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Quantum and LCD codes from skew constacyclic codes over a finite non-chain ring

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Abstract
For a prime p and a positive integer e, let $\mathbb{F}_{p^e}$ be the finite field and $\mathfrak{R} := \mathbb{F}_{p^e}[u, v]/\langle f(u), g(v), uv - vu\rangle$, where $f(u)$ and $g(v)$ are non-constant square-free polynomials of degree r and s, respectively. This paper constructs quantum and linear complementary dual (briefly, LCD) codes from skew constacyclic codes over the ring $\mathfrak{R}$. Toward this, we first discuss the explicit structure of skew constacyclic codes and their Euclidean as well as Hermitian duals over $\mathfrak{R}$. Then, we establish a necessary and sufficient condition for these codes to contain their Euclidean (Hermitian) duals. Further, by applying CSS (Hermitian) construction, many new quantum codes with better parameters are obtained. Moreover, a necessary and sufficient condition is established for these codes over $\mathfrak{R}$ to be Euclidean (Hermitian) LCD. Finally, many examples of MDS codes over $\mathbb{F}_{p^e}$ are provided under the gray images of the skew Euclidean LCD codes.

Keywords Constacyclic code · Skew constacyclic code · Additive code · Quantum code · Gray map

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1 Introduction

Cyclic codes over the finite field were introduced by Prange [49] in 1957 and further generalized as constacyclic codes by Berlekamp [7]. These codes are the most studied linear codes due to their rich algebraic structure and contain many classical codes such as quadratic residue codes, Reed–Solomon codes and BCH codes. But the study of codes over rings attracted researchers after the seminal work of Hammons et al. [33] in which they established a link between linear codes over $\mathbb{Z}_4$ and nonlinear binary codes. They [33] showed that certain good nonlinear codes (e.g., Kerdock, Preparata and Goethal codes) could be viewed as gray images of linear codes over $\mathbb{Z}_4$. In 1933, Ore [35] introduced a generalization of the polynomial ring as a skew polynomial ring which served as an example of a noncommutative ring. Recently, Boucher et al. [10] studied cyclic codes over a skew polynomial ring with non-trivial automorphism and called skew cyclic codes. Along with their algebraic richness, they [12] obtained many new codes whose minimum distances are comparatively larger than previously best-known codes. Again, skew cyclic codes are generalized as skew constacyclic codes in [11] over Galois rings and further studied by Jitman et al. [34] over chain rings. After that, many authors [8, 21, 27, 30, 47, 53, 60] studied skew codes such as skew constacyclic, skew quasi-cyclic, skew multi-twisted codes, etc.

In recent decades, the problem of constructing quantum codes with good parameters has become a very active research area due to their importance in quantum computation and quantum information transmission. After introducing the first quantum code by Shor [54] and a connection between classical binary codes and quantum codes by Calderbank et al. [14], the study of quantum error-correcting codes increased at an astonishing rate.

Let q be a prime power and $(\mathbb{C}^q)^{\otimes n} = \mathbb{C}^q \otimes \mathbb{C}^q \otimes \cdots \otimes \mathbb{C}^q$ ($n-$ times) be an n-dimensional complex Hilbert space. A q-ary quantum code $\mathcal{Q}$ of length n over $\mathbb{F}_q$ is a q^k-dimensional subspace of $(\mathbb{C}^q)^{\otimes n}$, denoted by $[[n, k, d]]_q$ where d is the minimum distance of $\mathcal{Q}$. A quantum code with the minimum distance d can correct both bit flip and phase shift type of errors up to $\lfloor \frac{d-1}{2} \rfloor$. As in [51], similar to the singleton bound for classical codes, quantum codes satisfy $k \leq n - 2d + 2$ and a quantum code attains this bound, known as the quantum maximum distance separable (MDS) code.

In 1999, Rains [52] obtained many non-binary quantum MDS codes of minimum distance 2 from classical linear codes over $\mathbb{F}_q$. Further, Grassl et al. [24] constructed many classes of quantum MDS codes over $\mathbb{F}_q$. In 2015, Grassl and Rötteler [25] obtained several classes of quantum MDS codes with magnificent minimum distances over small fields. Afterward, many authors [1, 2, 16, 25, 26, 28, 31, 37, 41, 56, 57] used different classes of linear codes over a finite field to construct optimal quantum codes.

It has also been noticed that classical linear (cyclic or constacyclic) codes over finite rings can be viewed as an excellent resource for producing many good quantum codes. In 2009, Qian et al. [50] obtained many binary quantum codes from cyclic codes over the ring $\mathbb{F}_2 + u\mathbb{F}_2$ where $u^2 = 0$. Further, Kai and Zhu [36] presented several new quantum codes from cyclic codes over the ring $\mathbb{F}_4 + u\mathbb{F}_4$ with $u^2 = 0$. In 2015, Ashraf and Mohammad [3] constructed quantum codes over $\mathbb{F}_p$ from cyclic codes over the non-chain ring $\mathbb{F}_p + v\mathbb{F}_p$. Dertli et al. [17] studied cyclic codes over the ring

$\mathbb{F}_2+u\mathbb{F}_2+v\mathbb{F}_2+uv\mathbb{F}_2$ and constructed some new binary quantum codes. Again, Ashraf and Mohammad [4] generalized their work over the ring $\mathbb{F}_q+u\mathbb{F}_q+v\mathbb{F}_q+uv\mathbb{F}_q$ and obtained many new non-binary quantum codes. Later, Gao and Wang [22] discussed the structural properties of u-constacyclic codes over the ring $\mathbb{F}_p+u\mathbb{F}_p$, where $u^2=1$ and constructed several new non-binary quantum codes. Further, Ma et al. [44] considered the ring $\mathbb{F}_p+v\mathbb{F}_p+v^2\mathbb{F}_p$ and obtained some new quantum codes. Recently, Habibul et al. [32] constructed quantum codes from constacyclic codes over the family of the non-chain rings $R_{k,m}$. For more literature on quantum codes from cyclic or constacyclic codes over different rings, we refer [20, 29, 40, 45, 58]. On the other hand, Ezerman et al. [19] constructed asymmetric quantum codes over $GF(4)$ from skew cyclic codes in 2010. Recently, Li et al. [39] constructed many quantum codes by considering skew constacyclic codes over $\mathbb{F}_{q^2}[u,v]/<u^2-u, v^2-v, uv-vu>$. In 2022, Verma et al. [55], obtained several new quantum codes and quantum MDS codes from additive skew constacyclic codes over $R_{l,m}$.

In 1992, Massey [46] introduced linear complementary dual (LCD) codes, a new class of linear codes. These codes had shown to be an optimal linear coding solution of a two-user binary adder channel. Later, Yang and Massey [59] gave a necessary and sufficient condition for cyclic codes over a finite field to be LCD. In 2015, Liu and Liu [43] studied LCD codes over finite chain ring. Afterward, Li [38] studied Hermitian LCD codes from cyclic codes. In 2018, Carlet et al. [15] studied Euclidean and Hermitian MDS LCD codes. Again, Liu and Wang [42] studied LCD codes over rings. Recently, Prakash et al. [48] discussed the structure of LCD codes over a non-chain ring $\mathbb{F}_q+u\mathbb{F}_q$ where $u^2=1$ and q is a power of an odd prime. More recently, Boulanouar et al. [6] obtained some results for skew constacyclic codes over the finite field to be LCD, and as an application, many MDS codes are presented.

Since skew polynomial rings are noncommutative and not a unique factorization domain (UFD), polynomials exhibit more factorizations and, consequently, more ideals than in the case of commutative rings. Recently, skew codes have been used to construct many new linear codes with better Hamming distance than already known linear codes with the same parameters [12].

In this paper, with the motivation of the above works, we focus on the following:

(i) Obtain the quantum codes from skew constacyclic codes over a non-chain ring $\mathfrak{R}$ using CSS and Hermitian constructions.
(ii) Criterion for skew constacyclic codes over a non-chain ring $\mathfrak{R}$ to be LCD and by using a gray map construct some MDS codes.

For a prime p and a positive integer e, the ring $\mathfrak{R}:=\mathbb{F}_{p^e}[u,v]/\langle f(u), g(v), uv-vu\rangle$ is a finite commutative non-chain ring, where $f(u)$ and $g(v)$ are non-constant square-free polynomials of degree r and s, respectively. From the earlier research on codes over rings [20, 29, 32, 39, 40, 45], we observe that codes over the finite non-chain ring are a good resource to produce codes with optimal parameters. But due to the fixed structure of non-chain rings, there are many restrictions on the parameters during the construction of quantum codes. Therefore, in this paper, we consider the general non-chain ring $\mathfrak{R}$ to avoid any constraints on the parameters of the codes during the construction of quantum codes. Mainly, in this paper, we focus on the construction of new quantum codes over qubits with good parameters from codes over rings with a large automorphism group. By this approach, we obtain codes with excellent minimum distance.

The paper is organized as follows: Section. 2 discusses the structure of ring $\mathfrak{R}$ along with some definitions and results related to skew constacyclic codes over $\mathfrak{R}$. Section 3 establishes a necessary and sufficient condition for skew constacyclic codes over $\mathfrak{R}$ to contain their duals. Then, by using CSS construction on these Euclidean dual containing skew constacyclic codes over $\mathfrak{R}$, we obtain some new quantum codes with better parameters than previously known codes. We also provide algorithms that are easy to implement in MAGMA computation software [9] to construct good quantum codes. Section 4 uses Hermitian construction to obtain quantum codes from Hermitian dual containing skew constacyclic codes over $\mathfrak{R}$. Section 5 contains the result on LCD codes with a necessary and sufficient condition for skew constacyclic codes over $\mathfrak{R}$ to be LCD (Euclidean/Hermitian). In Sect. 6, we obtain many new and better quantum codes mentioned in Tables 1 and 2 by using CSS and Hermitian constructions, respectively. We also get many linear MDS codes (Table 3) from gray images of Euclidean LCD skew constacyclic codes over $\mathfrak{R}$. Section 7 concludes the article.

2 Background

In this section, we will first describe the structure of the ring $\mathfrak{R}$ and the duality preserving gray map. Then, some known results for linear codes over the ring $\mathfrak{R}$ will be reviewed that are used in subsequent sections.

For a prime p and positive integer e, let $\mathbb{F}_{p^e}$ be a finite field and $\mathfrak{R} := \mathbb{F}_{p^e}[u, v]/ < f(u), g(v), uv-vu >$, where $f(u)$ and $g(v)$ are non-constant square-free polynomials of degree r and s, respectively. Here, we assume that at least one of r and s is greater than 1, or else $\mathfrak{R}$ is isomorphic to $\mathbb{F}_{p^e}$. Then, there exist $\mu_i, \nu_j \in \mathbb{F}_{p^e}$ with $\mu_i \neq \mu_i^{'}$ and $\nu_j \neq \nu_j^{'}$ for all $i \neq i^{'}$, $j \neq j^{'}$, $1 \leq i, i^{'} \leq r$ and $1 \leq j, j^{'} \leq s$ such that

$$f(u) = (u - \mu_1)(u - \mu_2) \cdots (u - \mu_r),$$
$$g(v) = (v - \nu_1)(v - \nu_2) \cdots (v - \nu_s).$$

Consider the elements ε_i and ς_j of the ring $\mathfrak{R}$ as

$$\varepsilon_i = \varepsilon_i(u) = \frac{(u - \mu_1)(u - \mu_2) \ldots (u - \mu_{i-1})(u - \mu_{i+1}) \cdots (u - \mu_r)}{(\mu_i - \mu_1)(\mu_i - \mu_2) \cdots (\mu_i - \mu_{i-1})(\mu_i - \mu_{i_1}) \cdots (\mu_i - \mu_r)},$$
$$\varsigma_j = \varsigma_j(v) = \frac{(v - \nu_1)(v - \nu_2) \ldots (v - \nu_{j-1})(v - \nu_{j+1}) \cdots (v - \nu_s)}{(\nu_j - \nu_1)(\nu_j - \nu_2) \cdots (\nu_j - \nu_{j-1})(\nu_j - \nu_{j_1}) \cdots (\nu_j - \nu_s)},$$

with the convention that $\varepsilon_1 = 1$ when $r = 1$ and $\varsigma_1 = 1$ when $s = 1$.

Clearly,

$$\varepsilon_i \varepsilon_{i'} = \begin{cases} \varepsilon_i, & \text{if } i = i^{'} \\ 0, & \text{if } i \neq i^{'} \end{cases} \quad \text{and} \quad \varsigma_j \varsigma_{j'} = \begin{cases} \varsigma_j, & \text{if } j = j^{'} \\ 0, & \text{if } j \neq j^{'}. \end{cases}$$

Also, $\sum_{i=1}^{r}\varepsilon_i = 1$ modulo $f(u)$ and $\sum_{j=1}^{s}\varsigma_j = 1$ modulo $g(v)$.

Now, for $1 \le i \le r$ and $1 \le j \le s$, define $\gamma_{ij} = \gamma_{ij}(u, v) = \varepsilon_i(u)\varsigma_j(v)$.

Hence, for $1 \le i, i^{'} \le r$ and $1 \le j, j^{'} \le s$,

$$\gamma_{ij}\gamma_{i^{'}j^{'}} = \begin{cases} \gamma_{ij}, & \text{if } (i, j) = (i^{'}, j^{'}) \\ 0, & \text{if } (i, j) \neq (i^{'}, j^{'}) \end{cases}$$

i.e., for $1 \le i, i^{'} \le r$ and $1 \le j, j^{'} \le s$, $\{\gamma_{ij}\}$ is the set of primitive orthogonal idempotents in $\mathfrak{R}$. Therefore, by using **Chinese Remainder Theorem**, we decompose the ring $\mathfrak{R}$ as

$$\mathfrak{R} = \bigoplus_{i,j}\gamma_{ij}\mathfrak{R} \cong \bigoplus_{i,j}\gamma_{ij}\mathbb{F}_{p^e}.$$

Also, any element $r \in \mathfrak{R}$ can be uniquely expressed as

$$r(u, v) = \sum_{i,j}\beta_{ij}\gamma_{ij},$$

where $\beta_{ij} \in \mathbb{F}_{p^e}$ for $1 \le i \le r$ and $1 \le j \le s$.

Let $GL_{rs}(\mathbb{F}_{p^e})$ be the set of all $rs \times rs$ nonsingular matrices over $\mathbb{F}_{p^e}$ and $A \in GL_{rs}(\mathbb{F}_{p^e})$ such that $A \cdot A^T = \kappa I$ where $\kappa \in \mathbb{F}_{p^e}^*$, A^T is the transpose of matrix A, I is the identity matrix and $\mathbb{F}_{p^e}^*$ is the set of nonzero elements of $\mathbb{F}_{p^e}$. Define the gray map associated with matrix A as

$$\varphi : \mathfrak{R} \to \mathbb{F}_{p^e}^{rs}$$

by

$$\begin{aligned}\varphi(r(u, v)) &= \varphi\left(\sum_{i,j}\beta_{ij}\gamma_{ij}\right)\\ &= (\beta_{11}, \beta_{12}, \ldots, \beta_{1s}, \ldots, \beta_{21}, \beta_{22}, \ldots, \beta_{2s}, \ldots, \beta_{r1}, \beta_{r2}, \ldots, \beta_{rs})A\\ &= \boldsymbol{r}A.\end{aligned}$$

Here, we enumerate the vector $(\beta_{11}, \beta_{12}, \ldots, \beta_{1s}, \ldots, \beta_{21}, \beta_{22}, \ldots, \beta_{2s}, \ldots, \beta_{r1}, \beta_{r2}, \ldots, \beta_{rs})$ as $\boldsymbol{r}$. This map φ can be extended from $\mathfrak{R}^n$ to $\mathbb{F}_{p^e}^{rsn}$ componentwise. Also, it is easy to check that φ is $\mathbb{F}_{p^e}$-linear and distance-preserving bijective map.

Recall that a linear code $\mathfrak{C}$ of length n over $\mathfrak{R}$ is a submodule of an $\mathfrak{R}$-module $\mathfrak{R}^n$ and elements of $\mathfrak{C}$ are called codewords. The Hamming weight $w_H(\mathfrak{a})$ for a codeword $\mathfrak{a} = (\mathfrak{a}_0, \mathfrak{a}_1, \ldots, \mathfrak{a}_{n-1}) \in \mathfrak{C}$ is defined as the number of nonzero components in $\mathfrak{a}$. Moreover, if $\mathfrak{a}, \mathfrak{d} \in \mathfrak{C}$, then the Hamming distance $d_H(\mathfrak{a}, \mathfrak{d})$ between $\mathfrak{a}$ and $\mathfrak{d}$ is defined as $w_H(\mathfrak{a} - \mathfrak{d})$ and the Hamming distance of the code $\mathfrak{C}$ is given as $d_H(\mathfrak{C}) = min\{d_H(\mathfrak{a}, \mathfrak{d}) \mid \mathfrak{a} \neq \mathfrak{d}, \forall\, \mathfrak{a}, \mathfrak{d} \in \mathfrak{C}\}$. Also, the Lee weight of an element $r(u, v) \in \mathfrak{R}$

is defined as the Hamming weight of $\varphi(r(u,v))$, i.e., $w_L(r(u,v)) = w_H(\varphi(r(u,v)))$ and for any element $\mathfrak{a} = (\mathfrak{a}_0, \mathfrak{a}_1, \ldots, \mathfrak{a}_{n-1}) \in \mathfrak{R}^n$ the Lee weight is defined as $w_L(\mathfrak{a}) = \sum_{i=0}^{n-1} w_L(\mathfrak{a}_i)$. Again, the Lee distance between any two codewords $\mathfrak{a}, \mathfrak{d}$ is define by $d_L(\mathfrak{a}, \mathfrak{d}) = w_L(\mathfrak{a} - \mathfrak{d}) = w_H(\varphi(\mathfrak{a} - \mathfrak{d})) = w_H(\varphi(\mathfrak{a}) - \varphi(\mathfrak{d}))$ and Lee distance for the code $\mathfrak{C}$ is $d_L(\mathfrak{C}) = min\{d_L(\mathfrak{a}, \mathfrak{d}) \mid \mathfrak{a} \neq \mathfrak{d}, \mathfrak{a}, \mathfrak{d} \in \mathfrak{C}\}$.

Let $\mathfrak{C}$ be a linear code of length n over the ring $\mathfrak{R}$. For each pair (i, j), $1 \leq i \leq r$, $1 \leq j \leq s$, define the set

$$\mathcal{D}_{ij} := \left\{ t_{ij} \in \mathbb{F}_{p^e}^n \mid \exists\, t_{i'j'} \in \mathbb{F}_{p^e}, (i,j) \neq (i',j') \text{ such that } t_{ij}\gamma_{ij} + \sum_{i',j'} t_{i'j'}\gamma_{i'j'} \in \mathfrak{C} \right\}.$$

Clearly, the sets $\mathcal{D}_{ij}$ are linear codes of length n over $\mathbb{F}_{p^e}$ for each pair (i, j), $1 \leq i \leq r$, $1 \leq j \leq s$. Moreover, if $\mathfrak{C}$ is any linear code of length n over $\mathfrak{R}$, then we can uniquely express it as $\mathfrak{C} = \bigoplus_{i,j} \gamma_{ij}\mathcal{D}_{ij}$. Further, the generator matrix for linear code $\mathfrak{C}$ over $\mathfrak{R}$ can be given as $M = \left(\gamma_{ij} M_{ij}\right)$ where the matrices M_{ij} are generator matrices of $\mathcal{D}_{ij}$ for each pair (i, j), $1 \leq i \leq r$, $1 \leq j \leq s$. Also, if $\mathfrak{C} = \bigoplus_{i,j} \gamma_{ij}\mathcal{D}_{ij}$ is a linear code over $\mathfrak{R}$ of length n, gray distance d_G, and $|\,\mathfrak{C}\,| = p^{e\sum_{i=1}^{r}\sum_{j=1}^{s} k_{ij}}$, then $\varphi(\mathfrak{C})$ is a linear $[rsn, K, d_H]$ code over $\mathbb{F}_{p^e}$ where $d_G = d_H$, $K = \sum_{i=1}^{r}\sum_{j=1}^{s} k_{ij}$ and k_{ij} denotes the dimension of the code $\mathcal{D}_{ij}$ for each pair (i, j), $1 \leq i \leq r$, $1 \leq j \leq s$, respectively.

The following Lemma illustrates the units of the ring $\mathfrak{R}$.

Lemma 1 *Let $\lambda = \lambda(u, v) = \sum_{i,j} \gamma_{ij}\beta_{ij} \in \mathfrak{R}$ where $\beta_{ij} \in \mathbb{F}_{p^e}$. Then, λ is a unit in $\mathfrak{R}$ if and only if for each pair (i, j), $1 \leq i \leq r$, $1 \leq j \leq s$, the corresponding element β_{ij} is a unit in $\mathbb{F}_{p^e}$.*

Proof Let $\lambda = \lambda(u, v) = \sum_{i,j} \gamma_{ij}\beta_{ij} \in \mathfrak{R}$ be a unit in $\mathfrak{R}$ where $\beta_{ij} \in \mathbb{F}_{p^e}$. Then, there exists an element $\xi = \sum_{i'j'} \gamma_{i'j'}\alpha_{i'j'} \in \mathfrak{R}$ such that $\lambda\xi = 1$. This implies that $\sum_{i,j} \gamma_{ij}\beta_{ij} \sum_{i'j'} \gamma_{i'j'}\alpha_{i'j'} = 1$. Since for each pair (i, j), $1 \leq i \leq r$, $1 \leq j \leq s$, the elements γ_{ij} are primitive orthogonal idempotents in $\mathfrak{R}$. Thus, $\beta_{ij}\alpha_{ij} = 1$, i.e., β_{ij} are units in $\mathbb{F}_{p^e}$ for all (i, j), $1 \leq i \leq r$, $1 \leq j \leq s$.

Conversely, for all (i, j), $1 \leq i \leq r$, $1 \leq j \leq s$, let β_{ij} be units in $\mathbb{F}_{p^e}$. Now, from the unique representation of $\lambda = \lambda(u, v) \in \mathfrak{R}$, we can write $\lambda = \sum_{i,j} \gamma_{ij}\beta_{ij}$. Consider an element $\mathfrak{t} = \sum_{i,j} \gamma_{ij}\beta_{ij}^{-1}$. Then $\lambda\mathfrak{t} = \sum_{i,j} \gamma_{ij}\beta_{ij} \sum_{i,j} \gamma_{ij}\beta_{ij}^{-1} = 1$. Thus, λ is a unit in $\mathfrak{R}$. □

The element $\lambda = \sum_{i,j} \gamma_{ij}\beta_{ij} \in \mathfrak{R}$ always denotes a unit in $\mathfrak{R}$, where β_{ij} is corresponding unit in $\mathbb{F}_{p^e}$. We denote the set of all units of the ring $\mathfrak{R}$ by $\mathfrak{R}^*$.

2.1 Skew constacyclic codes over the ring $\mathfrak{R}$ and their gray images

In this subsection, we give a sketch of the skew polynomial ring $\mathfrak{R}$ and a short review based on some results on skew constacyclic codes over the ring $\mathfrak{R}$ given in [8]. Moreover, we also find the gray images of these codes.

Let $\mathbb{F}_{p^e}$ be a finite field and $\mathcal{G} = Aut_{\mathbb{F}_p}(\mathbb{F}_{p^e})$ be the set of automorphisms σ of $\mathbb{F}_{p^e}$ given by $\sigma(z) = z^{\iota}$, for some $0 \leq \iota \leq e - 1$. Then $\mathcal{G}$ is a cyclic group of order

e generated by θ with fixed field $\mathbb{F}_p$, where θ is the Frobenius automorphism of $\mathbb{F}_{p^e}$ given by $z \longmapsto z^p$, for all $z \in \mathbb{F}_{p^e}$. Now, define a map

$$\Theta : \mathfrak{R} \longrightarrow \mathfrak{R}$$

as

$$\Theta(r(u, v)) = \Theta\left(\sum_{i,j} \beta_{ij}\gamma_{ij}\right) = \sum_{i,j} \theta(\beta_{ij})\gamma_{ij}.$$

Clearly, $\Theta \mid_{\mathbb{F}_{p^e}} = \theta$. Let us consider the set

$$\mathfrak{R}[x; \Theta] = \{b_0 + b_1 x + \cdots + b_n x^n \mid b_i \in \mathfrak{R} \,\forall\, i, n \in \mathbb{N}\}.$$

Then the set $\mathfrak{R}[x; \Theta]$ forms a ring under the usual addition of polynomials, and multiplication is defined by $(ax^i)(bx^j) = a\Theta^i(b)x^{i+j}$. This ring is known as a skew polynomial ring. Since in general, $(ax)(bx) = a\Theta(b)x^2 \neq b\Theta(a)x^2 = (bx)(ax)$. Therefore, $\mathfrak{R}[x; \Theta]$ is a noncommutative ring unless Θ is an identity automorphism.

Note that $\mathfrak{R}[x; \Theta]$ is neither left nor right Euclidean, but unitary skew polynomials satisfy the right division algorithm in the same way as it does for chain rings [34]. In particular, if $a(x)$ and $b(x)$ are two skew polynomials, where the leading coefficient of $b(x)$ is unit, then there exist unique skew polynomials $q(x)$ and $r(x)$ in $\mathfrak{R}[x; \Theta]$ such that $a(x) = q(x)b(x) + r(x)$, where $r(x) = 0$ or $deg(r(x)) < deg(b(x))$. If $r(x) = 0$, then we say $b(x)$ is a right divisor of $a(x)$. Also, the least common right multiple (lcrm) and the greatest common left divisors (gcld) can be defined similarly. Now, we define skew constacyclic codes over $\mathfrak{R}$.

Definition 1 Let $\lambda \in \mathfrak{R}^*$ and $\Theta \in Aut(\mathfrak{R})$. Let $\tau_{(\Theta,\lambda)} : \mathfrak{R}^n \longrightarrow \mathfrak{R}^n$ be a skew constacyclic shift operator defined by $\tau_{(\Theta,\lambda)}(\mathfrak{r}) = (\lambda\Theta(r_{n-1}), \Theta(r_0), \ldots, \Theta(r_{n-2}))$, for any $\mathfrak{r} = (r_0, r_1, \ldots, r_{n-1}) \in \mathfrak{R}^n$. Then a linear code $\mathfrak{C}$ of length n over $\mathfrak{R}$ is said to be a skew (Θ, λ)-constacyclic code if $\mathfrak{C}$ is closed under skew constacyclic shift operator $\tau_{(\Theta,\lambda)}$. In particular, a skew (Θ, λ)-constacyclic code is called a skew negacyclic if $\lambda = -1$ and skew cyclic if $\lambda = 1$. Moreover, if Θ is identity automorphism, then skew (Θ, λ)-constacyclic code is the usual λ-constacyclic code over $\mathfrak{R}$.

Suppose $\lambda \in \mathfrak{R}^*$, $\Theta \in Aut(\mathfrak{R})$. Define a map $\rho : \mathfrak{R}^n \longrightarrow \frac{\mathfrak{R}[x;\Theta]}{(x^n-\lambda)}$ as $\rho(\mathfrak{d}) = \rho(\mathfrak{d}_0, \mathfrak{d}_1, \ldots, \mathfrak{d}_{n-1}) \mapsto \mathfrak{d}(x) = (\mathfrak{d}_0 + \mathfrak{d}_1 x + \cdots + \mathfrak{d}_{n-1}x^{n-1}) \ mod\ (x^n - \lambda)$. Then similar to the polynomial representation of constacyclic codes, we can also identify each codeword $\mathfrak{c} = (c_0, c_1, \ldots, c_{n-1}) \in \mathfrak{C}$ by a polynomial $\mathfrak{c}(x) = c_0 + c_1 x + \cdots + c_{n-1}x^{n-1} \in \mathfrak{R}[x; \Theta]/\langle x^n - \lambda\rangle$. Note that the problem to find all λ-constacyclic codes of length n over the ring $\mathfrak{R}$ is equivalent to finding all the ideals of the quotient ring $\frac{\mathfrak{R}[x]}{\langle x^n-\lambda\rangle}$. However, being $\mathfrak{R}[x; \Theta]$ a noncommutative ring, $\frac{\mathfrak{R}[x;\Theta]}{\langle x^n-\lambda\rangle}$ need not be a ring. But it forms a left $\mathfrak{R}[x; \Theta]-$module structure under the scalar multiplication define by $u(x)(v(x) - (x^n - \lambda)) = u(x)v(x) + (x^n - \lambda)$. Thus, a skew (Θ, λ)-constacyclic code of length n over $\mathfrak{R}$ is define as a left $\mathfrak{R}[x; \Theta]$-submodule of the module $\frac{\mathfrak{R}[x;\Theta]}{\langle x^n-\lambda\rangle}$. Note that if $\Theta(\lambda) = \lambda$ and $o(\Theta) \mid n$, then $x(x^n - \lambda) = x^{n+1} - \Theta(\lambda)x = (x^n - \lambda)x$, where $o(\Theta)$ represents the order of Θ. Also, $(x^n - \lambda)\lambda' = \Theta^n(\lambda')x^n - \lambda'\lambda = \lambda'(x^n - \lambda)$, for all $\lambda^{'} \in \mathfrak{R}$. Therefore, $\langle x^n - \lambda \rangle$ forms two sided ideal in $\mathfrak{R}[x;\Theta]$ and in this case the quotient $\frac{\mathfrak{R}[x;\Theta]}{\langle x^n-\lambda \rangle}$ forms ring structure and any skew (Θ, λ)-constacyclic code of length n over $\mathfrak{R}$ can be viewed as a left ideal in $\frac{\mathfrak{R}[x;\Theta]}{\langle x^n-\lambda \rangle}$. Although, in both cases, these codes are generated by a monic right divisor of $x^n - \lambda$.

Theorem 1 ([8], Theorem 6) *Let* $\mathfrak{C} = \bigoplus_{i,j} \gamma_{ij}\mathcal{D}_{ij}$ *be a linear code of length n over* $\mathfrak{R}$. *Then,* $\mathfrak{C}$ *is a skew* (Θ, λ)*-constacyclic code if and only if* $\mathcal{D}_{ij}$ *is a skew* (θ, β_{ij})*-constacyclic code over* $\mathbb{F}_{p^e}$*, for each pair* (i, j), $1 \le i \le r$, $1 \le j \le s$*, respectively. Here,* β_{ij}*'s are units defined in Lemma 1.*

Theorem 2 ([8], Theorem 8) *Let* $\mathfrak{C} = \bigoplus_{i,j} \gamma_{ij}\mathcal{D}_{ij}$ *be a skew* (Θ, λ)*-constacyclic code of length n over* $\mathfrak{R}$ *and* $f_{ij}(x)$ *be the generator polynomial of skew* (θ, β_{ij})*-constacyclic code* $\mathcal{D}_{ij}$ *over* $\mathbb{F}_{p^e}$ *for each pair* (i, j), $1 \le i \le r$, $1 \le j \le s$*, respectively. Then*

1. *there exists a polynomial* $g(x) \in \mathfrak{R}[x;\Theta]$ *such that* $\mathfrak{C} = \langle f(x) \rangle$ *and* $(x^n - \lambda)$ *is right divisible by* $f(x)$*, where* $f(x) = \sum_{i=1}^{r}\sum_{j=1}^{s} \gamma_{ij} f_{ij}(x)$.
2. $\mathfrak{C} = \langle \gamma_{11} f_{11}(x), \dots, \gamma_{1s} f_{1s}(x), \dots, \gamma_{r1} f_{r1}(x), \dots, \gamma_{rs} f_{rs}(x) \rangle$ *and* $|\ \mathfrak{C}\ | = p^{ersn - \sum_{i=1}^{r}\sum_{j=1}^{s} deg f_{ij}(x)}$.

Next, we discuss the gray images of skew (Θ, λ)-constacyclic codes $\mathfrak{C}$ of length n over $\mathfrak{R}$. We begin with the following definition.

Definition 2 Suppose $n = st$ (where s, t are positive integers) and $\Upsilon_{\Theta,t} : \mathfrak{R}^n \longrightarrow \mathfrak{R}^n$ be a linear operator defined by

$$\begin{aligned}\Upsilon_{\Theta,t}(\mathfrak{c}) &= (c^1 \mid c^2 \mid \cdots \mid c^t)\\ &= (\tau_{\Theta,\lambda}(c^1) \mid \tau_{\Theta,\lambda}(c^2) \mid \cdots \mid \tau_{\Theta,\lambda}(c^t))\end{aligned}$$

where $c^i \in \mathfrak{R}^s$ for $i = 1, 2, \dots, t$, then $\mathfrak{C}$ is said to be a skew quasi-twisted code of length n and index t if $\Upsilon_{\Theta,t}(\mathfrak{C}) = \mathfrak{C}$. If Θ is the identity automorphism, then $\mathfrak{C}$ is a quasi-twisted code of length n and index t over $\mathfrak{R}$.

Theorem 3 ([8], Theorem 13) *Let* $\mathfrak{C} = \bigoplus_{i,j} \gamma_{ij}\mathcal{D}_{ij}$ *be a skew* (Θ, λ)*-constacyclic code of length n over* $\mathfrak{R}$. *Then,* $\varphi(\mathfrak{C})$ *is a skew quasi-twisted code of length rsn over* $\mathbb{F}_{p^e}$ *and index rs.*

We close this section with the following remarks.

Remark 1 Let $g(x) = \sum_{i=0}^{\ell} g_i x^i \in \mathbb{F}_{p^e}[x;\theta]$, $g_0 \neq 0$. The left monic skew-reciprocal polynomial $g^*(x)$ of $g(x)$ is define as $g^*(x) = \frac{1}{\theta^{\ell}(g_0)}(\sum_{i=0}^{\ell} \theta^i(g_{\ell-i})x^i)$. Also, the left monic skew Hermitian reciprocal polynomial $g^{\dagger}(x)$ of $g(x)$ is defined as $g^{\dagger}(x) = \theta(g^*(x))$. Boucher et al. [[13], Theorem 1] proved that if β is a fixed unit under the automorphism θ such that $\beta^2 = 1$ and $\mathfrak{C} = \langle f(x) \rangle$ is a skew β-constacyclic code of length n over $\mathbb{F}_{p^e}$, where n is multiple of $o(\theta)$, then $\mathfrak{C}^{\perp_E} = \langle h^*(x) \rangle$. If $o(\theta)$ is 2, then $\mathfrak{C}^{\perp_H} = \langle h^{\dagger}(x) \rangle$ where $f(x)h(x) = h(x)f(x) = x^n - \beta$. Here, $\mathfrak{C}^{\perp_E}$ and $\mathfrak{C}^{\perp_H}$ are skew β^{-1}-constacyclic codes and named as Euclidean and Hermitian dual of code $\mathfrak{C}$ defined in Sects. 3 and 4, respectively.

Remark 2 From here onwards, $\lambda \in \mathfrak{R}^*$ represents a unit fixed by the automorphism θ, and the length of skew λ-constacyclic code is the multiple of the order of Θ.

3 Quantum codes from Euclidean dual containing skew (Θ, λ)-constacyclic code over $\mathfrak{R}$

This section contains the construction of quantum codes from Euclidean dual containing skew (Θ, λ)-constacyclic code over $\mathfrak{R}$. Here, we begin with the following definition.

Definition 3 Let $\mathfrak{C}$ be a linear code $\mathfrak{C}$ of length n over $\mathfrak{R}$. Then, Euclidean dual $\mathfrak{C}^{\perp_E}$ of $\mathfrak{C}$ is defined as

$$\mathfrak{C}^{\perp_E} = \{\mathfrak{a} \in \mathfrak{R}^n \mid < \mathfrak{a}, \mathfrak{d} >_E = 0 \text{ for all } \mathfrak{d} \in \mathfrak{C}\},$$

where $< \mathfrak{a}, \mathfrak{d} >_E = \sum_{i=0}^{n-1} \mathfrak{a}_i \mathfrak{d}_i$ denotes the Euclidean inner product of vectors $\mathfrak{a} = (\mathfrak{a}_0, \mathfrak{a}_1, \cdots, \mathfrak{a}_{n-1})$ and $\mathfrak{d} = (\mathfrak{d}_0, \mathfrak{d}_1, \ldots, \mathfrak{d}_{n-1})$. Code $\mathfrak{C}$ is called Euclidean self-orthogonal, Euclidean self-dual and Euclidean dual containing if $\mathfrak{C} \subseteq \mathfrak{C}^{\perp_E}$, $\mathfrak{C} = \mathfrak{C}^{\perp_E}$ and $\mathfrak{C}^{\perp_E} \subseteq \mathfrak{C}$, respectively.

Recently, Jitman et al. [34] proved that the dual of a skew (Θ, λ)-constacyclic code of length n over a chain ring is a skew (Θ, λ^{-1})-constacyclic code provided n is a multiple of $o(\Theta)$ and $\Theta(\lambda) = \lambda$. It is also easy to prove the same result over a non-chain ring appeared in [8].

Lemma 2 ([8], Theorem 7) *A linear code* $\mathfrak{C} = \bigoplus_{i,j} \gamma_{ij} \mathcal{D}_{ij}$ *of length n over* $\mathfrak{R}$ *is skew* (Θ, λ)*-constacyclic code if and only if* $\mathfrak{C}^{\perp_E} = \bigoplus_{i,j} \gamma_{ij} \mathcal{D}_{ij}^{\perp_E}$ *is a skew* (Θ, λ^{-1})*-constacyclic code over* $\mathfrak{R}$ *and* $\mathcal{D}_{ij}^{\perp_E}$ *is a skew* $(\theta, \beta_{ij}^{-1})$*-constacyclic code over* $\mathbb{F}_{p^e}$. *Moreover,* $\mathfrak{C}$ *is Euclidean self-dual over* $\mathfrak{R}$ *if and only if* $\mathcal{D}_{ij}$ *is a Euclidean self-dual code over* $\mathbb{F}_{p^e}$ *for each pair* (i, j), $1 \le i \le r$, $1 \le j \le s$.

Lemma 3 ([8], Theorem 8) *Let* $\mathfrak{C} = \bigoplus_{i,j} \gamma_{ij} \mathcal{D}_{ij}$ *be a skew* (Θ, λ)*-constacyclic code of length n over* $\mathfrak{R}$. *If* $\mathcal{D}_{ij} = \langle f_{ij}(x) \rangle$ *such that* $(x^n - \beta_{ij}) = h_{ij}(x) f_{ij}(x)$ *for each pair* (i, j), $1 \le i \le r$, $1 \le j \le s$, *respectively, then*

1. $\mathfrak{C}^{\perp_E} = \bigoplus_{i,j} \gamma_{ij} \mathcal{D}_{ij}^{\perp_E}$ *is a skew* (Θ, λ^{-1})*-constacyclic code of length n over* $\mathfrak{R}$.
2. $\mathfrak{C}^{\perp_E} = \langle \sum_i \sum_j \gamma_{ij} h_{ij}^*(x) \rangle$, *where* $h_{ij}^*(x)$ *is skew-reciprocal polynomial of* $h_{ij}(x)$ *for each pair* (i, j), $1 \le i \le r$, $1 \le j \le s$.
3. $\mid \mathfrak{C}^{\perp_E} \mid = p^{e \sum_{i=1}^{r} \sum_{j=1}^{s} deg(f_{ij}(x))}$.

Now, we recall CSS construction (Lemma 4) and then establish a necessary and sufficient condition for skew (Θ, λ)-constacyclic codes to contain their Euclidean duals.

Lemma 4 ([14]) *Let* $\mathcal{C}_1 = [n, k_1, d_1]_{p^m}$ *and* $\mathcal{C}_2 = [n, k_2, d_2]_{p^m}$ *be two linear codes such that* $\mathcal{C}_2^{\perp_E} \subseteq \mathcal{C}_1$. *Then, there exists a quantum code* $[[n, k_1 + k_2 - n, d]]_{p^m}$ *where* $d = min\{w(\mathfrak{r}) : \mathfrak{r} \in (\mathcal{C}_1 \backslash \mathcal{C}_2^{\perp_E}) \cup (\mathcal{C}_2 \backslash \mathcal{C}_1^{\perp_E})\} \ge min\{d_1, d_2\}$. *Moreover, if* $\mathcal{C}_1^{\perp_E} \subseteq \mathcal{C}_1$, *then there exists a quantum code* $[[n, 2k_1 - n, d_1]]_{p^m}$.

Lemma 5 *Let* $\mathcal{D}_{ij} = \langle f_{ij}(x) \rangle$ *be a skew* (θ, β_{ij})*-constacyclic code of length n over* $\mathbb{F}_{p^e}$. *Then* $\mathcal{D}_{ij}$ *is a Euclidean dual containing code if and only if for each pair* (i, j), $1 \le i \le r$, $1 \le j \le s$, $h_{ij}^*(x) h_{ij}(x)$ *is right divisible by* $x^n - \beta_{ij}$, *where* $x^n - \beta_{ij} = h_{ij}(x) f_{ij}(x)$ *and* $h_{ij}^*(x)$ *is the skew-reciprocal polynomial of* $h_{ij}(x)$ *and* $\beta_{ij}^2 = 1$.

Proof Suppose $\mathcal{D}_{ij} = \langle f_{ij}(x)\rangle$ is a skew (θ, β_{ij})-constacyclic code of length n over $\mathbb{F}_{p^e}$ and for each pair (i, j), $1 \le i \le r$, $1 \le j \le s$, $h^*_{ij}(x)h_{ij}(x)$ is right divisible by $x^n - \beta_{ij}$. Consider $l_{ij}(x) \in \mathcal{D}_{ij}^{\perp_E}$. Then, there exists a $v_{ij}(x)$ in $\mathbb{F}_{p^m}[x;\theta]$ such that $l_{ij}(x) = v_{ij}(x)h^*_{ij}(x)$. Now, we multiply by $h_{ij}(x)$ from right, we get $l_{ij}(x)h_{ij}(x) = v_{ij}(x)h^*_{ij}(x)h_{ij}(x)$. As $h^*_{ij}(x)h_{ij}(x)$ is right divisible by $x^n - \beta_{ij}$, $h^*_{ij}(x)h_{ij}(x) = u_{ij}(x)(x^n - \beta_{ij}) = u_{ij}(x)h_{ij}(x)f_{ij}(x)$, for some skew polynomials $u_{ij}(x)$. Since n is multiple of $o(\theta)$ and $\theta(\beta_{ij}) = \beta_{ij}$, $h_{ij}(x)$ and $f_{ij}(x)$ commutes. Hence, $l_{ij}(x) = v_{ij}(x)u_{ij}(x)f_{ij}(x)$, i.e., $l_{ij} \in \mathcal{D}_{ij}$. Thus, $\mathcal{D}_{ij}$ is Euclidean dual containing skew (θ, β_{ij})-constacyclic code over $\mathbb{F}_{p^e}$.

Conversely, suppose $\mathcal{D}_{ij}^{\perp_E} \subseteq \mathcal{D}_{ij}$, for each pair (i, j), $1 \le i \le r$, $1 \le j \le s$. Then, $f_{ij}(x)$ divides $h^*_{ij}(x)$, where $x^n - \beta_{ij} = h_{ij}(x)f_{ij}(x) = f_{ij}(x)h_{ij}(x)$. Therefore, there exist some skew polynomials $t_{ij}(x)$ such that $h^*_{ij}(x) = t_{ij}(x)f_{ij}(x)$. Again, if we right multiply by $h_{ij}(x)$ in both sides, we get $h^*_{ij}(x)h_{ij}(x) = t_{ij}(x)f_{ij}(x)h_{ij}(x) = t_{ij}(x^n - \beta_{ij})$. Thus, $h^*_{ij}(x)h_{ij}(x)$ is right divisible by $x^n - \beta_{ij}$, for each pair (i, j), $1 \le i \le r$, $1 \le j \le s$. □

Theorem 4 *Let $\mathfrak{C} = \bigoplus_{i,j} \gamma_{ij}\mathcal{D}_{ij}$ be a skew (Θ, λ)-constacyclic code of length n over $\mathfrak{R}$ generated by $f(x) = \sum_{i=1}^{r}\sum_{j=1}^{s} \gamma_{ij} f_{ij}(x)$ where for each pair (i, j), $1 \le i \le r$, $1 \le j \le s$, polynomial $f_{ij}(x)$ is the generator polynomial of $\mathcal{D}_{ij}$ over $\mathbb{F}_{p^e}$ and $\beta_{ij}^2 = 1$. Then, $\mathfrak{C}^{\perp_E} \subseteq \mathfrak{C}$ if and only if $h^*_{ij}(x)h_{ij}(x)$ is right divisible by $x^n - \beta_{ij}$, where $x^n - \beta_{ij} = h_{ij}(x)f_{ij}(x)$, for each pair (i, j), $1 \le i \le r$, $1 \le j \le s$.*

Proof Let $\mathfrak{C} = \bigoplus_{i,j} \gamma_{ij}\mathcal{D}_{ij}$ be a skew (Θ, λ)-constacyclic code of length n over $\mathfrak{R}$ and $f(x)$ is a generator polynomial of $\mathfrak{C}$. Then, by Lemma 3, $\mathfrak{C}^{\perp_E} = \bigoplus_{i,j} \gamma_{ij}\mathcal{D}_{ij}^{\perp_E}$ is skew (Θ, λ^{-1})-constacyclic code over $\mathfrak{R}$ generated by $h^*(x) = \sum_i \sum_j \gamma_{ij} h^*_{ij}(x)\rangle$, where $h^*_{ij}(x)$ is skew-reciprocal polynomial of $h_{ij}(x)$ for each pair (i, j), $1 \le i \le r$, $1 \le j \le s$.

Now, if $\mathfrak{C}^{\perp_E} \subseteq \mathfrak{C}$, then $\bigoplus_{ij} \gamma_{ij}\mathcal{D}_{ij}^{\perp_E} \subseteq \bigoplus_{ij} \gamma_{ij}\mathcal{D}_{ij}$. Hence, for each pair (i, j), $1 \le i \le r$, $1 \le j \le s$ by taking modulo γ_{ij} on both sides, we have $\mathcal{D}_{ij}^{\perp_E} \subseteq \mathcal{D}_{ij}$. Thus, by Lemma 5, $h^*_{ij}(x)h_{ij}(x)$ is right divisible by $x^n - \beta_{ij}$ for each pair (i, j), $1 \le i \le r$, $1 \le j \le s$, where $x^n - \beta_{ij} = h_{ij}(x)f_{ij}(x)$ and $h^*_{ij}(x)$ is the skew-reciprocal polynomial of $h_{ij}(x)$.

Conversely, suppose for each pair (i, j), $1 \le i \le r$, $1 \le j \le s$, the polynomial $h^*_{ij}(x)h_{ij}(x)$ is right divisible by $x^n - \beta_{ij}$. Therefore, by Lemma 5, $\mathcal{D}_{ij}^{\perp_E} \subseteq \mathcal{D}_{ij}$. Consequently, $\bigoplus_{ij} \gamma_{ij}\mathcal{D}_{ij}^{\perp_E} \subseteq \bigoplus_{ij} \gamma_{ij}\mathcal{D}_{ij}$ or in other words, $\mathfrak{C}^{\perp_E} \subseteq \mathfrak{C}$. □

Corollary 1 *Let $\mathfrak{C} = \bigoplus_{i,j} \gamma_{ij}\mathcal{D}_{ij}$ be a skew (Θ, λ)-constacyclic code of length n over $\mathfrak{R}$. Then, $\mathfrak{C}^{\perp_E} \subseteq \mathfrak{C}$ if and only $\mathcal{D}_{ij}^{\perp_E} \subseteq \mathcal{D}_{ij}$ for each pair (i, j), $1 \le i \le r$, $1 \le j \le s$.*

Let d_G be the minimum distance of $\varphi(\mathfrak{C})$ and $K = \sum_{i=1}^{r}\sum_{j=1}^{s} k_{ij}$ be the dimension of $\varphi(\mathfrak{C})$ where k_{ij} denotes the dimension of the code $\mathcal{D}_{ij}$ for each pair (i, j), $1 \le i \le r$, $1 \le j \le s$. Now, with the help of Lemma 4 and Theorem 4, we present a result to construct the quantum codes.

Theorem 5 *If $\mathfrak{C} = \bigoplus_{i,j} \gamma_{ij}\mathcal{D}_{ij}$ is a skew (Θ, λ)-constacyclic code of length n over $\mathfrak{R}$ such that $\mathfrak{C}^{\perp_E} \subseteq \mathfrak{C}$, then there exists a quantum code with parameters $[[rsn, 2K - rsn, d_G]]$ over $\mathbb{F}_{p^e}$.*

Let $\lambda \in \mathfrak{R}^*$ and $\Theta \in Aut(\mathfrak{R})$ such that $\lambda^2 = 1$ and $\Theta(\lambda) = \lambda$. Let $\mathfrak{C}$ be a skew (Θ, λ)-constacyclic code of length n over $\mathfrak{R}$ where n is a multiple of $o(\Theta)$. Then, we use the following algorithms for quantum code construction.

Algorithm 1. Input: $(\mathbb{F}_{p^e}, n)$

- **Step 1:** Define skew polynomial ring over $\mathbb{F}_{p^e}$;
- **Step 2:** Find all k-degree skew polynomials f_{ij} which are right divisors of $x^n \pm 1$;
- **Step 3:** Find skew-reciprocal polynomial $h^*_{ij}(x)$ of $h_{ij}(x)$ where $x^n \pm 1 = h_{ij}(x) f_{ij}(x)$;
- **Step 4:** If $x^n \pm 1$ is a right divisor of $h^*_{ij}(x)h_{ij}(x)$, then Print $(f_{ij}(x))$;

Output: Generator polynomial of all dual containing skew negacyclic (cyclic) codes of length n and dimension $n - k$.

Let $\mathfrak{C} = \langle f(x)\rangle$ be a skew (Θ, λ)-constacyclic code of length n over $\mathfrak{R}$ where $\lambda^2 = 1$. Then, $f(x) = \sum_{i=1}^{r} \sum_{j=1}^{s} \gamma_{ij} f_{ij}(x)$ where f_{ij} are generator polynomials of skew cyclic or skew negacyclic codes $\mathcal{D}_{ij}$ of length n over $\mathbb{F}_{p^e}$, for each pair (i, j), $1 \leq i \leq r$, $1 \leq j \leq s$ and can be obtained by *Algorithm* 1. Then, for construction of quantum code from $\mathfrak{C}$ we use the following *Algorithm* :

Algorithm 2. Input: $(\mathbb{F}_{p^e},\ f(x))$

- **Step 1:** Compute generator matrix $M = \left(\gamma_{ij} M_{ij}\right)_{(i,j)}$ of $\mathfrak{C}$, where for each pair (i, j), $1 \leq i \leq r$, $1 \leq j \leq s$, M_{ij} is a generator matrix of code $\mathcal{D}_{ij} = \langle f_{ij}(x)\rangle$;
- **Step 2:** Find generator matrix, say, L of $\varphi(\mathfrak{C})$;
- **Step 3:** Compute Minimum Weight d of $\varphi(\mathfrak{C})$;
- **Step 3:** Define $Q := QuantumCode(\varphi(\mathfrak{C}))$ and compute Q;

Output: Quantum code with parameters $[[rsn, 2K - rsn, d]]_{p^e}$.

We implement Algorithms 1 and 2 in MAGMA computation software [9] to obtain quantum codes from skew (Θ, λ)-constacyclic code of length n over $\mathfrak{R}$.

Example 1 Let $f(u) = u^2 - 1$, $g(v) = v^2 - v$ and $\mathbb{F}_9 = \mathbb{F}_3(\mathfrak{t})$, where $\mathfrak{t}^2 = \mathfrak{t} + 1$. Then, $\mathfrak{R} = \mathbb{F}_9[u, v]/\langle u^2 - 1, v^2 - v, uv - vu\rangle$ and $\gamma_{11} = \frac{1}{2}(v + uv)$, $\gamma_{12} = \frac{1}{2}(v - uv)$, $\gamma_{21} = \frac{1}{2}(1 + u - v - uv)$ and $\gamma_{22} = \frac{1}{2}(1 - u - v + uv)$. Let Θ be an automorphism over $\mathfrak{R}$ given by $\Theta(r_0 + r_1 u + r_2 v + r_3 uv) = r_0^3 + r_1^3 u + r_2^3 v + r_3^3 uv$ and $\lambda = 1 - v - uv$. Let $\mathfrak{C}$ be a skew (Θ, λ)-constacyclic code of length 6 over $\mathfrak{R}$ generated by $f(x) = \sum_{i=1}^{1} \sum_{j=1}^{2} \gamma_{ij} f_{ij}(x)$ where $f_{11} = x + \mathfrak{t}$ is the generator polynomial of skew negacyclic code $\mathcal{D}_{11}$ and $f_{12} = x + 1$, $f_{21} = x + \mathfrak{t}^2$, $f_{22} = x^2 + \mathfrak{t}^7 x + \mathfrak{t}^6$ are generator polynomials of skew cyclic code $\mathcal{D}_{12}, \mathcal{D}_{21}, \mathcal{D}_{22}$, respectively. Let

$$M = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 1 & 2 \\ 1 & 1 & 2 & 2 \\ 1 & 2 & 2 & 1 \end{bmatrix} \in GL_4(\mathbb{F}_{3^2}),$$

satisfying $MM^T = I_4$. Then, the gray image $\varphi(\mathfrak{C})$ has parameters [24, 19, 4]. Here,

$$\begin{aligned}
h_{11}(x) &= x^5 + \mathfrak{t}^7x^4 + 2x^3 + \mathfrak{t}^3x^2 + x + \mathfrak{t}^7,\\
h_{12}(x) &= x^5 + 2x^4 + x^3 + 2x^2 + x + 2,\\
h_{21}(x) &= x^5 + \mathfrak{t}^2x^4 + x^3 + \mathfrak{t}^2x^2 + x + \mathfrak{t}^2,\\
h_{22}(x) &= x^4 + \mathfrak{t}^3x^3 + \mathfrak{t}x^2 + \mathfrak{t}^7x + \mathfrak{t}^6,
\end{aligned}$$

and

$$\begin{aligned}
h^*_{11}(x) &= x^5 + \mathfrak{t}^3x^4 + 2x^3 + \mathfrak{t}^7x^2 + x + \mathfrak{t}^3,\\
h^*_{12}(x) &= x^5 + 2x^4 + x^3 + 2x^2 + x + 2,\\
h^*_{21}(x) &= x^5 + \mathfrak{t}^2x^4 + x^3 + \mathfrak{t}^2x^2 + x + \mathfrak{t}^2,\\
h^*_{22}(x) &= x^4 + \mathfrak{t}^7x^3 + \mathfrak{t}^3x^2 + \mathfrak{t}^3x + \mathfrak{t}^2.
\end{aligned}$$

Also,

$$\begin{aligned}
h^*_{11}(x)h_{11}(x) &= (x^4 + \mathfrak{t}^2x^3 + \mathfrak{t}^7x^2 + \mathfrak{t}^2x + \mathfrak{t}^2)(x^6 + 1),\\
h^*_{12}(x)h_{12}(x) &= (x^4 + x^3 + 2x + 2)(x^6 - 1),\\
h^*_{21}(x)h_{21}(x) &= (x^4 + x^2 + 1)(x^6 - 1),\\
h^*_{22}(x)h_{22}(x) &= (x^2 + 2)(x^6 - 1).
\end{aligned}$$

Then, $h^*_{11}(x)h_{11}(x)$ and $h^*_{12}(x)h_{12}(x), h^*_{21}(x)h_{21}(x), h^*_{22}(x)h_{22}(x)$ are right divisible by (x^6+1) and (x^6-1), respectively. Hence, by Theorem 4, we have $\mathfrak{C}^{\perp_E} \subseteq \mathfrak{C}$, and by Theorem 5, there exists a quantum code with parameters $[[24, 14, 4]]_9$. We observe that our obtained quantum code has a better code rate than the known code $[[24, 8, 4]]_9$ appeared in [45]. They constructed this code over $\mathfrak{R}$ by using λ-constacyclic code, whereas we used skew (Θ, λ)-constacyclic code over $\mathfrak{R}$ and obtain better parameters.

Example 2 Suppose $deg(f(u))$ is 1, $g(v) = v^2 - v$ and $\mathbb{F}_{13^2} = \mathbb{F}_{13}(\mathfrak{t})$, where $\mathfrak{t}^2 = \mathfrak{t} + 11$, then $\mathfrak{R} = \mathbb{F}_{13^2}[v]/\langle v^2 - v\rangle$, $\gamma_{11} = v$ and $\gamma_{12} = 1 - v$. Let $\Theta \in Aut(\mathfrak{R})$ defined by $\Theta(r_0+r_1v) = r_0^{13}+r_1^{13}v$ and $\lambda = -1$. Let $\mathfrak{C}$ be a skew (Θ, λ)-constacyclic code of length 14 over $\mathfrak{R}$ generated by $f(x) = \gamma_{11}f_{11}(x) + \gamma_{12}f_{12}(x)$ where $f_{11} = x^2+\mathfrak{t}^{121}x+\mathfrak{t}^{144}$ and $f_{12} = x+5$ are generator polynomials of skew negacyclic codes $\mathcal{D}_{11}$ and $\mathcal{D}_{12}$, respectively. Here,

$$\begin{aligned}
h_{11}(x) &= x^{12}+\mathfrak{t}^{37}x^{11}+\mathfrak{t}^{115}x^{10}+\mathfrak{t}^{163}x^9 + \mathfrak{t}^{155}x^8 + \mathfrak{t}^{135}x^7 + \mathfrak{t}^{68}x^6 + \mathfrak{t}^{135}x^5\\
&\quad +\mathfrak{t}^{23}x^4 + \mathfrak{t}^{163}x^3 + \mathfrak{t}^7x^2 + \mathfrak{t}^{37}x + \mathfrak{t}^{24},\\
h^*_{11}(x) &= x^{12} + \mathfrak{t}^{121}x^{11} + \mathfrak{t}^{151}x^{10} + \mathfrak{t}^{79}x^9 + \mathfrak{t}^{167}x^8 + \mathfrak{t}^{51}x^7 + \mathfrak{t}^{44}x^6 + \mathfrak{t}^{51}x^5\\
&\quad +\mathfrak{t}^{131}x^4 + \mathfrak{t}^{79}x^3 + \mathfrak{t}^{91}x^2 + \mathfrak{t}^{121}x + \mathfrak{t}^{144},\\
h^*_{11}(x)h_{11}(x) &= (x^{10} + 5x^8 + 11x^6 + 11x^4 + 5x^2 + 1)(x^{14} + 1),
\end{aligned}$$

and

$$\begin{aligned} h_{12}(x) &= x^{13} + 8x^{12} + 12x^{11} + 5x^{10} + x^9 + 8x^8 + 12x^7 + 5x^6 + x^5 + 8x^4 \\ &\quad +12x^3 + 5x^2 + x + 8, \\ h^*_{12}(x) &= x^{13} + 5x^{12} + 12x^{11} + 8x^{10} + x^9 + 5x^8 + 12x^7 + 8x^6 + x^5 + 5x^4 \\ &\quad +12x^3 + 8x^2 + x + 5, \\ h^*_{12}(x)h_{12}(x) &= (x^{12} + 12x^{10} + x^8 + 12x^6 + x^4 + 12x^2 + 1)(x^{14} + 1). \end{aligned}$$

As both $h^*_{11}(x)h_{11}(x)$ and $h^*_{12}(x)h_{12}(x)$ are right divisible by $x^{14} + 1$, by Theorem 4, we have $\mathfrak{C}^{\perp_E} \subseteq \mathfrak{C}$. Again, let

$$M = \begin{bmatrix} 1 & 1 \\ 1 & 12 \end{bmatrix} \in GL_2(\mathbb{F}_{13^2}),$$

satisfying $MM^T = 2I_2$. Then, the gray image $\varphi(\mathfrak{C})$ is also dual containing and has parameters $[28, 25, 4]$. Hence, by Theorem 5, there exists a quantum code with parameters $[[28, 22, 4]]_{13^2}$. Note that constructed quantum code satisfies $n + 2 = k + 2d$. Hence, it is a quantum MDS code.

4 Quantum codes from Hermitian dual containing skew (Θ, λ)-constacyclic code over $\mathfrak{R}$

Let $q = p^e$ and $\mathfrak{R} := \mathbb{F}_{q^2}[u, v]/ < f(u), g(v), uv - vu >$ where p is a prime and $e > 0$. In this section, we construct quantum codes over $\mathbb{F}_q$ from Hermitian dual containing skew (Θ, λ)-constacyclic codes over the ring $\mathfrak{R}$. We begin with the following definition.

Definition 4 The Hermitian dual $\mathfrak{C}^{\perp_H}$ of a linear code $\mathfrak{C}$ of length n over $\mathfrak{R}$ is defined as $\mathcal{C}^{\perp_H} = \{\mathfrak{a} \in \mathfrak{R}^n \mid < \mathfrak{a}, \mathfrak{d} >_H = 0 \text{ for all } \mathfrak{d} \in \mathfrak{C}\}$, where $< \mathfrak{a}, \mathfrak{d} >_H = \sum_{i=0}^{n-1} \mathfrak{a}_i \Theta(\mathfrak{d}_i)$ represents the Hermitian inner product of vectors $\mathfrak{a} = (\mathfrak{a}_0, \mathfrak{a}_1, \cdots, \mathfrak{a}_{n-1})$ and $\mathfrak{d} = (\mathfrak{d}_0, \mathfrak{d}_1, \ldots, \mathfrak{d}_{n-1})$. $\mathfrak{C}$ is Hermitian self-orthogonal, Hermitian self-dual and Hermitian dual containing if $\mathfrak{C} \subseteq \mathfrak{C}^{\perp_H}$, $\mathfrak{C} = \mathfrak{C}^{\perp_H}$ and $\mathfrak{C}^{\perp_H} \subseteq \mathfrak{C}$, respectively.

Lemma 6 *Let $\mathfrak{C} = \bigoplus_{i,j} \gamma_{ij}\mathcal{D}_{ij}$ be a skew (Θ, λ)-constacyclic code of length n over the ring $\mathfrak{R}$. Then $\mathfrak{C}^{\perp_H} = \bigoplus_{i,j} \gamma_{ij}\mathcal{D}_{ij}^{\perp_H}$ is a skew (Θ, λ^{-1})-constacyclic code over $\mathfrak{R}$ where for each pair (i, j), $1 \le i \le r$, $1 \le j \le s$, $\mathcal{D}_{ij}^{\perp_H}$ is a skew $(\theta, \beta_{ij}^{-1})$-constacyclic code over $\mathbb{F}_q$. Moreover, for $\lambda^2 = 1$, $\mathfrak{C}$ is a skew (Θ, λ)-constacyclic code over $\mathfrak{R}$ if and only if $\mathfrak{C}^{\perp_H}$ is a skew (Θ, λ)-constacyclic code over $\mathfrak{R}$.*

Proof Let $\mathfrak{C} = \bigoplus_{i,j} \gamma_{ij}\mathcal{D}_{ij}$ be a skew (Θ, λ)-constacyclic code of length n over $\mathfrak{R}$. Consider two vectors $\mathfrak{c} = (\mathfrak{c}_0, \mathfrak{c}_1, \ldots, \mathfrak{c}_{n-1})$ and $\mathfrak{c}' = (\mathfrak{c}'_0, \mathfrak{c}'_1, \ldots, \mathfrak{c}'_{n-1})$ in $\mathfrak{C}$ and $\mathfrak{C}^{\perp_H}$, respectively. Then, $\langle \mathfrak{c}, \mathfrak{c}' \rangle_H = 0$. As $\mathfrak{c} \in \mathfrak{C}$, $\tau^{n-1}_{\lambda,\Theta}(\mathfrak{c}) =$

$(\lambda\Theta^{n-1}(\mathfrak{c}_1), \lambda\Theta^{n-1}(\mathfrak{c}_2), \ldots, \lambda\Theta^{n-1}(\mathfrak{c}_{n-1}), \Theta^{n-1}(\mathfrak{c}_0)) \in \mathfrak{C}$ and $\langle \tau_{\lambda,\Theta}(\mathfrak{c}), \mathfrak{c}' \rangle_H = 0$. Therefore,

$$\begin{aligned}
0 &= \langle (\lambda\Theta^{n-1}(\mathfrak{c}_1), \ldots, \lambda\Theta^{n-1}(\mathfrak{c}_{n-1}), \Theta^{n-1}(\mathfrak{c}_0)), (\mathfrak{c}'_0, \mathfrak{c}'_1, \ldots, \mathfrak{c}'_{n-1}) \rangle_H \\
&= (\lambda\Theta^{n-1}(\mathfrak{c}_1)\Theta(\mathfrak{c}'_0) + \cdots + \lambda\Theta^{n-1}(\mathfrak{c}_{n-1})\Theta(\mathfrak{c}'_{n-2}) + \Theta^{n-1}(\mathfrak{c}_0)\Theta(\mathfrak{c}'_{n-1}) \\
&= \lambda \left(\sum_{\ell=1}^{n-1} \Theta^{n-1}(\mathfrak{c}_\ell)\Theta(\mathfrak{c}'_{\ell-1}) + \lambda^{-1}\Theta^{n-1}(\mathfrak{c}_0)\Theta(\mathfrak{c}'_{n-1}) \right).
\end{aligned}$$

Since λ is fixed by Θ and $o(\Theta) \mid n$, by applying Θ on both sides, we get

$$\begin{aligned}
0 &= \lambda\Theta \left(\sum_{\ell=1}^{n-1} \Theta^{n-1}(\mathfrak{c}_\ell)\Theta(\mathfrak{c}'_{\ell-1}) + \lambda^{-1}\Theta^{n-1}(\mathfrak{c}_0)\Theta(\mathfrak{c}'_{n-1}) \right) \\
0 &= \lambda \left(\sum_{\ell=1}^{n-1} \mathfrak{c}_\ell \mathfrak{c}'_{\ell-1} + \lambda^{-1}\mathfrak{c}_0\mathfrak{c}'_{n-1} \right) \\
0 &= \sum_{\ell=1}^{n-1} \mathfrak{c}_\ell \mathfrak{c}'_{\ell-1} + \lambda^{-1}\mathfrak{c}_0\mathfrak{c}'_{n-1}.
\end{aligned}$$

On the other hand,

$$\begin{aligned}
\langle \mathfrak{c}, \tau_{(\lambda^{-1},\Theta)}(\mathfrak{c}') \rangle_H &= \langle \mathfrak{c}, \lambda^{-1}\Theta(\mathfrak{c}'_{n-1}), \Theta(\mathfrak{c}'_0), \ldots, \Theta(\mathfrak{c}'_{n-2}) \rangle_H \\
&= \mathfrak{c}_0\Theta(\lambda^{-1}\Theta(\mathfrak{c}'_{n-1})) + \mathfrak{c}_1\Theta(\Theta(\mathfrak{c}'_0)) + \cdots + \mathfrak{c}_{n-1}\Theta(\Theta(\mathfrak{c}'_{n-2})) \\
&= \lambda^{-1}\mathfrak{c}_0\mathfrak{c}'_{n-1} + \mathfrak{c}_1\mathfrak{c}'_0 + \cdots + \mathfrak{c}_{n-1}\mathfrak{c}'_{n-2} \\
&= 0.
\end{aligned}$$

Hence, for any element $\mathfrak{c}' \in \mathfrak{C}^{\perp_H}$, we have $\tau_{(\lambda^{-1},\Theta)}(\mathfrak{c}') \in \mathfrak{C}^{\perp_H}$. Thus, $\mathfrak{C}^{\perp_H}$ is a skew (Θ, λ^{-1}) constacyclic code over $\mathfrak{R}$. □

Now, we present the result on the generator polynomial for the Hermitian dual of a skew (θ, β_{ij})-constacyclic code over $\mathbb{F}_q$ given by Boucher et al. [13].

Lemma 7 ([13], Theorem 1) *Let $\beta \in \mathbb{F}_q^*$ and $\theta \in Aut(\mathbb{F}_q)$ such that $\theta(\beta) = \beta$. Let $\mathcal{C} = \langle f(x) \rangle$ be a skew (θ, λ)-constacyclic code of length n over $\mathbb{F}_q$ with $o(\theta) \mid n$. Then, there exists a polynomial $h(x) \in \mathbb{F}_q[x; \theta]$ such that $x^n - \beta = h(x)f(x)$ and $\mathcal{C}^{\perp_H}$ is generated by $h^\dagger(x)$ where $h^\dagger(x)$ is the skew Hermitian reciprocal polynomial of $h(x)$.*

In the below result, we provide the generator polynomial for the Hermitian dual of a skew (Θ, λ)-constacyclic code over $\mathfrak{R}$ by using Lemma 6 and Lemma 7.

Theorem 6 *Let $\mathfrak{C} = \bigoplus_{i,j} \gamma_{ij}\mathcal{D}_{ij}$ be a skew (Θ, λ)-constacyclic code of length n over $\mathfrak{R}$. If $\mathcal{D}_{ij} = \langle f_{ij}(x) \rangle$ such that $(x^n - \beta ij) = h_{ij}(x)f_{ij}(x)$ for each pair (i, j), $1 \le i \le r$, $1 \le j \le s$, respectively, then $\mathfrak{C}^{\perp_H} = \langle \sum_i \sum_j \gamma_{ij}h^\dagger_{ij}(x) \rangle$, where $h^\dagger_{ij}(x)$ is* *the skew Hermitian reciprocal polynomial of* $h_{ij}(x)$ *for each pair* (i, j), $1 \leq i \leq r$, $1 \leq j \leq s$.

We first recall the Hermitian construction (Lemma 8) for quantum code from Hermitian dual containing linear codes. Then, we establish a necessary and sufficient condition for skew (Θ, λ)-constacyclic code of length n over $\mathfrak{R}$ to contain their Hermitian dual in Theorem 7.

Lemma 8 [2] *Let* $\mathcal{C}$ *be a linear code* $\mathbb{F}_{q^2}$ *with parameters* $[n, k, d_H]$ *satisfying* $\mathcal{C}^{\perp_H} \subseteq \mathcal{C}$. *Then there exists a quantum code over* $\mathbb{F}_q$ *with parameters* $[[n, 2k - n, \geq d_H]]_q$.

Lemma 9 *Let* $\beta \in \mathbb{F}_{q^2}^*$ *such that* $\beta^2 = 1$. *Then, a skew* (θ, β)*-constacyclic code of length n over* $\mathbb{F}_{q^2}$ *with generator polynomial* $f(x)$ *contains its Hermitian dual if and only if* $h^\dagger(x)h(x)$ *is right divisible by* $x^n - \beta$, *where* $x^n - \beta = h(x)f(x)$ *and* $h^\dagger(x)$ *is the skew Hermitian reciprocal polynomial of* $h(x)$.

Proof We can get the desired result by following the same line of proof as in Lemma 5 and using the Hermitian inner product. □

Theorem 7 *Let* $\lambda \in \mathfrak{R}^*$ *and* $\mathfrak{C} = \bigoplus_{i,j} \gamma_{ij}\mathcal{D}_{ij}$ *be a skew* (Θ, λ)*-constacyclic code of length n over* $\mathfrak{R}$ *such that* $o(\Theta) \mid n$ *and* $\lambda^2 = 1$. *Then,* $\mathfrak{C}^{\perp_H} \subseteq \mathfrak{C}$ *if and only if for each pair* (i, j), $1 \leq i \leq r$, $1 \leq j \leq s$, *the polynomial* $h_{ij}^\dagger(x)h_{ij}(x)$ *is right divisible by* $x^n - \beta_{ij}$, *where* $x^n - \beta_{ij} = h_{ij}(x)f_{ij}(x)$ *and* $h_{ij}^\dagger(x)$ *is the skew Hermitian reciprocal polynomial of* $h_{ij}(x)$.

Proof Let $\mathfrak{C} = \bigoplus_{i,j} \gamma_{ij}\mathcal{D}_{ij}$ be a skew (Θ, λ)-constacyclic code of length n over $\mathfrak{R}$ such that $\mathfrak{C}^{\perp_H} \subseteq \mathfrak{C}$. Then by Lemma 6, $\bigoplus_{i,j} \gamma_{ij}\mathcal{D}_{ij}^{\perp_H} \subseteq \bigoplus_{i,j} \gamma_{ij}\mathcal{D}_{ij}$. Since $o(\Theta) \mid n$ and $\lambda^2 = 1$, for each pair (i, j), $1 \leq i \leq r$, $1 \leq j \leq s$ and by taking modulo γ_{ij} in the above equation, we have $\mathcal{D}_{ij}^{\perp_H} \subseteq \mathcal{D}_{ij}$. Therefore, by Lemma 9, $h_{ij}^\dagger h_{ij}$ is right divisible by $x^n - \beta_{ij}$ for each pair (i, j), $1 \leq i \leq r$, $1 \leq j \leq s$ where $x^n - \beta_{ij} = h_{ij}(x)f_{ij}(x)$ and $h_{ij}^\dagger(x)$ is the Hermitian skew-reciprocal polynomial of $h_{ij}(x)$.

Conversely, let $x^n - \beta_{ij}$ be a right divisor of $h_{ij}^\dagger(x)h_{ij}(x)$ for each pair (i, j), $1 \leq i \leq r$, $1 \leq j \leq s$. Then by Lemma 9, we get $\mathcal{D}_{ij}^{\perp_H} \subseteq \mathcal{D}_{ij}$. This implies that $\bigoplus_{i,j} \gamma_{ij}\mathcal{D}_{ij}^{\perp_H} \subseteq \bigoplus_{i,j} \gamma_{ij}\mathcal{D}_{ij}$. Thus, $\mathfrak{C}^{\perp_H} \subseteq \mathfrak{C}$. □

Now, using Hermitian construction (Lemma 8) and Theorem 7, we obtain quantum codes in the next result. Here, we follow the same notations as used in the Theorem 5.

Theorem 8 *Let* $\lambda \in \mathfrak{R}^*$ *and* $\mathfrak{C} = \bigoplus_{i,j} \gamma_{ij}\mathcal{D}_{ij}$ *be a skew* (Θ, λ)*-constacyclic code of length n over* $\mathfrak{R}$ *such that* $\lambda^2 = 1$. *If* $\mathfrak{C}^{\perp_H} \subseteq \mathfrak{C}$, *then there exists a quantum code with parameters* $[[rsn, 2K - rsn, d_G]]_q$.

Remark 3 To find quantum codes from Hermitian dual containing skew (Θ, λ)-constacyclic codes over $\mathfrak{R}$, we just need to change the finite field $\mathbb{F}_{p^e}$ by $\mathbb{F}_{q^2}$ where $q = p^e$ and $h^*(x)$ by $h^\dagger(x)$ in $Algorithm$ 1 and 2, respectively.

Example 3 Let $deg(f(u)) = 1$, $g(v) = v^2 - 1$ and $\mathbb{F}_{11^2} = \mathbb{F}_{11}(\mathfrak{t})$ where $\mathfrak{t}^2 = 4\mathfrak{t} + 9$. Then, $\mathfrak{R} = \mathbb{F}_{11^2}[v]/\langle v^2 - 1\rangle$, $\gamma_{11} = \frac{1}{2}(1+v)$ and $\gamma_{12} = \frac{1}{2}(1-v)$. Define $\Theta \in Aut(\mathfrak{R})$ by $\Theta(r_0 + r_1 v) = r_0^{11} + r_1^{11} v$ and $\lambda = -v$. Let $\mathfrak{C}$ be a skew (Θ, λ)-constacyclic code of length 12 over $\mathfrak{R}$ generated by $f(x) = \gamma_{11} f_{11}(x) + \gamma_{12} f_{12}(x)$ where $f_{11} = x^2 + \mathfrak{t}^2 x + 10$ and $f_{12} = x + \mathfrak{t}^5$ are generator polynomials of a skew negacyclic code $\mathcal{D}_{11}$ and a skew cyclic code $\mathcal{D}_{12}$, respectively. Here,

$$
\begin{aligned}
h_{11}(x) &= x^{10} + \mathfrak{t}^{62}x^9 + 5x^8 + \mathfrak{t}^{50}x^7 + 7x^6 + \mathfrak{t}^{74}x^5 + 4x^4 + \mathfrak{t}^{50}x^3 \\
&\quad +6x^2 + \mathfrak{t}^{62}x + 10, \\
h_{11}^{\dagger}(x) &= x^{10} + \mathfrak{t}^{2}x^9 + 5x^8 + \mathfrak{t}^{110}x^7 + 7x^6 + \mathfrak{t}^{14}x^5 + 4x^4 + \mathfrak{t}^{110}x^3 \\
&\quad +6x^2 + \mathfrak{t}^{2}x + 10, \\
h_{11}^{\dagger}(x)h_{11}(x) &= (x^8 + 6x^6 + 2x^4 + 6x^2 + 1)(x^{12} + 1),
\end{aligned}
$$

and

$$
\begin{aligned}
h_{12}(x) &= x^{11} + \mathfrak{t}^{115}x^{10} + 10x^9 + \mathfrak{t}^{55}x^8 + x^7 + \mathfrak{t}^{115}x^6 + 10x^5 + \mathfrak{t}^{55}x^4 \\
&\quad +x^3 + \mathfrak{t}^{115}x^2 + 10x + \mathfrak{t}^{55}, \\
h_{12}^{\dagger}(x) &= x^{11} + \mathfrak{t}^{5}x^{10} + 10x^9 + \mathfrak{t}^{65}x^8 + x^7 + \mathfrak{t}^{5}x^6 + 10x^5 + \mathfrak{t}^{65}x^4 + x^3 \\
&\quad +\mathfrak{t}^{5}x^2 + 10x + \mathfrak{t}^{65}, \\
h_{12}^{\dagger}(x)h_{12}(x) &= (x^{10} + 10x^8 + x^6 + 10x^4 + x^2 + 10)(x^{12} - 1).
\end{aligned}
$$

Since $h_{11}^{\dagger}(x)h_{11}(x)$ and $h_{12}^{\dagger}(x)h_{12}(x)$ are right divisible by $x^{12}+1$ and $x^{12}-1$, respectively. Thus, by Theorem 7, we have $\mathfrak{C}^{\perp_H} \subseteq \mathfrak{C}$. Also, let

$$
M = \begin{bmatrix} 1 & 1 \\ 1 & 10 \end{bmatrix} \in GL_2(\mathbb{F}_{11^2}),
$$

satisfying $MM^T = 2I_2$. Then, the gray image $\varphi(\mathfrak{C})$ is also a dual containing code with parameters $[24, 21, 4]_{11^2}$. Therefore, by Theorem 8, there exists a quantum code with parameters $[[24, 18, 4]]_{11}$. It is noted that our obtained code has a better code rate and minimum distance than the known code $[[24, 16, 3]]_{11}$ presented in [23].

5 LCD skew (Θ, λ)-constacyclic code over $\mathfrak{R}$ and their gray images

In this section, we begin with the definition of LCD codes introduced by Massey [46] and outline some known results for skew LCD codes. Then, we establish necessary and sufficient conditions for skew (Θ, λ)-constacyclic codes over $\mathfrak{R}$ to be LCD.

Definition 5 Let $\mathfrak{C}$ be a linear code of length n over $\mathfrak{R}$. Suppose $\mathfrak{C}^{\perp_E}$ and $\mathfrak{C}^{\perp_H}$ are the Euclidean and the Hermitian duals of $\mathfrak{C}$, respectively. Then, code $\mathfrak{C}$ is said to be

Euclidean LCD code if and only if $\mathfrak{C} \cap \mathfrak{C}^{\perp_E} = \{0\}$ and Hermitian LCD if and only if $\mathfrak{C} \cap \mathfrak{C}^{\perp_H} = \{0\}$.

Recently, Boulanouar et al. [6] established some conditions for skew constacyclic codes over the finite field to be LCD as follows:

Lemma 10 ([6], Theorem 2) *Let $\beta \in \mathbb{F}_{p^e}^*$ such that $\beta^2 = 1$ and $\theta \in Aut(\mathbb{F}_{p^e})$. Let $\mathcal{D}$ be a skew (θ, β)-constacyclic code of length n over $\mathbb{F}_{p^e}$ such that the order of θ divides n. Let $f(x)$ be a generator polynomial of $\mathcal{D}$ satisfies $x^n - \beta = h(x) f(x) = f(x) h(x)$. Then,*

1. *$\mathcal{D}$ is Euclidean LCD if and only if $gcrd(f(x), h^*(x)) = 1$.*
2. *When the order of θ is 2, then $\mathcal{D}$ is Hermitian LCD if and only if $gcrd(f(x), h^{\dagger}(x)) = 1$.*

Theorem 9 *Let $\lambda \in \mathfrak{R}^*$ and $\mathfrak{C} = \bigoplus_{i,j} \gamma_{ij} \mathcal{D}_{ij}$ be a skew (Θ, λ)-constacyclic code of length n over $\mathfrak{R}$ such that n is a multiple of $o(\Theta)$ and $\lambda^2 = 1$, where $\mathcal{D}_{ij} = \langle f_{ij}(x) \rangle$ is the skew (θ, β_{ij})-constacyclic code of length n over $\mathbb{F}_{p^e}$ for each pair (i, j), $1 \le i \le r$, $1 \le j \le s$. Then, $\mathfrak{C}$ is Euclidean LCD if and only if for each pair (i, j), $1 \le i \le r$, $1 \le j \le s$, $gcrd(f_{ij}(x), h_{ij}^*(x)) = 1$, where $x^n - \beta_{ij} = h_{ij}(x) f_{ij}(x)$ and $h_{ij}^*(x)$ is the skew-reciprocal polynomial of $h_{ij}(x)$.*

Proof Suppose $\mathfrak{C} = \bigoplus_{i,j} \gamma_{ij} \mathcal{D}_{ij}$ be a skew (Θ, λ)-constacyclic code of length n over $\mathfrak{R}$ which is Euclidean LCD i.e., $\mathfrak{C} \cap \mathfrak{C}^{\perp_E} = \{0\}$. Then, by Lemma 3,

$$\bigoplus_{i,j} \gamma_{ij} \mathcal{D}_{ij} \cap \bigoplus_{i,j} \gamma_{ij} \mathcal{D}_{ij}^{\perp_E} = \{0\}. \tag{1}$$

As n is a multiple of $o(\Theta)$ and $\lambda^2 = 1$, for each pair (i, j), $1 \le i \le r$, $1 \le j \le s$, under modulo γ_{ij} in Eq. 1, we have $\mathcal{D}_{ij} \cap \mathcal{D}_{ij}^{\perp_E} = \{0\}$. Therefore, by Lemma 10, for each pair (i, j), $1 \le i \le r$, $1 \le j \le s$, we have $gcrd(f_{ij}(x), h_{ij}^*(x)) = 1$, where $x^n - \beta_{ij} = h_{ij}(x) f_{ij}(x)$ and $h_{ij}^*(x)$ is the skew-reciprocal polynomial of $h_{ij}(x)$.

Conversely, let $gcrd(f_{ij}(x), h_{ij}^*(x)) = 1$ for each pair (i, j), $1 \le i \le r$, $1 \le j \le s$. Then again by Lemma 10, $\mathcal{D}_{ij} \cap \mathcal{D}_{ij}^{\perp_E} = \{0\}$, which implies $\bigoplus_{i,j} \gamma_{ij} \mathcal{D}_{ij} \cap \bigoplus_{i,j} \gamma_{ij} \mathcal{D}_{ij}^{\perp_E} = \{0\}$. Therefore, $\mathfrak{C} \cap \mathfrak{C}^{\perp_E} = \{0\}$. □

The following result gives the necessary and sufficient condition for skew (Θ, λ)-constacyclic code over $\mathfrak{R}$ to be Hermitian LCD. The proof follows the same line as in Theorem 9 with the Hermitian inner product.

Theorem 10 *Let $o(\Theta) = 2$ and $\lambda \in \mathfrak{R}^*$ such that $\lambda^2 = 1$. Let $\mathfrak{C} = \bigoplus_{i,j} \gamma_{ij} \mathcal{D}_{ij}$ be a skew (Θ, λ)-constacyclic code of length n over $\mathfrak{R}$ such that n is a multiple of $o(\Theta)$, where $\mathcal{D}_{ij} = \langle f_{ij}(x) \rangle$ is the skew (θ, β_{ij})-constacyclic code of length n over $\mathbb{F}_{p^e}$ for each pair (i, j), $1 \le i \le r$, $1 \le j \le s$. Then, $\mathfrak{C}$ is Hermitian LCD if and only if for each pair (i, j), $1 \le i \le r$, $1 \le j \le s$, $gcrd(f_{ij}(x), h_{ij}^{\dagger}(x)) = 1$ where $x^n - \beta_{ij} = h_{ij}(x) f_{ij}(x)$ and $h_{ij}^{\dagger}(x)$ is the skew Hermitian reciprocal polynomial of $h_{ij}(x)$.*

Corollary 2 *Let $\lambda \in \mathfrak{R}^*$ such that $\lambda^2 = 1$. Let $\mathfrak{C} = \bigoplus_{i,j} \gamma_{ij}\mathcal{D}_{ij}$ be a skew (Θ, λ)-constacyclic code of length n over $\mathfrak{R}$ such that n is a multiple of $o(\Theta)$, where $\mathcal{D}_{ij} = \langle f_{ij}(x)\rangle$ is the skew (θ, β_{ij})-constacyclic code of length n over $\mathbb{F}_{p^e}$ for each pair (i, j), $1 \le i \le r$, $1 \le j \le s$. Then,*

1. *$\mathfrak{C}$ is Euclidean LCD if and only if for each pair (i, j), $1 \le i \le r$, $1 \le j \le s$, code $\mathcal{D}_{ij}$ is Euclidean LCD.*
2. *If $o(\Theta) = 2$, then $\mathfrak{C}$ is Hermitian LCD if and only if for each pair (i, j), $1 \le i \le r$, $1 \le j \le s$, code $\mathcal{D}_{ij}$ is Hermitian LCD.*

Proof It follows from Lemma 10, Theorems 9 and 10. □

The following *Algorithm* provides us the generators of Euclidean LCD skew (Θ, λ)-constacyclic codes over $\mathfrak{R}$:

Algorithm 3. Input: $(\mathbb{F}_{p^e}, n)$

- **Step 1:** Define skew polynomial ring over $\mathbb{F}_{p^e}$;
- **Step 2:** Find all k-degree skew polynomials $f_{ij}(x)$ which are right divisors of $x^n \pm 1$;
- **Step 3:** Find skew-reciprocal polynomial $h^*_{ij}(x)$ of $h_{ij}(x)$ where $x^n \pm 1 = h_{ij}(x) f_{ij}(x)$;
- **Step 4:** If $gcrd(f_{ij}(x), h^*_{ij}(x)) = 1$, then Print$(f_{ij}(x))$;

Output: Generator polynomial of all Euclidean LCD skew negacyclic (cyclic) code of length n and dimension $n - k$ over $\mathbb{F}_{p^e}$.

Let $\mathfrak{C} = \langle f(x)\rangle$ be a skew (Θ, λ)-constacyclic code of length n over $\mathfrak{R}$ where $\lambda^2 = 1$. Then, $f(x) = \sum_{i=1}^{r} \sum_{j=1}^{s} \gamma_{ij} f_{ij}(x)$ where f_{ij} are generator polynomials of skew cyclic (or skew negacyclic) codes $\mathcal{D}_{ij}$ of length n over $\mathbb{F}_{p^e}$, for each pair (i, j), $1 \le i \le r$, $1 \le j \le s$ obtained by *Algorithm* 3. Then, $\mathfrak{C}$ is Euclidean LCD skew (Θ, λ)-constacyclic code over $\mathfrak{R}$. Now, by using *Algorithm* 2, we can obtain the minimum weight d_H of $\varphi(\mathfrak{C})$ and hence an LCD code with parameters $[rsn, K, d_H]$ over $\mathbb{F}_{p^e}$.

Remark 4 Note that for Hermitian LCD codes, we have to fix the order of automorphism 2 and $h^*_{ij}(x)$ by $h^\dagger_{ij}(x)$ in *Algorithm* 3.

Example 4 Let $\mathbb{F}_{3^2} = \mathbb{F}_3(\mathfrak{t})$, where $\mathfrak{t}^2 = \mathfrak{t} + 1$, $deg(f(u))$ is 1 and $g(v) = v^2 - 1$. Then, $\mathfrak{R} = \mathbb{F}_{3^2}[v]/\langle v^2 - 1\rangle$, $\gamma_{11} = \frac{1}{2}(1 + v)$ and $\gamma_{12} = \frac{1}{2}(1 - v)$. Consider $\Theta \in Aut(\mathfrak{R})$ is defined by $\Theta(r_0 + r_1 v) = r_0^3 + r_1^3 v$ and $\lambda = 1$. Let $\mathfrak{C}$ be a skew (Θ, λ)-constacyclic code of length 4 over $\mathfrak{R}$ generated by $f(x) = \gamma_{11} f_{11}(x) + \gamma_{12} f_{12}(x)$, where $f_{11} = x^2 + \mathfrak{t}^6 x + \mathfrak{t}^3$ and $f_{12} = x + \mathfrak{t}$ are generator polynomials of skew cyclic codes $\mathcal{D}_{11}$ and $\mathcal{D}_{12}$, respectively. Here, $h_{11}(x)$, $h^*_{11}(x)$ and $h^\dagger_{11}(x)$ are $x^2 + \mathfrak{t}^2 x + \mathfrak{t}$, $x^2 + \mathfrak{t}^5 x + \mathfrak{t}^7$ and $x^2 + \mathfrak{t}^7 x + \mathfrak{t}^5$, respectively. Also, $h_{12}(x)$, $h^*_{12}(x)$ and $h^\dagger_{12}(x)$ are $x^3 + \mathfrak{t}^7 x^2 + 2x + \mathfrak{t}^3$, $x^2 + \mathfrak{t}^5 x + \mathfrak{t}^7$ and $x^3 + \mathfrak{t}x^2 + 2x + \mathfrak{t}^5$, respectively.

Then, $gcrd(f_{11}(x), h^*_{11}(x)) = 1$ and $gcrd(f_{12}(x), h^*_{12}(x)) = 1$. However,

$gcrd(f_{11}(x), h^\dagger_{11}(x)) = x + \mathfrak{t}^7$ and $gcrd(f_{12}(x), h^\dagger_{12}(x)) = x + \mathfrak{t}$. Hence, by Theorem 10, $\mathfrak{C}$ is Euclidean LCD but not Hermitian LCD. Again, let

$$M = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} \in GL_2(\mathbb{F}_{3^2}),$$

satisfying $MM^T = 2I_2$. Then, the gray image $\varphi(\mathfrak{C})$ with parameters $[8, 5, 4]_{3^2}$ is an MDS Euclidean LCD skew 2-quasi-cyclic code.

Example 5 Let $\mathbb{F}_{17^2} = \mathbb{F}_{17}(\mathfrak{t})$, where $\mathfrak{t}^2 = \mathfrak{t} + 14$, $deg(f(u))$ is 1 and $g(v) = v^2 - v$. Then, $\mathfrak{R} = \mathbb{F}_{17^2}[v]/\langle v^2 - v\rangle$, $\gamma_{11} = v$ and $\gamma_{12} = 1 - v$. We define $\Theta \in Aut(\mathfrak{R})$ as $\Theta(r_0 + r_1 v) = r_0^{17} + r_1^{17} v$ and let $\lambda = 1$. Let $\mathfrak{C}$ be a skew (Θ, λ)-constacyclic code of length 18 over $\mathfrak{R}$ generated by $f(x) = \gamma_{11} f_{11}(x) + \gamma_{12} f_{12}(x)$, where $f_{11} = x^2 + \mathfrak{t}^{28} x + \mathfrak{t}^{32}$ and $f_{12} = x + \mathfrak{t}^{48}$ are generator polynomials of skew cyclic codes $\mathcal{D}_{11}$ and $\mathcal{D}_{12}$, respectively. Here,

$$\begin{aligned}
h_{11}(x) &= x^{16} + \mathfrak{t}^{172}x^{15} + \mathfrak{t}^{93}x^{14} + \mathfrak{t}^{82}x^{13} + \mathfrak{t}^{286}x^{12} + \mathfrak{t}^{46}x^{11} + \mathfrak{t}^{233}x^{10} + \mathfrak{t}^{154}x^{9} \\
&\quad + \mathfrak{t}^{83}x^{8} + \mathfrak{t}^{10}x^{7} + \mathfrak{t}^{41}x^{6} + \mathfrak{t}^{190}x^{5} + \mathfrak{t}^{78}x^{4} + \mathfrak{t}^{226}x^{3} + \mathfrak{t}^{253}x^{2} + \mathfrak{t}^{28}x + \mathfrak{t}^{112}, \\
h^*_{11}(x) &= x^{16} + \mathfrak{t}^{140}x^{15} + \mathfrak{t}^{93}x^{14} + \mathfrak{t}^{50}x^{13} + \mathfrak{t}^{286}x^{12} + \mathfrak{t}^{174}x^{11} + \mathfrak{t}^{153}x^{10} + \mathfrak{t}^{282}x^{9} \\
&\quad + \mathfrak{t}^{83}x^{8} + \mathfrak{t}^{266}x^{7} + \mathfrak{t}^{41}x^{6} + \mathfrak{t}^{30}x^{5} + \mathfrak{t}^{110}x^{4} + \mathfrak{t}^{66}x^{3} + \mathfrak{t}^{205}x^{2} + \mathfrak{t}^{156}x + \mathfrak{t}^{112}, \\
h^\dagger_{11}(x) &= x^{16} + \mathfrak{t}^{76}x^{15} + \mathfrak{t}^{141}x^{14} + \mathfrak{t}^{274}x^{13} + \mathfrak{t}^{254}x^{12} + \mathfrak{t}^{78}x^{11} + \mathfrak{t}^{9}x^{10} + \mathfrak{t}^{186}x^{9} \\
&\quad + \mathfrak{t}^{259}x^{8} + \mathfrak{t}^{202}x^{7} + \mathfrak{t}^{121}x^{6} + \mathfrak{t}^{222}x^{5} + \mathfrak{t}^{142}x^{4} + \mathfrak{t}^{258}x^{3} + \mathfrak{t}^{29}x^{2} + \mathfrak{t}^{60}x + \mathfrak{t}^{176};
\end{aligned}$$

and

$$\begin{aligned}
h_{12}(x) &= x^{17} + \mathfrak{t}^{96}x^{16} + x^{15} + \mathfrak{t}^{96}x^{14} + x^{13} + \mathfrak{t}^{96}x^{12} + x^{11} + \mathfrak{t}^{96}x^{10} + x^{9} \\
&\quad + \mathfrak{t}^{96}x^{8} + x^{7} + \mathfrak{t}^{96}x^{6} + x^{5} + \mathfrak{t}^{96}x^{4} + x^{3} + \mathfrak{t}^{96}x^{2} + x + \mathfrak{t}^{96}, \\
h^*_{12}(x) &= x^{17} + \mathfrak{t}^{192}x^{16} + x^{15} + \mathfrak{t}^{192}x^{14} + x^{13} + \mathfrak{t}^{192}x^{12} + \mathfrak{t}^{96}x^{11} + \mathfrak{t}^{192}x^{10} + x^{9} \\
&\quad + \mathfrak{t}^{192}x^{8} + x^{7} + \mathfrak{t}^{192}x^{6} + \mathfrak{t}^{96}x^{5} + \mathfrak{t}^{192}x^{4} + \mathfrak{t}^{96}x^{3} + \mathfrak{t}^{192}x^{2} + \mathfrak{t}^{96}x + \mathfrak{t}^{192}, \\
h^\dagger_{11}(x) &= x^{17} + \mathfrak{t}^{96}x^{16} + x^{15} + \mathfrak{t}^{96}x^{14} + x^{13} + \mathfrak{t}^{96}x^{12} + \mathfrak{t}^{192}x^{11} + \mathfrak{t}^{96}x^{10} + x^{9} \\
&\quad + \mathfrak{t}^{96}x^{8} + x^{7} + \mathfrak{t}^{96}x^{6} + \mathfrak{t}^{192}x^{5} + \mathfrak{t}^{96}x^{4} + \mathfrak{t}^{192}x^{3} + \mathfrak{t}^{96}x^{2} + \mathfrak{t}^{192}x + \mathfrak{t}^{96}.
\end{aligned}$$

Since $gcrd(f_{11}(x), h^*_{11}(x)) = 1$ and $gcrd(f_{12}(x), h^*_{12}(x)) = 1$ as well as $gcrd(f_{11}(x), h^\dagger_{11}(x)) = 1$ and $gcrd(f_{12}(x), h^\dagger_{12}(x)) = 1$. Therefore, by Theorem 10, $\mathfrak{C}$ is both Euclidean and Hermitian LCD codes. Again, let

$$M = \begin{bmatrix} 1 & 1 \\ 1 & 16 \end{bmatrix} \in GL_2(\mathbb{F}_{17^2}),$$

satisfying $MM^T = 2I_2$. Then, the gray image $\varphi(\mathfrak{C})$ with parameters $[36, 33, 4]_{17^2}$ is an MDS skew 2-quasi-cyclic code.

6 Computation table

Consider the finite field $\mathbb{F}_{p^2}$ and $deg(f(u)) = 1$ and $g(v) = v^2 - 1$. Then, $\mathfrak{R} := \mathbb{F}_{p^2}[v]/\langle v^2 - 1\rangle$, $\gamma_{11} = \frac{1}{2}(1 + v)$ and $\gamma_{12} = \frac{1}{2}(1 - v)$. Let $\Theta \in Aut(\mathfrak{R})$ defined by $\Theta(a + bv) = a^p + b^p v$ and $\lambda = a + bv \in \mathfrak{R}^*$. Then, $\beta_{11} = a + b$ and $\beta_{12} = a - b$ are corresponding units in $\mathbb{F}_{p^2}$. Let $\mathfrak{C}$ be a skew (Θ, λ)-constacyclic code of length n

Table 1 Quantum codes from Euclidean dual containing skew (Θ, λ)-constacyclic codes over $\mathfrak{R}$

p^m	n	λ	(β_{11}, β_{12})	$f_{11}(x)$	$f_{12}(x)$	$\Phi(\mathfrak{C})$	$[[n, k, d]]_{p^m}$
5^2	6	-1	$(-1, -1)$	$1\mathfrak{t}^{3}1$	2311	$[12, 7, 6]$	$[[12, 2, 6]]_{25}$
5^2	8	1	$(1, 1)$	$\mathfrak{t}^{19}\mathfrak{t}^{2}1$	$\mathfrak{t}^{21}\mathfrak{t}^{5}1$	$[16, 12, 4]$	$[[16, 8, 4]]_{25}$
5^2	8	1	$(1, 1)$	$\mathfrak{t}^{9}\mathfrak{t}1$	$\mathfrak{t}^{5}1$	$[16, 13, 3]$	$[[16, 10, 3]]_{25}$
5^2	8	$-v$	$(-1, 1)$	$\mathfrak{t}^{20}\mathfrak{t}^{19}\mathfrak{t}^{19}\mathfrak{t}^{7}1$	$\mathfrak{t}^{7}\mathfrak{t}^{3}1$	$[16, 10, 6]$	$[[16, 4, 6]]_{25}$
5^2	10	$-v$	$(-1, 1)$	$\mathfrak{t}^{2}\mathfrak{t}^{19}01$	11	$[20, 16, 4]$	$[[20, 12, 4]]_{25}$
5^2	12	-1	$(-1, -1)$	$\mathfrak{t}^{23}\mathfrak{t}^{11}\mathfrak{t}^{8}1$	$\mathfrak{t}1$	$[24, 20, 4]$	$[[24, 16, 4]]_{25}$
5^2	16	1	$(1, 1)$	$3\mathfrak{t}^{21}\mathfrak{t}^{23}1$	$\mathfrak{t}^{15}\mathfrak{t}^{16}1$	$[32, 27, 4]$	$[[32, 22, 4]]_{25}$
5^2	30	v	$(1, -1)$	$\mathfrak{t}^{20}1$	$\mathfrak{t}^{8}\mathfrak{t}^{17}1$	$[60, 57, 3]$	$[[60, 54, 3]]_{25}$
7^2	4	$-v$	$(-1, 1)$	$\mathfrak{t}^{42}\mathfrak{t}^{13}1$	$\mathfrak{t}^{12}1$	$[8, 5, 4]$	$[[8, 2, 4]]_{49}$
7^2	6	$-v$	$(-1, 1)$	$\mathfrak{t}^{30}\mathfrak{t}^{47}1$	$\mathfrak{t}^{36}1$	$[12, 9, 4]$	$[[12, 6, 4]]_{49}$
7^2	8	$-v$	$(-1, 1)$	$\mathfrak{t}^{6}\mathfrak{t}^{27}1$	$\mathfrak{t}^{36}1$	$[16, 13, 4]$	$[[16, 10, 4]]_{49}$
7^2	8	v	$(1, -1)$	$\mathfrak{t}^{12}\mathfrak{t}^{11}\mathfrak{t}^{23}1$	$\mathfrak{t}^{6}\mathfrak{t}^{3}1$	$[16, 11, 6]$	$[[16, 6, 6]]_{49}$
7^2	12	$-v$	$(-1, 1)$	$\mathfrak{t}^{4}01$	$\mathfrak{t}1$	$[24, 21, 3]$	$[[24, 18, 3]]_{49}$
7^2	12	$-v$	$(-1, 1)$	$\mathfrak{t}\mathfrak{t}^{4}11$	$\mathfrak{t}1$	$[24, 20, 4]$	$[[24, 16, 4]]_{49}$
7^2	16	1	$(1, 1)$	$\mathfrak{t}^{12}\mathfrak{t}^{4}21$	$\mathfrak{t}^{12}1$	$[32, 28, 4]$	$[[32, 24, 4]]_{49}$
7^2	20	$-v$	$(-1, 1)$	$\mathfrak{t}^{30}\mathfrak{t}^{7}\mathfrak{t}^{13}\mathfrak{t}^{5}1$	$\mathfrak{t}^{12}1$	$[40, 35, 4]$	$[[40, 30, 4]]_{49}$
11^2	4	$-v$	$(-1, 1)$	$\mathfrak{t}^{110}\mathfrak{t}^{19}1$	$\mathfrak{t}^{30}1$	$[8, 5, 4]$	$[[8, 2, 4]]_{121}$
11^2	10	v	$(1, -1)$	211	$\mathfrak{t}^{11}1$	$[20, 17, 4]$	$[[20, 14, 4]]_{121}$
11^2	10	v	$(1, -1)$	211	$\mathfrak{t}^{88}\mathfrak{t}^{2}1$	$[20, 16, 5]$	$[[20, 12, 5]]_{121}$
11^2	10	1	$(1, 1)$	211	$\mathfrak{t}^{78}1\mathfrak{t}^{54}1$	$[20, 15, 6]$	$[[20, 10, 6]]_{121}$
11^2	24	$-v$	$(-1, 1)$	$\mathfrak{t}^{5}01$	$\mathfrak{t}^{30}1$	$[48, 45, 3]$	$[[48, 42, 3]]_{121}$
11^2	24	$-v$	$(-1, 1)$	$\mathfrak{t}^{15}11$	$\mathfrak{t}^{30}01$	$[48, 44, 4]$	$[[48, 40, 4]]_{121}$
13^2	8	1	$(1, 1)$	81	$\mathfrak{t}^{3}1$	$[16, 14, 3]$	$[[16, 12, 3]]_{169}$
13^2	8	$-v$	$(-1, 1)$	$\mathfrak{t}^{9}\mathfrak{t}^{39}1$	$\mathfrak{t}^{99}1$	$[16, 13, 4]$	$[[16, 10, 4]]_{169}$
13^2	14	-1	$(-1, -1)$	$\mathfrak{t}^{144}\mathfrak{t}^{121}1$	51	$[28, 25, 4]$	$[[28, 22, 4]]_{169}$
17^2	4	-1	$(-1, -1)$	$\mathfrak{t}^{32}\mathfrak{t}^{25}1$	$\mathfrak{t}^{4}1$	$[8, 5, 4]$	$[[8, 2, 4]]_{289}$

over $\mathfrak{R}$ generated by $f(x) = \gamma_{11} f_{11}(x) + \gamma_{12} f_{12}(x)$, where f_{11} and f_{12} are generator polynomials of skew (θ, β_{11})-constacyclic code $\mathcal{D}_{11}$ and skew (θ, β_{12})-constacyclic code $\mathcal{D}_{12}$, respectively. In order to find out gray image of $\mathfrak{C}$, we use the matrix

$$M = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \in GL_2(\mathbb{F}_{p^2}),$$

which is satisfying $MM^T = 2I_2$. In this way, we obtain many new and better quantum codes presented in Tables 1 and 2 from skew (Θ, λ)-constacyclic code over $\mathfrak{R}$ by applying CSS and Hermitian constructions, respectively. Table 3 contains some MDS linear codes, which are the gray images of the Euclidean LCD skew (Θ, λ)-constacyclic codes over $\mathfrak{R}$. Note that, instead of writing the whole polynomial, we

Table 2 Quantum codes from Hermitian dual containing skew (Θ, λ)-constacyclic codes over $\mathfrak{R}$

n	λ	(β_{11}, β_{12})	$f_{11}(x)$	$f_{12}(x)$	$\Phi(\mathfrak{C})$	$[[n, k, d]]_p$	$[[n', k', d']]_p$
10	$-v$	$(-1, 1)$	$\mathfrak{t}^4\mathfrak{t}^{23}\mathfrak{t}^{14}\mathfrak{t}^{22}1$	$\mathfrak{t}^4\mathfrak{t}^3 1$	$[20, 14, 5]$	$[[20, 8, 5]]_5$	$[[20, 4, 5]]_5$ [32]
20	1	$(1, 1)$	1031	11	$[40, 36, 3]$	$[[40, 32, 3]]_5$	$[[40, 24, 3]]_5$ [45]
6	-1	$(-1, -1)$	$\mathfrak{t}1$	$\mathfrak{t}^3 1$	$[12, 10, 3]$	$[[12, 8, 3]]_7$	MDS
6	-1	$(-1, -1)$	$\mathfrak{t}^5 1$	$\mathfrak{t}^{10} 11$	$[12, 9, 4]$	$[[12, 6, 4]]_7$	MDS
12	1	$(1, 1)$	$\mathfrak{t}^4 101$	$\mathfrak{t}^{19} 11$	$[24, 19, 5]$	$[[24, 14, 5]]_7$	$[[24, 12, 5]]_7$ [39]
14	$-v$	$(-1, 1)$	$\mathfrak{t}^9\mathfrak{t}^{33} 11$	11	$[28, 24, 4]$	$[[28, 20, 4]]_7$	$[[28, 18, 4]]_7$ [32]
32	1	$(1, 1)$	$\mathfrak{t}^6\mathfrak{t}^{11}\mathfrak{t}^{10} 1$	$\mathfrak{t}^3\mathfrak{t}1$	$[64, 59, 4]$	$[[64, 54, 4]]_7$	$[[61, 51, 4]]_7$ [18]
56	1	$(1, 1)$	$\mathfrak{t}^3 1$	$\mathfrak{t}^{45} 11$	$[112, 109, 3]$	$[[112, 106, 3]]_7$	$[[112, 101, 3]]_7$ [18]
12	$-v$	$(-1, 1)$	$(10)\mathfrak{t}^2 1$	$\mathfrak{t}^5 1$	$[24, 21, 4]$	$[[24, 18, 4]]_{11}$	MDS
6	-1	$(-1, -1)$	$\mathfrak{t}^2 1$	911	$[12, 9, 4]$	$[[12, 6, 4]]_{13}$	MDS
8	1	$(1, 1)$	$\mathfrak{t}^3 1$	$\mathfrak{t}^9\mathfrak{t}1$	$[16, 13, 3]$	$[[16, 10, 3]]_{13}$	$[[16, 8, 3]]_{13}$ [32]
12	-1	$(-1, -1)$	$\mathfrak{t}1$	$\mathfrak{t}^{44} 11$	$[24, 21, 4]$	$[[24, 18, 4]]_{13}$	MDS
8	v	$(1, -1)$	$\mathfrak{t}^4 1$	$\mathfrak{t}^{152} 11$	$[16, 13, 4]$	$[[16, 10, 4]]_{17}$	MDS
24	$-v$	$(-1, 1)$	$\mathfrak{t}^4\mathfrak{t}^6 1$	$\mathfrak{t}^4 1$	$[48, 45, 3]$	$[[48, 42, 3]]_{17}$	$[[48, 40, 3]]_{17}$ [32]

Table 3 Gray images of LCD skew (Θ, λ)-constacyclic codes over $\mathfrak{R}$

n	λ	(β_{11}, β_{12})	$f_{11}(x)$	$f_{12}(x)$	$\Phi(\mathfrak{C})$	BKLC/MDS
4	1	(1, 1)	$\mathfrak{t}^3\mathfrak{t}^6 1$	$\mathfrak{t}1$	$[8, 5, 4]_9$	MDS
14	v	(1, −1)	11	$\mathfrak{t}1$	$[28, 26, 2]_9$	BKLC
12	v	(1, −1)	$\mathfrak{t}^5 1$	$1\mathfrak{t}^{37} 1$	$[24, 21, 4]_{121}$	MDS
8	1	(1, 1)	$\mathfrak{t}^3 1$	51	$[16, 14, 3]_{169}$	MDS
12	1	(1, 1)	11	$\mathfrak{t}^{18} 1$	$[24, 22, 3]_{169}$	MDS
12	1	(1, 1)	$\mathfrak{t}^6 1$	$\mathfrak{t}^{32} 31$	$[24, 21, 4]_{169}$	MDS
14	v	(1, −1)	$\mathfrak{t}^{12} 1$	$\mathfrak{t}^6 1$	$[28, 26, 3]_{169}$	MDS
8	1	(1, 1)	$\mathfrak{t}^4 1$	41	$[16, 14, 3]_{289}$	MDS
12	v	(1, −1)	$\mathfrak{t}^8 1$	$\mathfrak{t}^4 1$	$[24, 22, 3]_{289}$	MDS
12	1	(1, 1)	$\mathfrak{t}^4 1$	$\mathfrak{t}^{16}\mathfrak{t}1$	$[24, 21, 4]_{289}$	MDS
14	$-v$	(−1, 1)	$\mathfrak{t}^8 1$	$\mathfrak{t}^{16} 1$	$[28, 26, 3]_{289}$	MDS
8	1	(1, 1)	$\mathfrak{t}^9 1$	(18)1	$[16, 14, 3]_{361}$	MDS
12	1	(1, 1)	$\mathfrak{t}^9 1$	$\mathfrak{t}^{45}\mathfrak{t}1$	$[24, 21, 4]_{361}$	MDS

represent it by the coefficients in ascending order of the powers of the variable. For example, the polynomial $\mathfrak{t}^{21} + \mathfrak{t}^5 x + x^2$ is represented by $\mathfrak{t}^{21}\mathfrak{t}^5 1$.

7 Conclusion

In this paper, we have explored the structure of Euclidean and Hermitian duals of skew (Θ, λ)-constacyclic codes over the non-chain ring $\mathfrak{R} = \mathbb{F}_{p^e}[u, v]/\langle f(u), g(v), uv - vu\rangle$. We have also established a necessary and sufficient condition for these codes to contain their duals. As an application, several new quantum codes have been constructed by using the CSS construction (Table 1) and Hermitian construction (Table 2). Further, we have derived a necessary and sufficient condition for skew (Θ, λ)-constacyclic codes over $\mathfrak{R}$ to intersect their Euclidean (Hermitian) dual codes trivially. By applying a gray map on Euclidean LCD skew (Θ, λ)-constacyclic codes over $\mathfrak{R}$, many linear MDS codes have been obtained (Table 3). Our computational work shows that skew constacyclic codes over the non-chain ring are more promising to produce better quantum and linear codes than constacyclic codes. Recently, Aydin et al. [5] constructed many good classical and quantum codes from multi-twisted codes. Therefore, it would be interesting to work to obtain more good quantum and classical codes over $\mathbb{F}_{p^e}$ by considering other classes of linear codes such as skew quasi-twisted codes or skew multi-twisted codes.

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Data Availability All data generated or analyzed during this study are included in this paper.

Declarations

Conflict of interest On behalf of all authors, the corresponding author states that there is no conflict of interest.

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