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New asymmetric quantum codes over F_{q^2}

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Abstract
Two families of asymmetric quantum codes of length $n = q^{2m} - 1$ over $\mathbf{F}_{q^2}$ are constructed in this paper. By a detailed analysis of properties about q^2-ary cyclotomic cosets modulo n, Hermitian dual-containing conditions for a family of primitive narrow-sense BCH codes are presented. Consequently, a series of asymmetric quantum BCH codes are constructed via the CSS-like construction and pairs of nested BCH codes. The parameters of new asymmetric quantum codes presented here are better than those available in the literatures before, and the real Z-distance are much larger than $\delta_{max} + 1$.

Keywords Asymmetric quantum code · BCH code · CSS-like construction

1 Introduction

Asymmetric quantum codes are an efficient coding scheme against the qubit-flip errors σ_x, phase-flip errors σ_z and the combined qubit-phase flip errors σ_y in quantum communication, which occurs with different probability. In most cases, the phase-flip errors occur more frequently than qubit-flip errors [1–4]. Therefore, putting asymmetric quantum codes to use in the asymmetric quantum channels is an issue worth

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considering. In the past 15 years, researchers have focused on the construction of asymmetric quantum codes, and obtained many codes with good parameters. Aly [5–7] derived some families of asymmetric quantum codes from imprimitive non-narrow-sense BCH codes and RS codes over finite fields. Sarvepalli et al. [8] utilized the construction of asymmetric quantum codes from dual-containing BCH codes and LDPC codes. L. Wang et al. [9] extend the characterization of nonadditive symmetric quantum codes to the asymmetric case, established a relationship of asymmetric quantum codes with classical error-correcting codes and obtained an asymptotic bound on asymmetric quantum codes from algebraic geometry codes. Ezerman et al. [10, 11] proposed CSS-like constructions and exploited pairs of nested linear codes under one of the Euclidean, trace Euclidean, Hermitian and trace Hermitian inner products constructed good parameters CSS-like asymmetric quantum codes. La Guardia [12–14] constructed many good asymmetric quantum codes which derived from Euclidean (Hermitian) dual-containing BCH codes by CSS construction. Recently, the constructions of asymmetric quantum codes have been studied by many researchers, and some families of new asymmetric quantum codes are constructed by utilizing constacyclic codes, negacyclic codes and MDS codes et al. [15–20]. Especially, Grassl [21] present a new propagation rule for CSS codes, and this construction applies to asymmetric quantum codes from the CSS construction as well.

In [22], the authors have utilized Euclidean dual-containing BCH code to construct asymmetric quantum codes of length $n = q^m - 1$ over $\mathbf{F}_q$ where $q \geq 5$. Reference [23] presented a class of special code length asymmetric quantum codes of length $n = \frac{q^{2m}-1}{q^2-1}$ over $\mathbf{F}_{q^2}$. Inspired by the previous work mentioned [22, 23], in this paper, the construction of q^2-ary asymmetric quantum codes with code length $n = q^{2m} - 1(q \geq 3)$ from Hermitian dual-containing primitive narrow-sense BCH codes were studied. And the lower bound δ on Z-distance and X-distance of these codes were provided here. Furthermore, we exactly calculate the parameters of two families of asymmetric quantum codes with special Z-distance, where our Z-distance can be much larger than $\delta_{max} + 1$ given in Theorem 2.3 Ref. [24].

This paper is organized as follows. In Sect. 2, basic concepts on q^2-cyclotomic cosets, BCH codes and asymmetric quantum codes are reviewed. In Sect. 3, the conditions regarding Hermitian dual-containing BCH codes were discussed. In Sect. 4, a families of asymmetric quantum codes were constructed from Hermitian dual-containing BCH codes. In Sect. 5, we compared the parameters of the new codes with the ones available in the literature. Finally, the paper is summarized with a discussion in Sect. 6.

2 Preliminary

In this section, we will review the basic concepts on q^2-cyclotomic cosets of modulo n and BCH codes. Let q be a prime power, $\mathbf{F}_{q^2}$ be the finite field with q^2 elements. For $n = q^{2m} - 1$ denotes the code length, $\mathcal{B}^{\perp_h}$ denotes the Hermitian dual of BCH code $\mathcal{B}$, and an asymmetric quantum BCH code Q is denoted by $[[n, k, d_z/d_x]]$. For more details, refer to [6–9].

Definition 2.1 If $gcd(q, n) = 1$, the q^2-cyclotomic coset of modulo n containing x is defined by

$$C_x = \left\{x, xq^2, x(q^2)^2, ..., x(q^2)^{(k-1)}\right\} (\text{mod} n)$$

where k is the smallest positive integer such that $q^{2k}x \equiv x(\text{mod} n)$.

A cyclic code of length $n = q^{2m} - 1$ over $\mathbf{F}_{q^2}$ is called a BCH code with designed distance δ if its generator polynomial

$$g(x) = \prod_{z \in T}(x - \xi^z), T = C_b \cup C_{b+1} \cup \cdots \cup C_{b+\delta-2},$$

where C_x denotes the q^2-cyclotomic coset of modulo n containing x, ξ is a primitive element of $\mathbf{F}_{q^2}$ and $m = ord_n(q^2)$ is the multiplicative order of q modulo n. According to the concept of defining set, such a BCH code can also be defined, see following Definition 2.2.

Definition 2.2 Let $gcd(q, n) = 1$. If ξ is a primitive n-th root of unity in some field containing $\mathbf{F}_{q^2}$, $T = C_b \cup C_{b+1} \cup \cdots \cup C_{b+\delta-2} = T_{[b,b+\delta-2]}$, the cyclic code of length n with defining set T is called a BCH code of designed distance δ. If $b = 1$, $\mathcal{C}$ is called a narrow-sense BCH code, if $n = q^{2m} - 1$, $\mathcal{C}$ is called primitive.

Lemma 2.1 *If* $gcd(q, n) = 1$, $\mathcal{C}$ *is a cyclic code over* $\mathbf{F}_{q^2}$ *with defining set* T, $\mathcal{C}^{\perp_h} \subseteq \mathcal{C}$ *if and only if* $T \cap T^{-q} = \emptyset$, *where* $T^{-q} = \{n - qt(\text{mod} n) \mid t \in T\}$.

Let $\mathcal{B}_1$ *and* $\mathcal{B}_2$ *be* q^2*-ary BCH codes of length n, and with defining set* T_1 *and* T_2*, respectively. From above Lemma 3.1, we know* $\mathcal{B}_1^{\perp_h} \subseteq \mathcal{B}_2$ *if and only if* $T_1^{-q} \cap T_2 = \emptyset$. *Thus, we have*

Lemma 2.2 *Let* $\mathcal{B}_1$ *and* $\mathcal{B}_2$ *be* q^2*-ary BCH code with defining set* T_1 *and* T_2*, then* $\mathcal{B}_1^{\perp_h} \subseteq \mathcal{B}_2$ *if and only if* $T_1^{\perp_h} \supseteq T_2$.

According to [2, 5, 6, 8], an asymmetric quantum code $[[n, k, d_z/d_x]]$ *can control all* $\lfloor \frac{d_x-1}{2} \rfloor$ *qubit-flip errors and all* $\lfloor \frac{d_z-1}{2} \rfloor$ *phase-flip errors. At the same time, which can detect* $d_x - 1$ *qubit-flip errors and* $d_z - 1$ *phase-flip errors. Based on CSS construction, in 2013, Ezerman et al. proposed constructions are called CSS-like construction and utilized pairs of nested subfield linear codes under Hermitian inner products. The following Theorem 2.3 is CSS-like construction for asymmetric quantum codes.*

Theorem 2.3 *(CSS-Like Construction) For* $i = 1, 2$, *let* $\mathcal{C}_i$ *be a classical linear code with parameters* $[n, k_i, d_i]_{q^2}$. *If* $\mathcal{C}_1^{\perp_h} \subseteq \mathcal{C}_2$*, then there exists an asymmetric quantum code with parameters* $[[n, k(\delta_1) + k(\delta_2) - n, d_z/d_x]]_{q^2}$*, where* $\{d_x, d_z\} = \{d_1, d_2\}$.

3 Construction of Hermitian dual-containing BCH codes

In this section, the construction of asymmetric quantum codes derived from pairs of nested nonprimitive Hermitian dual-containing BCH codes were discussed as follow. If fixed n, denote $T = \bigcup_{i=1}^{r} C_i$, define $u = \min\left\{x \mid x \in T^{-q}\right\}$ and $v = \max\left\{y \mid y \in T^{-q}\right\}$, Similar to [22], we have following Lemma 2.2.

Lemma 3.1 *If $\mathcal{B}$ is a q^2-ary narrow-sense BCH code of length n with defining set $T = \cup_{i=1}^{r} C_i$ where $r < \delta_{max}$, and $T^{\perp_h} = Z_n \backslash T^{-q}$. Then $\mathcal{B}$ and $\mathcal{B}^{\perp_h}$ has design distance $\delta(\mathcal{B}) = r + 1$ and $\delta(\mathcal{B}^{\perp_h}) \le \max\{u, n - v - 1\}$, respectively.*

Proof The defining set of BCH code $\mathcal{B}$ is $T = \cup_{i=1}^{r} C_i = T_{[1,r]}$, so we have the maximal design distance of narrow sense BCH code $\mathcal{B}$ is $r + 1$, then we have $\delta(\mathcal{B}) = r + 1$.

Since $T^{\perp_h} = Z_n \setminus (T^{-q}) = Z_n - \{n - qx \mid x \in T\} = \{0, 1, 2, \cdots, n-1\} - \{u, u+s, \cdots, v-t, v\} = \{0, 1, 2, \cdots, u-1, \cdots, v+1, \cdots, n-1\}$, then $T^{\perp_h}$ contain u or $n - v - 1$ integer. From Definition 2.2, thus we have $\delta(\mathcal{B}^{\perp_h}) \le \max\{u, n - v - 1\}$.

For constructing asymmetric quantum codes via CSS-Like construction, let $q \ge 3$, the conditions regarding Hermitian dual-containing BCH codes of code length $n = q^{2m} - 1$ were discussed as follow Theorem 3.2. □

Theorem 3.2 *Let $n = q^{2m} - 1$, where $q \ge 3$ is prime power and $m \ge 4$.*

- (I) *If $1 \le i \le q-2$, $2 \le j \le q$. For $\delta_1 = (i \cdot q + j) \cdot q - 1$, $\delta_1 < \delta_2 \le q^{(2m-1)} - (i \cdot q + j)$, then there exist narrow sense BCH codes satisfying $\mathcal{B}_1^{\perp_h}(n, \delta_1) \subseteq \mathcal{B}_2(n, \delta_2)$.*
- (II) *If $2 \le i \le q^2 - q$. For $\delta_1 = q^3 - i$, $\delta_1 < \delta_2 \le i \cdot q^{(2m-3)} - 1$, then there exist narrow sense BCH codes satisfying $\mathcal{B}_1^{\perp_h}(n, \delta_1) \subseteq \mathcal{B}_2(n, \delta_2)$.*
- (III) *If $1 \le j \le q^2 - 1$*
  - (1) *For $\delta_1 = j \cdot q^{2s+1} - 1$, $\delta_1 < \delta_2 \le q^{2m-2s-1} - j$, where $1 \le s \le [\frac{m}{2}] - 1$, then there exist narrow sense BCH codes satisfying $\mathcal{B}_1^{\perp_h}(n, \delta_1) \subseteq \mathcal{B}_2(n, \delta_2)$;*
  - (2) *For $\delta_1 = q^{2s+1} - j$, $\delta_1 < \delta_2 \le j \cdot q^{2m-2s-1} - 1$, where $2 \le s \le [\frac{m+1}{2}] - 1$, then there exist narrow sense BCH codes satisfying $\mathcal{B}_1^{\perp_h}(n, \delta_1) \subseteq \mathcal{B}_2(n, \delta_2)$.*

Proof Here, we only show item (II) since the other cases are similar.

Let $n = q^{2m} - 1$, $q \ge 3$. Since $\delta_1 = q^3 - i$ where $2 \le i \le q^2 - q$, then narrow sense BCH code $\mathcal{B}_1(n, \delta_1)$ with defining set $T_1 = \cup_{t=1}^{\delta_1 - 1} C_t = \cup_{t=1}^{q^3 - i - 1} C_t$. Let $T_1^{-q} = \{n - qx_i \mid x_i \in T_1\}$, since $\mathcal{B}_1^{\perp_h}$ with defining set $T_1^{\perp_h} = Z_n \backslash T_1^{-q} = \{0, 1, 2, \cdots, n-1\} - \{n - qx_i \mid x_i \in T_1\}$.

If $\delta_1 < \delta_2 \le min\{n - qx_i \mid x_i \in T_1\} = i \cdot q^{(2m-3)} - 1$, then $\mathcal{B}_2(n, \delta_2)$ with defining set $T_2 = \cup_{t=1}^{\delta_2 - 1} C_t$. We can assume $T_2 = C_1 \cup C_2 \cup \cdots C_{\delta_2 - 1}$. If for any $j \in T_2$, from Lemma 2.2, we can deduce that $j \notin T_1^{-q}$, that is to say $T_1^{-q} \cap T_2 = \emptyset$, then $j \in Zn \backslash T_1^{-q}$, thus one can deduce that $T_2 \subseteq T_1^{\perp_h}$. From Lemma 2.1, we can conclude $\mathcal{B}_1^{\perp_h}(n, \delta_1) \subseteq \mathcal{B}_2(n, \delta_2)$ holds.

Theorem 3.2 provides the maximum designed distance of two nested BCH codes satisfying Hermitian dual-containing conditions. However, it is a well-known hard problem to calculate the dimensions of asymmetric quantum codes. Therefore, next, the dimensions of two families of BCH codes with special maximum designed distances can be computed. The following two cases discussed here because the definition set of these BCH codes are influenced by the parity of m. □

Case I. m **is even**

In this case, let $m \geq 4$ be even, $n = q^{2m} - 1$. For $q \leq i \leq q^2 - 1$, $1 \leq j_1 \leq i$ and $1 \leq j_2 \leq q^{2t-2}$, denote the defining set of BCH code is $T(\delta) = \cup_{i=1}^{\delta} C_i = T_{[1,\delta]}$, if

$$| T(\delta) | = \begin{cases} m(\delta - (q^{2t} + \sum_{k=0}^{q^2-3} k + 2(q^2-1)) + t & \delta = q^{2t+2} + q^2 \\ m(\delta - ((i+1)\cdot q^{2t-2} + \sum_{k=0}^{i-2} k + (i + j_1 - 1))) & \delta = (i+1)\cdot q^{2t} + j_1 \\ m(\delta - (i \cdot q^{2t-2} + \sum_{k=0}^{i-2} k + (i + j_2 - 1))) & \delta = i \cdot q^{2t} + j_2 \cdot q^2 \end{cases}$$

then there exists a narrow-sense BCH code with parameters $[n, n- | T(\delta) |, \geq \delta]_{q^2}$.

Example 3.1 Take $q = 5, m = 4$, so $n = 390624$. If $T(\delta) = 4\cdot(1251-51), 4\cdot(1877-78), 4\cdot(2503-106), 4\cdot(3129-135), 4\cdot(3755-165), 4\cdot(4381-196), 4\cdot(5007-228), 4\cdot(5633-261), 4\cdot(6259-295), 4\cdot(15649-925), 4\cdot(15650-926)+2$, then there exist BCH codes with parameters $[n, 385968, d \geq 1252]_{25}$, $[n, 383428, d \geq 1878]_{25}$, $[n, 381036, d \geq 2504]_{25}$, $[n, 378648, d \geq 3130]_{25}$, $[n, 376264, d \geq 3756]_{25}$, $[n, 373884, d \geq 4382]_{25}$, $[n, 371508, d \geq 5008]_{25}$, $[n, 369136, d \geq 5634]_{25}$, $[n, 366768, d \geq 6260]_{25}$, $[n, 331728, d \geq 15650]_{25}$, $[n, 331726, d \geq 15651]_{25}$, respectively.

Summarizing the analysis above, applying Theorem 3.2 and above the corresponding conclusion, then Corollary 3.3 follows.

Corollary 3.3 *Let $m = 2t (t \geq 2)$, $n = q^{2m} - 1$ with $q \geq 3$ be a power of a prime. If $\delta_1 = q^{2t-1} - q$, $\mathcal{B}_1 = [n, n - m\lceil(\delta_1 - 1)(1 - q^{-2})\rceil, \geq \delta_1]$, then there exist narrow sense BCH codes $\mathcal{B}_2$ with parameters*

(I) $[n, n - m(\delta_2 - (q^{2t} + \sum_{k=0}^{q^2-3} k + q^2 - 3)), \geq \delta_2]$ *satisfying* $\mathcal{B}_1^{\perp_h}(n, \delta_1) \subseteq \mathcal{B}_2(n, \delta_2)$, *where* $\delta_1 < \delta_2 \leq q^{2t+2} - 1$.
(II) $[n, n - m(\delta_2 - ((i+1)\cdot q^{2t-2} + \sum_{k=0}^{i-2} k + (i + j - 1))), \geq \delta_2]$ *satisfying* $\mathcal{B}_1^{\perp_h}(n, \delta_1) \subseteq \mathcal{B}_2(n, \delta_2)$, *where* $q \leq i \leq q^2 - 1$, $1 \leq j \leq i$ *and* $\delta_1 < \delta_2 \leq (i+1)\cdot q^{2t} + j$.
(III) $[n, n - m(\delta_2 - (i \cdot q^{2t-2} + \sum_{k=0}^{i-2} k + (i + j - 1))), \geq \delta_2]$ *satisfying* $\mathcal{B}_1^{\perp_h}(n, \delta_1) \subseteq \mathcal{B}_2(n, \delta_2)$, *where* $q \leq i \leq q^2 - 1$, $1 \leq j \leq q^{2t-2}$ *and* $\delta_1 < \delta_2 \leq i \cdot q^{2t} + j \cdot q^2$.

Case II. m **is odd**

In this case, let $m \geq 5$ be odd, $n = q^{2m} - 1$. For $\delta_0 = q^{2t+2}$; $1 \leq i_1 \leq q^2 - 1$, $1 \leq j_1 \leq q^{2t-2} - i$ and $\delta_1 = i_1 \cdot q^{2t+2} + (i_1 + j_1)\cdot q^2$; $1 \leq i_2 \leq q^2 - 2$, $1 \leq j_2 \leq q^2 - 1$, $1 \leq s \leq q^{2t-2}$ and $\delta_2 = i_2 \cdot q^{2t+2} + j_2 \cdot q^{2t} + s \cdot q^2$; $0 \leq i_3 \leq q^2 - 2$, $1 \leq j_3 \leq (i_3 + 1)\cdot q^2$ and $\delta_3 = (i_3+1)\cdot q^{2t+2} + j_3$, denote the defining set of BCH code is $T(\delta) = \cup_{i=1}^{\delta} C_i = T_{[1,\delta]}$, if

$$| T(\delta) | = \begin{cases} m(\delta_0 - q^{2t}) \\ m(\delta_1 - (i_1 \cdot q^{2t} + \frac{i_1^2 + i_1}{2}\cdot q^2 + \sum_{k=0}^{i_1-1} k \cdot (q^2 - 2) + j_1)) \\ m(\delta_2 - (i_2 \cdot q^{2t} + \frac{i_2^2 + i_2}{2}\cdot q^2 + j_2 \cdot q^{2t-2} \\ \quad + \sum_{k=0}^{i_2-1} k \cdot (q^2 - 2) + i_2 \cdot (j_2 - 1) + s)) \\ m(\delta_3 - ((i_3+1)\cdot q^{2t} + \frac{i_3^2 + i_3}{2}\cdot q^2 + (q^2 - 2)\cdot \sum_{k=0}^{i_3} k + j_3)) \end{cases}$$

then there exists a narrow-sense BCH code with parameters $[n, n- \mid T(\delta) \mid, \geq \delta]_{q^2}$.

Example 3.2 Take $q = 4, m = 5$, so $n = 1048575$. If $T(\delta) = 5 \cdot (4112 - 272), 5 \cdot (4353 - 288), 5 \cdot (4609 - 305), 5 \cdot (4865 - 322), 5 \cdot (5121 - 339), 5 \cdot (5377 - 356), 5 \cdot (5633 - 373), 5 \cdot (5889 - 390), 5 \cdot (6145 - 407), 5 \cdot (9986 - 698), 5 \cdot (10498 - 734), 5 \cdot (11010 - 770), 5 \cdot (11266 - 788), 5 \cdot (11522 - 806, 5 \cdot (11778 - 824), 5 \cdot (12336 - 906), 5 \cdot (61680 - 7230)$, then there exist BCH codes with parameters $[n, 1029375, d \geq 4113]_{16}$, $[n, 1028250, d \geq 4354]_{16}$, $[n, 1027055, d \geq 4610]_{16}$, $[n, 1025860, d \geq 4866]_{16}$, $[n, 1024665, d \geq 5122]_{16}$, $[n, 1023470, d \geq 5378]_{16}$, $[n, 1022275, d \geq 5634]_{16}$, $[n, 1021080, d \geq 5890]_{16}$, $[n, 1019885, d \geq 6146]_{16}$, $[n, 1002135, d \geq 9987]_{16}$, $[n, 999755, d \geq 10499]_{16}$, $[n, 997375, d \geq 11011]_{16}$, $[n, 996185, d \geq 11267]_{16}$, $[n, 994995, d \geq 11523]_{16}$, $[n, 993805, d \geq 11779]_{16}$, $[n, 991425, d \geq 12337]_{16}$, $[n, 776325, d \geq 61681]_{16}$, respectively.

Summarizing the analysis above, applying Theorem 3.2 and above the corresponding conclusion, then Corollary 3.4 follows.

Corollary 3.4 *Let $m = 2t + 1 (t \geq 2)$, $n = q^{2m} - 1$ with $q \geq 3$ be a power of a prime. If $\delta_1 = q^{2t-1} - q$, $\mathcal{B}_1 = [n, n - m\lceil(\delta_1 - 1)(1 - q^{-2})\rceil, \geq \delta_1]$, then there exist narrow sense BCH codes $\mathcal{B}_2$ with parameters*

(I) $[n, n - m(\delta_2 - q^{2t}), \geq \delta_2]$ *satisfying* $\mathcal{B}_1^{\perp h}(n, \delta_1) \subseteq \mathcal{B}_2(n, \delta_2)$, *where* $\delta_1 < \delta_2 \leq q^{2t+2}$.

(II) $[n, n - m(\delta_2 - (i \cdot q^{2t} + \frac{i^2+i}{2} \cdot q^2 + \sum_{k=0}^{i-1} k \cdot (q^2 - 2) + j)), \geq \delta_2]$ *satisfying* $\mathcal{B}_1^{\perp h}(n, \delta_1) \subseteq \mathcal{B}_2(n, \delta_2)$, *where* $1 \leq i \leq q^2 - 1$, $1 \leq j \leq q^{2t-2} - i$ *and* $\delta_1 < \delta_2 \leq i \cdot q^{2t+2} + (i + j) \cdot q^2$.

(III) $[n, n - m(\delta_2 - (i \cdot q^{2t} + \frac{i^2+i}{2} \cdot q^2 + j \cdot q^{2t-2} + \sum_{k=0}^{i-1} k \cdot (q^2 - 2) + i \cdot (j - 1) + s)), \geq \delta_2]$ *satisfying* $\mathcal{B}_1^{\perp h}(n, \delta_1) \subseteq \mathcal{B}_2(n, \delta_2)$, *where* $1 \leq i \leq q^2 - 2$, $1 \leq j \leq q^2 - 1$, $1 \leq s \leq q^{2t-2}$ *and* $\delta_1 < \delta_2 \leq i \cdot q^{2t+2} + j \cdot q^{2t} + s \cdot q^2$.

(IV) $[n, n - m(\delta_2 - ((i + 1) \cdot q^{2t} + \frac{i^2+i}{2} \cdot q^2 + (q^2 - 2) \cdot \sum_{k=0}^{i} k + j)), \geq \delta_2]$ *satisfying* $\mathcal{B}_1^{\perp h}(n, \delta_1) \subseteq \mathcal{B}_2(n, \delta_2)$, *where* $0 \leq i \leq q^2 - 2$, $1 \leq j \leq (i + 1) \cdot q^2$, *and* $\delta_1 < \delta_2 \leq (i + 1) \cdot q^{2t+2} + j$.

4 Construction of asymmetric quantum codes

In this section, applying CSS-like construction and the main results above section, some new asymmetric quantum codes can be constructed, and their dimension can be exactly calculated. In the following Theorem 4.1, our main construction results can be provided.

Theorem 4.1 *Let $n = q^{2m} - 1$, where $q \geq 3$ is a prime power and $m \geq 4$.*

(I) *For $\delta_1 = (i \cdot q + j) \cdot q - 1$, $\delta_1 < \delta_2 \leq q^{(2m-1)} - (i \cdot q + j)$, then there exist asymmetric quantum codes $[[n, n- \mid T(\delta_1) \mid - \mid T(\delta_2) \mid, d_z \geq \delta_2/d_x \geq \delta_1]]_{q^2}$, where $1 \leq i \leq q - 2$, $2 \leq j \leq q$.*

(II) *For* $\delta_1 = q^3 - i$, $\delta_1 < \delta_2 \le i \cdot q^{(2m-3)} - 1$, *then there exist asymmetric quantum codes* $[[n, n- \mid T(\delta_1) \mid - \mid T(\delta_2) \mid, d_z \ge \delta_2/d_x \ge \delta_1]]_{q^2}$, *where* $2 \le i \le q^2 - q$.

(III) *If* $1 \le j \le q^2 - 1$.

(1) *For* $\delta_1 = j \cdot q^{2s+1} - 1$, $\delta_1 < \delta_2 \le q^{2m-2s-1} - j$, *then there exist asymmetric quantum codes* $[[n, n- \mid T(\delta_1) \mid - \mid T(\delta_2) \mid, d_z \ge \delta_2/d_x \ge \delta_1]]_{q^2}$, *where* $1 \le s \le [\frac{m}{2}] - 1$;

(2) *For* $\delta_1 = q^{2s+1} - j$, $\delta_1 < \delta_2 \le j \cdot q^{2m-2s-1} - 1$, *then there exist asymmetric quantum codes* $[[n, n- \mid T(\delta_1) \mid - \mid T(\delta_2) \mid, d_z \ge \delta_2/d_x \ge \delta_1]]_{q^2}$, *where* $2 \le s \le [\frac{m+1}{2}] - 1$.

Proof We only prove item (II) since the other constructions are similar.

Let $\mathcal{B}_1$ be the narrow-sense BCH code over $\mathbf{F}_{q^2}$ of length $n = q^{2m} - 1$. Using $T(\delta_1) = \bigcup_{i=1}^{\delta_1 - 1} C_i$ to denote the defining set of BCH code, and the cardinality of $T(\delta_1)$ as $\mid T(\delta_1) \mid$. For $\delta_1 = q^3 - i$ where $2 \le i \le q^2 - q$, then there exists a narrow sense BCH code with parameters $[n, n- \mid T(\delta_1) \mid, q^3 - i]$. Next, consider another BCH code $\mathcal{B}_2$ with parameters $[n, n- \mid T(\delta_2) \mid, \delta_2]$.

According to Theorem 3.2, we know that if $\delta_1 < \delta_2 \le i \cdot q^{(2m-3)} - 1$, then there exist narrow-sense BCH codes satisfying $\mathcal{B}_1^{\perp h}(n, \delta_1) \subseteq \mathcal{B}_2(n, \delta_2)$. Hence, applying the CSS-Like construction in Theorem 2.3, and use the parameters of $B_1(n, \delta_1)$ and $B_2(n, \delta_2)$, q^2-ary asymmetric quantum codes $[[n, n- \mid T(\delta_1) \mid - \mid T(\delta_2) \mid, d_z \ge \delta_2/d_x \ge \delta_1]]_{q^2}$ can be constructed. □

Summarizing the above discussion, we can conclude (II) holds.

Remark 1 We presented the lower bound δ on Z-distance and X-distance of asymmetric quantum codes of length $n = q^{2m} - 1$ where $q \ge 3$. Obviously, our Z-distance are much larger than X-distance as well as are much larger than $\delta_{max} + 1$ in [22]. However, Theorem 4.1 does not give the exact dimensions of these code, owning to complex to calculate the exact dimensions for all δ. So we denote the cardinality of $T(\delta)$ as $\mid T(\delta) \mid$, and the dimension as $n- \mid T(\delta) \mid$. But, for fixed the special values of d_z and d_x, we will calculate the exact parameters of two families of asymmetric quantum codes in the following Corollary.

Corollary 4.2 *Let* $m = 2t (t \ge 2)$, $n = q^{2m} - 1$ *with* $q \ge 3$ *be a power of a prime. Then there exist asymmetric quantum codes with parameters*

(I) $[[n, n - m(\lceil (q^{2t-1} - q)(1 - q^{-2}) \rceil + (q^{2t-1} - 1)(q^2 - 1) - \sum_{k=0}^{q^2-3} k + 1), d_z \ge q^{2t+2} - 1/d_x \ge q^{2t-1} - q]]_{q^2}$.

(II) $[[n, n - m(\lceil (q^{2t-1} - q)(1 - q^{-2}) \rceil + (i+1) \cdot (q^{2t} - q^{2t-2}) - \sum_{k=0}^{i-2} k - i + 1), d_z \ge (i+1) \cdot q^{2t} + j/d_x \ge q^{2t-1} - q]]_{q^2}$, *where* $q \le i \le q^2 - 1$, $1 \le j \le i$.

(III) $[[n, n - m(\lceil (q^{2t-1} - q)(1 - q^{-2}) \rceil + i \cdot (q^{2t} - q^{2t-2} - 1) + j \cdot (q^2 - 1) - \sum_{k=0}^{i-2} k + 1), d_z \ge i \cdot q^{2t} + j \cdot q^2/d_x \ge q^{2t-1} - q]]_{q^2}$, *where* $q \le i \le q^2 - 1$, $1 \le j \le q^{2(t-1)}$.

Corollary 4.3 *Let* $m = 2t + 1 (t \ge 2)$, $n = q^{2m} - 1$ *with* $q \ge 3$ *be a power of a prime. Then there exist asymmetric quantum codes with parameters*

Table 1 Sample parameters of asymmetric quantum codes for $m = 4$

q	m	n	$[[n, k, d_z/d_x]]_{q^2}$	$\rho = \frac{d_z}{d_x}$
3	4	6560	$[[n, 1232, d_z \geq 2185/d_x \geq 6]]_9$	≈ 364.16
			$[[n, 1224, d_z \geq 2184/d_x \geq 10]]_9$	≈ 218.40
			$[[n, 1212, d_z \geq 2182/d_x \geq 15]]_9$	≈ 145.46
			$[[n, 1204, d_z \geq 2181/d_x \geq 19]]_9$	≈ 114.78
			$[[n, 2292, d_z \geq 1457/d_x \geq 22]]_9$	≈ 66.22
			$[[n, 2900, d_z \geq 1214/d_x \geq 23]]_9$	≈ 52.78
			$[[n, 3544, d_z \geq 971/d_x \geq 24]]_9$	≈ 40.45
			$[[n, 3992, d_z \geq 728/d_x \geq 25]]_9$	≈ 29.12
			$[[n, 4780, d_z \geq 485/d_x \geq 26]]_9$	≈ 18.65
			$[[n, 5604, d_z \geq 242/d_x \geq 28]]_9$	≈ 8.64

(I) $[[n, n - m(\lceil (q^{2t-1} - q)(1 - q^{-2}) \rceil + (q^{2t+2} - q^{2t})), d_z \geq q^{2t+2}/d_x \geq q^{2t-1} - q]]_{q^2}$.

(II) $[[n, n - m(\lceil (q^{2t-1} - q)(1 - q^{-2}) \rceil + i \cdot q^{2t}(q^2 - 1) + (i + j - \frac{i^2+i}{2}) \cdot q^2 - (q^2 - 2) \sum_{k=0}^{i-1} k - j), d_z \geq i \cdot q^{2t+2} + (i + j) \cdot q^2/d_x \geq q^{2t-1} - q]]_{q^2}$, *where* $1 \leq i \leq q^2 - 1$, $1 \leq j \leq q^{2t-2} - i$.

(III) $[[n, n - m(\lceil (q^{2t-1} - q)(1 - q^{-2}) \rceil + (i \cdot q^{2t} + j \cdot q^{2t-2})(q^2 - 1) + (s - \frac{i^2+i}{2}) \cdot q^2 - (q^2 - 2) \sum_{k=0}^{i-1} k + i \cdot (j - 1) - s, d_z \geq i \cdot q^{2t+2} + j \cdot q^{2t} + s \cdot q^2/d_x \geq q^{2t-1} - q]]_{q^2}$, *where* $1 \leq i \leq q^2 - 2$, $1 \leq j \leq q^2 - 1$, $1 \leq s \leq q^{2t-2}$.

(IV) $[[n, n - m(\lceil (q^{2t-1} - q)(1 - q^{-2}) \rceil + (i + 1) \cdot q^{2t}(q^2 - 1) - \frac{i^2+i}{2} \cdot q^2 - (q^2 - 2) \sum_{k=0}^{i} k), d_z \geq (i + 1) \cdot q^{2t+2} + j/d_x \geq q^{2t-1} - q]]_{q^2}$, *where* $1 \leq i \leq q^2 - 2$, $1 \leq j \leq (i + 1)q^2$.

Example 4.1 Take $q = 3, m = 4$, so $n = 6560$. The following Table 1 lists some asymmetric quantum BCH codes derived from Corollary 4.2.

Example 4.2 Take $q = 3, m = 5$, so $n = 59048$. Table 2 lists some asymmetric quantum BCH codes derived from Corollary 4.3.

Remark 2 Tables 1 and 2 listed some new asymmetric quantum codes which given in Corollary 4.2 and Corollary 4.4. For $q = 3$ and $m = 4, 5$, some of the Z-distances of our asymmetric quantum codes are much larger than X-distances. In general, researchers use the code rate to characterize the performance of a code. The so-called code rate is the ratio $\frac{k}{n}$, where n is code length, k is the dimension. However, in order to show the error-correcting ability to the phase-flip errors and qubit-flip errors, here we use the factor $\rho = \frac{d_z}{d_x}$ given in [5] to compare d_z and d_x. Therefore, if $d_z > d_x$, then the asymmetric quantum codes has a factor great than one. Hence, the phase-flip errors affect the quantum system more than qubit-flip errors do. In this paper, we would like to increase the factor ρ and dimension k of the codes.

Table 2 Sample parameters of asymmetric quantum codes for $m = 5$

q	m	n	$[[n, k, d_z/d_x]]_{q^2}$	$\rho = \frac{d_z}{d_x}$
3	5	59048	$[[n, 7748, d_z \geq 19681/d_x \geq 6]]_9$	≈ 3280.16
			$[[n, 7738, d_z \geq 19680/d_x \geq 10]]_9$	≈ 1968
			$[[n, 7723, d_z \geq 19678/d_x \geq 15]]_9$	≈ 1311.86
			$[[n, 7713, d_z \geq 19677/d_x \geq 19]]_9$	≈ 1053.63
			$[[n, 16708, d_z \geq 13121/d_x \geq 22]]_9$	≈ 596.40
			$[[n, 22163, d_z \geq 10934/d_x \geq 23]]_9$	≈ 475.39
			$[[n, 28248, d_z \geq 8747/d_x \geq 24]]_9$	≈ 364.45
			$[[n, 32658, d_z \geq 6560/d_x \geq 25]]_9$	≈ 262.40
			$[[n, 40693, d_z \geq 4374/d_x \geq 26]]_9$	≈ 168.23
			$[[n, 49448, d_z \geq 2186/d_x \geq 28]]_9$	≈ 78.07
			$[[n, 49333, d_z \geq 2185/d_x \geq 55]]_9$	≈ 39.72
			$[[n, 49218, d_z \geq 2184/d_x \geq 82]]_9$	≈ 26.63
			$[[n, 49103, d_z \geq 2183/d_x \geq 109]]_9$	≈ 20.02
			$[[n, 48988, d_z \geq 2182/d_x \geq 136]]_9$	≈ 16.04
			$[[n, 48873, d_z \geq 2181/d_x \geq 163]]_9$	≈ 13.38
			$[[n, 48758, d_z \geq 2180/d_x \geq 190]]_9$	≈ 11.47
			$[[n, 48643, d_z \geq 2179/d_x \geq 217]]_9$	≈ 10.04

Table 3 Compare the real Z-distances d_z and δ_{max}

q	m	d_z shown in Theorem 4.1	δ_{max} shown in [24]
3	4	2185	235
	5	19681	242
	6	177145	2179
4	4	16382	1009
	5	262142	1023
	6	4194302	16369
5	4	78123	3101
	5	1953123	3124
7	3	16805	342
	4	823541	16759
8	3	32766	511
	4	2097150	32705

For example, for $n = 6560$, if $d_x \geq 6, 10, 15, 19, 22, 23, 24, 25, 26$, our Z-distance can reach $d_z \geq 2185/d_x \geq 6, d_z \geq 2184/d_x \geq 10, d_z \geq 2182/d_x \geq 15, d_z \geq 2181/d_x \geq 19, d_z \geq 1457/d_x \geq 22, d_z \geq 1214/d_x \geq 23, d_z \geq 971/d_x \geq 24, d_z \geq 728/d_x \geq 25, d_z \geq 485/d_x \geq 26$. It is obvious that the factor ρ can reach 364.16, 218.40, 145.46, 114.78, 66.22, 52.78, 40.45, 29.12, 18.65, respectively. That is to say,

Table 4 Sample parameters of asymmetric quantum codes $[[n, k, d_z/d_x]]_{q^2}$

q	m	n	$[[n, k, d_z/d_x]]_{q^2}$	$[[n, k', d_{z'}/d_{x'}]]_{q^2}$ shown in [14]
4	4	65535	$[[n, 65411, d_z \geq 18/d_x \geq 17]]_{16}$	$[[n, 65409, d_{z'} \geq 18/d_{x'} \geq 17]]_{16}$
			$[[n, 65407, d_z \geq 19/d_x \geq 17]]_{16}$	$[[n, 65405, d_{z'} \geq 19/d_{x'} \geq 17]]_{16}$
			$[[n, 65403, d_z \geq 20/d_x \geq 17]]_{16}$	$[[n, 65401, d_{z'} \geq 20/d_{x'} \geq 17]]_{16}$
			$[[n, 65399, d_z \geq 21/d_x \geq 17]]_{16}$	$[[n, 65397, d_{z'} \geq 21/d_{x'} \geq 17]]_{16}$
			$[[n, 65395, d_z \geq 22/d_x \geq 17]]_{16}$	$[[n, 65393, d_{z'} \geq 22/d_{x'} \geq 17]]_{16}$
			$[[n, 65391, d_z \geq 23/d_x \geq 17]]_{16}$	$[[n, 65389, d_{z'} \geq 23/d_{x'} \geq 17]]_{16}$
			$[[n, 65387, d_z \geq 24/d_x \geq 17]]_{16}$	$[[n, 65385, d_{z'} \geq 24/d_{x'} \geq 17]]_{16}$
			$[[n, 65363, d_z \geq 30/d_x \geq 17]]_{16}$	$[[n, 65361, d_{z'} \geq 30/d_{x'} \geq 17]]_{16}$
			$[[n, 20571, d_z \geq 16380/d_x \geq 17]]_{16}$	—
			$[[n, 65355, d_z \geq 25/d_x \geq 24]]_{16}$	$[[n, 65353, d_{z'} \geq 25/d_{x'} \geq 24]]_{16}$
			$[[n, 65351, d_z \geq 26/d_x \geq 24]]_{16}$	$[[n, 65349, d_{z'} \geq 26/d_{x'} \geq 24]]_{16}$
			$[[n, 65347, d_z \geq 27/d_x \geq 24]]_{16}$	$[[n, 65345, d_{z'} \geq 27/d_{x'} \geq 24]]_{16}$
			$[[n, 65343, d_z \geq 28/d_x \geq 24]]_{16}$	$[[n, 65341, d_{z'} \geq 28/d_{x'} \geq 24]]_{16}$
			$[[n, 65339, d_z \geq 29/d_x \geq 24]]_{16}$	$[[n, 65337, d_{z'} \geq 29/d_{x'} \geq 24]]_{16}$
			$[[n, 65335, d_z \geq 30/d_x \geq 24]]_{16}$	$[[n, 65333, d_{z'} \geq 30/d_{x'} \geq 24]]_{16}$
			$[[n, 20551, d_z \geq 16378/d_x \geq 24]]_{16}$	—
			$[[n, 20527, d_z \geq 16376/d_x \geq 33]]_{16}$	—
			$[[n, 20483, d_z \geq 16372/d_x \geq 49]]_{16}$	—
			$[[n, 50367, d_z \geq 4095/d_x \geq 61]]_{16}$	—
			$[[n, 61467, d_z \geq 1023/d_x \geq 65]]_{16}$	—
			$[[n, 61231, d_z \geq 1022/d_x \geq 129]]_{16}$	—
			$[[n, 58175, d_z \geq 1009/d_x \geq 961]]_{16}$	—
			...	...
4	5	1048575	$[[n, 1048400, d_z \geq 22/d_x \geq 17]_{16}$	$[[n, 1048398, d_{z'} \geq 22/d_{x'} \geq 17]]_{16}$
			$[[n, 1048395, d_z \geq 23/d_x \geq 17]_{16}$	$[[n, 1048393, d_{z'} \geq 23/d_{x'} \geq 17]]_{16}$
			$[[n, 1048390, d_z \geq 24/d_x \geq 17]_{16}$	$[[n, 1048388, d_{z'} \geq 24/d_{x'} \geq 17]]_{16}$
			$[[n, 1048385, d_z \geq 25/d_x \geq 17]_{16}$	$[[n, 1048383, d_{z'} \geq 25/d_{x'} \geq 17]]_{16}$
			$[[n, 1048380, d_z \geq 26/d_x \geq 17]_{16}$	$[[n, 1048378, d_{z'} \geq 26/d_{x'} \geq 17]]_{16}$
			$[[n, 1048375, d_z \geq 27/d_x \geq 17]_{16}$	$[[n, 1048373, d_{z'} \geq 27/d_{x'} \geq 17]]_{16}$
			$[[n, 1048370, d_z \geq 28/d_x \geq 17]_{16}$	$[[n, 1048368, d_{z'} \geq 28/d_{x'} \geq 17]]_{16}$
			$[[n, 1048365, d_z \geq 29/d_x \geq 17]_{16}$	$[[n, 1048363, d_{z'} \geq 29/d_{x'} \geq 17]]_{16}$
			$[[n, 1048360, d_z \geq 30/d_x \geq 17]_{16}$	$[[n, 1048358, d_{z'} \geq 30/d_{x'} \geq 17]]_{16}$
			$[[n, 248805, d_z \geq 262131/d_x \geq 17]_{16}$	—

the error-correcting ability to the phase-flip errors of our asymmetric quantum BCH codes can be much better then qubit-flip errors.

Table 4 continued

q	m	n	$[[n,k,d_z/d_x]]_{q^2}$	$[[n,k',d_{z'}/d_{x'}]]_{q^2}$ shown in [14]
			$[[n, 1048350, d_z \geq 25/d_x \geq 24]]_{16}$	$[[n, 1048348, d_{z'} \geq 25/d_{x'} \geq 24]]_{16}$
			$[[n, 1048345, d_z \geq 26/d_x \geq 24]]_{16}$	$[[n, 1048343, d_{z'} \geq 26/d_{x'} \geq 24]]_{16}$
			$[[n, 1048340, d_z \geq 27/d_x \geq 24]]_{16}$	$[[n, 1048338, d_{z'} \geq 27/d_{x'} \geq 24]]_{16}$
			$[[n, 1048335, d_z \geq 28/d_x \geq 24]]_{16}$	$[[n, 1048333, d_{z'} \geq 28/d_{x'} \geq 24]]_{16}$
			$[[n, 1048330, d_z \geq 29/d_x \geq 24]]_{16}$	$[[n, 1048328, d_{z'} \geq 29/d_{x'} \geq 24]]_{16}$
			$[[n, 1048325, d_z \geq 30/d_x \geq 24]]_{16}$	$[[n, 1048323, d_{z'} \geq 30/d_{x'} \geq 24]]_{16}$
			$[[n, 248770, d_z \geq 262131/d_x \geq 24]]_{16}$	—
			$[[n, 1048310, d_z \geq 29/d_x \geq 28]]_{16}$	$[[n, 1048308, d_{z'} \geq 29/d_{x'} \geq 28]]_{16}$
			$[[n, 1048305, d_z \geq 30/d_x \geq 28]]_{16}$	$[[n, 1048303, d_{z'} \geq 30/d_{x'} \geq 28]]_{16}$
			$[[n, 248730, d_z \geq 262131/d_x \geq 33]]_{16}$	—
			$[[n, 972375, d_z \geq 16383/d_x \geq 65]]_{16}$	—
			$[[n, 972080, d_z \geq 16382/d_x \geq 129]]_{16}$	—
			$[[n, 971785, d_z \geq 16381/d_x \geq 193]]_{16}$	—
			$[[n, 971490, d_z \geq 16380/d_x \geq 257]]_{16}$	—
			$[[n, 982155, d_z \geq 13311/d_x \geq 1011]]_{16}$	—
5	3	15624	$[[n, 15477, d_z \geq 27/d_x \geq 26]]_{25}$	$[[n, 15475, d_{z'} \geq 27/d_{x'} \geq 26]]_{25}$
			$[[n, 15474, d_z \geq 28/d_x \geq 26]]_{25}$	$[[n, 15472, d_{z'} \geq 28/d_{x'} \geq 26]]_{25}$
			$[[n, 15471, d_z \geq 29/d_x \geq 26]]_{25}$	$[[n, 15469, d_{z'} \geq 29/d_{x'} \geq 26]]_{25}$
			$[[n, 15468, d_z \geq 30/d_x \geq 26]]_{25}$	$[[n, 15466, d_{z'} \geq 30/d_{x'} \geq 26]]_{25}$
			$[[n, 15414, d_z \geq 48/d_x \geq 26]]_{25}$	$[[n, 15412, d_{z'} \geq 48/d_{x'} \geq 26]]_{25}$
			$[[n, 7932, d_z \geq 3120/d_x \geq 26]]_{25}$	—
			$[[n, 15387, d_z \geq 48/d_x \geq 35]]_{25}$	$[[n, 15385, d_{z'} \geq 48/d_{x'} \geq 35]]_{25}$
			$[[n, 7911, d_z \geq 3118/d_x \geq 35]]_{25}$	—
			$[[n, 7899, d_z \geq 3117/d_x \geq 40]]_{25}$	—
			$[[n, 7887, d_z \geq 3116/d_x \geq 45]]_{25}$	—
			$[[n, 7875, d_z \geq 3115/d_x \geq 51]]_{25}$	—
			$[[n, 8928, d_z \geq 2451/d_x \geq 106]]_{25}$	—
			$[[n, 9264, d_z \geq 2376/d_x \geq 107]]_{25}$	—
			$[[n, 14190, d_z \geq 376/d_x \geq 123]]_{25}$	—
			$[[n, 14547, d_z \geq 251/d_x \geq 124]]_{25}$	—
5	4	390624	$[[n, 390428, d_z \geq 27/d_x \geq 26]]_{25}$	$[[n, 390426, d_{z'} \geq 27/d_{x'} \geq 26]]_{25}$
			$[[n, 390424, d_z \geq 28/d_x \geq 26]]_{25}$	$[[n, 390422, d_{z'} \geq 28/d_{x'} \geq 26]]_{25}$
			$[[n, 390420, d_z \geq 29/d_x \geq 26]]_{25}$	$[[n, 390418, d_{z'} \geq 29/d_{x'} \geq 26]]_{25}$
			$[[n, 390416, d_z \geq 30/d_x \geq 26]]_{25}$	$[[n, 390414, d_{z'} \geq 30/d_{x'} \geq 26]]_{25}$
			$[[n, 390412, d_z \geq 31/d_x \geq 26]]_{25}$	$[[n, 390410, d_{z'} \geq 31/d_{x'} \geq 26]]_{25}$
			$[[n, 390352, d_z \geq 46/d_x \geq 26]]_{25}$	$[[n, 390350, d_{z'} \geq 46/d_{x'} \geq 26]]_{25}$
			$[[n, 390348, d_z \geq 47/d_x \geq 26]]_{25}$	$[[n, 390346, d_{z'} \geq 47/d_{x'} \geq 26]]_{25}$
			$[[n, 390344, d_z \geq 48/d_x \geq 26]]_{25}$	$[[n, 390342, d_{z'} \geq 48/d_{x'} \geq 26]]_{25}$
			$[[n, 161720, d_z \geq 77353/d_x \geq 35]]_{25}$	—

Table 4 continued

q	m	n	$[[n,k,d_z/d_x]]_{q^2}$	$[[n,k',d_{z'}/d_{x'}]]_{q^2}$ shown in [14]
			$[[n,390316,d_z\geq 41/d_x\geq 40]]_{25}$	$[[n,390314,d_{z'}\geq 41/d_{x'}\geq 40]]_{25}$
			$[[n,390288,d_z\geq 48/d_x\geq 40]]_{25}$	$[[n,390286,d_{z'}\geq 48/d_{x'}\geq 40]]_{25}$
			$[[n,161760,d_z\geq 77353/d_x\geq 26]]_{25}$	—
			$[[n,390276,d_z\geq 46/d_x\geq 45]]_{25}$	$[[n,390274,d_{z'}\geq 46/d_{x'}\geq 45]]_{25}$
			$[[n,390272,d_z\geq 47/d_x\geq 45]]_{25}$	$[[n,390270,d_{z'}\geq 47/d_{x'}\geq 45]]_{25}$
			$[[n,390268,d_z\geq 48/d_x\geq 45]]_{25}$	$[[n,390266,d_{z'}\geq 48/d_{x'}\geq 45]]_{25}$
			$[[n,161680,d_z\geq 77353/d_x\geq 45]]_{25}$	—
			$[[n,331264,d_z\geq 15624/d_x\geq 121]]_{25}$	—
			$[[n,342840,d_z\geq 12499/d_x\geq 122]]_{25}$	—
			$[[n,354516,d_z\geq 9374/d_x\geq 123]]_{25}$	—
			$[[n,366292,d_z\geq 6249/d_x\geq 124]]_{25}$	—
			$[[n,378168,d_z\geq 3124/d_x\geq 126]]_{25}$	—
			$[[n,375788,d_z\geq 3119/d_x\geq 751]]_{25}$	—
			$[[n,367244,d_z\geq 3101/d_x\geq 3001]]_{25}$	—

5 Parameters analysis

In this section, we compare the parameters of the new asymmetric quantum codes and the ones available in the literature. In the following tables, the parameters of the asymmetric quantum codes shown in [14](Theorem 4.7) are denoted by $[[n,k',d_{z'}/d_{x'}]]_{q^2}$, and the new code parameters are denoted by $[[n,k,d_z/d_x]]_{q^2}$. Furthermore, our Z-distances are denoted by d_z, δ_{max} are the maximal designed distance of dual containing narrow-sense BCH code in [24].

Remark 3 Table 3 has showed some Z-distances which given in Theorem 4.1. For example, for $q=3, m=4,5,6$, if $\delta_{max}=235, 242, 2179$, our Z-distance can reach 2185, 19681 and 177145, respectively, it is obviously larger than δ_{max}. In a word, we use Table 4 to present evidences of the real Z-distance of our asymmetric quantum codes, which are much larger than $\delta_{max}+1$.

Remark 4 Table 4 listed some new asymmetric quantum codes which given in Corollary 4.2 and Corollary 4.3. For $m=3,4,5$ and $q=4,5$, some of the parameters of our asymmetric quantum codes are better than those available in [14]. For example, let $q=5, m=4, n=390624$, for $d_x\geq 26$ and $d_z\geq 27, 28, 29, 30, 31, 46, 47, 48$, the dimensions of our asymmetric quantum codes are larger than those available in [14]. What is more, some of the asymmetric quantum codes are new ones and are not included in the literature. For example, if $d_x\geq 26$, our Z-distance can reach 77353, which are new and are much greater than the results in the literature. Hence, the error-correcting ability to the phase-flip errors can be further improved. Furthermore, in order to calculate the dimensions, we restrict $d_z\geq 77353/d_x\geq 35$ and $d_z\geq 77353/d_x\geq 45$, then one can easily construct two asymmetric quantum codes $[[n,161720,d_z\geq 77353/d_x\geq 35]]_{25}$ and $[[n,161680,d_z\geq 77353/d_x\geq 45]]_{25}$.

However, if the value of $d_x \geq 35, 45$, then our Z-distance can reach 78118, 78116, respectively.

6 Conclusion

Using the CSS-like construction, we have constructed several families of q^2-ary asymmetric quantum codes of length $n = q^{2m} - 1$ derived from Hermitian dual-containing primitive narrow-sense BCH codes. Some of these codes have parameters better than the ones available in the literature. Furthermore, the real Z-distance are much larger than X-distance and $\delta_{max} + 1$. And others are not included in the literature, which are new ones. Unfortunately, for fixed values of the length n, we only give partial results, the discussions of asymmetric quantum codes constructed from pairs of nested BCH codes for all δ may be a little complex. It would be interesting to construct good asymmetric quantum codes from cyclic codes of other lengths.

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Data availibility The authors confirm that the data supporting the findings of this study are available within the article.

Declarations

Conflict of interest The authors declare that they have no conflicts of interest.

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