phi_2}$ are the Choi states (density operators) corresponding to $\phi_1$ and $\phi_2$. $M_{\widetilde{\phi}}$, the Choi state corresponding to $\widetilde{\phi}$, can be written as

$$M_{\widetilde{\phi}} = q_1 M_{\widetilde{\phi}_1} \oplus q_2 M_{\widetilde{\phi}_2},$$

where $q_1, q_2 > 0$ with $q_1 + q_2 = 1$, and $M_{\widetilde{\phi}_1}$ and $M_{\widetilde{\phi}_2}$ are the Choi states (density operators) corresponding to $\widetilde{\phi}_1$ and $\widetilde{\phi}_2$. Denote by $\Delta$ either max or min. Let $t_m = \Delta_{M_{\widetilde{\phi}_m}} \text{Tr}\left(M_{\widetilde{\phi}_m}^{\frac{1-\alpha}{2z}} M_{\phi_m}^{\frac{\alpha}{z}} M_{\widetilde{\phi}_m}^{\frac{1-\alpha}{2z}}\right)^z$, $m = 1, 2$. It can be derived that

$$\Delta_{M_{\widetilde{\phi}}\in\mathcal{I}}\text{Tr}\left(M_{\widetilde{\phi}}^{\frac{1-\alpha}{2z}} M_{\phi}^{\frac{\alpha}{z}} M_{\widetilde{\phi}}^{\frac{1-\alpha}{2z}}\right)^z = \Delta_{q_1,q_2}(q_1^{1-\alpha} p_1^{\alpha} t_1 + q_1^{1-\alpha} p_2^{\alpha} t_2).$$

Using the Hölder inequality with $0 < \alpha < 1$, we have

$$q_1^{1-\alpha} p_1^{\alpha} t_1 + q_2^{1-\alpha} p_2^{\alpha} t_2 \leq \left( \sum_{m=1,2} p_m t_m^{\frac{1}{\alpha}} \right)^{\alpha},$$

where the equality holds if and only if $q_1 = l p_1 t_1^{\frac{1}{\alpha}}$ and $q_2 = l p_2 t_2^{\frac{1}{\alpha}}$ with $l = \left( p_1 t_1^{\frac{1}{\alpha}} + p_2 t_2^{\frac{1}{\alpha}} \right)^{-1}$. Consequently

$$\max_{q_1, q_2} (q_1^{1-\alpha} p_1^{\alpha} t_1 + q_2^{1-\alpha} p_2^{\alpha} t_2) = \left( \sum_{m=1,2} p_m t_m^{\frac{1}{\alpha}} \right)^{\alpha}.$$

Similarly, it is not difficult to obtain that when $\alpha > 1$,

$$q_1^{1-\alpha} p_1^{\alpha} t_1 + q_2^{1-\alpha} p_2^{\alpha} t_2 \geq \left( \sum_{m=1,2} p_m t_m^{\frac{1}{\alpha}} \right)^{\alpha},$$

and the equality holds when $q_1 = l p_1 t_1^{\frac{1}{\alpha}}$ and $q_2 = l p_2 t_2^{\frac{1}{\alpha}}$, which yields

$$\min_{q_1, q_2} (q_1^{1-\alpha} p_1^{\alpha} t_1 + q_2^{1-\alpha} p_2^{\alpha} t_2) = \left( \sum_{m=1,2} p_m t_m^{\frac{1}{\alpha}} \right)^{\alpha}.$$

We have further

$$\Delta_{M_{\widetilde{\phi}} \in \mathcal{I}} f_{\alpha,z}^{\frac{1}{\alpha}}(M_{\phi}, M_{\widetilde{\phi}}) = p_1 \Delta_{M_{\widetilde{\phi}_1} \in \mathcal{I}} f_{\alpha,z}^{\frac{1}{\alpha}}(M_{\phi_1}, M_{\widetilde{\phi}_1}) + p_2 \Delta_{M_{\widetilde{\phi}_2} \in \mathcal{I}} f_{\alpha,z}^{\frac{1}{\alpha}}(M_{\phi_2}, M_{\widetilde{\phi}_2}).$$

Thus

$$C_{\alpha,z}(\phi) = p_1 C_{\alpha,z}(\phi_1) + p_2 C_{\alpha,z}(\phi_2),$$

which implies that $C_{\alpha,z}(\phi)$ satisfies Eq. (4). This completes the proof. □

## 3 An alternative coherence measure of quantum channels based on the generalized $\alpha$-$z$-relative Rényi entropy

In this section, we present a coherence measure of quantum channels through an alternative method by quantifying the commutativity between the channels and the completely dephasing channels via the generalized $\alpha$-$z$-relative Rényi entropy. Furthermore, by utilizing the properties of the generalized $\alpha$-$z$-relative Rényi entropy [6], we discuss some properties of this coherence measure.

**Definition 3** The completely dephasing channel $\Delta^A \in \mathcal{C}_{\mathrm{AB}}$ is defined as [45]

$$\Delta^A(\rho^A) = \sum_i \langle i|\rho^A|i\rangle|i\rangle\langle i|, \quad \rho^A \in \mathcal{D}(H_A). \tag{9}$$

A state $\sigma^A \in \mathcal{D}(H_A)$ is called incoherent if $\Delta^A(\sigma^A) = \sigma^A$. Otherwise, we say that it is coherent.

**Definition 4** For a channel $\phi \in \mathcal{C}_{\mathrm{AB}}$, we define an alternative coherence measure $\widetilde{C}_{\alpha,z}(\phi)$ of $\phi$,

$$\widetilde{C}_{\alpha,z}(\phi) = \sup_\rho D_{\alpha,z}(\phi \circ \Delta^A(\rho), \Delta^B \circ \phi(\rho)), \tag{10}$$

where $D_{\alpha,z}(\cdot,\cdot)$ is the generalized $\alpha$-$z$-relative Rényi entropy, and the supremum in Eq. (10) is taken over all quantum states.

**Theorem 2** *$\widetilde{C}_{\alpha,z}(\phi)$ has the following elegant properties:*

*(i) (Extremal property) for $\sup_\rho D_{\alpha,z}(\phi \circ \Delta(\rho), \Delta \circ \phi(\rho))$, there exists a pure state $|\psi\rangle\langle\psi|$ such that the supremum in Eq. (10) is attained when $\rho = |\psi\rangle\langle\psi|$.*
*(ii) (Monotonicity) for any quantum channel $\phi$, if $\phi_0$ is a quantum channel satisfying $\widetilde{C}_{\alpha,z}(\phi_0) = 0$, then $\widetilde{C}_{\alpha,z}(\phi_0 \circ \phi) \le \widetilde{C}_{\alpha,z}(\phi)$ and $\widetilde{C}_{\alpha,z}(\phi \circ \phi_0) \le \widetilde{C}_{\alpha,z}(\phi)$.*
*(iii) (Convexity) for some quantum channels $\phi_m$, and some positive real number $\lambda_m$ such that $\sum_m \lambda_m = 1$, we have $\widetilde{C}_{\alpha,z}\left(\sum_m \lambda_m \phi_m\right) \le \sum_m \lambda_m \widetilde{C}_{\alpha,z}(\phi_m)$.*

**Proof** *Suppose that the spectral decomposition of $\rho$ is $\rho = \sum_m \mu_m |\psi_m\rangle\langle\psi_m|$. We have*

$$\begin{aligned}
&D_{\alpha,z}(\phi \circ \Delta(\rho), \Delta \circ \phi(\rho)) \\
=&D_{\alpha,z}\left(\phi \circ \Delta\left(\sum_m \mu_m |\psi_m\rangle\langle\psi_m|\right), \Delta \circ \phi\left(\sum_m \mu_m |\psi_m\rangle\langle\psi_m|\right)\right) \\
=&D_{\alpha,z}\left(\sum_m \mu_m \phi \circ \Delta(|\psi_m\rangle\langle\psi_m|), \sum_m \mu_m \Delta \circ \phi(|\psi_m\rangle\langle\psi_m|)\right) \\
\le& \sum_m \mu_m D_{\alpha,z}(\phi \circ \Delta(|\psi_m\rangle\langle\psi_m|), \Delta \circ \phi(|\psi_m\rangle\langle\psi_m|)) \\
\le& \sum_m \mu_m \sup_{|\psi\rangle} D_{\alpha,z}(\phi \circ \Delta(|\psi\rangle\langle\psi|), \Delta \circ \phi(|\psi\rangle\langle\psi|)) \\
=& \sup_{|\psi\rangle} D_{\alpha,z}(\phi \circ \Delta(|\psi\rangle\langle\psi|), \Delta \circ \phi(|\psi\rangle\langle\psi|)),
\end{aligned}$$

*where the first inequality follows from the joint convexity of $D_{\alpha,z}(\cdot,\cdot)$. Thus,*

$$\widetilde{C}_{\alpha,z}(\phi) \le \sup_{|\psi\rangle} D_{\alpha,z}(\phi \circ \Delta(|\psi\rangle\langle\psi|), \Delta \circ \phi(|\psi\rangle\langle\psi|)).$$

*It follows from Eq. (10) that*

$$\widetilde{C}_{\alpha,z}(\phi) = \sup_{|\psi\rangle} D_{\alpha,z}(\phi \circ \Delta(|\psi\rangle\langle\psi|), \Delta \circ \phi(|\psi\rangle\langle\psi|)). \tag{11}$$

*Therefore, item (i) holds.*

*Using the monotonicity of $D_{\alpha,z}$ under the CPTP maps, we have*

$$\begin{aligned} & D_{\alpha,z}(\phi_0 \circ \phi \circ \Delta(|\psi\rangle\langle\psi|), \Delta \circ \phi_0 \circ \phi(|\psi\rangle\langle\psi|)) \\ =& D_{\alpha,z}(\phi_0 \circ \phi \circ \Delta(|\psi\rangle\langle\psi|), \phi_0 \circ \Delta \circ \phi(|\psi\rangle\langle\psi|)) \\ \le& D_{\alpha,z}(\phi \circ \Delta(|\psi\rangle\langle\psi|), \Delta \circ \phi(|\psi\rangle\langle\psi|)), \end{aligned}$$

*where the first equality holds due to $\widetilde{C}_{\alpha,z}(\phi_0) = 0$ and Definition 1 in [52]. Then, by Eq. (11), we obtain $\widetilde{C}_{\alpha,z}(\phi_0 \circ \phi) \le \widetilde{C}_{\alpha,z}(\phi)$. On the other hand,*

$$\begin{aligned} & \widetilde{C}_{\alpha,z}(\phi \circ \phi_0) \\ =& \sup_{\rho} D_{\alpha,z}(\phi \circ \phi_0 \circ \Delta(\rho), \Delta \circ \phi \circ \phi_0(\rho)) \\ =& \sup_{\rho} D_{\alpha,z}(\phi \circ \Delta \circ \phi_0(\rho), \Delta \circ \phi \circ \phi_0(\rho)) \\ =& \sup_{\sigma=\phi_0(\rho)} D_{\alpha,z}(\phi \circ \Delta(\sigma), \Delta \circ \phi(\sigma)) \\ \le& \sup_{\rho} D_{\alpha,z}(\phi \circ \Delta(\rho), \Delta \circ \phi(\rho)) \\ =& \widetilde{C}_{\alpha,z}(\phi), \end{aligned}$$

*which implies that $\widetilde{C}_{\alpha,z}(\phi \circ \phi_0) \le \widetilde{C}_{\alpha,z}(\phi)$. Hence, item (ii) is proved.*

*By utilizing the joint convexity of $D_{\alpha,z}(\cdot, \cdot)$, we can further obtain*

$$\begin{aligned} & \widetilde{C}_{\alpha,z}\left(\sum_m \lambda_m \phi_m\right) \\ =& \sup_{|\psi\rangle} D_{\alpha,z}\left(\sum_m \lambda_m \phi_m \circ \Delta(|\psi\rangle\langle\psi|), \Delta \circ \sum_m \lambda_m \phi_m(|\psi\rangle\langle\psi|)\right) \\ =& \sup_{|\psi\rangle} D_{\alpha,z}\left(\sum_m \lambda_m \phi_m \circ \Delta(|\psi\rangle\langle\psi|), \sum_m \lambda_m \Delta \circ \phi_m(|\psi\rangle\langle\psi|)\right) \\ \le& \sum_m \lambda_m \sup_{|\psi\rangle} D_{\alpha,z}\left(\phi_m \circ \Delta(|\psi\rangle\langle\psi|), \Delta \circ \phi_m(|\psi\rangle\langle\psi|)\right) \\ =& \sum_m \lambda_m \widetilde{C}_{\alpha,z}(\phi_m). \end{aligned}$$

*Therefore,*

$$\widetilde{C}_{\alpha,z}\left(\sum_m \lambda_m \phi_m\right) \le \sum_m \lambda_m \widetilde{C}_{\alpha,z}(\phi_m), \tag{12}$$

*and the item (iii) is derived.* □

*From Eq.* (10)*, it can be easily seen that* $\widetilde{C}_{\alpha,z}(\phi) = 0$ *when the quantum channel* $\phi$ *is detection-creation-incoherent* [52]*, i.e.,* $\phi \circ \Delta^A = \Delta^B \circ \phi$*. Comparing the two quantifiers of the coherence of quantum channels in Eqs.* (8) *and* (10)*, it can be found that* $C_{\alpha,z}(\phi) \geq \widetilde{C}_{\alpha,z}(\phi)$ *always holds in this special case. From the examples in the next section and numerical results, it is conjectured that* $C_{\alpha,1}(\phi) \geq \widetilde{C}_{\alpha,1}(\phi)$ *holds for all quantum channels* $\phi$*, but we have not yet found a proof.*

## 4 Examples

In this section, we choose several typical channels to calculate the coherence measures defined in Eqs. (8) and (10).

***Example 1*** Consider the phase flip channel $\phi_{\mathrm{PF}}(\rho) = \sum_{n=1}^{2} K_n \rho K_n^\dagger$ with the Kraus operators

$$K_1 = \sqrt{p}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad K_2 = \sqrt{1-p}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad 0 \leq p \leq 1.$$

Direct calculation shows that

$$C_{\alpha,1}(\phi_{\mathrm{PF}}) = \frac{\sum_{i,\beta=0}^{1} \langle i\beta | M^{\alpha}_{\phi_{\mathrm{PF}}} | i\beta \rangle^{\frac{1}{\alpha}} - 1}{\alpha - 1} = \frac{2^{1-\frac{1}{\alpha}}[p^\alpha + (1-p)^\alpha]^{\frac{1}{\alpha}} - 1}{\alpha - 1}. \tag{13}$$

However, if we calculate the values of the coherence measure given in Eq. (10), we can clearly see that $\widetilde{C}_{\alpha,z}(\phi_{\mathrm{PF}}) \equiv 0$ regardless of the values of $\alpha$ and $z$. In fact, for any pure state $|\psi\rangle = a|0\rangle + b|1\rangle$ with $|a|^2 + |b|^2 = 1$, we have

$$\begin{aligned} \Delta \circ \phi_{\mathrm{PF}}(|\psi\rangle\langle\psi|) &= \Delta(\phi_{\mathrm{PF}}(|\psi\rangle\langle\psi|)) \\ &= \Delta(K_1(|\psi\rangle\langle\psi|)K_1^\dagger + K_2(|\psi\rangle\langle\psi|)K_2^\dagger), \end{aligned}$$

where $K_1|\psi\rangle = a\sqrt{p}|0\rangle + b\sqrt{p}|1\rangle$ and $K_2|\psi\rangle = a\sqrt{1-p}|0\rangle - b\sqrt{1-p}|1\rangle$. It can be shown that

$$\begin{aligned} &\phi_{\mathrm{PF}}(|\psi\rangle\langle\psi|) = |a|^2|0\rangle\langle 0| + (2p-1)a\bar{b}|0\rangle\langle 1| + (2p-1)b\bar{a}|1\rangle\langle 0| + |b|^2|1\rangle\langle 1|, \\ &\Delta \circ \phi_{\mathrm{PF}}(|\psi\rangle\langle\psi|) = |a|^2|0\rangle\langle 0| + |b|^2|1\rangle\langle 1|, \\ &\phi_{\mathrm{PF}} \circ \Delta(|\psi\rangle\langle\psi|) = \phi_{\mathrm{PF}}(|a|^2|0\rangle\langle 0| + |b|^2|1\rangle\langle 1|) = |a|^2|0\rangle\langle 0| + |b|^2|1\rangle\langle 1|, \end{aligned}$$

which implies that $\widetilde{C}_{\alpha,z}(\phi_{\mathrm{PF}}) = 0$.

In Fig. 1, we plot the surfaces of $\widetilde{C}_{\alpha,z}(\phi_{\mathrm{PF}})$ and $C_{\alpha,1}(\phi_{\mathrm{PF}})$ given in Eqs. (10) and (13). By calculation, it is found that $\lim_{\alpha \to 1} C_{\alpha,1}(\phi_{\mathrm{PF}}) = \ln 2 + p\ln p + \ln(1-p) - p\ln(1-p)$,

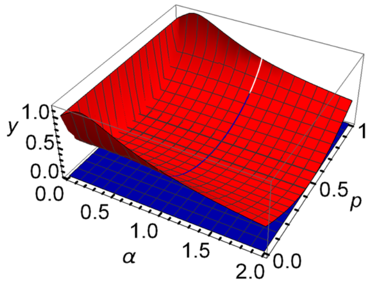


**Fig. 1** Surfaces of $\widetilde{C}_{\alpha,z}(\phi_{\text{PF}})$ and $C_{\alpha,1}(\phi_{\text{PF}})$. The blue (red) surface represents the values of $\widetilde{C}_{\alpha,z}(\phi_{\text{PF}})$ ($C_{\alpha,1}(\phi_{\text{PF}})$) in Eq. (10) (Eq. 13)

which reaches its minimum value 0 when $p = \frac{1}{2}$, and reaches its maximum value ln2 when $p = 0$. When $\alpha = \frac{1}{2}$, $C_{\frac{1}{2},1}(\phi_{\text{PF}}) = 1 - 2\sqrt{p(1-p)}$. Its minimum value 0 is obtained when $p = \frac{1}{2}$ and its maximum value 1 is obtained when $p = 0$. It can be shown that $C_{\alpha,1}(\phi_{\text{PF}}) \geq \widetilde{C}_{\alpha,z}(\phi_{\text{PF}})$ when $\alpha \in (0,1) \cup (1,2]$, $0 \leq p \leq 1$.

***Example 2*** Consider the depolarizing channel $\phi_D(\rho) = \sum_{n=1}^{4} K_n \rho K_n^\dagger$ with the Kraus operators

$$K_1 = \sqrt{1 - \frac{3}{4}p}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad K_2 = \frac{\sqrt{p}}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$
$$K_3 = \frac{\sqrt{p}}{2}\begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}, \quad K_4 = \frac{\sqrt{p}}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad 0 \leq p \leq 1.$$

Hence, $C_{\alpha,1}(\phi)$ defined in Eq. (8) is given by

$$C_{\alpha,1}(\phi_D) = \frac{\sum_{i,\beta=0}^{1} \langle i\beta | M_{\phi_D}^{\alpha} | i\beta \rangle^{\frac{1}{\alpha}} - 1}{\alpha - 1} = \frac{2\left[\frac{p^\alpha}{2^{2\alpha+1}} + \frac{\left(1-\frac{3}{4}p\right)^\alpha}{2}\right]^{\frac{1}{\alpha}} + \frac{p}{2} - 1}{\alpha - 1}. \tag{14}$$

Similar to the phase flip channel, $\widetilde{C}_{\alpha,z}(\phi_D) \equiv 0$ regardless of the values of $\alpha$ and $z$.

In Fig. 2, we plot the surfaces of $\widetilde{C}_{\alpha,z}(\phi_D)$ and $C_{\alpha,1}(\phi_D)$ in Eqs. (10) and (14). Direct calculation shows that $\lim_{\alpha\to 1} C_{\alpha,1}(\phi_D) = \frac{1}{4}[(4-3p)\ln(4-3p) + 2(p-2)\ln(2-p) + p\ln p]$, which reaches its minimum value 0 when $p = 1$, and reaches its maximum value ln2 when $p = 0$. When $\alpha = \frac{1}{2}$, we have $C_{\frac{1}{2},1}(\phi_D) = 1 - \frac{\sqrt{p(4-3p)}+p}{2}$. Its minimum value 0 is attained when $p = 1$, and its maximum value of 1 is attained when $p = 0$. It can be found that $C_{\alpha,1}(\phi_D) \geq \widetilde{C}_{\alpha,z}(\phi_D)$ when $\alpha \in (0,1) \cup (1,2]$, $0 \leq p \leq 1$.

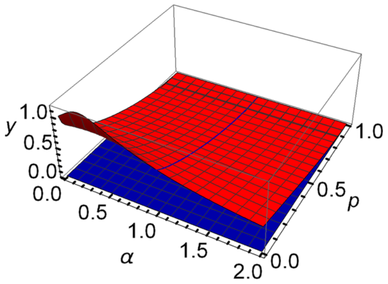


**Fig. 2** Surfaces of $\widetilde{C}_{\alpha,z}(\phi_D)$ and $C_{\alpha,1}(\phi_D)$. The blue (red) surface represents the values of $\widetilde{C}_{\alpha,z}(\phi_D)$ ($C_{\alpha,1}(\phi_D)$) in Eq. (10) (Eq. 14)

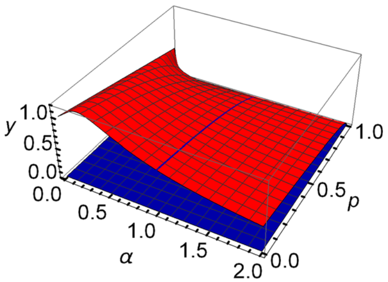


**Fig. 3** Surfaces of $\widetilde{C}_{\alpha,z}(\phi_{\mathrm{AD}})$ and $C_{\alpha,1}(\phi_{\mathrm{AD}})$. The blue (red) surface represents the values of $\widetilde{C}_{\alpha,z}(\phi_{\mathrm{AD}})$ ($C_{\alpha,1}(\phi_{\mathrm{AD}})$) in Eq. (10) (Eq. 15)

***Example 3*** Consider the amplitude damping channel $\phi_{\mathrm{AD}}(\rho) = \sum_{n=1}^{2} K_n \rho K_n^{\dagger}$ with the Kraus operators

$$K_1 = \begin{pmatrix} 1 & 0 \\ 0 & \sqrt{1-p} \end{pmatrix}, \quad K_2 = \begin{pmatrix} 0 & \sqrt{p} \\ 0 & 0 \end{pmatrix}, \quad 0 \leq p \leq 1.$$

It follows from Eq. (8) that

$$C_{\alpha,1}(\phi_{\mathrm{AD}}) = \frac{\sum_{i,\beta=0}^{1} \langle i\beta | M_{\phi_{\mathrm{AD}}}^{\alpha} | i\beta \rangle^{\frac{1}{\alpha}} - 1}{\alpha - 1} = \frac{\left(\frac{1}{2} + \frac{1}{2}(1-p)^{\frac{1}{\alpha}}\right)(2-p)^{1-\frac{1}{\alpha}} + \frac{p}{2} - 1}{\alpha - 1}. \tag{15}$$

Similarly, $\widetilde{C}_{\alpha,z}(\phi_{\mathrm{AD}}) \equiv 0$ regardless of the values of $\alpha$ and $z$.

In Fig. 3, we plot the surfaces of $\widetilde{C}_{\alpha,z}(\phi_{\mathrm{AD}})$ and $C_{\alpha,1}(\phi_{\mathrm{AD}})$ in Eqs. (10) and (15). It is found that $\lim_{\alpha\to 1} C_{\alpha,1}(\phi_{\mathrm{AD}}) = \frac{1}{2}[(p-1)\ln(1-p)-(p-2)\ln(2-p)]$. $\lim_{\alpha\to 1} C_{\alpha,1}(\phi_{\mathrm{AD}})$ reaches its minimum value 0 when $p = 1$. $\lim_{\alpha\to 1} C_{\alpha,1}(\phi_{\mathrm{AD}})$ reaches its maximum value ln2 when $p = 0$. When $\alpha = \frac{1}{2}$, we have $C_{\frac{1}{2},1}(\phi_{\mathrm{AD}}) = \frac{2p-2}{p-2}$. Its minimum value 0 is obtained when $p = 1$ and its maximum value 1 is obtained when $p = 0$. It can be shown that $C_{\alpha,1}(\phi_{\mathrm{AD}}) \geq \widetilde{C}_{\alpha,z}(\phi_{\mathrm{AD}})$ when $\alpha \in (0, 1) \cup (1, 2]$, $0 \leq p \leq 1$.

***Example 4*** Consider the isotropic channel $\phi_\Lambda$ for $t \in [\frac{-1}{d^2-1}, 1]$ [53]

$$\phi_\Lambda(\rho) = tU\rho U^\dagger + (1-t)\frac{\mathbb{I}_d}{d}, \tag{16}$$

where $U$ is an unitary operation, $\mathbb{I}_d$ is $d \times d$ identity matrix, and $d$ is the dimension of the Hilbert space. In particular, taking $U = H$, where $H = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$ is the Hadamard gate, we have

$$\phi_\Lambda^H(\rho) = tH\rho H^\dagger + (1-t)\frac{\mathbb{I}_2}{2} = \sum_{n=1}^{5} K_n \rho K_n^\dagger, \quad -\frac{1}{3} \le t \le 1, \tag{17}$$

where $\mathbb{I}_2$ is $2 \times 2$ identity matrix, and

$$K_1 = \sqrt{t}H = \sqrt{\frac{t}{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, \qquad K_2 = \frac{\sqrt{1-t}}{2}X = \frac{\sqrt{1-t}}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$
$$K_3 = \frac{\sqrt{1-t}}{2}Y = \frac{\sqrt{1-t}}{2}\begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}, \quad K_4 = \frac{\sqrt{1-t}}{2}Z = \frac{\sqrt{1-t}}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$
$$K_5 = \frac{\sqrt{1-t}}{2}\mathbb{I}_2 = \frac{\sqrt{1-t}}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

By Eq. (8), it can be easily deduced that

$$C_{\alpha,1}(\phi_\Lambda^H) = \frac{\sum_{i,\beta=0}^{1} \langle i\beta | M_{\phi_\Lambda^H}^\alpha | i\beta \rangle^{\frac{1}{\alpha}} - 1}{\alpha - 1} = \frac{4^{-\frac{1}{\alpha}}[3(1-t)^\alpha + (1+3t)^\alpha]^{\frac{1}{\alpha}} - 1}{\alpha - 1}. \tag{18}$$

According to Eq. (18), we obtain

$$\lim_{\alpha \to 1} C_{\alpha,1}(\phi_\Lambda^H) = \frac{3(1-t)\ln(1-t) + (1+3t)\ln(1+3t)}{4}, \tag{19}$$

$$C_{\frac{1}{2},1}(\phi_\Lambda^H) = \frac{3t-5}{4} - \frac{3}{4}\sqrt{(1-t)(1-3t)} + 2. \tag{20}$$

Set $\alpha = \frac{1}{2}$ and $z = 1$. Then,

$$\widetilde{C}_{\frac{1}{2},1}(\phi_\Lambda^H) = \sup_{|\psi\rangle} D_{\frac{1}{2},1}(\phi_\Lambda^H \circ \Delta(|\psi\rangle\langle\psi|), \Delta \circ \phi_\Lambda^H(|\psi\rangle\langle\psi|)),$$

where

$$
\begin{aligned}
&D_{\frac{1}{2},1}\left(\phi_\Lambda^H\circ\Delta(|\psi\rangle|\langle\psi|),\Delta\circ\phi_\Lambda^H(|\psi\rangle|\langle\psi|)\right)\\
=&-2\left[f_{\frac{1}{2},1}^2\left(\phi_\Lambda^H\circ\Delta(|\psi\rangle|\langle\psi|),\Delta\circ\phi_\Lambda^H(|\psi\rangle|\langle\psi|)\right)-1\right]\\
=&-2\left[\left[\mathrm{Tr}\left((\phi_\Lambda^H\circ\Delta(|\psi\rangle|\langle\psi|))^{\frac{1}{2}}(\Delta\circ\phi_\Lambda^H(|\psi\rangle|\langle\psi|))^{\frac{1}{2}}\right)\right]^2-1\right]\\
=&-2\left[\left(1+\sqrt{1-4t^2\mathrm{Re}^2(ab^*)}\right)\left(\frac{1}{4}+\frac{1}{4}\sqrt{1-t^2(|a|^2-|b|^2)^2}\right)-1\right]\\
\leq&-2\left[\left(1+\sqrt{1-4t^2|a|^2|b|^2}\right)\left(\frac{1}{4}+\frac{1}{4}\sqrt{1-t^2(|a|^2-|b|^2)^2}\right)-1\right]\\
\leq&-2\left[\left(1+\sqrt{1-t^2}\right)\left(\frac{1}{4}+\frac{1}{4}\sqrt{1-t^2(|a|^2-|b|^2)^2}\right)-1\right]\\
\leq&-2\left[\left(1+\sqrt{1-t^2}\right)\left(\frac{1}{4}+\frac{1}{2}|a||b|\right)-1\right]\\
\leq&-2|a||b|\left(1+\sqrt{1-t^2}\right)+2\\
\leq&1-\sqrt{1-t^2}.
\end{aligned}
$$

The above inequalities hold due to the facts that $0\leq|a|^2|b|^2\leq\frac{1}{4}$ and $(|a|^2-|b|^2)^2=1-4|a|^2|b|^2$. It follows from item (i) that $\widetilde{C}_{\frac{1}{2},1}(\phi_\Lambda^H)\leq1-\sqrt{1-t^2}$. Meanwhile, for the classical pure state $|0\rangle$ or $|1\rangle$, the maximum value of $D_{\frac{1}{2},1}(\Delta\circ\phi_\Lambda^H(\rho),\phi_\Lambda^H\circ\Delta(\rho))$ can be obtained directly. It is easy to see that

$$D_{\frac{1}{2},1}(\Delta\circ\phi_\Lambda^H(|0\rangle\langle0|),\phi_\Lambda^H\circ\Delta(|0\rangle\langle0|))=1-\sqrt{1-t^2}.$$

Thus, we get

$$\widetilde{C}_{\frac{1}{2},1}(\phi_\Lambda^H)=1-\sqrt{1-t^2}. \tag{21}$$

According to the above results, it is found that $\widetilde{C}_{\alpha,z}(\phi_\Lambda^H)$ is not an incoherent channel when $\alpha=\frac{1}{2}$ and $z=1$.

Setting $t=1$ in Eq. (17), $\phi_\Lambda^H$ becomes the unitary channel $\phi_H$ induced by the Hadamard gate $H$. Then, it follows from Eq. (18) that

$$C_{\alpha,1}(\phi_H)=\frac{\sum_{i,\beta=0}^{1}\langle i\beta|M_{\phi_H}^\alpha|i\beta\rangle^{\frac{1}{\alpha}}-1}{\alpha-1}=\frac{4^{1-\frac{1}{\alpha}}-1}{\alpha-1}. \tag{22}$$

According to Eq. (22), we obtain that $\lim\limits_{\alpha\to1}C_{\alpha,1}(\phi_H)=\ln4$ and $C_{\frac{1}{2},1}(\phi_H)=\frac{3}{2}$. From the deduction of $\widetilde{C}_{\frac{1}{2},1}(\phi_\Lambda^H)$, we can also infer that $\widetilde{C}_{\frac{1}{2},1}(\phi_H)=1$ by letting $t=1$.

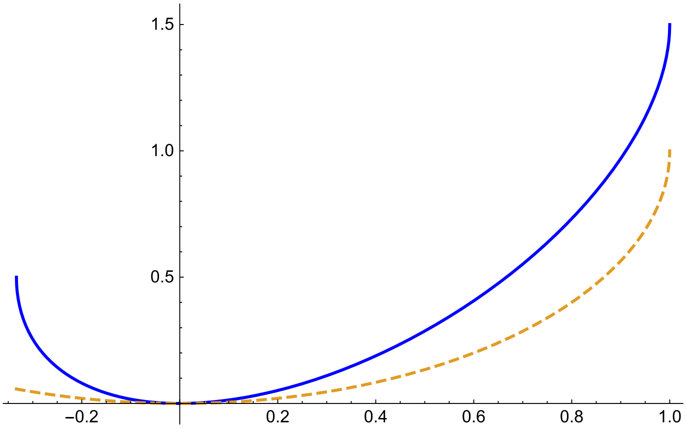


**Fig. 4** The values of $C_{\frac{1}{2},1}(\phi^H_\Lambda)$ and $\widetilde{C}_{\frac{1}{2},1}(\phi^H_\Lambda)$. The blue (orange) curve represents the values of $C_{\frac{1}{2},1}(\phi^H_\Lambda)$ ($\widetilde{C}_{\frac{1}{2},1}(\phi^H_\Lambda)$) in Eq. (20) (Eq. 21)

It can be seen that $C_{\frac{1}{2},1}(\phi^H_\Lambda) \geq \widetilde{C}_{\frac{1}{2},1}(\phi^H_\Lambda)$ holds when $-\frac{1}{3} \leq t \leq 1$. And as a special case of $t = 1$, we get $C_{\frac{1}{2},1}(\phi_H) \geq \widetilde{C}_{\frac{1}{2},1}(\phi_H)$. In Fig. 4, we plot the values of $C_{\frac{1}{2},1}(\phi^H_\Lambda)$ and $\widetilde{C}_{\frac{1}{2},1}(\phi^H_\Lambda)$ in Eqs. (20) and (21).

***Example 5*** Consider the unitary channels $\phi_S$ and $\phi_T$ induced by the phase gate $S$ and $\frac{\pi}{8}$ gate $T$, i.e., $\phi_S(\rho) = S\rho S^\dagger$ and $\phi_T(\rho) = T\rho T^\dagger$, where

$$S = \begin{pmatrix} 1 & 0 \\ 0 & \mathrm{i} \end{pmatrix} \quad \text{and} \quad T = \begin{pmatrix} 1 & 0 \\ 0 & e^{\frac{\mathrm{i}\pi}{4}} \end{pmatrix}.$$

By Eq. (8), we have $C_{\alpha,1}(\phi_S) = C_{\alpha,1}(\phi_T) = \frac{2^{1-\frac{1}{\alpha}}-1}{\alpha-1}$. It is obvious that $\lim_{\alpha\to 1} C_{\alpha,1}(\phi_S) = \lim_{\alpha\to 1} C_{\alpha,1}(\phi_T) = \ln 2$, and $C_{\frac{1}{2},1}(\phi_S) = C_{\frac{1}{2},1}(\phi_T) = 1$. By Eq. (10), we obtain $\widetilde{C}_{\alpha,z}(\phi_S) = \widetilde{C}_{\alpha,z}(\phi_T) = 0$. Note that the two quantifiers of the coherence $C_{\alpha,1}(\cdot)$ and $\widetilde{C}_{\alpha,1}(\cdot)$ for the quantum channels induced by $S$ and $T$ are the same.

From Examples 4 and 5, it can be seen that $C_{\frac{1}{2},1}(\phi) > \widetilde{C}_{\frac{1}{2},1}(\phi)$, where $\phi$ is the unitary channel induced by $H$, $S$ or $T$.

The above results are based on the channels of single qubits. We now turn to discuss the channels of entangled qubits. The corresponding Choi–Jamiołkowski states for the channels of entangled qubits are too complicated to be calculated for general two-qubit unitaries. For simplicity, we take $S \otimes S$ and $T \otimes T$.

***Example 6*** Consider the unitary channels $\phi_{S\otimes S}$ and $\phi_{T\otimes T}$ induced by $S\otimes S$ and $T\otimes T$, i.e., $\phi_{S\otimes S}(\rho_{\mathrm{AB}}) = (S\otimes S)\rho_{\mathrm{AB}}(S\otimes S)^\dagger$ and $\phi_{T\otimes T}(\rho_{\mathrm{AB}}) = (T\otimes T)\rho_{\mathrm{AB}}(T\otimes T)^\dagger$, where $S$ is the phase gate and $T$ is the $\frac{\pi}{8}$ gate defined in Example 5.

**Table 1** Comparisons of the values of $C_{\alpha,1}(\phi)$ defined in Eq. (8) with $\alpha \to 1$ and $\alpha = \frac{1}{2}$ and $\widetilde{C}_{\alpha,z}(\phi)$ defined in Eq. (10)

| Channels | $p_{\max}$ | $\max \lim_{\alpha\to 1} C_{\alpha,1}(\phi)$ | $p_{\min}$ | $\min \lim_{\alpha\to 1} C_{\alpha,1}(\phi)$ | $\widetilde{C}_{\alpha,z}(\phi)$ |
|---|---|---|---|---|---|
| $\phi_{\mathrm{PF}}$ | 0 | ln2 | $\frac{1}{2}$ | 0 | $0, \forall\alpha, z$ |
| $\phi_D$ | 0 | ln2 | 1 | 0 | $0, \forall\alpha, z$ |
| $\phi_{\mathrm{AD}}$ | 0 | ln2 | 1 | 0 | $0, \forall\alpha, z$ |
| Channels | $p_{\max}$ | $\max C_{\frac{1}{2},1}(\phi)$ | $p_{\min}$ | $\min C_{\frac{1}{2},1}(\phi)$ | $\widetilde{C}_{\alpha,z}(\phi)$ |
| $\phi_{\mathrm{PF}}$ | 0 | 1 | $\frac{1}{2}$ | 0 | $0, \forall\alpha, z$ |
| $\phi_D$ | 0 | 1 | 1 | 0 | $0, \forall\alpha, z$ |
| $\phi_{\mathrm{AD}}$ | 0 | 1 | 1 | 0 | $0, \forall\alpha, z$ |

The first column represents the channels, $p_{\min}$ and $p_{\max}$ represent the values of $p$ where the maximum and minimum values are attained, respectively. $\max \lim_{\alpha\to 1} C_{\alpha,1}(\phi)$ and $\min \lim_{\alpha\to 1} C_{\alpha,1}(\phi)$ represent the maximum and minimum values of $\lim_{\alpha\to 1} C_{\alpha,1}(\phi)$, respectively, while $\max C_{\frac{1}{2},1}(\phi)$ and $\min C_{\frac{1}{2},1}(\phi)$ represent the maximum and minimum values of $C_{\frac{1}{2},1}(\phi)$, respectively. The last column represents the values of $\widetilde{C}_{\alpha,z}(\phi)$ defined in Eq. (10)

By Eq. (8), it follows that

$$C_{\alpha,1}(\phi_{S\otimes S}) = C_{\alpha,1}(\phi_{T\otimes T}) = \frac{4^{1-\frac{1}{\alpha}} - 1}{\alpha - 1}. \tag{23}$$

It is obvious that $\lim_{\alpha\to 1} C_{\alpha,1}(\phi_{S\otimes S}) = \lim_{\alpha\to 1} C_{\alpha,1}(\phi_{T\otimes T}) = \ln 4$, and $C_{\frac{1}{2},1}(\phi_{S\otimes S}) = C_{\frac{1}{2},1}(\phi_{T\otimes T}) = \frac{3}{2}$. On the other hand, by using Eq. (10) we obtain $\widetilde{C}_{\alpha,z}(\phi_{S\otimes S}) = \widetilde{C}_{\alpha,z}(\phi_{T\otimes T}) = 0$.

It can be found from Table 1 that under the three quantum channels $\phi_{\mathrm{PF}}$, $\phi_D$ and $\phi_{\mathrm{AD}}$, for either $\alpha \to 1$ or $\alpha = \frac{1}{2}$, $C_{\alpha,1}(\phi) \geq \widetilde{C}_{\alpha,1}(\phi)$ and $C_{\alpha,1}(\phi)$ reaches the maximum value when $p = 0$. The minimum values 0 are attained at the same $p$ for each quantum channel $\phi_{\mathrm{PF}}$, $\phi_D$ and $\phi_{\mathrm{AD}}$. The coherence of $\phi_{\mathrm{PF}}$, $\phi_D$ and $\phi_{\mathrm{AD}}$ have the same maximum values ln2 when $\alpha \to 1$, and the same maximum values 1 when $\alpha = \frac{1}{2}$.

## 5 Conclusion

Utilizing the coherence measure of quantum states induced by the generalized $\alpha$-$z$-relative Rényi entropy, we have studied the quantifications of the coherence of quantum channels by using two different approaches. Following the idea in [45], we have introduced a coherence measure of quantum channels by utilizing the Choi–Jamiołkowski isomorphism. We have also verified that $C_{\alpha,z}(\phi)$ defined in Eq. (7) is a well-defined coherence measure. On the other hand, inspired by the idea in [52], we have presented an alternative coherence measure by quantifying the commutativity between the channels and the completely dephasing channels with the generalized $\alpha$-$z$-relative Rényi entropy. The extremal property, monotonicity and convexity of $\widetilde{C}_{\alpha,z}(\phi)$ defined in Eq. (10) have been explored in detail.

Furthermore, the coherence measures defined in Eqs. (8) and (10) have been calculated for some typical channels, respectively. Analytical formulas of $C_{\alpha,1}(\phi)$ defined in Eq. (8) for the phase flip channel, depolarizing channel and amplitude damping channel have been derived and analyzed for the case of $\alpha \to 1$ and $\alpha = \frac{1}{2}$. According to Eq. (10), it can be found that $\phi_{\text{PF}}$, $\phi_D$ and $\phi_{\text{AD}}$ are all incoherent channels. A table has been presented to compare different values of coherence measures for $\phi_{\text{PF}}$, $\phi_D$ and $\phi_{\text{AD}}$. In addition, we have also considered the unitary channels induced by three quantum gates. The coherence measures defined in Eqs. (8) and (10) for isotropic channels $\phi_{\Lambda}^{H}$ with $t \in [-\frac{1}{3}, 1]$ induced by Hadamard gate have been derived. The quantifiers defined in Eqs. (8) and (10) for unitary channel $\phi_H$ induced by Hadamard gate have been deduced as a special case when $t = 1$. The unitary channels induced by $S$ gate and $T$ gate are all incoherent channels according to Eq. (10), and they have the same expressions of $C_{\alpha,1}(\phi)$ as Eq. (8). Finally, we have calculated the coherence of quantum channels induced by $S \otimes S$ and $T \otimes T$ for entangled qubits, and presented the analytical formulae of the coherence measures.

Detailed examples and numerical results show that $C_{\alpha,1}(\phi) \geq \widetilde{C}_{\alpha,1}(\phi)$ for specific quantum channels $\phi$, so we conjecture that $C_{\alpha,1}(\phi) \geq \widetilde{C}_{\alpha,1}(\phi)$ holds for any quantum channel, while a rigorous proof is missing. Our results may shed some new light on the exploration of quantification of coherence for quantum channels. The regime of coherence quantifiers on the level of quantum channels needs further study in the future.

**Acknowledgements** The authors would like to thank the anonymous referees for their valuable suggestions which greatly improved this paper. This work was supported by National Natural Science Foundation of China (Grant Nos. 12161056, 12075159, 12171044) and Natural Science Foundation of Jiangxi Province of China (Grant No. 20232ACB211003); the Academician Innovation Platform of Hainan Province.

**Data availability** Data sharing is not applicable to this article as no datasets were generated or analyzed during the current study.

## Declarations

**Conflict of interest** The authors declare no competing interests.

## Appendix A. Calculation of $C_{\alpha,1}(\phi_{\text{PF}})$

According to the Kraus operators of $\phi_{\text{PF}}$ given in Example 1, we have

$$
\begin{aligned}
M_{\phi_{\mathrm{PF}}} &= (\mathbb{I}_2 \otimes K_1)\left(\frac{1}{2}\sum_{i,j=0}^{1} |ii\rangle\langle jj|\right)(\mathbb{I}_2 \otimes K_1)^\dagger \\
&\quad + (\mathbb{I}_2 \otimes K_2)\left(\frac{1}{2}\sum_{i,j=0}^{1} |ii\rangle\langle jj|\right)(\mathbb{I}_2 \otimes K_2)^\dagger \\
&= \frac{1}{2}\begin{pmatrix} 1 & 0 & 0 & 2p-1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 2p-1 & 0 & 0 & 1 \end{pmatrix},
\end{aligned}
$$

where $\mathbb{I}_2$ denotes the $2 \times 2$ identity matrix. Furthermore, we have

$$
M_{\phi_{\mathrm{PF}}}^{\alpha} = \begin{pmatrix} \frac{p^\alpha+(1-p)^\alpha}{2} & 0 & 0 & \frac{p^\alpha-(1-p)^\alpha}{2} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \frac{p^\alpha-(1-p)^\alpha}{2} & 0 & 0 & \frac{p^\alpha+(1-p)^\alpha}{2} \end{pmatrix}.
$$

Based on $M_{\phi_{\mathrm{PF}}}^{\alpha}$, we get $C_{\alpha,1}(\phi_{\mathrm{PF}})$ in Eq. (13) from Eq. (8).

## Appendix B. Calculation of $C_{\alpha,1}(\phi_D)$

Direct calculation shows that

$$
\begin{aligned}
M_{\phi_D} &= (\mathbb{I}_2 \otimes K_1)\left(\frac{1}{2}\sum_{i,j=0}^{1} |ii\rangle\langle jj|\right)(\mathbb{I}_2 \otimes K_1)^\dagger \\
&\quad + (\mathbb{I}_2 \otimes K_2)\left(\frac{1}{2}\sum_{i,j=0}^{1} |ii\rangle\langle jj|\right)(\mathbb{I}_2 \otimes K_2)^\dagger \\
&\quad + (\mathbb{I}_2 \otimes K_3)\left(\frac{1}{2}\sum_{i,j=0}^{1} |ii\rangle\langle jj|\right)(\mathbb{I}_2 \otimes K_3)^\dagger \\
&\quad + (\mathbb{I}_2 \otimes K_4)\left(\frac{1}{2}\sum_{i,j=0}^{1} |ii\rangle\langle jj|\right)(\mathbb{I}_2 \otimes K_4)^\dagger \\
&= \begin{pmatrix} \frac{1}{2}-\frac{p}{4} & 0 & 0 & \frac{1}{2}-\frac{p}{2} \\ 0 & \frac{p}{4} & 0 & 0 \\ 0 & 0 & \frac{p}{4} & 0 \\ \frac{1}{2}-\frac{p}{2} & 0 & 0 & \frac{1}{2}-\frac{p}{4} \end{pmatrix},
\end{aligned}
$$

where $\mathbb{I}_2$ denotes the $2 \times 2$ identity matrix. Then,

$$M_{\phi_D}^{\alpha} = \begin{pmatrix} \frac{1}{2^{2\alpha+1}} p^{\alpha} + \frac{1}{2}\left(1 - \frac{3}{4}p\right)^{\alpha} & 0 & 0 & \frac{1}{2}\left(1 - \frac{3}{4}p\right)^{\alpha} - \frac{1}{2^{2\alpha+1}} p^{\alpha} \\ 0 & 4^{-\alpha} p^{\alpha} & 0 & 0 \\ 0 & 0 & 4^{-\alpha} p^{\alpha} & 0 \\ \frac{1}{2}\left(1 - \frac{3}{4}p\right)^{\alpha} - \frac{1}{2^{2\alpha+1}} p^{\alpha} & 0 & 0 & \frac{1}{2^{2\alpha+1}} p^{\alpha} + \frac{1}{2}\left(1 - \frac{3}{4}p\right)^{\alpha} \end{pmatrix},$$

from which we get $C_{\alpha,1}(\phi_D)$ in Eq. (14) by using Eq. (8).

## Appendix C. Calculation of $C_{\alpha,1}(\phi_{\mathrm{AD}})$

According to the Kraus operators of $\phi_{\mathrm{AD}}$ given in Example 3, we have

$$\begin{aligned} M_{\phi_{\mathrm{AD}}} &= (\mathbb{I}_2 \otimes K_1)\left(\frac{1}{2}\sum_{i,j=0}^{1} |ii\rangle\langle jj|\right)(\mathbb{I}_2 \otimes K_1)^{\dagger} \\ &\quad + (\mathbb{I}_2 \otimes K_2)\left(\frac{1}{2}\sum_{i,j=0}^{1} |ii\rangle\langle jj|\right)(\mathbb{I}_2 \otimes K_2)^{\dagger} \\ &= \frac{1}{2}\begin{pmatrix} 1 & 0 & 0 & \sqrt{1-p} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & p & 0 \\ \sqrt{1-p} & 0 & 0 & 1-p \end{pmatrix}. \end{aligned}$$

Then,

$$M_{\phi_{\mathrm{AD}}}^{\alpha} = \begin{pmatrix} 2^{-\alpha}(2-p)^{\alpha-1} & 0 & 0 & 2^{-\alpha}\sqrt{(1-p)}\,(2-p)^{\alpha-1} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 2^{-\alpha} p^{\alpha} & 0 \\ 2^{-\alpha}\sqrt{(1-p)}\,(2-p)^{\alpha-1} & 0 & 0 & 2^{-\alpha}(1-p)(2-p)^{\alpha-1} \end{pmatrix}.$$

Utilizing $M_{\phi_{\mathrm{AD}}}^{\alpha}$, we derive the formulas of $C_{\alpha,1}(\phi_{\mathrm{AD}})$ in Eq. (15) via Eq. (8).

## Appendix D. Calculation of $C_{\alpha,1}(\phi_\Lambda^H)$

Noting that

$$\begin{aligned} M_{\phi_\Lambda^H} =& (\mathbb{I}_2 \otimes K_1) \left( \frac{1}{2} \sum_{i,j=0}^{1} |ii\rangle\langle jj| \right) (\mathbb{I}_2 \otimes K_1)^\dagger \\ &+ (\mathbb{I}_2 \otimes K_2) \left( \frac{1}{2} \sum_{i,j=0}^{1} |ii\rangle\langle jj| \right) (\mathbb{I}_2 \otimes K_2)^\dagger \\ &+ (\mathbb{I}_2 \otimes K_3) \left( \frac{1}{2} \sum_{i,j=0}^{1} |ii\rangle\langle jj| \right) (\mathbb{I}_2 \otimes K_3)^\dagger \\ &+ (\mathbb{I}_2 \otimes K_4) \left( \frac{1}{2} \sum_{i,j=0}^{1} |ii\rangle\langle jj| \right) (\mathbb{I}_2 \otimes K_4)^\dagger \\ &+ (\mathbb{I}_2 \otimes K_5) \left( \frac{1}{2} \sum_{i,j=0}^{1} |ii\rangle\langle jj| \right) (\mathbb{I}_2 \otimes K_5)^\dagger = \frac{1}{4} \begin{pmatrix} 1 & t & t & -t \\ t & 1 & t & -t \\ t & t & 1 & -t \\ -t & -t & -t & 1 \end{pmatrix}, \end{aligned}$$

where $\mathbb{I}_2$ denotes the $2 \times 2$ identity matrix, we have

$$M^\alpha_{\phi_\Lambda^H} = 4^{-1-\alpha} \begin{pmatrix} 3(1-t)^\alpha + (1+3t)^\alpha & -(1-t)^\alpha + (1+3t)^\alpha & -(1-t)^\alpha + (1+3t)^\alpha & (1-t)^\alpha - (1+3t)^\alpha \\ -(1-t)^\alpha + (1+3t)^\alpha & 3(1-t)^\alpha + (1+3t)^\alpha & -(1-t)^\alpha + (1+3t)^\alpha & (1-t)^\alpha - (1+3t)^\alpha \\ -(1-t)^\alpha + (1+3t)^\alpha & -(1-t)^\alpha + (1+3t)^\alpha & 3(1-t)^\alpha + (1+3t)^\alpha & (1-t)^\alpha - (1+3t)^\alpha \\ (1-t)^\alpha - (1+3t)^\alpha & (1-t)^\alpha - (1+3t)^\alpha & (1-t)^\alpha - (1+3t)^\alpha & 3(1-t)^\alpha + (1+3t)^\alpha \end{pmatrix}.$$

Making use of $M^\alpha_{\phi_\Lambda^H}$, the quantity $C_{\alpha,1}(\phi_\Lambda^H)$ in Eq. (18) follows immediately from Eq. (8).

## Appendix E. Calculations of $C_{\alpha,1}(\phi_{S\otimes S})$ and $C_{\alpha,1}(\phi_{T\otimes T})$

Direct calculation shows that

$$
\begin{aligned}
M_{\phi_{S\otimes S}} &= (\mathbb{I}_4 \otimes (S \otimes S)) \left( \frac{1}{2} \sum_{i,j=0}^{1} |ii\rangle\langle jj| \otimes \frac{1}{2} \sum_{i,j=0}^{1} |ii\rangle\langle jj| \right) (\mathbb{I}_4 \otimes (S \otimes S))^{\dagger} \\
&= \begin{pmatrix}
\frac{1}{4} & 0 & 0 & -\frac{1}{4} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{4} & 0 & 0 & -\frac{1}{4} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-\frac{1}{4} & 0 & 0 & \frac{1}{4} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{1}{4} & 0 & 0 & \frac{1}{4} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\frac{1}{4} & 0 & 0 & -\frac{1}{4} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{4} & 0 & 0 & -\frac{1}{4} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-\frac{1}{4} & 0 & 0 & \frac{1}{4} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{1}{4} & 0 & 0 & \frac{1}{4}
\end{pmatrix}
\end{aligned}
$$

and

$$
\begin{aligned}
M_{\phi_{T\otimes T}} &= (\mathbb{I}_4 \otimes (T \otimes T)) \left( \frac{1}{2} \sum_{i,j=0}^{1} |ii\rangle\langle jj| \otimes \frac{1}{2} \sum_{i,j=0}^{1} |ii\rangle\langle jj| \right) (\mathbb{I}_4 \otimes (T \otimes T))^{\dagger} \\
&= \begin{pmatrix}
\frac{1}{4} & 0 & 0 & \frac{e^{-2i\pi}}{4} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{4} & 0 & 0 & \frac{e^{-2i\pi}}{4} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\frac{e^{2i\pi}}{4} & 0 & 0 & \frac{1}{4} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{e^{2i\pi}}{4} & 0 & 0 & \frac{1}{4} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\frac{1}{4} & 0 & 0 & \frac{e^{-2i\pi}}{4} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{4} & 0 & 0 & \frac{e^{-2i\pi}}{4} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\frac{e^{2i\pi}}{4} & 0 & 0 & \frac{1}{4} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{e^{2i\pi}}{4} & 0 & 0 & \frac{1}{4}
\end{pmatrix},
\end{aligned}
$$

where $\mathbb{I}_4$ denotes the $4 \times 4$ identity matrix. Then,

$$
M^{\alpha}_{\phi_{S\otimes S}} = \begin{pmatrix}
\frac{1}{4} & 0 & 0 & -\frac{1}{4} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{4} & 0 & 0 & -\frac{1}{4} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-\frac{1}{4} & 0 & 0 & \frac{1}{4} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{1}{4} & 0 & 0 & \frac{1}{4} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\frac{1}{4} & 0 & 0 & -\frac{1}{4} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{4} & 0 & 0 & -\frac{1}{4} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-\frac{1}{4} & 0 & 0 & \frac{1}{4} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{1}{4} & 0 & 0 & \frac{1}{4}
\end{pmatrix}
$$

and

$$
M^{\alpha}_{\phi_{T\otimes T}} = \begin{pmatrix}
\frac{1}{4} & 0 & 0 & \frac{e^{-2i\pi}}{4} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{4} & 0 & 0 & \frac{e^{-2i\pi}}{4} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\frac{e^{2i\pi}}{4} & 0 & 0 & \frac{1}{4} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{e^{2i\pi}}{4} & 0 & 0 & \frac{1}{4} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\frac{1}{4} & 0 & 0 & \frac{e^{-2i\pi}}{4} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{4} & 0 & 0 & \frac{e^{-2i\pi}}{4} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\frac{e^{2i\pi}}{4} & 0 & 0 & \frac{1}{4} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{e^{2i\pi}}{4} & 0 & 0 & \frac{1}{4}
\end{pmatrix},
$$

By Eq. (8), we can thus deduce $C_{\alpha,1}(\phi_{S\otimes S})$ and $C_{\alpha,1}(\phi_{T\otimes T})$ in Eq. (23) based on $M^{\alpha}_{\phi_{S\otimes S}}$ and $M^{\alpha}_{\phi_{T\otimes T}}$.

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