



Gaussian dynamics equation in normal product form

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Abstract

In this paper, we discuss the normal product form of the density operator of multimode Gaussian states and obtain the correlation equation between the kernel matrix \mathbf{R} of the Gaussian density operator in the normal product form and its kernel matrix \mathbf{G} in the standard quadratic form. Further, we explore the time evolution mechanism of \mathbf{R} and obtain the Gaussian dynamical equation under the normal product $\dot{\mathbf{R}} = i(\mathbf{R}\mathbf{J}\mathbf{H} - \mathbf{H}\mathbf{J}\mathbf{R})$. Our work is devoted to searching for another mechanism for Gaussian dynamics. By exploring the description of the normal ordered density matrix under the coherent state representation, we find that our mechanism is feasible and easy to operate.

Keywords Gaussian dynamics · Normal product form · Covariance matrix

1 Introduction

Quantum information science with continuous variable systems is developing rapidly, presenting many exciting prospects in both its experimental realization and theoretical research. Concepts and protocols, such as entanglement and teleportation, initially intended only for discrete quantum systems, have been extended to continuous variable systems, allowing more efficient implementation and measurements. In this context, Gaussian states, as continuous variable quantum states, play an important role in both the experimental and theoretical fields. Gaussian states are defined as quantum states that have Gaussian Wigner functions, while Gaussian dynamics studies the time evolution mechanism of Gaussian state under Gaussian unitary transformation. Two points should be paid special attention to here, one is that the Gaussian state itself must be of Gaussian type, and the other is that the Hamiltonian of the dynamical system in which the Gaussian state evolves is of standard quadratic form.

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There are many works on the dynamics mechanism of Gaussian state evolution in quadratic systems [1–7]. However, many studies focused on the evolution mechanism of the covariance matrix of the Gaussian state, which almost became the paradigm of Gaussian dynamics, and most of the research was done in this way. Here, let us make a brief introduction to this mechanism. For a standard quadratic system, its Hamiltonian can be written as follows:

$$\widehat{H} = \frac{1}{2} \widehat{A}^T \mathbf{H} \widehat{A} + h^T \widehat{A}, \tag{1}$$

where T represents the transpose of the matrix, h^T is a real row matrix and \mathbf{H} is a positive definite, Hermitian and symmetric $2n \times 2n$ matrix, while $\widehat{A} = (\widehat{a}_1, \dots, \widehat{a}_n, \widehat{a}_1^\dagger, \dots, \widehat{a}_n^\dagger)^T$, in which \widehat{a}_i and \widehat{a}_i^\dagger represents the creation and annihilation operators for n -mode Gaussian bosonic systems, satisfying the usual bosonic commutation relations $[\widehat{a}_i, \widehat{a}_j] = [\widehat{a}_i^\dagger, \widehat{a}_j^\dagger] = 0$ and $[\widehat{a}_i, \widehat{a}_j^\dagger] = \delta_{ij}$. Without loss of generality, and for the convenience of discussing problems, we adopts $\widehat{H} = \frac{1}{2} \widehat{A}^T \mathbf{H} \widehat{A}$ in the following. Then, for a Gaussian state, its time-evolution covariance matrix $\sigma(t)$ is according to the following rules [8, 9]

$$\dot{\sigma}(t) = \frac{d\sigma(t)}{dt} = (\mathbf{J}\mathbf{H})\sigma + \sigma(\mathbf{J}\mathbf{H})^T, \tag{2}$$

where $\mathbf{J} = \begin{pmatrix} \mathbf{0} & \mathbf{I}_n \\ -\mathbf{I}_n & \mathbf{0} \end{pmatrix}$, \mathbf{I}_n is $n \times n$ identity matrix. Thus, by solving Eq. (2), the time evolution of the Gaussian state can be mapped as

$$\sigma(t) \rightarrow \mathbf{S}(t)\sigma(0)\mathbf{S}^T(t). \tag{3}$$

Note that $\mathbf{S}(t) \equiv \exp(\mathbf{J}\mathbf{H}t)$, which is a symplectic matrix and satisfies with

$$\mathbf{S}^T \mathbf{J} \mathbf{S} = \mathbf{J} \mathbf{S} \mathbf{S}^T = \mathbf{J}. \tag{4}$$

However, can we directly give the law of the time evolution of the Gaussian state $\rho_G(t)$ itself? This is the main topic to be studied in the present paper. In short, we give the law of the time evolution of the kernel \mathbf{R} of the Gaussian density matrix in the normal product form through effective theoretical derivation, which is an important development of the Gaussian dynamics mechanism. Compared with the previous work, our work is dedicated to directly giving the time evolution of the Gaussian density matrix, breaking the previous theoretical paradigm with the covariance matrix as a bridge. Moreover, due to the operational simplicity of the normal ordered operator in the coherent state representation, we can in principle solve analytically many problems related to the evolution of density matrices, such as the evolution of von Neumann entropy.

Our work is arranged as follows: In Sect. 2, we first give a brief review of the Gaussian state and its covariance matrix. Then, we use the covariance matrix of the Gaussian state $\rho_G(t)$ as a bridge to obtain the algebraic relationship between the kernel \mathbf{G} of the Gaussian state density matrix and the kernel \mathbf{R} of the normal form of the density matrix, so that once we get \mathbf{R} , we can give \mathbf{G} , vice versa. In Sect. 3, we

introduce the coherent state representation description of the Gaussian state, which is the basis for our follow-up work. In Sect. 4, we will show the time evolution law of the kernel matrix \mathbf{R} of the normal product of $\rho_G(t)$

$$\dot{\mathbf{R}} = i(\mathbf{R}\mathbf{J}\mathbf{H} - \mathbf{H}\mathbf{J}\mathbf{R}). \tag{5}$$

2 Gaussian state and its covariance matrix

The density of a Gaussian state can generally be written as [10]

$$\rho_G = \frac{e^{-\hat{G}}}{Tr(e^{-\hat{G}})}. \tag{6}$$

Note that $\hat{G} = \frac{1}{2}\hat{A}^T \mathbf{G} \hat{A}$. By Williamson’s theorem [11], for a positive definite, Hermitian and symmetric $2n \times 2n$ matrix \mathbf{G} , it can be decomposed into the following form

$$\mathbf{G} = \mathbf{S}^T \tilde{\mathbf{K}} \mathbf{S}, \tag{7}$$

where, \mathbf{S} denotes a symplectic matrix, $\tilde{\mathbf{K}} = \begin{pmatrix} \mathbf{K} & \mathbf{0} \\ \mathbf{0} & \mathbf{K} \end{pmatrix}$ and $\mathbf{K} = \text{diag}(\omega_1, \dots, \omega_n)$. According to [10], for the Gaussian state given by Eq. (6), its covariance matrix can be written as

$$\sigma = \mathbf{S}^{-1} \tilde{\nu} \mathbf{S}^{-T}, \tag{8}$$

in which, $\tilde{\nu} = \begin{pmatrix} \nu & \mathbf{0} \\ \mathbf{0} & \nu \end{pmatrix}$, $\nu = \text{diag}(\nu_1, \dots, \nu_n)$, and $\nu_i = \frac{1+e^{-\omega_i}}{1-e^{-\omega_i}}$. Then [12]

$$\sigma = \frac{\mathbf{I} + e^{-\Omega G}}{\mathbf{I} - e^{-\Omega G}} \Omega = \coth\left(\frac{\Omega G}{2}\right) \Omega, \tag{9}$$

where, $\Omega = \begin{pmatrix} \mathbf{I}_n & \mathbf{0} \\ \mathbf{0} & -\mathbf{I}_n \end{pmatrix}$.

We also know that the characteristic function of any Gaussian state can be written as [13–15]

$$C(\mathbf{Z}) = e^{-\frac{1}{2} \mathbf{Z}^\dagger \mathbf{C} \mathbf{Z}}. \tag{10}$$

Note that $\mathbf{Z} = (z_1, \dots, z_n, z_1^*, \dots, z_n^*)^T$. Because its characteristic function in phase space is Gaussian, we call it Gaussian state. By using

$$e^{\mathbf{Z}^\dagger \Omega \hat{A}} =: e^{\mathbf{Z}^\dagger \Omega \hat{A} - \frac{1}{4} \mathbf{Z}^\dagger \mathbf{Z}} : , \tag{11}$$

where, $: \dots :$ represents normal ordering. Then,

$$\begin{aligned}
 \rho_G &= \int (d\mathbf{Z}) e^{\mathbf{Z}^\dagger \Omega \widehat{A}} C(\mathbf{Z}) \\
 &= \int (d\mathbf{Z}) : e^{\mathbf{Z}^\dagger \Omega \widehat{A} - \frac{1}{4} \mathbf{Z}^\dagger \mathbf{Z}} : e^{-\frac{1}{2} \mathbf{Z}^\dagger \mathbf{C} \mathbf{Z}} \\
 &= \int (d\mathbf{Z}) : e^{-\frac{1}{2} \mathbf{Z}^\dagger (\mathbf{C} + \frac{1}{2} \mathbf{I}) \mathbf{Z}} e^{\mathbf{Z}^\dagger \Omega \widehat{A}} : .
 \end{aligned}
 \tag{12}$$

By using the technique of integration within ordered product (IWOP) [16] and the integral formula

$$\int (d\mathbf{Z}) e^{-\frac{1}{2} \mathbf{Z}^\dagger \mathbf{V} \mathbf{Z}} e^{\mathbf{Z}^\dagger \mathbf{X}} = \frac{1}{\sqrt{\det \mathbf{V}}} e^{-\frac{1}{2} \mathbf{X}^T \mathbf{E} \mathbf{V}^{-1} \mathbf{X}},
 \tag{13}$$

where, $\mathbf{E} = \begin{pmatrix} \mathbf{0} & \mathbf{I}_n \\ \mathbf{I}_n & \mathbf{0} \end{pmatrix}$, let us continue our derivation

$$\begin{aligned}
 \rho_G &= \frac{1}{\sqrt{\det (\mathbf{C} + \frac{1}{2} \mathbf{I})}} : \exp \left[-\frac{1}{2} (\Omega \widehat{A})^T \mathbf{E} (\mathbf{C} + \frac{1}{2} \mathbf{I})^{-1} (\Omega \widehat{A}) \right] : \\
 &= \frac{1}{\sqrt{\det (\mathbf{C} + \frac{1}{2} \mathbf{I})}} : \exp \left[-\frac{1}{2} \widehat{A}^T \Omega \mathbf{E} (\mathbf{C} + \frac{1}{2} \mathbf{I})^{-1} \Omega \widehat{A} \right] : .
 \end{aligned}
 \tag{14}$$

Here, we emphasize that in the normal product form, the bosonic operators are all numbers, so we can perform integral and differential operations without any obstacles, which is the most unique feature of IWOP technique. Now, we can set $\mathbf{R} \equiv \Omega \mathbf{E} (\mathbf{C} + \frac{1}{2} \mathbf{I})^{-1} \Omega$, then

$$\rho_G = \sqrt{\det \mathbf{R}} : \exp \left(-\frac{1}{2} \widehat{A}^T \mathbf{R} \widehat{A} \right) : .
 \tag{15}$$

Since the Wigner function of the Gaussian state ρ_G can be written as

$$W(\mathbf{Z}) = \frac{1}{\sqrt{\det \sigma}} \exp(-\mathbf{Z}^\dagger \sigma^{-1} \mathbf{Z}).
 \tag{16}$$

Note that σ here is the covariance matrix in Eq. (2). According to the Fourier transform relationship between $C(\mathbf{Z})$ and $W(\mathbf{Z})$, we can get

$$\frac{\sigma^{-1}}{2} = \Omega C^{-1} \Omega
 \tag{17}$$

or

$$\mathbf{C} = \frac{1}{2} \Omega \sigma \Omega.
 \tag{18}$$

Substituting Eq. (9) into Eq. (18), we have

$$\mathbf{C} = \frac{\Omega \mathbf{I} + e^{-\Omega G}}{2 \mathbf{I} - e^{-\Omega G}}. \tag{19}$$

Then, taking Eq. (19) into Eq. (14), we can get

$$\begin{aligned} \mathbf{R} &= \Omega \mathbf{E} \left(\frac{\mathbf{I}}{2} + \frac{\Omega \mathbf{I} + e^{-\Omega G}}{2 \mathbf{I} - e^{-\Omega G}} \right)^{-1} \Omega \\ &= -2 \mathbf{E}^* \left(\mathbf{I} + \Omega \frac{\mathbf{I} + e^{-\Omega G}}{\mathbf{I} - e^{-\Omega G}} \right)^{-1} \Omega \\ &= -2 \left(\mathbf{E} + \frac{\mathbf{I} + e^{-\Omega G}}{\mathbf{I} - e^{-\Omega G}} \mathbf{J} \right)^{-1} \\ &= -2 \left(\mathbf{E} + \mathbf{J} \mathbf{J}^{-1} \frac{\mathbf{I} + e^{-\Omega G}}{\mathbf{I} - e^{-\Omega G}} \mathbf{J} \right)^{-1} \\ &= -2 \left(\mathbf{E} + \mathbf{J} \frac{\mathbf{I} + e^{-\mathbf{J}^{-1} \Omega G \mathbf{J}}}{\mathbf{I} - e^{-\mathbf{J}^{-1} \Omega G \mathbf{J}}} \right)^{-1} \\ &= -2 \left(\mathbf{E} + \mathbf{J} \frac{\mathbf{I} + e^{-\mathbf{E} \mathbf{G} \mathbf{J}}}{\mathbf{I} - e^{-\mathbf{E} \mathbf{G} \mathbf{J}}} \right)^{-1}. \end{aligned} \tag{20}$$

In this way, we obtain the relationship of the kernel matrix \mathbf{R} of the normal product of $\rho_G(t)$ and \mathbf{G} , which is exactly the same results as in [17]. In Gaussian dynamics, as long as we know the time evolution of \mathbf{R} , we can infer the evolution of \mathbf{G} from Eq. (20). That is to say, we can directly calculate the time evolution of the density matrix of the Gaussian state by using this method. Moreover, according to the above calculation, we can also deduce the relationship between \mathbf{R} and σ

$$\mathbf{R} = -2 \mathbf{E}(\sigma + I)^{-1}. \tag{21}$$

As far as we know, there is currently no similar work that draws the same conclusion as Eq. (21).

3 Coherent state representation of Gaussian state

Now we introduce n -mode coherent states $|\mathbf{Z}\rangle \equiv |z_1, \dots, z_n\rangle$ and suppose that $\rho(\mathbf{Z}) = \langle \mathbf{Z} | \rho_G | \mathbf{Z} \rangle$. In normal product form, bosonic creation and annihilation operators could be replaced by the complex parameter of the coherent state, thus, we have

$$\begin{aligned} \rho(\mathbf{Z}) &= \sqrt{\det \mathbf{R}} \langle \mathbf{Z} | : \exp \left(-\frac{1}{2} \widehat{\mathbf{A}}^T \mathbf{R} \widehat{\mathbf{A}} \right) : | \mathbf{Z} \rangle \\ &= \sqrt{\det \mathbf{R}} e^{-\frac{1}{2} \mathbf{Z}^T \mathbf{R} \mathbf{Z}}. \end{aligned} \tag{22}$$

For a single-mode coherent state $|z\rangle$, we have

$$|z\rangle\langle z|\widehat{a} = \left(z + \frac{\partial}{\partial z^*}\right) |z\rangle\langle z|, \tag{23}$$

$$\widehat{a}^\dagger |z\rangle\langle z| = \left(z^* + \frac{\partial}{\partial z}\right) |z\rangle\langle z|. \tag{24}$$

We can generalize the relationship given by the above two equations to the multimode case and have

$$\begin{aligned} \widehat{A}|\mathbf{Z}\rangle\langle\mathbf{Z}| &= \begin{pmatrix} \widehat{a}_1 \\ \vdots \\ \widehat{a}_n \\ \widehat{a}_1^\dagger \\ \vdots \\ \widehat{a}_n^\dagger \end{pmatrix} |\mathbf{Z}\rangle\langle\mathbf{Z}| = \left[\begin{pmatrix} z_1 \\ \vdots \\ z_n \\ z_1^* \\ \vdots \\ z_n^* \end{pmatrix} + \begin{pmatrix} 0 \\ \vdots \\ 0 \\ \frac{\partial}{\partial z_1} \\ \vdots \\ \frac{\partial}{\partial z_n} \end{pmatrix} \right] |\mathbf{Z}\rangle\langle\mathbf{Z}| \\ &= \left(\mathbf{Z} + \frac{\mathbf{E} - \mathbf{J}}{2} \frac{\partial}{\partial \mathbf{Z}^T}\right) |\mathbf{Z}\rangle\langle\mathbf{Z}|. \end{aligned} \tag{25}$$

Similarly, the following formula can be derived

$$|\mathbf{Z}\rangle\langle\mathbf{Z}|\widehat{A}^T = \left(\mathbf{Z}^T + \frac{\mathbf{E} + \mathbf{J}}{2} \frac{\partial}{\partial \mathbf{Z}}\right) |\mathbf{Z}\rangle\langle\mathbf{Z}|. \tag{26}$$

Taking into account Eqs. (25) and (26), in the coherent state representation, we obtain

$$\begin{aligned} \langle\mathbf{Z}|\rho_G\widehat{A}|\mathbf{Z}\rangle &= \langle\mathbf{Z}|\rho_G|\mathbf{Z}\rangle\left(\mathbf{Z} + \overleftarrow{\frac{\partial}{\partial \mathbf{Z}^T}} \frac{\mathbf{E} - \mathbf{J}}{2}\right) \\ &= \rho\left(\mathbf{Z}\left(\mathbf{Z} + \overleftarrow{\frac{\partial}{\partial \mathbf{Z}^T}} \frac{\mathbf{E} - \mathbf{J}}{2}\right)\right) \end{aligned} \tag{27}$$

and

$$\begin{aligned} \langle\mathbf{Z}|\widehat{A}^T\rho_G|\mathbf{Z}\rangle &= \left(\mathbf{Z}^T + \frac{\mathbf{E} + \mathbf{J}}{2} \frac{\partial}{\partial \mathbf{Z}}\right) \langle\mathbf{Z}|\rho_G|\mathbf{Z}\rangle \\ &= \left(\mathbf{Z}^T + \frac{\mathbf{E} + \mathbf{J}}{2} \frac{\partial}{\partial \mathbf{Z}}\right) \rho(\mathbf{Z}), \end{aligned} \tag{28}$$

where, we have set $\rho(\mathbf{Z}) \equiv \langle\mathbf{Z}|\rho_G|\mathbf{Z}\rangle$, which is actually a Husimi-Q function in the phase space representation. Eqs. (27) and (28) are the results obtained by referring to Eq. (6.1) in Ref. [17], but they are different from the form of Eq. (6.1) and we explicitly describe them in the form of coherent state representation.

4 Gaussian dynamics equation in normal product form

For an open dynamic system, the time evolution mechanism of the system is determined by the following Lindblad equation [8, 18]

$$\dot{\rho}(t) = -i[\widehat{H}, \rho(t)] + \sum_i \left[\widehat{c}_i \rho(t) \widehat{c}_i^\dagger - \frac{1}{2} \widehat{c}_i^\dagger \widehat{c}_i \rho(t) - \frac{1}{2} \rho(t) \widehat{c}_i^\dagger \widehat{c}_i \right], \tag{29}$$

where \widehat{H} is quadratic, \widehat{c}_i and \widehat{c}_i^\dagger are the linear forms of the creation and annihilation operators. *The first term of Eq. (29) is the Schrödinger term of unitary evolution. The following summation term represents the effect of the interaction between the system and the environment on the evolution of the system state, which describes the possible transitions, dissipation, and decoherence that the system undergoes due to the interaction between the system and the environment. In practical applications, for closed systems, the influence of environmental factors can be ignored, and the internal Hamiltonian dominates the process of system evolution. Quantum ensembles evolve based on the Schrödinger equation. On the contrary, when the internal Hamiltonian can be ignored and the environment dominates, the Lindblad operator plays a major role, and we are concerned about the interaction between the system and the environment. Therefore, in order to study the stochastic dynamic characteristics in the open quantum system, that is, the laws of dissipation and decoherence of the system, we can first ignore the unitary evolution part of the system and only study the Lindblad operator term of the system. Therefore, the dynamic master equation of the reduced density matrix of the system with the time evolution is as follows:*

$$\dot{\rho}(t) = \sum_i \left[\widehat{c}_i \rho(t) \widehat{c}_i^\dagger - \frac{1}{2} \widehat{c}_i^\dagger \widehat{c}_i \rho(t) - \frac{1}{2} \rho(t) \widehat{c}_i^\dagger \widehat{c}_i \right]. \tag{30}$$

Although the content discussed in this paper can be fully extended to the case where the quantum system is affected by the coherent environment, that is, considering the second term on the right side of Eq. (29), for the sake of brevity and beauty of the text, we only analyze the time evolution mechanism of Gaussian states in quadratic Hamiltonian systems independent of the environment. That is to say, we only discuss the quantum Liouville equation

$$\dot{\rho}_G(t) = i[\rho_G(t), \widehat{H}]. \tag{31}$$

Note that here $\widehat{H} = \frac{1}{2} \widehat{A}^T \mathbf{H} \widehat{A}$ (For simplicity, we did not adopt the more general form given by Eq. (1)) and $\rho_G = \frac{e^{-\widehat{G}}}{Tr(e^{-\widehat{G}})} = \sqrt{\det \mathbf{R}} : \exp(-\frac{1}{2} \widehat{A}^T \mathbf{R} \widehat{A}) :$. Substituting \widehat{H} and ρ_G into Eq. (31), we get

$$\frac{d[\sqrt{\det \mathbf{R}} : \exp(-\frac{1}{2} \widehat{A}^T \mathbf{R} \widehat{A}) :]}{dt} = -\frac{i}{2} \sqrt{\det \mathbf{R}} \left[\widehat{A}^T \mathbf{H} \widehat{A}, : \exp\left(-\frac{1}{2} \widehat{A}^T \mathbf{R} \widehat{A}\right) : \right]. \tag{32}$$

By using the commutation formula $[AB, C] = A[B, C] + [A, C]B$, we obtain

$$\begin{aligned} \frac{d[\sqrt{\det \mathbf{R}} : \exp(-\frac{1}{2}\widehat{A}^T \mathbf{R} \widehat{A}) :]}{dt} &= -\frac{i}{2} \sqrt{\det \mathbf{R}} \widehat{A}^T \mathbf{H} \left[\widehat{A}, : \exp\left(-\frac{1}{2}\widehat{A}^T \mathbf{R} \widehat{A}\right) : \right] \\ &\quad -\frac{i}{2} \sqrt{\det \mathbf{R}} \widehat{A}^T \left[\mathbf{H}, : \exp\left(-\frac{1}{2}\widehat{A}^T \mathbf{R} \widehat{A}\right) : \right] \widehat{A} \\ &\quad -\frac{i}{2} \sqrt{\det \mathbf{R}} \left[\widehat{A}^T, : \exp\left(-\frac{1}{2}\widehat{A}^T \mathbf{R} \widehat{A}\right) : \right] \widehat{\mathbf{H}} \widehat{A}. \end{aligned} \tag{33}$$

Considering the following normal product properties [19]

$$: \frac{\partial}{\partial \widehat{a}} f(\widehat{a}, \widehat{a}^\dagger) : = [: f(\widehat{a}, \widehat{a}^\dagger) : , \widehat{a}^\dagger], \tag{34}$$

$$: \frac{\partial}{\partial \widehat{a}^\dagger} f(\widehat{a}, \widehat{a}^\dagger) : = [\widehat{a}, : f(\widehat{a}, \widehat{a}^\dagger) :], \tag{35}$$

and the derivation rule of quadratic matrix

$$\frac{d(X^T A X)}{dX} = 2X^T A, \tag{36}$$

$$\frac{d(X^T A X)}{dX^T} = 2AX, \tag{37}$$

under the condition $A = A^T$ (A is a symmetric matrix), we can simplify Eq. (33) into the following form

$$\begin{aligned} \frac{d[\sqrt{\det \mathbf{R}} : \exp(-\frac{1}{2}\widehat{A}^T \mathbf{R} \widehat{A}) :]}{dt} &= -\frac{i}{2} \sqrt{\det \mathbf{R}} \widehat{A}^T \mathbf{H} : \mathbf{J} \frac{\partial}{\partial \widehat{A}^T} \exp\left(-\frac{1}{2}\widehat{A}^T \mathbf{R} \widehat{A}\right) : \\ &\quad +\frac{i}{2} \sqrt{\det \mathbf{R}} : \frac{\partial}{\partial \widehat{A}} \exp\left(-\frac{1}{2}\widehat{A}^T \mathbf{R} \widehat{A}\right) \mathbf{J} : \widehat{\mathbf{H}} \widehat{A} \\ &\quad -\frac{i}{2} \sqrt{\det \mathbf{R}} \widehat{A}^T \left[\mathbf{H}, : \exp\left(-\frac{1}{2}\widehat{A}^T \mathbf{R} \widehat{A}\right) : \right] \widehat{A} \\ &= \frac{i}{2} \sqrt{\det \mathbf{R}} \widehat{A}^T \mathbf{H} : \mathbf{J} \widehat{\mathbf{R}} \widehat{A} \exp\left(-\frac{1}{2}\widehat{A}^T \mathbf{R} \widehat{A}\right) : \\ &\quad -\frac{i}{2} \sqrt{\det \mathbf{R}} : \widehat{A}^T \mathbf{R} \exp\left(-\frac{1}{2}\widehat{A}^T \mathbf{R} \widehat{A}\right) \mathbf{J} : \widehat{\mathbf{H}} \widehat{A} \\ &\quad -\frac{i}{2} \sqrt{\det \mathbf{R}} \widehat{A}^T \left[\mathbf{H}, : \exp\left(-\frac{1}{2}\widehat{A}^T \mathbf{R} \widehat{A}\right) : \right] \widehat{A}. \end{aligned} \tag{38}$$

In the dynamics of phase space, the time evolution formula of Husimi-Q function $\rho(\mathbf{Z})$ can be derived as follows:

$$\begin{aligned} \frac{d\rho(\mathbf{Z})}{dt} &= Tr(\dot{\rho}|\mathbf{Z}\rangle\langle\mathbf{Z}|) \\ &= -iTr(\rho\hat{H}|\mathbf{Z}\rangle\langle\mathbf{Z}| - \hat{H}\rho|\mathbf{Z}\rangle\langle\mathbf{Z}|) \\ &= -i\langle\mathbf{Z}|\rho\hat{H}|\mathbf{Z}\rangle + i\langle\mathbf{Z}|\hat{H}\rho|\mathbf{Z}\rangle, \end{aligned} \tag{39}$$

In fact, we just need to average the coherent states on both sides of the Liouville equation. By calculating the average value of the coherent states on both sides of Eq. (38), we have

$$\begin{aligned} &\frac{d[\sqrt{\det \mathbf{R}}\langle\mathbf{Z}| : \exp(-\frac{1}{2}\hat{A}^T \mathbf{R} \hat{A}) : |\mathbf{Z}\rangle]}{dt} \\ &= \frac{i}{2}\sqrt{\det \mathbf{R}}\langle\mathbf{Z}|\hat{A}^T \mathbf{H} \mathbf{J} \mathbf{R} : \exp(-\frac{1}{2}\hat{A}^T \mathbf{R} \hat{A}) \hat{A} : |\mathbf{Z}\rangle \\ &\quad - \frac{i}{2}\sqrt{\det \mathbf{R}}\langle\mathbf{Z}| : \hat{A}^T \exp(-\frac{1}{2}\hat{A}^T \mathbf{R} \hat{A}) : \mathbf{R} \mathbf{J} \mathbf{H} \hat{A} |\mathbf{Z}\rangle \\ &\quad - \frac{i}{2}\sqrt{\det \mathbf{R}}\langle\mathbf{Z}|\hat{A}^T \left[\mathbf{H}, : \exp(-\frac{1}{2}\hat{A}^T \mathbf{R} \hat{A}) : \right] \hat{A} |\mathbf{Z}\rangle. \end{aligned} \tag{40}$$

We first calculate the third part of the right-hand side of Eq. (40) and have

$$\begin{aligned} &-\frac{i}{2}\sqrt{\det \mathbf{R}}\langle\mathbf{Z}|\hat{A}^T [\mathbf{H}, : \exp(-\frac{1}{2}\hat{A}^T \mathbf{R} \hat{A}) :] \hat{A} |\mathbf{Z}\rangle \\ &= -\frac{i}{2}\sqrt{\det \mathbf{R}}\left(\mathbf{Z}^T + \frac{\mathbf{E} + \mathbf{J}}{2} \frac{\partial}{\partial \mathbf{Z}}\right) \\ &\quad \langle\mathbf{Z}|\mathbf{H}, : \exp(-\frac{1}{2}\hat{A}^T \mathbf{R} \hat{A}) :] |\mathbf{Z}\rangle \left(\mathbf{Z} + \frac{\overleftarrow{\partial}}{\partial \mathbf{Z}^T} \frac{\mathbf{E} - \mathbf{J}}{2}\right) \\ &= -\frac{i}{2}\sqrt{\det \mathbf{R}}\left(\mathbf{Z}^T + \frac{\mathbf{E} + \mathbf{J}}{2} \frac{\partial}{\partial \mathbf{Z}}\right) (\mathbf{H}\langle\mathbf{Z}| : \exp(-\frac{1}{2}\hat{A}^T \mathbf{R} \hat{A}) : |\mathbf{Z}\rangle) \\ &\quad - \langle\mathbf{Z}| : \exp(-\frac{1}{2}\hat{A}^T \mathbf{R} \hat{A}) : |\mathbf{Z}\rangle \mathbf{H} (\mathbf{Z} + \frac{\overleftarrow{\partial}}{\partial \mathbf{Z}^T} \frac{\mathbf{E} - \mathbf{J}}{2}) \\ &= -\frac{i}{2}\left(\mathbf{Z}^T + \frac{\mathbf{E} + \mathbf{J}}{2} \frac{\partial}{\partial \mathbf{Z}}\right) (\mathbf{H}\rho(\mathbf{Z}) - \rho(\mathbf{Z})\mathbf{H}) \left(\mathbf{Z} + \frac{\overleftarrow{\partial}}{\partial \mathbf{Z}^T} \frac{\mathbf{E} - \mathbf{J}}{2}\right). \end{aligned} \tag{41}$$

Since $\rho(\mathbf{Z})$ is a number, $\mathbf{H}\rho(\mathbf{Z}) - \rho(\mathbf{Z})\mathbf{H} = \mathbf{0}$. So we show $-\frac{i}{2}\sqrt{\det \mathbf{R}}\langle\mathbf{Z}|\hat{A}^T [\mathbf{H}, : \exp(-\frac{1}{2}\hat{A}^T \mathbf{R} \hat{A}) :] \hat{A} |\mathbf{Z}\rangle = 0$. We continue to calculate the first two terms on the right-hand side of Eq. (40),

$$\begin{aligned} & \frac{i}{2} \sqrt{\det \mathbf{R}} \langle \mathbf{Z} | \widehat{A}^T \mathbf{HJR} : \exp \left(-\frac{1}{2} \widehat{A}^T \mathbf{R} \widehat{A} \right) \widehat{A} : | \mathbf{Z} \rangle \\ & = \frac{i}{2} \sqrt{\det \mathbf{R}} \left(\mathbf{Z}^T + \frac{\mathbf{E} + \mathbf{J}}{2} \frac{\partial}{\partial \mathbf{Z}} \right) [\mathbf{HJR} \langle \mathbf{Z} | : \exp \left(-\frac{1}{2} \widehat{A}^T \mathbf{R} \widehat{A} \right) \widehat{A} : | \mathbf{Z} \rangle] \end{aligned} \quad (42)$$

and

$$\begin{aligned} & -\frac{i}{2} \sqrt{\det \mathbf{R}} \langle \mathbf{Z} | : \widehat{A}^T \exp \left(-\frac{1}{2} \widehat{A}^T \mathbf{R} \widehat{A} \right) : \mathbf{R} \mathbf{J} \mathbf{H} \widehat{A} | \mathbf{Z} \rangle \\ & = -\frac{i}{2} \sqrt{\det \mathbf{R}} \langle \mathbf{Z} | : \widehat{A}^T \exp \left(-\frac{1}{2} \widehat{A}^T \mathbf{R} \widehat{A} \right) : \mathbf{R} \mathbf{J} \mathbf{H} | \mathbf{Z} \rangle \left(\mathbf{Z} + \frac{\overleftarrow{\partial}}{\partial \mathbf{Z}^T} \frac{\mathbf{E} - \mathbf{J}}{2} \right). \end{aligned} \quad (43)$$

Then,

$$\begin{aligned} \frac{d\rho(\mathbf{Z})}{dt} & = \frac{d[\sqrt{\det \mathbf{R}} \langle \mathbf{Z} | : \exp \left(-\frac{1}{2} \widehat{A}^T \mathbf{R} \widehat{A} \right) : | \mathbf{Z} \rangle]}{dt} \\ & = -\frac{1}{2} \sqrt{\det \mathbf{R}} \mathbf{Z}^T \dot{\mathbf{R}} \tilde{\rho}(\mathbf{Z}) + \frac{d\sqrt{\det \mathbf{R}}}{dt} \tilde{\rho}(\mathbf{Z}) \\ & = \frac{i}{2} \sqrt{\det \mathbf{R}} \left(\mathbf{Z}^T + \frac{\mathbf{E} + \mathbf{J}}{2} \frac{\partial}{\partial \mathbf{Z}} \right) [\mathbf{HJR} \langle \mathbf{Z} | : \exp \left(-\frac{1}{2} \widehat{A}^T \mathbf{R} \widehat{A} \right) \widehat{A} : | \mathbf{Z} \rangle] \\ & \quad - \frac{i}{2} \sqrt{\det \mathbf{R}} \langle \mathbf{Z} | : \widehat{A}^T \exp \left(-\frac{1}{2} \widehat{A}^T \mathbf{R} \widehat{A} \right) : \mathbf{R} \mathbf{J} \mathbf{H} | \mathbf{Z} \rangle \left(\mathbf{Z} + \frac{\overleftarrow{\partial}}{\partial \mathbf{Z}^T} \frac{\mathbf{E} - \mathbf{J}}{2} \right) \\ & = \frac{i}{2} \sqrt{\det \mathbf{R}} \left(\mathbf{Z}^T + \frac{\mathbf{E} + \mathbf{J}}{2} \frac{\partial}{\partial \mathbf{Z}} \right) [\mathbf{HJR} \tilde{\rho}(\mathbf{Z}) \mathbf{Z}] \\ & \quad - \frac{i}{2} \sqrt{\det \mathbf{R}} [\mathbf{Z}^T \tilde{\rho}(\mathbf{Z}) \mathbf{R} \mathbf{J} \mathbf{H}] \left(\mathbf{Z} + \frac{\overleftarrow{\partial}}{\partial \mathbf{Z}^T} \frac{\mathbf{E} - \mathbf{J}}{2} \right) \\ & = \frac{i}{2} \sqrt{\det \mathbf{R}} \mathbf{Z}^T (\mathbf{HJR} - \mathbf{R} \mathbf{J} \mathbf{H}) \mathbf{Z} \tilde{\rho}(\mathbf{Z}) + \frac{i}{2} \sqrt{\det \mathbf{R}} \frac{\mathbf{E} + \mathbf{J}}{2} \frac{\partial}{\partial \mathbf{Z}} [\mathbf{HJR} \tilde{\rho}(\mathbf{Z}) \mathbf{Z}] \\ & \quad - \frac{i}{2} \sqrt{\det \mathbf{R}} [\mathbf{Z}^T \tilde{\rho}(\mathbf{Z}) \mathbf{R} \mathbf{J} \mathbf{H}] \frac{\overleftarrow{\partial}}{\partial \mathbf{Z}^T} \frac{\mathbf{E} - \mathbf{J}}{2}. \end{aligned} \quad (44)$$

Note that here we have set $\tilde{\rho}(\mathbf{Z}) = \rho(\mathbf{Z}) / \sqrt{\det \mathbf{R}}$. Multiplying $\mathbf{E} + \mathbf{J}$ on the left-hand side of Eq. (44) and $\mathbf{E} - \mathbf{J}$ on its right-hand side and noting that $(\mathbf{E} + \mathbf{J})^2 = 0$ and $(\mathbf{E} - \mathbf{J})^2 = 0$, we obtain

$$\begin{aligned} & (\mathbf{E} + \mathbf{J}) \mathbf{Z}^T \dot{\mathbf{R}} \mathbf{Z} (\mathbf{E} - \mathbf{J}) - 2(\mathbf{E} + \mathbf{J}) \frac{1}{\sqrt{\det \mathbf{R}}} \frac{d\sqrt{\det \mathbf{R}}}{dt} (\mathbf{E} - \mathbf{J}) \\ & = -i(\mathbf{E} + \mathbf{J}) \mathbf{Z}^T (\mathbf{HJR} - \mathbf{R} \mathbf{J} \mathbf{H}) \mathbf{Z} (\mathbf{E} - \mathbf{J}). \end{aligned} \quad (45)$$

Because $\mathbf{Z}^T \dot{\mathbf{R}} \mathbf{Z}$, $\frac{d\sqrt{\det \mathbf{R}}}{dt}$ and $\mathbf{Z}^T (\mathbf{H} \mathbf{J} \mathbf{R} - \mathbf{R} \mathbf{J} \mathbf{H}) \mathbf{Z}$ are all numbers, Eq. (45) can be written as

$$\begin{aligned} & (\mathbf{E} + \mathbf{J}) (\mathbf{E} - \mathbf{J}) \mathbf{Z}^T \dot{\mathbf{R}} \mathbf{Z} - 2 (\mathbf{E} + \mathbf{J}) (\mathbf{E} - \mathbf{J}) \frac{d \ln \sqrt{\det \mathbf{R}}}{dt} \\ & = -i (\mathbf{E} + \mathbf{J}) (\mathbf{E} - \mathbf{J}) \mathbf{Z}^T (\mathbf{H} \mathbf{J} \mathbf{R} - \mathbf{R} \mathbf{J} \mathbf{H}) \mathbf{Z}. \end{aligned} \tag{46}$$

Obviously, we have

$$\mathbf{Z}^T [\dot{\mathbf{R}} - i(\mathbf{R} \mathbf{J} \mathbf{H} - \mathbf{H} \mathbf{J} \mathbf{R})] \mathbf{Z} = \frac{d \ln \det \mathbf{R}}{dt}. \tag{47}$$

For any \mathbf{R} , \mathbf{H} and \mathbf{Z} , Eq. (47) always holds, then we get Eq. (5) given in the introduction and $\frac{d \ln \det \mathbf{R}}{dt} = 0$. In this way, we derive the Gaussian dynamics equation in the normal product form. At the same time, there is reason to believe that $\ln \det \mathbf{R}$ is a constant that does not change with time. According to the formula $\det e^A = e^{Tr(A)}$, we can obtain $\ln \det \mathbf{R} = Tr(\ln \mathbf{R})$.

In addition, since $\ln \det \mathbf{R} = Tr(\ln \mathbf{R})$, then

$$\begin{aligned} \frac{d \ln \det \mathbf{R}}{dt} &= \frac{d Tr(\ln \mathbf{R})}{dt} \\ &= Tr(\dot{\mathbf{R}} \mathbf{R}^{-1}) \\ &= Tr[i(\mathbf{R} \mathbf{J} \mathbf{H} - \mathbf{H} \mathbf{J} \mathbf{R}) \mathbf{R}^{-1}] \\ &= i Tr(\mathbf{R} \mathbf{J} \mathbf{H} \mathbf{R}^{-1} - \mathbf{H} \mathbf{J}) \\ &= i [Tr(\mathbf{R} \mathbf{J} \mathbf{H} \mathbf{R}^{-1}) - Tr(\mathbf{H} \mathbf{J})] \\ &= i [Tr(\mathbf{J} \mathbf{H}) - Tr(\mathbf{H} \mathbf{J})] \\ &= 0. \end{aligned} \tag{48}$$

So, we show that if $\dot{\mathbf{R}} = i(\mathbf{R} \mathbf{J} \mathbf{H} - \mathbf{H} \mathbf{J} \mathbf{R})$, then $\frac{d \ln \det \mathbf{R}}{dt} = 0$ naturally satisfies.

Compared with Eqs. (2) and (5), it is not difficult to draw

$$\mathbf{R}(t) = \mathbf{U}(t) \mathbf{R}(0) \mathbf{U}^T(t), \tag{49}$$

where $\mathbf{U}(t) \equiv \exp(-i \mathbf{J} \mathbf{H} t)$. In this way, we get the solution of Eq. (5) smoothly.

In continuous variable systems, in order to provide a more complete description of the quantum state ρ , it is necessary to indirectly introduce the characteristic function [6].

$$\chi(\rho) = Tr(\rho \widehat{D}_\zeta), \tag{50}$$

where \widehat{D}_ζ is Weyl displacement operator and $\zeta \in \mathbf{R}^{2n}$. According to the Weyl–Fourier relationship,

$$\rho = \frac{1}{(2\pi)^n} \int d^{2n} \zeta \chi(\rho) \widehat{D}_\zeta. \tag{51}$$

Therefore, each characteristic function corresponds to a unique quantum state. That is to say, quantum states can be obtained based on their characteristic functions. At the same time, we can obtain the correlation between the Wigner function and the characteristic function of the quasi-probability distribution through Fourier transform

$$W(\zeta) = \frac{1}{(2\pi)^n} \int d^{2n} \kappa e^{i\kappa^T \Omega \zeta} \chi(\kappa). \quad (52)$$

As mentioned earlier, a Gaussian state is a quantum state whose characteristic function is a Gaussian function. These Gaussian states play a key role in the processing of continuous variable quantum information. The Wigner function of the Gaussian state can be written as [6]

$$W(\mathbf{Z}) = \frac{1}{\sqrt{\det \sigma}} \exp \left[-(\mathbf{Z} - \bar{\mathbf{Z}})^\dagger \sigma^{-1} (\mathbf{Z} - \bar{\mathbf{Z}}) \right]. \quad (53)$$

From the above equation, the general Gaussian state ρ can be completely determined by the first moment $\bar{\mathbf{Z}}$ and the covariance matrix σ . It should be noted that in our work, for processing convenience, we write the Wigner function in the form of Eq. (16). In general, any quantum state ρ in a Hilbert space must be determined by a different order correlation, but the Gaussian state can be completely determined by using only the first-order correlation $\bar{\mathbf{Z}}$ and the second-order correlation σ .

The significance of our work is that we do not need introduce phase space functions such as characteristic function and Wigner function to indirectly determine the Gaussian state and characterize its evolution. If we know the R-matrix in the normal product form, we can directly write the quantum state ρ from Eq. (15); Next, it can be determined from the evolution with time of R in Eq. (5); Furthermore, the R(t) that evolves with time can be calculated according to Eq. (49) to determine the evolution with time of $\rho(t)$. The discovery of Eqs. (5) and (49) is very important, and in principle, they can be applied to any physical systems where Gaussian state evolves in a Gaussian channel.

5 Conclusion

The time evolution mechanism of Gaussian states is a long-standing and ever-new topic. This paper mainly provides another mechanism for dealing with the dynamics of Gaussian states. Different from the previous covariance mechanism, our work gives the equation for the time evolution of the kernel matrix \mathbf{R} of Gaussian states in the normal product form, which provides a new perspective for Gaussian quantum information processing.

The advantage of writing the density matrix of the Gaussian state in the normal product form is that the specific functional form of the density matrix under the coherent state representation can be directly given, which can be done simply by replacing bosonic operators in the density matrix with the complex parameters of the coherent state. This processing method will bring us convenience to solve some problems.

Moreover, in the normal product, we regard bosonic operators as numbers, so we can perform integration and differentiation operations without any obstacles, which cannot be replaced by conventional methods. This processing method undoubtedly has great potential and has the value of further research.

Following the theoretical ideas proposed in this paper, in principle, the closed evolution of the Gaussian state that does not interact with the environment can be extended to the case in which the system is coherent with the environment; that is, the Lindblad equation can be solved smoothly, which will be our follow-up work.

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Data availability The data that support the findings of this study are available from the corresponding author upon reasonable request.

Declarations

Conflict of interest The author does not have any possible conflicts of interest.

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