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Some new quantum codes from constacyclic codes

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Abstract
In this paper, let q be an odd prime power. Based on new constacyclic codes which contain their Hermitian duals and Hermitian construction, we construct some classes of quantum MDS codes and quantum codes. When $q \equiv 1 \bmod 4$, x and y are a divisor of $q-1$ and $q+1$, respectively, we can construct a class of new quantum codes of length $n = 2xy\frac{q^{2m}-1}{q^2-1}$ for odd $x, y, m \geq 3$. These codes have larger dimensions than existing codes. In addition, for q with the form $2am \pm \sqrt{(x^2+y^2)a-1}$ and odd x, y, a with $gcd(x, y) = 1$, we get some quantum MDS codes of length $n = \frac{q^2+1}{a}$.

Keywords Hermitian construction · Constacyclic codes · Quantum codes · Quantum MDS codes

1 Introduction

Quantum error-correcting codes (quantum codes) are useful in quantum computing and quantum communication. Given a prime power q, an $[[n, k, d]]_q$ quantum code is a q^k-dimensional vector subspace of the Hilbert space $(\mathbb{C}^q)^{\otimes n}$ with minimal distance d [1]. Especially, if a quantum code reaches the quantum Singleton bound, i.e. $k = n-2d+2$, it is called a quantum maximum-distance-separable (MDS) code. A quantum code can also be denoted by $((n, K, d))_q$, where $k = log_q K$.

In [2], Calderbank et al. presented the first systematic and effective mathematical method for constructing quantum codes and thus established the connection between classical error-correcting codes and quantum error-correcting codes. Since then, the mathematical study of quantum codes has progressed rapidly. Many good quantum codes have been constructed by using different approaches [1, 3–39]. Among these methods, the most commonly used methods are Euclidean construction and Hermitian construction. We list the quantum MDS codes constructed by these two methods

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in Table 1. In [40], Goyeneche et al. builded a relationship between an irredundant orthogonal array (IrOA), v-uniform state, and quantum code $((n, K, d))_q$ for $K = 1$. Based on Hamming distances and construction methods of orthogonal arrays (OAs), Pang et al. constructed infinite classes of v-uniform states for $v = 2, 3$ in [41] and infinite classes of v-uniform states for $v \geq 4$ in [42]. Besides, Pang et al. generalized construction method of uniform states for homogeneous systems to heterogeneous systems [43]. Moreover, Pang et al. extended methods of constructing quantum codes $((n, K, d))_q$ for $K = 1$ to $K > 1$ [44, 45]. And a large number of quantum codes including quantum MDS codes can be obtained. Part of these codes are listed in Table 2. Even so, there are still some good quantum codes that remain unknown.

In this paper, let $q \equiv 1 \bmod 4$ be an odd prime power, $m \geq 3$ be odd, $x \geq 3$ be an odd divisor of $q - 1$, and $y \geq 3$ be an odd divisor of $q + 1$. Using negacyclic codes over F_{q^2}, we construct new quantum codes of length $n = 2xy\frac{q^{2m}-1}{q^2-1}$ with parameters $[[n, n-2m(\delta_1-\lfloor\frac{\delta_1}{q^2}\rfloor+\delta_2-\lfloor\frac{\delta_2}{q^2}\rfloor)-2, \geq \delta_1+\delta_2+2]]_q$, where $0 \leq \delta_1, \delta_2 \leq y\lfloor\frac{q^m-1}{q^2-1}\rfloor$. Besides, for smaller odd x, y, a with $gcd(x, y) = 1$, let $q = 2am \pm \sqrt{(x^2 + y^2)a - 1}$ be an odd prime power, where m is a positive integer. We get some q-ary quantum MDS codes of length $n = \frac{q^2+1}{a}$ from ω^{q-1}-constacyclic codes over F_{q^2}.

The paper is organized as follows. In Sect. 2, we state the basic notations and review the results about constacyclic codes and quantum codes used in this work. In addition, we present some lemmas for constructing quantum codes. In Sect. 3, some new quantum codes and quantum MDS codes are constructed by using constacyclic codes. This paper is summarised in Sect. 4.

2 Preliminaries

In this section, we state some basic notations and review some results about constacyclic codes and quantum codes [18, 48, 49].

Throughout this paper, assume that q is an odd prime power. Define $\bar{\alpha} = \alpha^q$ for any element $\alpha \in F_{q^2}$. For any two vectors $a = (a_0, a_1, \ldots, a_{n-1})$ and $b = (b_0, b_1, \ldots, b_{n-1}) \in F_{q^2}^n$, their Hermitian inner product is defined as

$$\langle a, b\rangle = a_0\bar{b}_0 + a_1\bar{b}_1 + \ldots + a_{n-1}\bar{b}_{n-1} \in F_{q^2}.$$

The vectors a and b are called orthogonal with respect to the Hermitian inner product if $\langle a, b\rangle = 0$. For a q^2-ary linear code C of length n, the Hermitian dual code of C is defined as

$$C^{\perp h} = \{a \in F_{q^2}^n | \langle a, b\rangle = 0, b \in C\}.$$

Table 1 Quantum MDS codes $[[n, k, d]]_q$

Class	Length n	Distance d	References
1	$3 \le n \le q+1$	$1 \le d \le \lfloor \frac{n}{2} \rfloor + 1$	[6–8]
2	$mq-l,\ 0 \le l < m,\ 1 < m < q$	$d \le m-l+1$	[13, 17]
3	$mq-l,\ 0 \le l \le q-1,\ 1 \le m \le q$	$3 \le d \le \frac{q+1-\lfloor l/m \rfloor}{2}$	[10]
4	$r(q-1)+1,\ q \equiv r-1 \bmod 2r$	$d \le \frac{q+r+1}{2}$	[11]
5	$q^2-s,\ 0 \le s < \frac{q}{2}-1$	$\frac{q}{2}+1 < d \le q-s$	[11]
6	$\frac{q^2+1}{2}-s,\ 0 \le s < \frac{q}{2}-1$	$\frac{q}{2}+1 < d \le q-s$	[11]
7	$4 \le n \le q^2+1,\ q \ne 2,\ n \ne 4$	3	[3, 10, 14]
8	$q^2-l,\ 0 \le l \le q-2$	$d \le q-l$	[7, 13]
9	$\lambda(q-1),\ q+1 = \lambda r,\ r$ even	$2 \le d \le \frac{q+1}{2}+\lambda-1$	[21]
10	$\lambda(q-1),\ q+1 = \lambda r,\ r$ odd	$2 \le d \le \frac{q+1}{2}+\frac{\lambda}{2}-1$	[21]
11	$m(q-1),\ 1 \le m \le q$	$2 \le d \le \lfloor \frac{mq-1}{q+1} \rfloor + 1$	[35]
12	$4(q-1),\ q \equiv 1 \bmod 4$	$d = \frac{q+1}{2}$	[22]
13	$s(q+1),\ 1 \le s \le q-1$	$2 \le d \le s$	[35]
14	$2^f s(q+1),\ 2^e \| (q-1)$	$2 \le d \le \frac{q+1}{2}+2^f s$	[19]
	$0 \le f < e,\ s \mid (q-1),\ s$ odd		
15	q^2+1	$2 \le d \le q+1$	[10–12]
16	$\frac{q^2+1}{2},\ q$ odd	$3 \le d \le q,\ d$ odd	[18]
17	$\frac{q^2+1}{5},\ q = 10m \pm 2$	$3 \le d \le 6m \pm 1,\ d$ odd	[1]
18	$\frac{q^2+1}{a},\ q = 2am \pm t,\ t = \sqrt{2a-1},\ a$ odd	$2 \le d \le 2tm \pm 2,\ d$ even	[30]
19	$\frac{q^2+1}{a},\ a = m^2+1,\ m \ge 1,\ q$ odd	$2 \le d \le \frac{(m+1)q \pm (m-1)}{a},\ d$ even, m even	[31]
	$a \mid (q+m)$ or $a \mid (q-m)$	$3 \le d \le \frac{(m+1)q \pm (m-1)}{a},\ d$ odd, m odd	

Table 1 continued

Class	Length n	Distance d	References
20	$\frac{q^2+1}{a},\ a=\frac{m^2+1}{2},\ q$ odd	$2\leq d\leq\frac{mq\pm1}{a},\ d$ even	[31]
	$a\mid(q+m)$ or $a\mid(q-m)$, odd $m\geq3$		
21	$\frac{q^2+1}{a},\ a=\frac{m^2+1}{5},\ q$ odd	$2\leq d\leq\frac{(3m-1)q\pm(m+3)}{5a},\ d$ even, $m=10h+2$	[31]
	$a\mid(q+m)$ or $a\mid(q+a-m)$	$3\leq d\leq\frac{(3m+1)q\pm(m-3)}{5a},\ d$ odd, $m=10h+3$	
		$3\leq d\leq\frac{(3m-1)q\pm(m+3)}{5a},\ d$ odd, $m=10h+7$	
		$2\leq d\leq\frac{(3m+1)q\pm(m-3)}{5a},\ d$ even, $m=10h+8$	
22	$\frac{q^2+1}{a},\ a=\frac{m^2+1}{10},\ q$ odd	$2\leq d\leq\frac{(2m-1)q\pm(m+2)}{5a},\ d$ even, $m=10h+3$	[31]
	$a\mid(q+m)$ or $a\mid(q+a-m)$	$2\leq d\leq\frac{(2m+1)q\pm(m-2)}{5a},\ d$ even, $m=10h+7$	
23	$\frac{q^2-1}{h}$	$2\leq d\leq\frac{q+1}{h_1}+\frac{q-1}{h_2},\ h_1h_2=2h$	[1]
	$h_1=gcd(h,q+1),\ h_2=gcd(h,q-1)$	$2\leq d\leq min\{\frac{q-1}{h_2},\frac{q+1}{2h_1}+\frac{q-1}{2h_2}\},\ h_1h_2=h$	
24	$\frac{q^2-1}{h}$, odd $h\geq3$, $q=hbm-1,\ m,b\geq2$	$2\leq d\leq\frac{(h+1)(q+1)}{2h}-1$	[26]
25	$\frac{q^2-1}{2}$	$2\leq d\leq q,\ q=2bm\pm1,\ m>0,\ b\geq2$	[26]
26	$\frac{q^2-1}{h},\ q=2ht+1,\ h$ odd	$3\leq d\leq 2t(h+1)+1,\ d$ odd	[29]
27	$\frac{q^2-1}{15},\ q=30m-11$	$3\leq d\leq 8(2m-1)+1,\ d$ odd	[29]
28	$\frac{q^2-1}{21},\ q=42m-29$	$3\leq d\leq 4(5m-4)+1,\ d$ odd	[29]
29	$\frac{q^2-1}{35},\ q=70m-41$	$3\leq d\leq 8(3m-2)+1,\ d$ odd	[29]
30	$q^2-1,\ q$ odd	$3\leq d\leq 2q-1,\ d$ odd	[29]
31	$\frac{q^2-1}{2t},\ q=2tm+1$	$2\leq d\leq(t+1)m+1$	[46]

Table 1 continued

Class	Length n	Distance d	References
32	$\frac{q^2-1}{3t},\ t \equiv 5 \bmod 9,\ q = 16t^2 - 12t + 1$	$2 \leq d \leq \frac{q+1}{2} + \frac{2t-1}{3}$	[22]
33	$\frac{q^2-1}{m} + 1,\ m\|(q+1),\ m$ odd	$2 \leq d \leq \frac{q+1}{2} + \frac{q-1}{2m}$	[22]
34	$tq,\ 1 \leq t \leq q$	$2 \leq d \leq \lfloor \frac{tq+q-1}{4+1} \rfloor + 1$	[23]
35	$t(q+1)+2,\ q = p^s$	$2 \leq d \leq t+2$	[23]
	$1 \leq t \leq q-1,\ (p,t,d) \neq (2,q-1,q)$		
36	$(2t+2)\frac{q^2-1}{h},\ 0 \leq t \leq \frac{h-3}{2},\ h\|(q+1),\ h$ odd	$2 \leq d \leq \frac{(h+2t+3)(q+1)}{2h} - 1$	[24]
37	$(q+1)(q-\delta-1),\ 0 \leq \delta \leq q-3$, even $q > 2$	$2 \leq d \leq q - \delta - 1$	[26]
38	$(q+1)(q-2\delta-1),\ 0 \leq \delta \leq \frac{q-5}{2},\ q > 3$	$2 \leq d \leq q - 2\delta - 2$	[26]
	$q \equiv 3 \bmod 4$		
39	$2(q-1)(2\delta+1),\ 0 \leq \delta \leq \frac{q-1}{4}$	$2 \leq d \leq 4\delta + 2$	[26]
	$q \equiv 1 \bmod 4$		
40	$\frac{r(q^2-1)}{s} + 1,\ s\|(q-1),\ 1 \leq r \leq s$	$2 \leq d \leq \frac{r(q-1)}{s} + 1$	[28]
41	$\frac{r(q^2-1)}{2s} + 1,\ q > 2$	$2 \leq d \leq \frac{(s+1)(q+1)}{2s}$	[28]
	$2s\|(q+1),\ 2 \leq r \leq 2s$		
42	$\frac{(2t+2)(q^2-1)}{2s} + 1,\ 2s\|(q+1),\ 0 \leq t \leq s-2$	$2 \leq d \leq \frac{(s+t+1)(q+1)}{2s}$	[28]
43	$\frac{q^2-1}{4},\ q \equiv 3 \bmod 4$	$2 \leq d \leq \frac{3q-1}{4}$	[20]
44	$\frac{2(r_1+1)(r_2+1)(q^2-1)}{h},\ h = h_1h_2 \geq 9$	$2 \leq d \leq min\{\frac{(2r_1-1)(q-1)}{h_1}, \frac{2r_2(q+1)}{h_2}\} + 2$	[27]
	$h_1\|(q-1),\ h_2\|(q+1),\ \frac{q+1}{h_2}$ odd		
	$1 \leq r_1 \leq \frac{h_1+1}{2},\ 0 \leq r_2 \leq \frac{h_2-3}{2}$		

Table 1 continued

Class	Length n	Distance d	References
45	$\frac{(r_1+1)(r_2+1)(q^2-1)}{h},\ q \equiv 3 \bmod 4,\ h = h_1h_2 \geq 9$	$2 \leq d \leq min\{\frac{(2r_1-1)(q-1)}{2h_1}, \frac{2r_2(q+1)}{h_2}\} + 2$	[27]
	$h_1 \mid \frac{q-1}{2},\ h_2 \mid (q+1),\ \frac{q+1}{h_2}$ odd		
	$1 \leq r_1 \leq h_1 - 1,\ 0 \leq r_2 \leq \frac{h_2-2}{2}$		
46	$\frac{(r_1+1)r_2(q^2-1)}{h},\ q \equiv 1 \bmod 4,\ h = h_1h_2 \geq 6$	$2 \leq d \leq max\{\frac{(r_1-1)(q-1)}{h_1}, \frac{(r_2-1)(q+1)}{2h_2}\} + 2$	[27]
	$h_1 \mid (q-1),\ h_2 \mid \frac{q+1}{2}$		
	$1 \leq r_1 \leq h_1 - 1,\ 1 \leq r_2 \leq \frac{h_2-1}{2}$		
47	$\frac{2(q^2-1)}{m}$, q odd, m odd, $m \mid (q+1),\ (m-1) \mid (q-1)$	$2 \leq d \leq \frac{q-1}{2} + \frac{q+1}{2m}$	[22]
48	$\frac{(m_1+m_2-1)(q^2-1)}{2m_1m_2},\ m_1 = 2k_1+1 < m_2 = 2k_2+1$	$2 \leq d \leq \frac{q+1}{2} + \frac{q-1}{2(2k_2+1)}$	[22]
	$gcd(m_1, m_2) = 1$		
49	$\frac{r(q^2-1)}{2t+1},\ t \geq 1,\ 1 \leq r \leq 2t+1$	$2 \leq d \leq \frac{(q+1)(t+1)}{2t+1}$	[25]
	$gcd(r, q) > 1,\ (2t+1) \mid (q+1)$		
50	$\frac{r(q^2-1)}{s},\ 1 \leq r \leq s-1,\ s \mid (q+1)$, s even	$2 \leq d \leq \frac{r(q+1)}{s} - 1$	[36]
51	$\frac{r(q^2-1)}{s+1} + 1,\ 1 \leq r \leq s,\ (s+1) \mid (q+1)$, s even	$2 \leq d \leq \frac{r(q+1)}{s+1}$	[36]
52	$\frac{r(q^2-1)}{s} + \frac{l(q^2-1)}{t} - \frac{2rl(q^2-1)}{st}$, s even, t even	$2 \leq d \leq min\{\frac{r(q+1)}{s} - 1, \frac{q+1}{2} + \frac{q-1}{t} - 1\}$	[36]
	$s \mid (q+1),\ t \mid (q-1),\ st > 2(q+1)$		
	$1 \leq r \leq s-1,\ t \geq 2,\ 1 \leq l \leq t$		

Table 1 continued

Class	Length n	Distance d	References
53	$\frac{r(q^2-1)}{s+1}+\frac{l(q^2-1)}{t}-\frac{rl(q^2-1)}{(s+1)t}+1,\ s$ even, t even $(s+1)\|(q+1),\ t\|(q-1), (s+1)t>q+1$ $1\le r\le s,\ t\ge 2,\ 1\le l\le t$	$2\le d\le min\{\frac{r(q+1)}{s+1}, \frac{q+1}{2}+\frac{q-1}{t}-1\}$	[36]
54	$\frac{q^2-1}{4}+\frac{q^2-1}{h},\ h,\tau>0,\ \frac{2(q-1)}{h}=2\tau+1$	$2\le d\le \frac{q-1}{2}+\tau$	[32]
55	$\frac{q^2-1}{4}+\frac{2(q^2-1)}{h},\ h,\tau>0,\ h\ne 4$ $\frac{2(q-1)}{h}=2\tau+1,\ q\equiv 3 \bmod 4$	$2\le d\le \frac{q-1}{2}+2\tau+1$	[32]
56	$1+\frac{h(q^2-1)}{s}+\frac{r(q^2-1)}{t}-\frac{hr(q^2-1)}{st},\ s$ odd, t even $s\|(q+1),\ t\|(q-1)$, odd $h\le s-1$ $t\ge 2,\ r\le t,\ q-1\ge \frac{q^2-1}{st}hr$	$2\le d\le min\{\frac{s+h}{2}\frac{q+1}{s}, \frac{q+1}{2}+\frac{q-1}{t}\}$	[33]
57	$\frac{h(q^2-1)}{s}+\frac{r(q^2-1)}{t}-\frac{hr(q^2-1)}{st},\ s$ odd, t even $s\|(q+1),\ t\|(q-1),\ h\le s-1$ $t\ge 2,\ r\le t,\ q-1\ge \frac{q^2-1}{st}hr$	$2\le d\le min\{\lfloor\frac{s+h}{2}\rfloor\frac{q+1}{s}-1, \frac{q+1}{2}+\frac{q-1}{t}\}$	[33]
58	$\frac{h(q^2-1)}{s}+\frac{r(q^2-1)}{t},\ s$ even, t even $s\|(q+1),\ t\|(q-1)$ $t\ge 2,\ h\le\frac{s}{2},\ r\le\frac{t}{2}$	$2\le d\le min\{\lfloor\frac{s+h}{2}\rfloor\frac{q+1}{s}-1, \frac{q+1}{2}+\frac{q-1}{t}\}$	[33]
59	$bm(q+1),\ m\|\frac{q-1}{2},\ bm\le q-1$	$2\le d\le \frac{q+1}{2}+m$	[34]
60	$bm(q-1),\ m\|\frac{q+1}{2},\ bm\le q+1$	$2\le d\le \frac{q-1}{2}+m$	[34]
61	$c(q\pm 1),\ q=2am\pm 1,\ gcd(a,m)=1$ $1\le c\le 2(a+m-1)$	$2\le d\le \frac{q\pm 1}{2}+c_1$ $c_1=c$, if $1\le c\le a+m-1$ $c_1=\lfloor\frac{c}{2}\rfloor$, if $a+m\le c\le 2(a+m-1)$	[34]

Table 1 continued

Class	Length n	Distance d	References
62	$(bm + c(m-1))(q+1),\ m\|\frac{q-1}{2}$ $(b+c)m \le q-1,\ b, c \ge 0,\ b \ge 1$ or $m \ge 2$	$2 \le d \le \frac{q-1}{2} + m$	[34]
63	$(bm + c(m-1))(q-1),\ m\|\frac{q+1}{2}$ $(b+c)m \le q+1,\ b, c \ge 0,\ b \ge 1$ or $m \ge 2$	$2 \le d \le \frac{q-3}{2} + m$	[34]
64	$(c_1(2m-1) + (c_2+c_3)m)(q-1),\ m\|\frac{q+1}{2}$ $0 \le c_1 + c_2 \le \frac{q+1}{2m},\ 0 \le c_1 + c_3 \le \frac{q+1}{2m}$ $c_1, c_2, c_3 \ge 0,\ c_1 + c_2 + c_3 \ge 1$	$2 \le d \le \frac{q-1}{2} + m$	[34]

Table 2 Quantum codes $((n, K, d))_q$

Class	Length n	Dimension K	Distance d	References
1	$2t - l \le n \le q - l + 1,\ t \ge l \ge 1$	q^l	$t - l + 1$	[45]
2	$t,\ t \ge 1$	q^t	1	[45]
3	$q + 2,\ q = 2^m,\ m > 1$	1	4	[45]
4	$t + 1$, odd $t \ge 2,\ q = 2$	$1 \le K \le 2^{t-1}$	2	[44]
5	$n \equiv 1 \bmod 4,\ q = 2$	$K = 1 + C_N^2 + C_N^4 + \cdots + C_N^{\frac{N-5}{2}} + C_{N-1}^{\frac{N-3}{2}}$	2	[44]
6	$n \equiv 3 \bmod 4,\ q = 2$	$K = 1 + C_N^2 + C_N^4 + \cdots + C_N^{\frac{N-3}{2}}$	2	[44]
7	$3p,\ 2^{l-1} \le p \le 2^l - 1,\ l \ge 3,\ q = 2$	2^{p-l}	3	[44]
8	$4p,\ 2^{l-1} \le p \le 2^l - 1,\ l \ge 3,\ q = 2$	2^{p-l+1}	3	[44]
9	$4p,\ 2^{l-2} + 1 \le p \le 2^{l-1},\ l \ge 4,\ q = 2$	2^{p-l+1}	4	[44]
10	$2(m_d + 1)(d - 1),\ q = 2$	1	$d \ge 5$	[44]
	$2^{m_d - 1} + 2 \le d \le 2^{m_d} + 1$			
11	$2(m_d + 1)(d - 1) \le n \le 2d(m_d + 1) - 1$	1	$d \ge 5$	[44]
	$2^{m_d - 1} + 2 \le d \le 2^{m_d},\ q = 2$			
12	$n \ge 2,\ q = 4$	$1 \le K \le 4^{n-2}$	2	[47]
13	$3p,\ l \ge 3,\ q = 4$	4^{p-l+1}	3	[47]
	$\frac{4^{l-1}+2}{3} \le p \le \frac{4^l - 1}{3}$			
14	$2m(d - 1) \le n \le (4^m + 1)m,\ q = 4$	1	$\frac{4^{m-1}+3}{2} < d \le \frac{4^m+3}{2}$	[47]

Definition 2.1 ([49] Constacyclic Code) A q^2-ary linear code C of length n is said to be constacyclic if C is closed under the η-constacyclic shift τ_η on $F_{q^2}^n$

$$\tau_\eta(a_0, a_1, \ldots, a_{n-1}) = (\eta a_{n-1}, a_0, \ldots, a_{n-2}),$$

where η is a nonzero element of F_{q^2}. In particular, if $\eta = -1$, then C is said to be negacyclic.

Let ω be a primitive element of F_{q^2}. Assume that $gcd(n, q) = 1$ and $\eta = \omega^{v(q-1)}$ for some $v \in \{0, 1, \ldots, q\}$. Then, $C^{\perp h}$ of an η-constacyclic code C over F_{q^2} is also η-constacyclic. And there exists a unique monic divisor $g(x)$ of $x^n - \eta$ such that $C = \langle g(x)\rangle$, where $\langle g(x)\rangle = \{r(x)g(x)|r(x) \in F_{q^2}[x]/\langle x^n - \eta\rangle\}$. The polynomial $g(x)$ is called the generator polynomial of C.

Let r be the order of η in $F_{q^2}^* = F_{q^2} - \{0\}$ and δ be a primitive rn-th root of unity in some extension field of F_{q^2} such that $\delta^n = \eta$. Then, the roots of $x^n - \eta$ are δ^{1+rj}, $0 \leq j \leq n-1$. Denote $\Omega = \{1 + rj|0 \leq j \leq n-1\}$. For $i \in \Omega$, let $C_i = \{iq^{2j} \bmod rn,\ j \in N\}$ be the q^2-cyclotomic coset modulo rn containing i. The set $Z = \{z \in \Omega|g(\delta^z) = 0\}$ is called the defining set of C. Let $g(x) = \prod_{z\in Z}(x - \delta^z)$ be the generator polynomial of C. Then, $C^{\perp h}$ has generator polynomial $g^{\perp h}(x) = \prod_{z\in\Omega\setminus Z}(x - \delta^{-qz})$. Hence, $C^{\perp h}$ has defining set $Z^{\perp h} = \{-qz \bmod rn|z \in \Omega\setminus Z\}$.

Lemma 2.1 ([49] The BCH Bound for Constacyclic Codes) *Assume that $gcd(n, q) = 1$. Let C be an η-constacyclic code of length n over F_{q^2}, and let the generator polynomial $g(x)$ have the elements $\{\delta^{1+rj}|0 \leq j \leq d-2\}$ as the roots, where δ is a primitive rn-th root of unity. Then, the minimum distance of C is at least d.*

Lemma 2.2 ([50] Hermitian Construction) *If C is a q^2-ary $[n, k, d]$ linear code such that $C^{\perp h} \subseteq C$, then there exists a q-ary $[[n, 2k - n, \geq d]]$ quantum code.*

Lemma 2.3 [18, 48] *Let C be an η-constacyclic code of length n over F_{q^2} with defining set $Z \subseteq \Omega$. Then, $C^{\perp h} \subseteq C$ if and only if $Z \bigcap Z^{-q} = \emptyset$, where $Z^{-q} = -qZ = \{-qz \bmod rn|z \in Z\}$.*

Let $\lfloor x\rfloor$ denote the largest integer not exceeding x. To construct new quantum codes, we give the following lemmas.

Lemma 2.4 *Let $x, y, m \geq 3$ be odd, $x|(q-1)$, $y|(q+1)$, and let $n = 2xy\frac{q^{2m}-1}{q^2-1}$. Then $gcd(n, q) = 1$.*

Proof Note that $n = 2xy\frac{q^{2m}-1}{q^2-1} = 2xy(q^{2(m-1)} + q^{2(m-2)} + \cdots + q^2 + 1)$, so $gcd(n, q) = gcd(2xy, q)$. Since $2x|(q-1)$, we can assume that $q - 1 = 2xm$, i.e. $q = 2xm + 1$, $m \geq 1$. Then, we have $gcd(2xy, q) = gcd(2xy, 2xm + 1) = gcd(y, 2xm + 1) = gcd(y, q)$. Note that $2y|(q + 1)$, we can assume that $q + 1 = 2yl$, i.e. $q = 2yl - 1$, $l \geq 1$. It follows that $gcd(y, q) = gcd(y, 2yl - 1) = gcd(y, 1) = 1$. Thus, we can conclude that $gcd(n, q) = 1$. □

Lemma 2.5 *Under the conditions of Lemma 2.4, let* $\zeta = y\frac{q^{2m}-1}{q^2-1}$. *Then, for integers* $i, j,\ 1 \le i, j \le y\lfloor\frac{q^m-1}{q^2-1}\rfloor$, *we have the following results.*
(1) *The* q^2*-cyclotomic coset* C_ζ *modulo* $2n$ *is* $C_\zeta = \{\zeta\}$;
(2) *The* q^2*-cyclotomic coset* $C_{\zeta-2i}$ *and* $C_{\zeta+2i}$ *modulo* $2n$ *have cardinality* m;
(3) $C_{\zeta-2i} = C_{\zeta+2j}$ *if and only if there exists* $t \in [0, \frac{m-1}{2}]$ *such that* $i + jq^{2t} \equiv 0 \bmod n$ *or* $j + iq^{2t} \equiv 0 \bmod n$;
(4) *If* $i < j$, *then* $C_{\zeta-2i} = C_{\zeta-2j}$ *if and only if* $j = iq^{2t}$ *for some* $t \in [1, \frac{m-1}{2}]$;
(5) *If* $i < j$, *then* $C_{\zeta+2i} = C_{\zeta+2j}$ *if and only if* $j = iq^{2t}$ *for some* $t \in [1, \frac{m-1}{2}]$;
(6) $C_{\zeta-2i} = -qC_{\zeta+2j}$ *if and only if there exists* $t \in [0, \frac{m-1}{2}]$ *such that* $\zeta - 2i \equiv -(\zeta+2j)q^{2t+1} \bmod 2n$ *or* $\zeta + 2j \equiv -(\zeta - 2i)q^{2t+1} \bmod 2n$;
(7) $C_{\zeta-2i} = -qC_{\zeta-2j}$ *if and only if there exists* $t \in [0, \frac{m-1}{2}]$ *such that* $\zeta - 2i \equiv -(\zeta-2j)q^{2t+1} \bmod 2n$ *or* $\zeta - 2j \equiv -(\zeta - 2i)q^{2t+1} \bmod 2n$;
(8) $C_{\zeta+2i} = -qC_{\zeta+2j}$ *if and only if there exists* $t \in [0, \frac{m-1}{2}]$ *such that* $\zeta + 2i \equiv -(\zeta+2j)q^{2t+1} \bmod 2n$ *or* $\zeta + 2j \equiv -(\zeta + 2i)q^{2t+1} \bmod 2n$;
(9) $C_\zeta \ne -qC_\zeta$;
(10) $C_{\zeta-2i} \ne C_{\zeta+2j}$.

Proof Here, we only prove (1), (2), (9) and (10). Other proofs are similar to that of Lemma 3 and Lemma 5 of [51], so we omit.
(1) Obviously, $2n | (\zeta(q^2-1))$. Thus, we have $C_\zeta = \{\zeta\}$.
(2) Since $2n|(q^{2m}-1)$, we have $(\zeta-2i)q^{2m} \equiv \zeta - 2i \bmod 2n$. Suppose that $|C_{\zeta-2i}| = t < m$. Then, we have $(\zeta-2i)q^{2t} \equiv \zeta - 2i \bmod 2n$. Because $m \ge 3$ is odd and $t|m$, we can get $1 \le t \le \frac{m}{3} \le \frac{m-1}{2}$. By $\zeta q^2 \equiv \zeta \bmod 2n$, we have $i(q^{2t}-1) \equiv 0 \bmod n$. It follows from $1 \le i, j \le y\lfloor\frac{q^m-1}{q^2-1}\rfloor$ and $1 \le t \le \frac{m-1}{2}$ that $0 < q^2 - 1 \le i(q^{2t}-1) \le y\lfloor\frac{q^m-1}{q^2-1}\rfloor(q^{m-1}-1) < y\frac{q^m-1}{q^2-1}(q^m+1) < n$. This contradicts $i(q^{2t}-1) \equiv 0 \bmod n$. Therefore, $|C_{\zeta-2i}| = m$. Similarly, we can also get $|C_{\zeta+2i}| = m$.
(9) Suppose that $C_\zeta = -qC_\zeta$. Then, we have $\zeta \equiv -q\zeta \bmod 2n$, i.e. $\zeta(q+1) \equiv 0 \bmod 2n$. This shows that $2n|(\zeta(q+1))$, which is equivalent to $4x|(q+1)$. It contradicts the fact that $2x|(q-1)$. Hence, $C_\zeta \ne -qC_\zeta$.
(10) Suppose that $C_{\zeta-2i} = C_{\zeta+2j}$. Then, by (3), we can assume that there exists $t \in [0, \frac{m-1}{2}]$ such that $i + jq^{2t} \equiv 0 \bmod n$. Note that $0 < i + jq^{2t} \le y\lfloor\frac{q^m-1}{q^2-1}\rfloor(q^{m-1}+1) < y\frac{q^m-1}{q^2-1}(q^m+1) < n$. This yields a contradiction. Hence, $C_{\zeta-2i} \ne C_{\zeta+2j}$. □

Lemma 2.6 *For* q *with the form* $2am \pm t$, *where* a, m, t *are positive integers, let* $n = \frac{q^2+1}{a}$ *be an integer. Then* $gcd(n, q) = 1$.

Proof Here, we only prove the case of $q = 2am + t$, the case of $q = 2am - t$ is similar. Since $q^2 + 1 = (2am+t)^2 + 1 = 4a^2m^2 + 4amt + t^2 + 1$, it follows that $n = \frac{q^2+1}{a} = 4am^2 + 4mt + \frac{t^2+1}{a}$. Thus, $gcd(n, q) = gcd(4am^2 + 4mt + \frac{t^2+1}{a}, 2am + t) = gcd(2m(2am+t) + 2mt + \frac{t^2+1}{a}, 2am+t) = gcd(2mt + \frac{t^2+1}{a}, 2am+t)$. Suppose that $gcd(2mt + \frac{t^2+1}{a}, 2am+t) = s$ and $s \ne 1$. Let $2mt + \frac{t^2+1}{a} = sx$

and $2am + t = sy$, where x, y are integers. Then, we can obtain $s(ax - ty) = 1$. Obviously, this is impossible. Hence $gcd(n, q) = 1$. □

Lemma 2.7 *[30] For q with the form $2am \pm t$, where m is a positive integer and a, t are odd, let $n = \frac{q^2+1}{a}$, $s = \frac{q^2+1}{2}$. Then, for any integer $i \in \Omega = \{1 + (q+1)j | 0 \leq j \leq n-1\}$, the q^2-cyclotomic coset C_i modulo $(q+1)n$ is given by*
(1) $C_s = \{s\}$, $C_{s+\frac{n(q+1)}{2}} = \{s + \frac{n(q+1)}{2}\}$.
(2) $C_{s-(q+1)j} = \{s - (q+1)j, s + (q+1)j\}$, $1 \leq j \leq \frac{n}{2} - 1$.

3 New quantum codes from constacyclic codes

3.1 Length $n = 2xy\frac{q^{2m}-1}{q^2-1}$

Based on the lemmas in Sect. 2, we can give a sufficient condition for the existence of negacyclic codes over F_{q^2} of length $2xy\frac{q^{2m}-1}{q^2-1}$ which contain their Hermitian duals.

Lemma 3.1 *Let $q \equiv 1 \bmod 4$, $x, y, m \geq 3$ be odd, $x|(q-1)$, $y|(q+1)$, and let $n = 2xy\frac{q^{2m}-1}{q^2-1}$, $\zeta = y\frac{q^{2m}-1}{q^2-1}$. If C is a q^2-ary negacyclic code of length n with defining set $Z = \bigcup_{i=-\delta_1}^{\delta_2} C_{\zeta+2i}$, where $0 \leq \delta_1, \delta_2 \leq y\lfloor\frac{q^m-1}{q^2-1}\rfloor$, then $C^{\perp h} \subseteq C$.*

Proof By Lemma 2.3, we only need to prove that $Z \bigcap Z^{-q} = \emptyset$. Suppose that $Z \bigcap Z^{-q} \neq \emptyset$. By Lemma 2.5, we have $C_\zeta \bigcap -qC_\zeta = \emptyset$. Then, for integers i, j, $1 \leq i, j \leq y\lfloor\frac{q^m-1}{q^2-1}\rfloor$, we can obtain a contradiction by considering the following three cases:
Case 1 $C_{\zeta-2i} = -qC_{\zeta-2j}$, which means that there exists $t \in [0, \frac{m-1}{2}]$ such that $\zeta - 2i \equiv -(\zeta - 2j)q^{2t+1} \bmod 2n$. By $q \equiv 1 \bmod 4$ and $x|(q-1)$, we can get $2n|(\zeta(q-1))$, i.e. $q\zeta \equiv \zeta \bmod 2n$. Thus, we have $\zeta - i - jq^{2t+1} \equiv 0 \bmod n$. It follows from $1 \leq i, j \leq y\lfloor\frac{q^m-1}{q^2-1}\rfloor$ that $0 < y\frac{q^{2m}-1}{q^2-1} - (q^m+1)y\lfloor\frac{q^m-1}{q^2-1}\rfloor \leq \zeta - i - jq^{2t+1} \leq y\frac{q^{2m}-1}{q^2-1} - (q+1) < n$. This contradicts the fact that $\zeta - i - jq^{2t+1} \equiv 0 \bmod n$.
Case 2 $C_{\zeta+2i} = -qC_{\zeta+2j}$, which means that there exists $t \in [0, \frac{m-1}{2}]$ such that $\zeta + 2i \equiv -(\zeta + 2j)q^{2t+1} \bmod 2n$. This is equivalent to $\zeta + i + jq^{2t+1} \equiv 0 \bmod n$. Note that $0 < \zeta + i + jq^{2t+1} \leq y\frac{q^{2m}-1}{q^2-1} + (q^m+1)y\lfloor\frac{q^m-1}{q^2-1}\rfloor < 2y\frac{q^{2m}-1}{q^2-1} < n$. This gives a contradiction.
Case 3 $C_{\zeta-2i} = -qC_{\zeta+2j}$, which means that there exists $t \in [0, \frac{m-1}{2}]$ such that $\zeta - 2i \equiv -(\zeta + 2j)q^{2t+1} \bmod 2n$. This is equivalent to $\zeta - i + jq^{2t+1} \equiv 0 \bmod n$. Since $1 \leq i, j \leq y\lfloor\frac{q^m-1}{q^2-1}\rfloor$, it follows that $0 < y\frac{q^{2m}-1}{q^2-1} + q - y\lfloor\frac{q^m-1}{q^2-1}\rfloor \leq \zeta - i + jq^{2t+1} \leq y\frac{q^{2m}-1}{q^2-1} - 1 + y\lfloor\frac{q^m-1}{q^2-1}\rfloor q^m < y\frac{q^{2m}-1}{q^2-1} - 1 + y\frac{q^m-1}{q^2-1}(q^m+1) = 2y\frac{q^{2m}-1}{q^2-1} - 1 < n$. It contradicts the fact that $\zeta - i + jq^{2t+1} \equiv 0 \bmod n$.

Therefore, $Z \bigcap Z^{-q} = \emptyset$, i.e. $C^{\perp h} \subseteq C$. □

Using the aforementioned lemma, some new q-ary quantum codes of length $2xy\frac{q^{2m}-1}{q^2-1}$ can be constructed.

Theorem 3.1 *Under the conditions of Lemma 3.1, there exist quantum codes with parameters* $[[n, n-2m(\delta_1 - \lfloor\frac{\delta_1}{q^2}\rfloor + \delta_2 - \lfloor\frac{\delta_2}{q^2}\rfloor) - 2, \geq \delta_1 + \delta_2 + 2]]_q$.

Proof Consider the negacyclic code C over F_{q^2} of length n with defining set $Z = \bigcup_{i=-\delta_1}^{\delta_2} C_{\zeta+2i}$, where $0 \leq \delta_1, \delta_2 \leq y\lfloor\frac{q^m-1}{q^2-1}\rfloor$. By Lemma 2.1, we have $d(C) \geq \delta_1 + \delta_2 + 2$. By Lemma 2.5, for $1 \leq i \leq y\lfloor\frac{q^m-1}{q^2-1}\rfloor$, we have $C_{\zeta-2i} = C_{\zeta-2iq^{2t}}$ and $C_{\zeta+2i} = C_{\zeta+2iq^{2t}}$ for some $t \in [1, \frac{m-1}{2}]$. Hence, the number of cosets is reduced by $\lfloor\frac{\delta_1}{q^2}\rfloor + \lfloor\frac{\delta_2}{q^2}\rfloor$. Therefore, C is a negacyclic code over F_{q^2} with parameters $[n, n-m(\delta_1 - \lfloor\frac{\delta_1}{q^2}\rfloor + \delta_2 - \lfloor\frac{\delta_2}{q^2}\rfloor) - 1, \geq \delta_1 + \delta_2 + 2]$. By Lemma 3.1, $C^{\perp h} \subseteq C$. Applying the Hermitian construction to C obtains q-ary $[[n, n-2m(\delta_1 - \lfloor\frac{\delta_1}{q^2}\rfloor + \delta_2 - \lfloor\frac{\delta_2}{q^2}\rfloor) - 2, \geq \delta_1 + \delta_2 + 2]]$ quantum codes. □

We list some new quantum codes in Table 3. When the distance is equal, the dimension of quantum codes we construct is better than those in [52]. We give Theorem 21 of [52] as follows.

Theorem 3.2 *([53], [52, Theorem 21]) Let* $n = r\frac{q^{2m}-1}{q^2-1}$*, where* $m \geq 2$ *and* q *is a prime power. For* $2 \leq \delta \leq \lfloor r\frac{q^m-1}{q^2-1}\rfloor$*, then there exists a quantum code with parameters* $[[n, n-2m\lceil(\delta-1)(1-\frac{1}{q^2})\rceil, \geq \delta]]_q$.

3.2 Length $n = \frac{q^2+1}{a}$ with odd a

In this section, for q with the form $2am \pm t$, we will use ω^{q-1}-constacyclic codes over F_{q^2} to construct some q-ary quantum MDS codes of length $\frac{q^2+1}{a}$.

Lemma 3.2 (1) *For* q *with the form* $178m + 55$*, where* m *is a positive integer, let* $s = \frac{q^2+1}{2}$, $n = \frac{q^2+1}{89}$*. If* C *is an* ω^{q-1}*-constacyclic code over* F_{q^2} *of length* n *with defining set* $Z = \bigcup_{j=0}^{\delta} C_{s-(q+1)j}$*, where* $0 \leq \delta \leq 13m + 3$*, then* $C^{\perp h} \subseteq C$*;*
(2) *For* q *with the form* $178m - 55$*, where* m *is a positive integer, let* $s = \frac{q^2+1}{2}$, $n = \frac{q^2+1}{89}$*. If* C *is an* ω^{q-1}*-constacyclic code over* F_{q^2} *of length* n *with defining set* $Z = \bigcup_{j=0}^{\delta} C_{s-(q+1)j}$*, where* $0 \leq \delta \leq 13m - 5$*, then* $C^{\perp h} \subseteq C$.

Proof (1) By Lemma 2.3, we only need to prove that $Z \bigcap Z^{-q} = \emptyset$. Suppose that $Z \bigcap Z^{-q} \neq \emptyset$. Then, by Lemma 2.7, there exist two integers i, j, $0 \leq i, j \leq 13m+3$, such that $s - (q+1)i \equiv -[s-(q+1)j]q^{\epsilon} \bmod (q+1)n$ for $\epsilon = 1, 3$.

If $\epsilon = 1$, then $s - (q+1)i \equiv -[s-(q+1)j]q \bmod (q+1)n$. This is equivalent to $s \equiv jq + i \bmod n$, which means

$$q^2 + 1 \equiv 178jq + 178i \bmod 2(q^2+1).$$

As $0 \leq i, j \leq 13m + 3 = \frac{13q-181}{178}$, it follows that $0 \leq 178i, 178j \leq 13q - 181$. We can obtain a contradiction by considering the following two cases:

Table 3 Comparisons of quantum codes with length $n = r\frac{q^{2m}-1}{q^2-1}$

(q, m, r)	δ_1, δ_2	δ	$[[n, k, d]]_q$ from Theorem 3.1	$[[n, k, d]]_q$ in [52]
(29,3,42)	$0 \le \delta_1, \delta_2 \le 87$	$2 \le \delta \le 176$	$[[n, n-6\delta+10, \ge \delta]]_{29}$	$[[n, n-6\delta+6, \ge \delta]]_{29}$
(29,5,42)	$0 \le \delta_1, \delta_2 \le 840$	$2 \le \delta \le 841$	$[[n, n-10\delta+18, \ge \delta]]_{29}$	$[[n, n-10\delta+10, \ge \delta]]_{29}$
	$841 \le \delta_1, \delta_2 \le 1681$	$1684 \le \delta \le 2523$	$[[n, n-10\delta+38, \ge \delta]]_{29}$	$[[n, n-10\delta+30, \ge \delta]]_{29}$
	$1682 \le \delta_1, \delta_2 \le 2522$	$3366 \le \delta \le 4205$	$[[n, n-10\delta+58, \ge \delta]]_{29}$	$[[n, n-10\delta+50, \ge \delta]]_{29}$
	$2523 \le \delta_1, \delta_2 \le 3363$	$5048 \le \delta \le 5887$	$[[n, n-10\delta+78, \ge \delta]]_{29}$	$[[n, n-10\delta+70, \ge \delta]]_{29}$
	$3364 \le \delta_1, \delta_2 \le 4204$	$6730 \le \delta \le 7569$	$[[n, n-10\delta+98, \ge \delta]]_{29}$	$[[n, n-10\delta+90, \ge \delta]]_{29}$
	$4205 \le \delta_1, \delta_2 \le 5045$	$8412 \le \delta \le 9251$	$[[n, n-10\delta+118, \ge \delta]]_{29}$	$[[n, n-10\delta+110, \ge \delta]]_{29}$
	...	...	...	...

(i) $0 \le 178i \le 12q-1$. Then, $0 \le 178jq+178i \le (13q-181)q+12q-1 = 13q^2-169q-1 < 13q^2$. Assume that $178i = eq+u$, where $0 \le e \le 11$ and $0 \le u \le q-1$ are integers. Then, by $q^2+1 \equiv 178jq+178i \bmod 2(q^2+1)$, we have $178jq+178i = (178j+e)q+u = h(q^2+1) = hq^2+h$, where $1 \le h \le 11$ is odd. This implies that $q|(u-h)$. Since $-q < -h \le u-h \le q-1-h < q$, we have $u-h=0$, i.e. $u=h$. It follows that $178j+e=hq$, where $1 \le e \le 11$ is odd. Then, $j = \frac{hq-e}{178} = \frac{h(178m+55)-e}{178} = hm + \frac{55h-e}{178}$. Obviously, when $1 \le e, h \le 11$ are odd, j is not an integer. This gives a contradiction.
(ii) $12q \le 178i \le 13q-181$. Then, $12q \le 178jq+178i \le (13q-181)(q+1) = 13q^2-168q-181 < 13q^2$. Assume that $178i = 12q+u$, where $0 \le u \le q-181$. Then, we have $178jq+178i = (178j+12)q+u = h(q^2+1) = hq^2+h$, where $1 \le h \le 11$ is odd. Hence $q|(u-h)$. Similar to (i), we can get $u=h$. Thus $178j+12=hq$. Note that $178j+12$ is even and hq is odd. This gives a contradiction.

If $\epsilon = 3$, then $s-(q+1)i \equiv -[s-(q+1)j]q^3 \bmod (q+1)n$. This is equivalent to $s-(q+1)i \equiv -sq-(q+1)qj \bmod (q+1)n$, which means

$$178jq+q^2+1 \equiv 178i \bmod 2(q^2+1).$$

As $0 \le 178i, 178j \le 13q-181$, it follows that $q^2+1 \le 178jq+q^2+1 \le (13q-181)q+q^2+1 = 14q^2-181q+1$. We can obtain a contradiction by considering the following two cases:
(i) $q^2+1 \le 178jq+q^2+1 \le 2q^2+1$. Then, $178jq+q^2+1 = 178i$. It follows that $q|(178i-1)$. Note that $-1 \le 178i-1 \le 13q-181-1 < 13q$. Hence, we can assume $178i-1 = hq$, where $1 \le h \le 11$ is odd. Then, $178i = hq+1 = h(178m+55)+1 = 178mh+55h+1$, which implies that $178|(55h+1)$. Assume that $55h+1 = 178p = 2\frac{55^2+1}{34}p,\ p \ge 1$. This is equivalent to $34 \cdot 55h+34 = 2 \cdot 55^2p+2p$, which means $55|(2p-34)$. Note that $2p-34 > -55$, so we can assume that $2p-34 = 55c,\ c \ge 0$. Then, $2p = 55c+34 \ge 34$, i.e. $p \ge 17$. So $h = \frac{178p-1}{55} \ge 55$. It contradicts the fact that $1 \le h \le 11$.
(ii) $2(q^2+1) \le 178jq+q^2+1 \le 14q^2-181q+1 < 14q^2$. Then, we have $178jq+q^2+1-178i = h(q^2+1)$, where $2 \le h \le 12$ is even. It follows that $178jq-(h-1)q^2 = 178i+h-1$. Obviously, $q|(178i+h-1)$. Note that $1 \le 178i+h-1 \le 13q-170 < 13q$, so we can assume that $178i+h-1 = h'q$, where $1 \le h' \le 11$ is odd. Then, $i = \frac{h'q-(h-1)}{178} = \frac{h'(178m+55)-(h-1)}{178} = h'm + \frac{55h'-(h-1)}{178}$. Similar to the case of $\epsilon = 1$, we can also get a contradiction here.
Therefore, $Z \bigcap Z^{-q} = \emptyset$, i.e. $C^{\perp h} \subseteq C$.
(2) It is similar to the proof of (1). □

Now, we can construct some q-ary quantum MDS codes by using the above lemma.

Theorem 3.3 *For q with the form $178m+55$ $(178m-55)$, where m is a positive integer, let $n = \frac{q^2+1}{89}$. Then, there exist quantum MDS codes with parameters $[[n, n-2d+2, d]]_q$, where $2 \le d \le 26m+8$ $(2 \le d \le 26m-8)$ is even.*

Table 4 Some quantum MDS codes for $m \leq 9$

m	q	n	d
1	233	610	$2 \leq d \leq 34,\ d$ even
3	479	2578	$2 \leq d \leq 70,\ d$ even
6	1013	11,530	$2 \leq d \leq 148,\ d$ even
6	1123	14,170	$2 \leq d \leq 164,\ d$ even
7	1301	19,018	$2 \leq d \leq 190,\ d$ even
9	1657	30,850	$2 \leq d \leq 242,\ d$ even

Proof Suppose that $q = 178m + 55$. Let $s = \frac{q^2+1}{2}$. Consider the ω^{q-1}-constacyclic code C over F_{q^2} of length n with defining set $Z = \bigcup_{j=0}^{\delta} C_{s-(q+1)j}$, where $0 \leq \delta \leq 13m + 3$. By Lemma 2.1 and Singleton bound, $d(C) = 2\delta + 2$. Hence, C is an ω^{q-1}-constacyclic code over F_{q^2} with parameters $[n, n - (2\delta + 1), 2\delta + 2]$. By Lemma 3.2, $C^{\perp h} \subseteq C$. Applying the Hermitian construction and quantum Singleton bound to C obtains a q-ary quantum MDS code with parameters $[[n, n - 4\delta - 2, 2\delta + 2]]$. The desired quantum MDS code follows. The case $q = 178m - 55$ is similar. □

Some quantum MDS codes obtained from Theorem 3.3 are listed in Table 4.

Lemma 3.3 (1) *For q with the form $250m + 57$, where m is a positive integer, let $s = \frac{q^2+1}{2}$, $n = \frac{q^2+1}{125}$. If C is an ω^{q-1}-constacyclic code over F_{q^2} of length n with defining set $Z = \bigcup_{j=0}^{\delta} C_{s-(q+1)j}$, where $0 \leq \delta \leq 13m + 2$, then $C^{\perp h} \subseteq C$;*
(2) *For q with the form $250m - 57$, where m is a positive integer, let $s = \frac{q^2+1}{2}$, $n = \frac{q^2+1}{125}$. If C is an ω^{q-1}-constacyclic code over F_{q^2} of length n with defining set $Z = \bigcup_{j=0}^{\delta} C_{s-(q+1)j}$, where $0 \leq \delta \leq 13m - 4$, then $C^{\perp h} \subseteq C$.*

Proof (1) Suppose that $Z \bigcap Z^{-q} \neq \emptyset$. Then, by Lemma 2.7, there exist two integers $i,\ j,\ 0 \leq i,\ j \leq 13m + 2$, such that $s - (q+1)i \equiv -[s - (q+1)j]q^{\epsilon} \bmod (q+1)n$ for $\epsilon = 1, 3$.

If $\epsilon = 1$, then $s - (q+1)i \equiv -[s - (q+1)j]q \bmod (q+1)n$. This is equivalent to

$$q^2 + 1 \equiv 250jq + 250i \bmod 2(q^2 + 1).$$

As $0 \leq i, j \leq 13m + 2 = \frac{13q-241}{250}$, it follows that $0 \leq 250i, 250j \leq 13q - 241$. We can obtain a contradiction by considering the following two cases:
(i) $0 \leq 250i \leq 12q - 1$. Then, $0 \leq 250jq + 250i \leq (13q - 241)q + 12q - 1 = 13q^2 - 229q - 1 < 13q^2$. Assume that $250i = eq + u$, where $0 \leq e \leq 11$ and $0 \leq u \leq q - 1$ are integers. By $q^2 + 1 \equiv 250jq + 250i \bmod 2(q^2 + 1)$, we have $250jq + 250i = (250j + e)q + u = h(q^2 + 1) = hq^2 + h$, where $1 \leq h \leq 11$ is odd. Similar to the proof of Lemma 3.2, we can get $250j + e = hq$. Thus, $j = \frac{hq-e}{250} = \frac{h(250m+57)-e}{250} = hm + \frac{57h-e}{250}$. Obviously, when $1 \leq e, h \leq 11$ are odd, j is not an integer. This gives a contradiction.

(ii) $12q \le 250i \le 13q - 241$. Then, $12q \le 250jq + 250i \le (13q-241)(q+1) = 13q^2 - 228q - 241 < 13q^2$. Assume that $250i = 12q + u$, where $0 \le u \le q - 241$. Hence, $250jq + 250i = (250j+12)q + u = h(q^2+1) = hq^2 + h$, where $1 \le h \le 11$ is odd. Similarly, we have $250j + 12 = hq$. This is impossible since hq is odd.

If $\epsilon = 3$, then $s - (q+1)i \equiv -[s - (q+1)j]q^3 \bmod (q+1)n$. This is equivalent to

$$250jq + q^2 + 1 \equiv 250i \bmod 2(q^2+1).$$

As $0 \le 250i, 250j \le 13q - 241$, it follows that $q^2 + 1 \le 250jq + q^2 + 1 \le (13q-241)q + q^2 + 1 = 14q^2 - 241q + 1$. We can obtain a contradiction by considering the following two cases:
(i) $q^2 + 1 \le 250jq + q^2 + 1 \le 2q^2 + 1$. Then $250jq + q^2 + 1 = 250i$, which implies that $q|(250i - 1)$. Note that $-1 \le 250i - 1 \le 13q - 241 - 1 < 13q$, so we can assume that $250i - 1 = hq$, where $1 \le h \le 11$ is odd. Then $250i = hq + 1 = h(250m + 57) + 1 = 250mh + 57h + 1$. Obviously, $250|(57h + 1)$. Assume that $57h + 1 = 250p = 2\frac{57^2+1}{26}p,\ p \ge 1$, it follows that $26 \cdot 57h + 26 = 2 \cdot 57^2 p + 2p$. Then, we can get $57|(2p - 26)$. Note that $2p - 26 > -57$, so we can assume that $2p - 26 = 57c,\ c \ge 0$. Then, $2p = 57c + 26 \ge 26$, i.e. $p \ge 13$. Thus $h = \frac{250p-1}{57} \ge 57$. It contradicts the fact that $1 \le h \le 11$.
(ii) $2(q^2+1) \le 250jq + q^2 + 1 \le 14q^2 - 241q + 1 < 14q^2$. Then, we have $250jq + q^2 + 1 - 250i = h(q^2+1)$, where $2 \le h \le 12$ is even. It follows that $250jq - (h-1)q^2 = 250i + h - 1$. This gives that $q|(250i + h - 1)$. Note that $1 \le 250i + h - 1 \le 13q - 230 < 13q$, so we can assume that $250i + h - 1 = h'q$, where $1 \le h' \le 11$ is odd. Hence, $i = \frac{h'q-(h-1)}{250} = \frac{h'(250m+57)-(h-1)}{250} = h'm + \frac{57h'-(h-1)}{250}$. Similar to the case of $\epsilon = 1$, we can also get a contradiction here.
Therefore, $Z \bigcap Z^{-q} = \emptyset$, i.e. $C^{\perp h} \subseteq C$.
(2) The proof is similar to that of (1). □

Theorem 3.4 *For q with the form $250m + 57$ ($250m - 57$), where m is a positive integer, let $n = \frac{q^2+1}{125}$. Then, there exist quantum MDS codes with parameters $[[n, n - 2d + 2, d]]_q$, where $2 \le d \le 26m + 6$ ($2 \le d \le 26m - 6$) is even.*

Proof Here, we only prove the case of $q = 250m + 57$, the case of $q = 250m - 57$ is similar. Let $s = \frac{q^2+1}{2}$. Consider the ω^{q-1}-constacyclic code C over F_{q^2} of length n with defining set $Z = \bigcup_{j=0}^{\delta} C_{s-(q+1)j}$, where $0 \le \delta \le 13m + 2$. By Lemma 2.1 and Singleton bound, $d(C) = 2\delta + 2$. Hence, C is an ω^{q-1}-constacyclic code over F_{q^2} with parameters $[n, n - (2\delta+1), 2\delta + 2]$. By Lemma 3.3, $C^{\perp h} \subseteq C$. Applying the Hermitian construction and quantum Singleton bound to C obtains a q-ary quantum MDS code with parameters $[[n, n - 4\delta - 2, 2\delta + 2]]$. The desired quantum MDS code follows. □

In Table 5, we list some quantum MDS codes obtained from Theorem 3.4.

Lemma 3.4 (1) *For q with the form $298m + 105$, where m is a positive integer, let $s = \frac{q^2+1}{2}$, $n = \frac{q^2+1}{149}$. If C is an ω^{q-1}-constacyclic code over F_{q^2} of length n with defining set $Z = \bigcup_{j=0}^{\delta} C_{s-(q+1)j}$, where $0 \le \delta \le 17m + 5$, then $C^{\perp h} \subseteq C$;*

Table 5 Some quantum MDS codes for $m \leq 10$

m	q	n	d
1	193	298	$2 \leq d \leq 20,\ d$ even
1	307	754	$2 \leq d \leq 32,\ d$ even
2	443	1570	$2 \leq d \leq 46,\ d$ even
2	557	2482	$2 \leq d \leq 58,\ d$ even
5	1193	11,386	$2 \leq d \leq 124,\ d$ even
5	1307	13,666	$2 \leq d \leq 136,\ d$ even
7	1693	22,930	$2 \leq d \leq 176,\ d$ even
10	2557	52,306	$2 \leq d \leq 266,\ d$ even

Table 6 Some quantum MDS codes for $m \leq 4$

m	q	n	d
1	193	250	$2 \leq d \leq 22,\ d$ even
2	491	1618	$2 \leq d \leq 56,\ d$ even
2	701	3298	$2 \leq d \leq 80,\ d$ even
4	1087	7930	$2 \leq d \leq 124,\ d$ even
4	1297	11290	$2 \leq d \leq 148,\ d$ even

(2) *For q with the form $298m - 105$, where m is a positive integer, let $s = \frac{q^2+1}{2}$, $n = \frac{q^2+1}{149}$. If C is an ω^{q-1}-constacyclic code over F_{q^2} of length n with defining set $Z = \bigcup_{j=0}^{\delta} C_{s-(q+1)j}$, where $0 \leq \delta \leq 17m - 7$, then $C^{\perp h} \subseteq C$.*

Proof It is similar to the proofs of Lemma 3.2 and Lemma 3.3. □

Theorem 3.5 *For q with the form $298m + 105$ $(298m - 105)$, where m is a positive integer, let $n = \frac{q^2+1}{149}$. Then, there exist quantum MDS codes with parameters $[[n, n - 2d + 2, d]]_q$, where $2 \leq d \leq 34m + 12$ $(2 \leq d \leq 34m - 12)$ is even.*

Proof The proof is similar to that of Theorems 3.3 and 3.4. □

Applying Theorem 3.5 obtains some quantum MDS codes in Table 6.

Lemma 3.5 (1) *For q with the form $338m + 99$, where m is a positive integer, let $s = \frac{q^2+1}{2}$, $n = \frac{q^2+1}{169}$. If C is an ω^{q-1}-constacyclic code over F_{q^2} of length n with defining set $Z = \bigcup_{j=0}^{\delta} C_{s-(q+1)j}$, where $0 \leq \delta \leq 17m + 4$, then $C^{\perp h} \subseteq C$;*

(2) *For q with the form $338m - 99$, where m is a positive integer, let $s = \frac{q^2+1}{2}$, $n = \frac{q^2+1}{169}$. If C is an ω^{q-1}-constacyclic code over F_{q^2} of length n with defining set $Z = \bigcup_{j=0}^{\delta} C_{s-(q+1)j}$, where $0 \leq \delta \leq 17m - 6$, then $C^{\perp h} \subseteq C$.*

Proof The proof is similar to that of Lemmas 3.2 and 3.3. □

Theorem 3.6 *For q with the form $338m + 99$ $(338m - 99)$, where m is a positive integer, let $n = \frac{q^2+1}{169}$. Then, there exist quantum MDS codes with parameters $[[n, n - 2d + 2, d]]_q$, where $2 \leq d \leq 34m + 10$ $(2 \leq d \leq 34m - 10)$ is even.*

Table 7 Some quantum MDS codes for $m \leq 8$

m	q	n	d
1	239	338	$2 \leq d \leq 24,\ d$ even
2	577	1970	$2 \leq d \leq 58,\ d$ even
4	1451	12,458	$2 \leq d \leq 146,\ d$ even
5	1789	18,938	$2 \leq d \leq 180,\ d$ even
7	2267	30,410	$2 \leq d \leq 228,\ d$ even
8	2803	46,490	$2 \leq d \leq 282,\ d$ even

Table 8 Some quantum MDS codes with smaller x, y, a

x	y	a	q	k	n	d
1	5	125	$250m \pm 57$	13	$(q^2+1)/125$	$2 \leq d \leq 26m \pm 6$
3	5	89	$178m \pm 55$	13	$(q^2+1)/89$	$2 \leq d \leq 26m \pm 8$
3	7	169	$338m \pm 99$	17	$(q^2+1)/169$	$2 \leq d \leq 34m \pm 10$
5	7	149	$298m \pm 105$	17	$(q^2+1)/149$	$2 \leq d \leq 34m \pm 12$
1	7	173	$346m \pm 93$	15	$(q^2+1)/173$	$2 \leq d \leq 30m \pm 8$
3	5	193	$386m \pm 81$	19	$(q^2+1)/193$	$2 \leq d \leq 38m \pm 8$

Proof The proof is similar to that of Theorems 3.3 and 3.4. □

Some quantum MDS codes obtained from Theorem 3.6 are listed in Table 7.

According to the proofs of the above lemmas and theorems, for smaller odd x, y, a with $gcd(x, y) = 1$ and q with the form $2am + \sqrt{(x^2+y^2)a-1}$ ($2am - \sqrt{(x^2+y^2)a-1}$), where m is a positive integer such that q is a prime power, let $n = \frac{q^2+1}{a}$. We can construct quantum MDS codes $[[n, n-2d+2, d]]_q$ from constacyclic MDS codes over F_{q^2}, where $2 \leq d \leq 2km + x + y$ ($2 \leq d \leq 2km - x - y$) is even and some k are given in Table 8.

4 Conclusion

Let $q \equiv 1 \bmod 4$ be an odd prime power, and let $x, y, m \geq 3$ be odd, $x|(q-1),\ y|(q+1)$. Based on negacyclic codes over F_{q^2}, we construct new quantum codes of length $n = 2xy\frac{q^{2m}-1}{q^2-1}$ with parameters $[[n, n-2m(\delta_1 - \lfloor\frac{\delta_1}{q^2}\rfloor + \delta_2 - \lfloor\frac{\delta_2}{q^2}\rfloor) - 2, \geq \delta_1+\delta_2+2]]_q$, where $0 \leq \delta_1, \delta_2 \leq y\lfloor\frac{q^m-1}{q^2-1}\rfloor$.

Besides, for smaller odd x, y, a with $gcd(x, y) = 1$ and odd prime power q with the form $2am + t$ ($2am - t$), where $t = \sqrt{(x^2+y^2)a-1}$ and m is a positive integer, let $n = \frac{q^2+1}{a}$. Based on ω^{q-1}-constacyclic codes, we construct some quantum MDS codes with parameters $[[n, n-2d+2, d]]_q$, where $2 \leq d \leq 2km + x + y$ ($2 \leq d \leq 2km - x - y$) is even and k is given in Table 8. Although Theorem 3.4 in [10] is very powerful, its construction method based on GRS codes is not simple. Hence, it is of certain significance to construct quantum MDS codes by using constacyclic codes here.

In the future, we want to explore the proofs of the case that for all odd x, y, a with $gcd(x, y) = 1$ and odd prime power q with the form $2am + t$ $(2am - t)$, where $t = \sqrt{(x^2 + y^2)a - 1}$ and m is a positive integer, let $n = \frac{q^2+1}{a}$, then there exist quantum MDS codes with parameters $[[n, n - 2d + 2, d]]_q$, where $2 \leq d \leq 2km + x + y$ $(2 \leq d \leq 2km - x - y)$ is even and

$$k = \begin{cases} \frac{(x+y)t+(x-y)}{x^2+y^2}; \; if \; \frac{(x+y)t+(x-y)}{x^2+y^2} \; is \; an \; integer, \\ \\ \frac{(x+y)t-(x-y)}{x^2+y^2}; \; if \; \frac{(x+y)t-(x-y)}{x^2+y^2} \; is \; an \; integer. \end{cases}$$

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Data availability Not applicable.

Declarations

Conflict of interest All the authors declare that they have no conflict of interest.

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