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Many-party controlled remote implementations of multiple partially unknown quantum operations

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Abstract

In order to study the controlled remote implementation of quantum operation (RIO for short) for multiple partially unknown quantum operations, we first propose a scheme in the traditional sense for RIO of a partially unknown operation via the control of many agents in a network, which triggers that a new RIO scheme to teleporting multiple partially unknown quantum operations to a distant receiver via the control of one agent is put forwards. After that, we extend the above new method to the RIO of multiple partially unknown quantum operations via the control of many agents in a network. In the extended protocol, as long as all agents cooperate, the receiver can restore the partially unknown quantum operation acting on each qubit. However, even if one agent does not cooperate, the receiver cannot completely restore the partially unknown quantum operation acting on each qubit. This method works essentially through entangling quantum information during implementation, which greatly reduces the required auxiliary qubit resources, local operations and classical communication. Finally, the above scheme is further generalized to transmitting multiple partially unknown quantum operation-string for many distant receivers via the control of many agents in a network.

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1 Introduction

As one of the remarkable characteristics of quantum mechanics, quantum entanglement is a very important resource of quantum computation and quantum information. On the one hand, entanglement can improve the overall stability and against fluctuations of the measurement settings in qubit sytems [1–3]. On the other hand, entanglement can be utilized to transfer informations. A canonical example of entanglement assisted processes is provided by quantum state teleportation (QST) [4], where an unknown quantum state is transmitted from the sender to the remote receiver in a completely different way compared with a classical state, without teleporting the state carrier itself. Another important application of quantum entanglement, which correlates closely to QST, is quantum operation teleportation (QOT). The first QOT protocol was proposed by Huelga et al. [5], where QOT is also called the remote implementation of quantum operation (RIO), and may be understood as that a sender transfers an unknown quantum operation belonging to a local system to a receiver in a remote system without physically sending the device. Remote implementation of a quantum operation means that this unknown quantum operation performed on the local system is teleported and simultaneously acts on an unknown state belonging to the remote system. Obviously, the RIO is different from simple teleportation of quantum operation without action, and it is also not an implementation of nonlocal quantum operation [6, 7]. After Huelga et al's pioneering work, a series of works on RIO have appeared both in theoretical [8–17] and experimental [18–20].

In 2002, Huelga, Plenio and Vaccaro (HPV) [21] proposed the idea of teleportation of angles, which is a special remote control of restricted sets of quantum operations. Subsequently, Wang [8] proposed and proved protocols of remote implementations of partially unknown quantum operations of multiqubits via deducing the general restricted sets of quantum operations and found the unified recovery operations. It is useful and interesting to investigate the remote implementations of partially unknown quantum operations because they consume less overall resources than the ones of the completely unknown quantum operations do, and such RIOs can satisfy the requirements of some practical applications. Here, the "partially unknown"quantum operations are thought of as those belonging to some restricted sets that satisfy some given restricted conditions. Note that the restricted sets of quantum operations are still very large sets of unitary transformations because their unknown elements take continuous values, which had been seen in Refs. [8, 10, 11, 21]. In the simplest case of one qubit, two kinds of restricted sets of quantum operations are, respectively, a set of diagonal operations and a set of off-diagonal operations [21]. For the case of multiqubits, the general forms of restricted sets of quantum operations were obtained in Ref. [8] and every row and column of these operations have only one nonvanishing element.

At present, some significant progress has been made on the remote implementation of partially unknown quantum operations [8, 10, 11, 13, 20–25].

It must be emphasized that the literatures [10–12, 16, 22, 24] mentioned above are also RIO controlled schemes. In these controlled RIO protocols, not only does a controller play such a role that the quantum channel between sender and receiver is opened by his or her operations, but also the controller's measurement (classical information) affects the form of the sender's operations or the receiver's operations. This implies that the controller's action contains two aspects of "start up"and "authorization"so that the RIOs can be faithfully and determinedly completed. Based on this fact, we can say the controlled RIO protocol definitely enhances the security of remote quantum information processing and communications. Startup of quantum channel in the controlled RIO protocol is easy to understand. However, from our point of view, the necessity of the authorization from controllers and the variations of its ways in the controlled RIO protocol, that is, why, how, and when to distribute the passwords (carry out authorization) by the controllers, this need to be carefully studied in order to faithfully and determinedly complete the protocol in the different cases and for the different purposes of RIOs. It will be seen that they are not trivial or simple, and they have practical significance and applications in engineering. For example, when the controller trusts the sender or receiver, it is easy for controller to authorize and communicate with others; when the controller hopes to "say the last words", he or she only authorizes the receiver at a chosen stage of the protocols. In addition, it should be pointed out that a controller has only a qubit here. When there are many controllers, they can form one or several controlled parties. Every controlled party is made of m controllers, and then the length of its distributing password will be to m c-bits. Obviously, more number of controllers implies higher security. The reason is that, if only one controller does not collaborate with other, the rest participants will not be able to complete the task. In other words, each controller possesses a part of 'key'. Only if they work together at the same time can the safe box' be opened.

Both HPVs and Wang's protocol of RIOs use Bell states as a quantum channel. However, it is well known that Greenberger–Horne–Zeilinger (GHZ) states [26] are also a very important quantum resource in quantum information processing and communication. Motivated by the scheme of teleportation of multi-qubit quantum information using GHZ states, we would like to investigate the remote implementations of quantum operations using GHZ state(s). Specifically, using the GHZ type state(s) in our RIO schemes can enhance security (since it's particles can distribute to multiple controllers), increase variety, extend applications, as well as advance efficiency via fetching in many controllers.

In this paper, we restrict ourselves to an issue, i.e., remotely implementing multiple partially unknown quantum operations from a sender to a distant receiver via the control of many agents in a network. We wish that the receiver can successfully get access to the original operation of each qubit, as long as all the agents collaborate through local operation and classical communication. However, even if one agent does not cooperate, the original operation of each qubit can not fully be recovered by the receiver. The topic here might be of particular interest, since the controlled remote implementation of quantum operations should have some remarkable applications in the remote quantum information including the future quantum internet servers.

Besides Sect. 1 written as introduction, this paper are arranged as follows: in Sect. 2, we present a RIO scheme using GHZ state and introduce our restricted sets of quantum operations via the control of many agents, which is a protocol in the traditional sense of RSP. In Sect. 3, inspired by the RIO scheme in Sect. 1, we propose a method to implement multiple partially unknown quantum operations for the distant receiver through the control of one agent. Then we compare our scheme with the scheme in Sect. 2. In Sect. 4, we discuss how to decompose multi-qubit GHZ State, and then extend the scheme in Sect. 3 to multi-agent controlled RIO. In Sect. 5, we further extend to implementing multiple partially unknown quantum operation-strings for many distant receivers, via the control of many agents in a network. A brief discussion and summary are given in Sect. 6.

2 RIO of a partially unknown quantum operation to a distant receiver via the control of many agents

In order to embody the influence of the controller, the number of the operation transferred and the receiver is reduced to one in this section, respectively. Then, a standard RIO scheme is proposed for implementing a partially unknown quantum operation to a distant receiver via the control of many agents. For concreteness, suppose that the quantum operation to be remotely implemented belongs to one of the two restricted sets defined by [8]

$$\mathcal{U}_0 = \begin{pmatrix} u_0 & 0 \\ 0 & u_1 \end{pmatrix}, \quad \mathcal{U}_1 = \begin{pmatrix} 0 & u_0 \\ u_1 & 0 \end{pmatrix}. \tag{1}$$

We can say that they are partially unknown in the sense that the values of their matrix elements are unknown, but their structures, that is, the positions of their nonzero matrix elements, are known. In our notation, a restricted set of one-qubit operations is denoted by $\mathcal{U}_d$, where $d = 0$ or 1 indicates, respectively, this operation belonging to a diagonal or off-diagonal restricted set.

Assuming that Alice wants to apply $\mathcal{U}_d$ ($d \in \{0, 1\}$) on Bob's qubit B, where qubit B is in an unknown state

$$|\varphi\rangle_B = (\alpha|0\rangle + \beta|1\rangle)_B, \tag{2}$$

where the coefficients α and β are complex numbers with $\alpha^2 + \beta^2 = 1$, and $\{|0\rangle, |1\rangle\}$ are the eigenstates of the $\sigma^{(1,1)} = |0\rangle\langle 0| - |1\rangle\langle 1|$ Pauli operator of the respective qubit. The above task can only be done under the control of n agents and their role of n agents is that to take the responsibility to decide whether or not and when the task should be done. The quantum channel linking Alice, Bob, and the n agents is a $(n+2)$-qubit GHZ state, which is given by

$$|\text{GHZ}\rangle = \frac{1}{\sqrt{2}}(|0\rangle_a|0\rangle_b|0\rangle^{\otimes n} + |1\rangle_a|1\rangle_b|1\rangle^{\otimes n}), \tag{3}$$

where GHZ qubits a belong to Alice, and Bob holds GHZ qubit b, while the other n GHZ qubits belong to the n agents. The joint system of Alice, Bob and the n agents initially reads

$$\begin{aligned}|\mathcal{T}\rangle &= |\mathrm{GHZ}\rangle \otimes |\varphi\rangle_B \\ &= \frac{1}{\sqrt{2}}(|0\rangle_a|0\rangle_b|0\rangle^{\otimes n} + |1\rangle_a|1\rangle_b|1\rangle^{\otimes n})(\alpha|0\rangle + \beta|1\rangle)_B.\end{aligned} \tag{4}$$

Our RIO scheme is made of the following six steps.

Step 1: Bob's preparing. As a receiver, Bob first performs a controlled-NOT gate $\mathcal{N}_{Bb}$ using his qubit B occupied by unknown state (to be acted state) as a control, his shared part b (the second qubit in the above initial state) of the GHZ state as a target, here the unitary operation $\mathcal{N}$ is defined as $\mathcal{N} = |0\rangle\langle 0| \otimes \sigma^{(0,0)} + |1\rangle\langle 1| \otimes \sigma^{(0,1)}$ with Pauli operations $\sigma^{(0,0)} = |0\rangle\langle 0| + |1\rangle\langle 1|$ and $\sigma^{(0,1)} = |0\rangle\langle 1| + |1\rangle\langle 0|$. Complementarily, the other Pauli operations are $\sigma^{(1,0)} = |0\rangle\langle 1| - |1\rangle\langle 0|$ and $\sigma^{(1,1)} = |0\rangle\langle 0| - |1\rangle\langle 1|$. This operation transforms the state $|\mathcal{T}\rangle$ into

$$\begin{aligned}\mathcal{N}_{Bb}|\mathcal{T}\rangle = \frac{1}{\sqrt{2}}[&(\alpha|0\rangle_a|0\rangle_B|0\rangle^{\otimes n} + \beta|1\rangle_a|1\rangle_B|1\rangle^{\otimes n})|0\rangle_b \\ &+ (\alpha|1\rangle_a|0\rangle_B|1\rangle^{\otimes n} + \beta|0\rangle_a|1\rangle_B|0\rangle^{\otimes n})|1\rangle_b].\end{aligned} \tag{5}$$

Subsequently, Bob measures his shared part of GHZ state in the computational basis $|c\rangle_b\langle c|$ $(c = 0, 1)$, then the corresponding collapsed state of residual qubits will be written as

$$\alpha|c\rangle_a|0\rangle_B|c\rangle^{\otimes n} + \beta|c \oplus 1\rangle_a|1\rangle_B|c \oplus 1\rangle^{\otimes n} \quad (c = 0, 1), \tag{6}$$

where $\oplus$ is an addition mod 2. The purpose of this step is to let the unknown state be correlated with Alice's local qubit a. This is a precondition that Alice is able to remotely implement a quantum operation belonging to the restricted sets.

Step 2: First classical communication. Bob sends the classical message c to Alice. The aim of this step is that the receiver Bob tells Alice that he is ready to receive the remote operation, as well as his prepared way.

It must be emphasized that Bob's preparing has two equivalent ways with respect to $c = 0$ or 1, respectively. If Bob first fixes the value of c and tells Alice before the beginning of protocol, this step can be saved. In particular, when c is just taken as 0, the sender Alice does not need the transformation $\sigma^{(0,c)}$ in the next steps, since $\sigma^{(0,0)}$ is trivial.

Step 3: Alice's sending. Alice's operation includes four parts. After receiving Bob's classical bit c, Alice first performs a prior transformation $\sigma^{(0,c)}$ dependent on c, then the state in Eq. (6) becomes

$$\alpha|0\rangle_a|0\rangle_B|c\rangle^{\otimes n} + \beta|1\rangle_a|1\rangle_B|c \oplus 1\rangle^{\otimes n} \quad (c = 0, 1). \tag{7}$$

Second, she carries out the quantum operation $\mathcal{U}_d$ ($d \in \{0, 1\}$) on her qubit a, which transforms the state in Eq. (7) to

$$\alpha u_d |d\rangle_a |0\rangle_B |c\rangle^{\otimes n} + \beta u_{d\oplus 1} |d \oplus 1\rangle_a |1\rangle_B |c \oplus 1\rangle^{\otimes n} \quad (c, d = 0, 1). \tag{8}$$

Third, she executes a Hadamard transformation $\mathcal{H}$ on her qubit a, where Hadamard transformation $\mathcal{H}$ has the form:

$$\mathcal{H}|d\rangle = \frac{1}{\sqrt{2}}[|0\rangle + (-1)^d |1\rangle], \quad d \in \{0, 1\},$$

which transform the state shown in (8) into

$$\begin{aligned} &\frac{1}{\sqrt{2}}\{|0\rangle_a[\alpha u_d |0\rangle_B |c\rangle^{\otimes n} + \beta u_{d\oplus 1} |1\rangle_B |c \oplus 1\rangle^{\otimes n}] \\ &\quad + (-1)^d |1\rangle_a [\alpha u_d |0\rangle_B |c\rangle^{\otimes n} - \beta u_{d\oplus 1} |1\rangle_B |c \oplus 1\rangle^{\otimes n}]\}, \end{aligned} \tag{9}$$

where $c, d \in \{0, 1\}$. Finally, she measures her qubit a in the computational basis $|l\rangle_a \langle l|$ ($l = 0, 1$). After that, When a possible global phase factor is ignored, the state in Eq. (9) collapses into

$$\alpha u_d |0\rangle_B |c\rangle^{\otimes n} + (-1)^l \beta u_{d\oplus 1} |1\rangle_B |c \oplus 1\rangle^{\otimes n} \quad (c, d, l = 0, 1). \tag{10}$$

The action of the first part $\sigma^{(0,c)}$ is to perfectly prepare the state of the joint system, this is a superposition state that the basis in the locally acted system (belonging to the Alice's subsystem) of every component state is the same as its basis in the remotely operated system (belonging to the space of the unknown state in the receiver's subsystems) and the corresponding coefficients are ones of the unknown state. The second part of the sending step is an operation belonging to the restricted sets, which will be remotely implemented in the protocol. The third part of the sending step, the Hadamard gate, is often seen in quantum computation and quantum communication. Its action is similar to the cases in the teleportation of states. The fourth part of the sending step is a measurement on the computational basis whose aim is to project to the needed result.

Step 4: Second classical communication. Alice sends the classical messages d, l to Bob.

The communication to Bob is that the sender Alice tells the receiver Bob what measurement (denote by l) has been done and which kind of operations (denote by d) has been transferred. In the protocol, Alice has a one-to-one mapping table to indicate a kind of restricted set by a value of classical information. It can be encoded by one c-bit, in which 0 denotes a restricted set of diagonal operations and 1 denotes a restricted set of off-diagonal operations. This communication is necessary in order to faithfully and determinedly finish the protocol.

Step 5: Agents' operating. Each agent's operation includes controlling step and allowing step. In controlling step, each agent j ($j = 1, 2, \ldots, n$) first performs a

Hadamard transformation $\mathcal{H}$ on her/his GHZ qubit A_j, then the state in Eq. (10) becomes

$$\begin{aligned}&\alpha u_d|0\rangle_B\left\{\frac{1}{\sqrt{2}}[|0\rangle+(-1)^c|1\rangle]\right\}^{\otimes n}\\&\quad+(-1)^l\beta u_{d\oplus 1}|1\rangle_B\left\{\frac{1}{\sqrt{2}}[|0\rangle+(-1)^{c+1}|1\rangle]\right\}^{\otimes n}\quad(c,d,l=0,1).\end{aligned}\tag{11}$$

Then he/she makes a measurement on her/his shared GHZ qubit A_j in a single-qubit computational basis $|\varepsilon_j\rangle_{A_j}\langle\varepsilon_j|$ $(\varepsilon_j=0,1)$, the corresponding collapse state is

$$\alpha u_d|0\rangle_B+(-1)^{l+\sum_{j=1}^{n}\varepsilon_j}\beta u_{d\oplus 1}|1\rangle_B\ \ (d,l,\varepsilon_j=0,1).\tag{12}$$

This step is a key matter in the controlled RIO protocol. In fact, when one agent has not done it, there are no feasible remote implementations of quantum operations. Only if each agent agrees or wishes that the receiver implements the RIO protocol does he or she carry out this operation and measurement. Its action is to open the quantum channel between the sender and receiver that is necessary for the remote implementation of quantum operations belonging to the restricted sets.

In allowing step, it still has to be completed by each agent j, that is, he or she transfers one c-bit ε_j $(j=1,2,\ldots,n)$ to the receiver Bob.

This step can be understood figuratively as that each agent j distributes the "password"ε_j to the receiver Bob, or gives an authorization to Bob, or says the last word to the receiver in the protocol. This indicates that the role of each agent is very important and indispensable. In other words, this is not trivial in engineering because the above means useful in the controlled process and imply potential applications in practice. Without the password distribution by each agent, the sender and receiver cannot faithfully and determinedly complete the RIO.

Step 6: Bob's recovering. Based on $n+2$ classical bits d, l and ε_j $(j=1,2,\ldots,n)$, respectively, from Alice and n agents, Bob performs the following unitary transformation on his qubit B

$$(1-d)\sigma^{(f,f)}+d\sigma^{(f,1\oplus f)},\tag{13}$$

where the remainder of $l+\sum_{j=1}^{n}\varepsilon_j$ divided by 2 is f, and $d,l,\varepsilon_j=0,1$. Then he can successfully recover the quantum operation $\mathcal{U}_d$ by using the transformation (13), that is

$$\begin{aligned}&[(1-d)\sigma^{(f,f)}+d\sigma^{(f,1\oplus f)}][\alpha u_d|0\rangle_B\\&\quad+(-1)^{l+\sum_{j=1}^{n}\varepsilon_j}\beta u_{d\oplus 1}|1\rangle_B]=\mathcal{U}_d(|\varphi\rangle_B),\quad d=0,1.\end{aligned}\tag{14}$$

Obviously, we use the relations $\mathcal{U}_d|0\rangle=u_d|d\rangle$ and $\mathcal{U}_d|1\rangle=u_{d\oplus 1}|d\oplus 1\rangle$ $(d=0,1)$ in Eq. (14), and use notation rules to describe the general formulae of operations in our scheme, so the probability of success of our scheme is 100%.

3 RIO of multiple partially unknown quantum operations to a distant receiver via the control of one agent

Inspired by the scheme in Sect. 2 and in order to embody that how to realize a protocol with multiple operations, we consider that RIO of multiple partially unknown quantum operations to a distant receiver via the control of one agent in this section.

Assume that Alice holds a string of partially unknown quantum operations labeled by $1, 2, \ldots, m$, which come from the two restricted sets and is given by as follows

$$\mathcal{U}_{d_1} \otimes \mathcal{U}_{d_2} \otimes \cdots \otimes \mathcal{U}_{d_m} \quad (d_1, d_2, \ldots, d_m \in \{0, 1\}). \tag{15}$$

Bob has a string of message qubits labeled by $1, 2, \ldots, m$, which is initially in unknown quantum state

$$\bigotimes_{k=1}^{m} (\alpha_k |0\rangle_k + \beta_k |1\rangle_k), \tag{16}$$

where, the complex numbers α_k and β_k satisfy $|\alpha|_k^2 + |\beta|_k^2 = 1\,(k = 1, 2, \ldots, m)$. Alice wants to send the m partially unknown quantum operations to the distant receiver Bob via the control of one agent Charlie, such that Bob can get the complete information of each partially unknown quantum operation carried by his corresponding message qubit (in unknown quantum state) only if Charlie collaborates. This can be done by the following procedure.

Step 1 Alice constructs the following EPR entangled state through local logic gates

$$\begin{aligned} |\mathcal{G}\rangle = \frac{1}{\sqrt{2^{m+2}}} \Bigg\{ &\bigotimes_{k=1}^{m} (|00\rangle_{k'k''} + |11\rangle_{k'k''})(|00\rangle_{AC} + |11\rangle_{AC}) \\ &+ \bigotimes_{k=1}^{m} (|00\rangle_{k'k''} - |11\rangle_{k'k''})(|00\rangle_{AC} - |11\rangle_{AC}) \Bigg\} \end{aligned} \tag{17}$$

as the quantum channel, and then sends an EPR qubit C to the controller Charlie and m EPR qubits $(1'', 2'', \ldots, m'')$ to the receiver Bob, while keeping the other $m+1$ EPR qubits $(1', 2', \ldots, m')$ and an EPR qubit A to herself.

The process of constructing quantum channel $|\mathcal{G}\rangle$ is given as follows: The input state is the $2(m+1)$-qubit product state $|\psi_1\rangle$ as

$$|\psi_1\rangle = |00\rangle_{1'1''} |00\rangle_{2'2''} \cdots |00\rangle_{m'm''} |00\rangle_{AC}. \tag{18}$$

Firstly, Alice performs the Hadamard operation ($\mathcal{H}$) on qubit $1'$ and rewrite the $\mathcal{H}|\psi_1\rangle$ as

$$|\psi_2\rangle = \mathcal{H}|\psi_1\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |10\rangle)_{1'1''} |00\rangle_{2'2''} \cdots |00\rangle_{m'm''} |00\rangle_{AC}. \tag{19}$$

Then, she operates CNOT $\mathcal{N}$ operations on the qubit pairs $(1', 2')$, $(1', 3')$, …, $(1', m')$ and $(1', A)$, respectively, where qubit $1'$ is used as controlled qubit and each of m qubits $2', 3', \ldots, m', A$ are used as target qubit. The transformed $|\psi_2\rangle$ can be expressed as

$$\begin{aligned}|\psi_3\rangle = \frac{1}{\sqrt{2}}(&|00\rangle_{1'1''}|00\rangle_{2'2''}\cdots|00\rangle_{m'm''}|00\rangle_{AC}\\ &+|10\rangle_{1'1''}|10\rangle_{2'2''}\cdots|10\rangle_{m'm''}|10\rangle_{AC}).\end{aligned} \tag{20}$$

After implementing the Hadamard operations on $1', 2', \ldots, m', A$, Alice executes CNOT operations on the qubit pairs $(1', 1'')$, $(2', 2'')$, …, (m', m''), (A, C), respectively, where qubits $1', 2', \ldots, m', A$ are used as controlled qubit and each of $2m+1$ qubits $1'', 2'', \ldots, m'', C$ is used as target qubit. The $2(m+1)$-qubit state can be generated as

$$|\psi_4\rangle = \frac{1}{\sqrt{2}}[\bigotimes_{k=1}^{m}|\phi^+\rangle_{k'k''}|\phi^+\rangle_{AC} + \bigotimes_{k=1}^{m}|\phi^-\rangle_{k'k''}|\phi^-\rangle_{AC}], \tag{21}$$

where $|\phi^{\pm}\rangle = \frac{1}{\sqrt{2}}(|00\rangle \pm |11\rangle)$. Obviously, $|\mathcal{G}\rangle = |\psi_4\rangle$.

The state of the whole system is given by

$$\begin{aligned}&\bigotimes_{k=1}^{m}(\alpha_k|0\rangle_k + \beta_k|1\rangle_k)\otimes|\mathcal{G}\rangle\\ &= \frac{1}{\sqrt{2^{m+2}}}\{\bigotimes_{k=1}^{m}[(\alpha_k|0\rangle_k+\beta_k|1\rangle_k)(|00\rangle_{k'k''}+|11\rangle_{k'k''})](|00\rangle_{AC}+|11\rangle_{AC})\\ &\quad+\bigotimes_{k=1}^{m}[(\alpha_k|0\rangle_k+\beta_k|1\rangle_k)(|00\rangle_{k'k''}-|11\rangle_{k'k''})](|00\rangle_{AC}-|11\rangle_{AC})\}.\end{aligned} \tag{22}$$

Step 2 Bob performs a series of CNOT operations respectively on qubit pairs $(1, 1'')$, $(2, 2'')$, …, (m, m''). For simplicity, normalized factors throughout the following paper are omitted. After Bob's operating, the state in Eq. (22) becomes

$$\begin{aligned}&\bigotimes_{k=1}^{m}[(\alpha_k|00\rangle_{kk'}+\beta_k|11\rangle_{kk'})|0\rangle_{k''}+(\alpha_k|01\rangle_{kk'}+\beta_k|10\rangle_{kk'})|1\rangle_{k''}]\\ &(|00\rangle_{AC}+|11\rangle_{AC})\\ &+\bigotimes_{k=1}^{m}[(\alpha_k|00\rangle_{kk'}-\beta_k|11\rangle_{kk'})|0\rangle_{k''}-(\alpha_k|01\rangle_{kk'}-\beta_k|10\rangle_{kk'})|1\rangle_{k''}]\\ &(|00\rangle_{AC}-|11\rangle_{AC}).\end{aligned} \tag{23}$$

Now, Bob carries out a series of single-qubit measurements respectively on his EPR qubit $1'', 2'', \ldots, m''$ in basis $|c\rangle_{k''}\langle c|$ $(c = 0, 1)$, and sends the sender Alice all of his measurement results (the corresponding classical message of each measurement result is c) via classical communication. Then the corresponding collapse state is

$$\begin{aligned}&\bigotimes_{k=1}^{m}(\alpha_k|0, c\rangle_{kk'} + \beta_k|1, c \oplus 1\rangle_{kk'})(|00\rangle_{AC} + |11\rangle_{AC})\\&\quad+\bigotimes_{k=1}^{m}(-1)^c(\alpha_k|0, c\rangle_{kk'} - \beta_k|1, c \oplus 1\rangle_{kk'})(|00\rangle_{AC} - |11\rangle_{AC}).\end{aligned} \tag{24}$$

Step 3 After hearing the measurements from Bob, Alice performs a series of Pauli transformations $\sigma^{(0,c)}$ (dependent on $c, c = 0, 1$) on her particles $1', 2', \ldots, m'$, respectively, then the state in Eq. (24) becomes

$$\begin{aligned}&\bigotimes_{k=1}^{m}(\alpha_k|00\rangle_{kk'} + \beta_k|11\rangle_{kk'})(|00\rangle_{AC} + |11\rangle_{AC})\\&\quad+\bigotimes_{k=1}^{m}(-1)^c(\alpha_k|00\rangle_{kk'} - \beta_k|11\rangle_{kk'})(|00\rangle_{AC} - |11\rangle_{AC}).\end{aligned} \tag{25}$$

Second, she carries out the quantum operation $\mathcal{U}_{d_k}$ $(d_k \in \{0, 1\})$ on her each EPR qubit k', which transforms the state in Eq. (25) to

$$\begin{aligned}&\bigotimes_{k=1}^{m}(\alpha_k u_{d_k}|0, d_k\rangle_{kk'} + \beta_k u_{d_k\oplus 1}|1, d_k \oplus 1\rangle_{kk'})(|00\rangle_{AC} + |11\rangle_{AC})\\&\quad+\bigotimes_{k=1}^{m}(-1)^c(\alpha_k u_{d_k}|0, d_k\rangle_{kk'} - \beta_k u_{d_k\oplus 1}|1, d_k \oplus 1\rangle_{kk'})(|00\rangle_{AC} - |11\rangle_{AC}).\end{aligned} \tag{26}$$

Third, she executes a Hadamard transformation $\mathcal{H}$ on each EPR qubit, which transform the state shown in Eq. (26) into

$$\begin{aligned}&\bigotimes_{k=1}^{m}[(\alpha_k u_{d_k}|0\rangle + \beta_k u_{d_k\oplus 1}|1\rangle)_k|0\rangle_{k'} + (-1)^{d_k}(\alpha_k u_{d_k}|0\rangle - \beta_k u_{d_k\oplus 1}|1\rangle)_k|1\rangle_{k'}]\\&(|00\rangle + |11\rangle)_{AC}\\&+\bigotimes_{k=1}^{m}(-1)^c[(\alpha_k u_{d_k}|0\rangle - \beta_k u_{d_k\oplus 1}|1\rangle)_k|0\rangle_{k'} + (-1)^{d_k}(\alpha_k u_{d_k}|0\rangle + \beta_k u_{d_k\oplus 1}|1\rangle)_k|1\rangle_{k'}]\\&(|00\rangle - |11\rangle)_{AC}.\end{aligned} \tag{27}$$

where $c, d_k \in \{0, 1\}$. Finally, Alice measures her each EPR qubit in the computational basis $|l\rangle_{k'}\langle l|$ $(l = 0, 1)$, and and sends the messages d_k of each $\mathcal{U}_{d_k}$ used by her and l of each EPR qubit k' measured by her to the receiver Bob. After that, the state in

Eq. (27) collapses into

$$\begin{aligned}&\bigotimes_{k=1}^{m}(-1)^{d_k l}[(\alpha_k u_{d_k}|0\rangle+(-1)^l\beta_k u_{d_k\oplus 1}|1\rangle)_k](|00\rangle+|11\rangle)_{AC}\\&\quad+\bigotimes_{k=1}^{m}(-1)^{c+d_k l}[(\alpha_k u_{d_k}|0\rangle+(-1)^{l+1}\beta_k u_{d_k\oplus 1}|1\rangle)_k](|00\rangle-|11\rangle)_{AC},\end{aligned} \tag{28}$$

and one has

$$|\zeta\rangle(|00\rangle+|11\rangle)_{AC}+|\zeta'\rangle(|00\rangle-|11\rangle)_{AC}, \tag{29}$$

with

$$|\zeta\rangle=\bigotimes_{k=1}^{m}|\zeta\rangle_k,\quad |\zeta'\rangle=\bigotimes_{k=1}^{m}|\zeta'\rangle_k, \tag{30}$$

where $|\zeta\rangle$ and $|\zeta'\rangle$ are the states for the m qubits $(1,2,\ldots,m)$ belonging to Bob, while $|\zeta\rangle_k$ and $|\zeta'\rangle_k$ are the states of Bob's qubit k. From Eq. (28), one can see that the states $|\zeta\rangle_k$ and $|\zeta'\rangle_k$ depend on the d_k of Alice's applied $\mathcal{U}_{d_k}$ and the outcome l of Alice's single-qubit measurement on the qubits k', and given by

$$|\zeta\rangle_k=(-1)^{d_k\cdot l}(\alpha_k u_{d_k}|0\rangle+(-1)^l\beta_k u_{d_k\oplus 1}|1\rangle)\quad \text{for}\quad d_k,l=0,1 \tag{31}$$

$$|\zeta'\rangle_k=(-1)^{c\oplus(d_k\cdot l)}(\alpha_k u_{d_k}|0\rangle+(-1)^{l\oplus 1}\beta_k u_{d_k\oplus 1}|1\rangle)\quad \text{for}\quad d_k,l=0,1 \tag{32}$$

where c is a output of Bob's measurement result on his qubit k''.

The results (31) and (32) show that according to the Alice's operating $\mathcal{U}_{d_k}$ on qubit k' and the outcome of Alice's measurement on qubit k', Bob can always recover the state $\mathcal{U}_{d_k}(\alpha_k|0\rangle_k+\beta_k|1\rangle_k)$ of the message qubit k from the state $|\zeta\rangle_k$ or $|\zeta'\rangle_k$ of his qubit k, by performing a single-qubit unitary operation $U_k=(1-d_k)\sigma^{(l,l)}+d_k\sigma^{(l,l\oplus 1)}$ or $U'_k=(1-d_k)\sigma^{(l\oplus 1,l\oplus 1)}+d_k\sigma^{(l\oplus 1,l)}$.

Step 4 Alice and Charlie execate a Hadamard transformation $\mathcal{H}$ on their respective qubits A and C. As a result, the state $|00\rangle_{AC}-|11\rangle_{AC}$ goes to $|01\rangle_{AC}+|10\rangle_{AC}$ while the state $|00\rangle_{AC}+|11\rangle_{AC}$ remains unchanged. after Alice's and Charlie's Hadamard transformations, the state (29) will change into

$$|\zeta\rangle(|00\rangle+|11\rangle)_{AC}+|\zeta'\rangle(|01\rangle+|10\rangle)_{AC}. \tag{33}$$

Subsequently, Alice and Charlie perform a measurement on their respective qunits A and C in the computational basis, and then each sends the measurement outcome to Bob. One can see from Eq. (33) that if Bob knows that Alice and Charlie both measured their qubits in the sate $|0\rangle$ or $|1\rangle$, he can judge that his m qubits $(1,2,\ldots,m)$ must be in the state $|\zeta\rangle$. On the other hand, expression (33) shows that if Bob knows that Alice measured her qubit in the state $|0\rangle$ ($|1\rangle$) while Charlie measured his qubit in state $|1\rangle$ ($|0\rangle$), he knows that his m qubits $(1,2,\ldots,m)$ must be in the state $|\zeta'\rangle$. That is to say, according to the measurement outcomes from Alice and Charlie, Bob can determine whether his m qubits are in $|\zeta\rangle$ or $|\zeta'\rangle$.

Step 5 Note that the state $|\zeta\rangle$ ($|\zeta'\rangle$), described in Eq. (30), is a product of individual-qubit states $|\zeta\rangle_1, |\zeta\rangle_2, \ldots, |\zeta\rangle_m$ ($|\zeta'\rangle_1, |\zeta'\rangle_2, \ldots, |\zeta'\rangle_m$) for the qubits $(1, 2, \ldots, m)$. And, as mentioned above, Bob can always recover the state $\mathcal{U}_{d_k}(\alpha_k|0\rangle_k + \beta_k|1\rangle_k)$ of the message qubit k from the state $|\zeta\rangle_k$ or $|\zeta'\rangle_k$ of his message qubit k, based on the outcomes of Alice's measurements on the qubits k' as well as the message d_k of $\mathcal{U}_{d_k}$ applied by Alice on qunit k', and via a single-qubit unitary operation U_k or U'_k on his qubit k. Thus, Bob can always reconstruct the corresponding state $\bigotimes_{k=1}^{m}\mathcal{U}_{d_k}(\alpha_k|0\rangle_k + \beta_k|1\rangle_k)$ of m message qubits $(1, 2, \ldots, m)$ from the state $|\zeta\rangle$ or $|\zeta'\rangle$ of his m qubits $(1, 2, \ldots, m)$, according to the outcomes of Alice's measurements on the qubits $(1', 2', \ldots, m')$ and m messages $(d_1, d_2, \ldots, d_m)$ of $(\mathcal{U}_{d_1}, \mathcal{U}_{d_2}, \ldots, \mathcal{U}_{d_m})$ performed by Alice, and through his series $\{U_1, U_2, \ldots, U_m\}$ or $\{U'_1, U'_2, \ldots, U'_m\}$ of local single-qubit unitary operations.

Remark 1 (i) Now, we show that Bob can't get the full information of quantum operations when Charlie doesn't cooperate. Examining the state (29), one can see that when only Alice executes a Hadamard transformation $\mathcal{H}$ on her qubit A, the state (29) well become

$$\begin{aligned}&[(|\zeta\rangle + |\zeta'\rangle)|0\rangle_C + (|\zeta\rangle - |\zeta'\rangle)|1\rangle_C]|0\rangle_A\\ &\quad +[(|\zeta\rangle + |\zeta'\rangle)|0\rangle_C - (|\zeta\rangle - |\zeta'\rangle)|1\rangle_C]|1\rangle_A,\end{aligned} \tag{34}$$

which implies that whether Alice measures her qubit A in state $|0\rangle$ or $|1\rangle$, the density operator of the m qubits $(1, 2, \ldots, m)$ belonging to Bob well, after tracing over Charlie's qubit C, be given by

$$\rho = (|\zeta\rangle + |\zeta'\rangle)(\langle\zeta| + \langle\zeta'|) + (|\zeta\rangle - |\zeta'\rangle)(\langle\zeta| - \langle\zeta'|). \tag{35}$$

The result (35) shows that the m qubits $(1, 2, \ldots, m)$ are in a mixed state, in which they are in a superposition state $|\zeta\rangle + |\zeta'\rangle$ with a probability $p_1 = ||\zeta\rangle + |\zeta'\rangle|^2/2(\langle\zeta|\zeta\rangle + \langle\zeta'|\zeta'\rangle)$ while being in the other superposition state $|\zeta\rangle - |\zeta'\rangle$ with a probability $p_2 = ||\zeta\rangle - |\zeta'\rangle|^2/2(\langle\zeta|\zeta\rangle + \langle\zeta'|\zeta'\rangle)$.

Based on Eqs. (30)–(32), one can express the states $|\zeta\rangle + |\zeta'\rangle$ and $|\zeta\rangle - |\zeta'\rangle$ involved in Eq. (35) as follows

$$\begin{aligned}|\zeta\rangle \pm |\zeta'\rangle = &(-1)^{d_s\cdot l}|\widetilde{\zeta}\rangle(\alpha_s u_{d_s}|0\rangle + (-1)^{l}\beta_s u_{d_s\oplus 1}|1\rangle)_s\\ &\pm (-1)^{c\oplus(d_s\cdot l)}|\widetilde{\zeta'}\rangle(\alpha_s u_{d_s}|0\rangle + (-1)^{l\oplus 1}\beta_s u_{d_s\oplus 1}|1\rangle)_s \quad \text{for} \quad d_s, l = 0, 1\end{aligned} \tag{36}$$

where the subscript s represents any one of the m qubits $(1, 2, \ldots, m)$ belonging to Bob, and the notations d_s and l denote the value of nonvanishing elements of matrix $\mathcal{U}_{d_s}$ used by Alice and Alice's measurement message on her EPR qubit s' (corresponding to Bob's qubit s), respectively. In Eq. (36), we further note that $|\widetilde{\zeta}\rangle = \bigotimes_{t\neq s}|\zeta\rangle_t$ and $|\widetilde{\zeta'}\rangle = \bigotimes_{t\neq s}|\zeta'\rangle_t$ are the states of the remaining $m-1$ qubits belonging to Bob (after excluding the qubit s). Here, $|\zeta\rangle_t$ and $|\zeta'\rangle_t$ are the states of qubit t ($t \neq s$), which depend on the outcomes of Alice's measurements on the qubits t' and the value of nonvanishing elements of matrix $\mathcal{U}_{d_t}$ used by Alice, and take the form of Eqs. (31) and (32), respectively.

Obviously, $\mathcal{U}_{d_s}^{\dagger}\mathcal{U}_{d_s} = I$ because $\mathcal{U}_{d_s}$ is a unitary matrix. It follows that $|u_{d_s}|^2 = |u_{d_s\oplus 1}|^2 = 1$. From Eqs. (35) and (36), it is easily shown that for each $(d_s, l) \in \{(0,0), (0,1), (1,0), (1,1)\}$ of Alice's implementing $\mathcal{U}_{d_s}$ and Alice's measurement on qubit s', and for each $c \in \{0, 1\}$, the density operator for the remaining $m-1$ qubits belonging to Bob is, after tracing over the qubit s, given by

$$\begin{aligned}\widetilde{\rho} &= tr_s(\rho)\\ &= |\alpha_s|^2|u_{d_s}|^2(|\widetilde{\zeta}\rangle + |\widetilde{\zeta'}\rangle)(\langle\widetilde{\zeta}| + \langle\widetilde{\zeta'}|) + |\beta_s|^2|u_{d_s\oplus 1}|^2(|\widetilde{\zeta}\rangle - |\widetilde{\zeta'}\rangle)(\langle\widetilde{\zeta}| - \langle\widetilde{\zeta'}|)\\ &\quad + |\alpha_s|^2|u_{d_s}|^2(|\widetilde{\zeta}\rangle - |\widetilde{\zeta'}\rangle)(\langle\widetilde{\zeta}| - \langle\widetilde{\zeta'}|) + |\beta_s|^2|u_{d_s\oplus 1}|^2(|\widetilde{\zeta}\rangle + |\widetilde{\zeta'}\rangle)(\langle\widetilde{\zeta}| + \langle\widetilde{\zeta'}|)\\ &= (|\widetilde{\zeta}\rangle + |\widetilde{\zeta'}\rangle)(\langle\widetilde{\zeta}| + \langle\widetilde{\zeta'}|) + (|\widetilde{\zeta}\rangle - |\widetilde{\zeta'}\rangle)(\langle\widetilde{\zeta}| - \langle\widetilde{\zeta'}|),\end{aligned} \tag{37}$$

where we have used $|\alpha_s|^2 + |\beta_s|^2 = 1$. Equation (37) implies that the density operator, for the remaining $m-1$ "non-traced" qubits belonging to Bob, has the same form as Eq. (35). Therefore, repeating the above single-qubit tracing procedure, one finds that the density operator for any qubit r belonging to Bob ($r = 1, 2, \ldots$, or m) can, after tracing over Bob's other $m-1$ qubits, be written as

$$\rho_r = |\alpha_r|^2|0\rangle\langle 0| + |\beta_r|^2|1\rangle\langle 1| \tag{38}$$

for any $(d_r, l) \in \{(0,0), (0,1), (1,0), (1,1)\}$. The above process demonstrates that the density operator (38) depends only on the outcome of Alice's measurement on the qubit r', but independent of the outcome of Alice's measurement on all other qubits.

Equation (38) demonstrates that the following results. Firstly, any qubit r belonging to Bob is in a mixed state, in which it is in the state $|0\rangle$ with a probability $|\alpha_r|^2$, while being in the state $|1\rangle$ with a probability $|\beta_r|^2$. Secondly, Bob doesn't know anything about $\mathcal{U}_{d_r}$. Therefore, Bob can not fully restore the state $\mathcal{U}_{d_r}(\alpha_r|0\rangle_r + \beta_r|1\rangle_r)$ for any message qubit r belonging to Bob with the agent does not cooperate with him.

(ii) It is necessary for us to compare the present method with the method in the Sect. 2. From the description above, we conclude that to teleport m partially unknown quantum operations to a distant receiver via the control of one agent, the present method requires only: $2m+2$ auxiliary qubits for the generation of the state (17), one qubit being assigned to the controller, one single-qubit Hadamard transformation and one single-qubit measurement being preformed by the controller, and one-bit classical message being sent by the controller.

However, as addressed in the Sect. 2, to implement the present task, the method in the Sect. 2 will require: $3m$ auxiliary qubits for preparing m copies of a three-qubit GHZ state when the number of controller is one, m qubits being assigned to the controller, m single-qubit Hadamard transformations and m single-qubit measurements being preformed by the controller, and m-bit classical messages being sent to the receiver by the controller.

The above analysis demonstrates that for the case of $m = 1$, the present scheme is trivial, since it requires 4 auxiliary qubits while the method in the Sect. 2 only needs 3 auxiliary qubits. However, the advantage for the present protocol appears when $m = 2$, because it requires the same number of auxiliary qubits but less local operation and classical communication, compared with the method in the Sect. 2. Moreover, when $m = 3$, the number of auxiliary qubits required in present method becomes smaller than that using the method in the Sect. 2. One can clearly see that the advantage of the present method becomes apparent with the increment of m. Especially, when m is a large number, the required auxiliary qubit resources, local operations by the controller and classical communication between the controller and the receiver are greatly reduced in the present approach.

On a final note, we point out that excluding Alice's measurement on qubit A and the number of Alice's single-qubit measurements needed in the present scheme is the same as that required by the method in the Sect. 2. This is obvious, since using the method in the Sect. 2, Alice also needs to perform a series of single-qubit measurements, each acting on one GHZ qubit.

4 RIO of multiple partially unknown quantum operations to a distant receiver via the control of many agents

This section demonstrates the influence of many operations and agents in RIO protocol. Thus, the proposed protocol only contains one receiver. For the convenience of the discussion in this section, we first consider the decomposition of multi-qubit GHZ states. GHZ states play an important role in quantum information processing. Many theoretical protocols have appeared for the generation of multi-qubit GHZ states. Moreover, it has reported that up to four-qubit GHZ states were experimentally prepared with polarized-state photons [27] and trapped ions [28]. As especially relevant to this work, we consider the following two types of $(n + 1)$-qubit GHZ states:

$$|GHZ\rangle_+ = |00\cdots 0\rangle + |11\cdots 1\rangle, \tag{39}$$

$$|GHZ\rangle_- = |00\cdots 0\rangle - |11\cdots 1\rangle. \tag{40}$$

We find that if a Hadamard transformation $\mathcal{H}$ is performed on each qubit, the states (39) and (40) will be decomposed, respectively, into

$$|GHZ\rangle_+ \to \sum_{\{x_t\}} |\{x_t\}\rangle|0\rangle + \sum_{\{y_t\}} |\{y_t\}\rangle|1\rangle, \tag{41}$$

$$|GHZ\rangle_- \to \sum_{\{x_t\}} |\{x_t\}\rangle|1\rangle + \sum_{\{y_t\}} |\{y_t\}\rangle|0\rangle, \tag{42}$$

where $|\{x_t\}\rangle = |x_1 x_2 \cdots x_n\rangle$ and $|\{y_t\}\rangle = |y_1 y_2 \cdots y_n\rangle$ are computational basis states of the first n qubits ($x_t, y_t \in \{0, 1\}; t = 1, 2, \ldots, n$), and $\sum_{\{x_t\}} |\{x_t\}\rangle$ ($\sum_{\{y_t\}} |\{y_t\}\rangle$) is a sum over all possible basis states $|\{x_t\}\rangle$ ($|\{y_t\}\rangle$) each containing an even (odd) number of "1"s. For instance, when $n = 4$, $\sum_{\{x_t\}} |\{x_t\}\rangle = |0000\rangle + |0011\rangle + |0101\rangle + |0110\rangle +$

$|1001\rangle + |1010\rangle + |1100\rangle + |1111\rangle$ and $\sum_{\{y_t\}} |\{y_t\}\rangle = |0001\rangle + |0010\rangle + |0100\rangle + |1000\rangle + |0111\rangle + |1011\rangle + |1101\rangle + |1110\rangle$. Note that the number of the basis states $|\{x_t\}\rangle$ is the same as that of the basis states $|\{y_t\}\rangle$, thus the two states (41) and (42) on the right side both have the same normalized factors.

Now, we generalize the method in Sect. 3 to the RIO of multiple partially unknown quantum operations via the control of many agents. Suppose that Bob has a string of message qubits labeled by $1, 2, \ldots, m$, which is initially in the unknown state (16) (i.e., $\bigotimes_{k=1}^{m}(\alpha_k|0\rangle_k + \beta_k|1\rangle_k)$). Alice wishes to teleport the m partially unknown quantum operations in Eq. (15) (i.e., $\bigotimes_{k=1}^{m}\mathcal{U}_{d_k}$) to Bob via the control of n agents $(C_1, C_2, \ldots, C_n)$ in a network, such that Bob can get the complete information of each partially unknown quantum operation carried by his corresponding message qubit only if all the agents collaborate. This can be done by the following four steps.

Step 1 Alice generates the following EPR-GHZ entangled state through local gates

$$|\mathcal{G}'\rangle = \bigotimes_{k=1}^{m}(|00\rangle_{k'k''} + |11\rangle_{k'k''})|GHZ\rangle_{+} + \bigotimes_{k=1}^{m}(|00\rangle_{k'k''} - |11\rangle_{k'k''})|GHZ\rangle_{-}, \quad (43)$$

where $|\mathrm{GHZ}\rangle_{\pm} = |00\cdots0\rangle \pm |11\cdots1\rangle$ are $(n+1)$-qubit GHZ states, and then she sends the first n GHZ qubits to the n agents and the m EPR qubits $(1'', 2'', \ldots, m'')$ to Bob, while keeping the last GHZ qubit and the other m EPR qubits $(1', 2', \ldots, m')$ to herself.

The process of generating the $|\mathcal{G}'\rangle$ is given as follows: The input state is the $(2m + n + 1)$-qubit product state $|\phi_1\rangle$ as

$$|\phi_1\rangle = |00\rangle_{1'1''}|00\rangle_{2'2''}\cdots|00\rangle_{m'm''}|00\cdots0\rangle_{C_1\cdots C_n A}. \quad (44)$$

Firstly, a Hadamard operation $(\mathcal{H})$ is performed on qubit $1'$ and rewrite the $\mathcal{H}|\phi_1\rangle$ as

$$|\phi_2\rangle = (|00\rangle + |10\rangle)_{1'1''}|00\rangle_{2'2''}\cdots|00\rangle_{m'm''}|00\cdots0\rangle_{C_1\cdots C_n A}. \quad (45)$$

Then, she operates CNOT operations on the qubit pairs $(1', 2')$, $(1', 3')$, …, $(1', m')$ and $(1', A)$, respectively, where qubit $1'$ is used as controlled qubit and each of m qubits $2', 3', \ldots, m', A$ are used as target qubit. The transformed $|\phi_2\rangle$ can be expressed as

$$\begin{aligned}|\phi_3\rangle = \frac{1}{\sqrt{2}}(&|00\rangle_{1'1''}|00\rangle_{2'2''}\cdots|00\rangle_{m'm''}|00\cdots0\rangle_{C_1\cdots C_n A}\\ &+|10\rangle)_{1'1''}|10\rangle_{2'2''}\cdots|10\rangle_{m'm''}|0\cdots01\rangle_{C_1\cdots C_n A}).\end{aligned} \quad (46)$$

After implementing the Hadamard operations on $1', 2', \ldots, m', A$, Alice executes CNOT operations on the qubit pairs $(1', 1'')$, $(2', 2'')$, $\ldots$, (m', m''), (A, C_1), (A, C_2), …, (A, C_n), respectively, where qubits $1', 2', \ldots, m', A$ are used as controlled qubit and each of $m + n$ qubits $1'', 2'', \ldots, m'', C_1, C_1, \ldots, C_n$ is used as target qubit. The

$(2m+n+1)$-qubit state can be generated as

$$|\phi_4\rangle = \bigotimes_{k=1}^{m}(|00\rangle + |11\rangle)_{k'k''}|GHZ\rangle_+ + \bigotimes_{k=1}^{m}(|00\rangle - |11\rangle)_{k'k''}|GHZ\rangle_-.$$

The state of the whole system is given by

$$\begin{aligned}
&\bigotimes_{k=1}^{m}(\alpha_k|0\rangle_k + \beta_k|1\rangle_k)|\mathcal{G}'\rangle \\
&\quad = \bigotimes_{k=1}^{m}[(\alpha_k|0\rangle_k + \beta_k|1\rangle_k)(|00\rangle_{k'k''} + |11\rangle_{k'k''})]|GHZ\rangle_+ \\
&\qquad + \bigotimes_{k=1}^{m}[(\alpha_k|0\rangle_k + \beta_k|1\rangle_k)(|00\rangle_{k'k''} - |11\rangle_{k'k''})]|GHZ\rangle_-.
\end{aligned} \tag{47}$$

Step 2 After doing exactly the same as Steps 2 and 3 in Sect. 3, we have

$$|\zeta\rangle|GHZ\rangle_+ + |\zeta'\rangle|\mathrm{GHZ}\rangle_-, \tag{48}$$

where $|\zeta\rangle$ and $|\zeta'\rangle$ are the states for the m message qubits $(1, 2, \ldots, m)$ belonging to Bob. Note that the left part of the first (second) product term in the right side of Eq. (47) is the same as that of the first (second) product term in the right side of Eq. (22). Hence, the two states $|\zeta\rangle$ and $|\zeta'\rangle$ here take the same form as $|\zeta\rangle$ and $|\zeta'\rangle$ described by (30), respectively.

Step 3 Each agent and Alice perform a Hadamard transformation $\mathcal{H}$ on their respective GHZ qubits. After that, based on (41) and (42), one gets from (48)

$$|\zeta\rangle\{\sum_{\{x_t\}}|\{x_t\}\rangle|0\rangle + \sum_{\{y_t\}}|\{y_t\}\rangle|1\rangle\} + |\zeta'\rangle\{\sum_{\{x_t\}}|\{x_t\}\rangle|1\rangle + \sum_{\{y_t\}}|\{y_t\}\rangle|0\rangle\}. \tag{49}$$

Step 4 Each agent and Alice make a measurement on their respective GHZ qubits, and then send their measurement results to Bob. Recall the notation of $|\{x_t\}\rangle$ and $|\{y_t\}\rangle$ described above, i.e., each basis state $|\{x_t\}\rangle$ ($|\{y_t\}\rangle$) contains an even (odd) number of "1"s. Therefore, one sees from (49) that Bob can predict that the m message qubits $(1, 2, \ldots, m)$ belonging to him must be in the state $|\zeta\rangle$ ($|\zeta'\rangle$), if he knows that the outcome of the n agents' measurement on their n GHZ qubits contains an even number of "1"s and that Alice measured her GHZ qubit in the state $|0\rangle$ ($|1\rangle$). On the other hand, the result (49) shows that Bob knows that his m message qubits $(1, 2, \ldots, m)$ must be in the state $|\zeta\rangle$ ($|\zeta'\rangle$), if he knows that the outcome of n agents' measurement includes an odd number of "1"s and that Alice measured her GHZ qubit in the state $|1\rangle$ ($|0\rangle$). Thus, according to the measurement outcomes from the n agents and Alice, Bob can predict whether his m message qubits are in $|\zeta\rangle$ or $|\zeta'\rangle$.

As addressed in the Sect. 3, Bob can always restore the corresponding state $\bigotimes_{k=1}^{m}\mathcal{U}_{d_k}(\alpha_k|0\rangle_k + \beta_k|1\rangle_k)$ of m message qubits $(1, 2, \ldots, m)$ from the state $|\zeta\rangle$

or $|\zeta'\rangle$ of his m qubits $(1, 2, \ldots, m)$, according to all of Alice's measurements on the qubits $(1', 2', \ldots, m')$ and information about all the matrices that Alice has used, and through his series of local single-qubit unitary operations.

Remark 2 (i) Even if only one controller does not cooperate, Bob will not be able to recover the operations that Alice wanted to transmit to him. The corresponding analysis which caculates the reduced density matrix of every qubit is the same as the Remark1.(i) and is omitted. We have shown from (48) that the partially unknown quantum operations $\bigotimes_{k=1}^{m} \mathcal{U}_{d_k}$ carried by the m message qubits $(1, 2, \ldots, m)$ can be recovered by Bob, as long as each agent performs a Hadamard transformation and then a measurement on her/his qubit. Now let us focus on the problem that Bob can not gain the full quantum messages about these partially unknown quantum operations even if one agent does not collaborate. To see this, let us go back to the state (48). This state can be rewritten as

$$\begin{aligned} &|\zeta\rangle[(|\varsigma^+\rangle + |\varsigma^-\rangle)|0\rangle_{C_t} + (|\varsigma^+\rangle - |\varsigma^-\rangle)|1\rangle_{C_t}] \\ &\quad + |\zeta'\rangle[(|\varsigma^+\rangle + |\varsigma^-\rangle)|0\rangle_{C_t} - (|\varsigma^+\rangle - |\varsigma^-\rangle)|1\rangle_{C_t}], \end{aligned} \tag{50}$$

where $|0\rangle_{C_t}$ and $|1\rangle_{C_t}$ are the two logic states of the GHZ qubit belonging to agent C_t $(t = 1, 2, \ldots, n-1, \text{ or } n)$, while $|\varsigma^+\rangle$ and $|\varsigma^-\rangle$, taking the form of (39) and (40) respectively, are the GHZ states of the remaining n GHZ qubits belonging to other $n-1$ agents and Alice.

Suppose that the agent C_t does not collaborate with Bob. When the other $n-1$ agents and Alice perform a Hadamard transformation on their respective GHZ qubit, it follows from (41) and (42) that the states $|\varsigma^+\rangle$ and $|\varsigma^-\rangle$ will be transformed into

$$|\varsigma^+\rangle \to \sum_{\{x_r'\}} |\{x_r'\}\rangle|0\rangle_A + \sum_{\{y_r'\}} |\{y_r'\}\rangle|1\rangle_A, \quad |\varsigma^-\rangle \to \sum_{\{x_r'\}} |\{x_r'\}\rangle|1\rangle_A + \sum_{\{y_r'\}} |\{y_r'\}\rangle|0\rangle_A, \tag{51}$$

where the subscript A represents the GHZ qubit belonging to Alice; $|\{x_r'\}\rangle = |x_1'x_2' \cdots x_{n-1}'\rangle$ and $|\{y_r'\}\rangle = |y_1'y_2' \cdots y_{n-1}'\rangle$ are computational basis states of the $n-1$ GHZ qubits belonging to the other $n-1$ agents $(x_r', y_r' \in \{0, 1\}; r = 1, 2, \ldots, n-1)$. Further, $\sum_{\{x_r'\}} |\{x_r'\}\rangle$ $(\sum_{\{y_r'\}} |\{y_r'\}\rangle)$ represents a sum over all possible basis states $|\{x_r'\}\rangle$ $(|\{y_r'\}\rangle)$ each containing an even (odd) number of "1"s. The state (50) will, after replacing $|\varsigma^\pm\rangle$ by (51), change into

$$\begin{aligned} &[(|\zeta\rangle + |\zeta'\rangle)|0\rangle_{C_t} + (|\zeta\rangle - |\zeta'\rangle)|1\rangle_{C_t}] \sum_{\{x_r'\}} |\{x_r'\}\rangle|0\rangle_A \\ &+[(|\zeta\rangle + |\zeta'\rangle)|0\rangle_{C_t} - (|\zeta\rangle - |\zeta'\rangle)|1\rangle_{C_t}] \sum_{\{x_r'\}} |\{x_r'\}\rangle|1\rangle_A \\ &+[(|\zeta\rangle + |\zeta'\rangle)|0\rangle_{C_t} - (|\zeta\rangle - |\zeta'\rangle)|1\rangle_{C_t}] \sum_{\{y_r'\}} |\{y_r'\}\rangle|0\rangle_A \end{aligned} \tag{52}$$

$$+[(|\zeta\rangle + |\zeta'\rangle)|0\rangle_{C_t} + (|\zeta\rangle - |\zeta'\rangle)|1\rangle_{C_t}]\sum_{\{y'_r\}}|\{y'_r\}\rangle|1\rangle_A,$$

which implies that if the other $n-1$ agents and Alice perform a measurement on their respective GHZ qubits, the m message qubits $(1, 2, \ldots, m)$ belonging to Bob well be entangled with agent C_t's GHZ qubit.

From (52), it is easily seen that for ever outcome $|\{x'_r\}\rangle|0\rangle_A$, $|\{x'_r\}\rangle|1\rangle_A$, $|\{y'_r\}\rangle|0\rangle_A$, or $|\{y'_r\}\rangle|1\rangle_A$ of the other $n-1$ agents' and Alice's measurements on their GHZ qubits, the density operator of the m message qubits $(1, 2, \ldots, m)$ belonging to Bob is, after tracing over agent C_t's GHZ qubit, given by

$$\rho = (|\zeta\rangle + |\zeta'\rangle)(\langle\zeta| + \langle\zeta'|) + (|\zeta\rangle - |\zeta'\rangle)(\langle\zeta| - \langle\zeta'|). \tag{53}$$

Since Eq. (53) takes the same form as (35), one can obtain the same results (36), (37) and (38) as described above. Therefore, Bob can full restore the corresponding state $\bigotimes_{k=1}^{m}\mathcal{U}_{d_k}(\alpha_k|0\rangle_k + \beta_k|1\rangle_k)$ of m message qubits $(1, 2, \ldots, m)$, only if all the agents collaborate with him.

(ii) To prepare m-qubit information to a distant receiver via the control of n agents, the present method requires only: $2m + n + 1$ auxiliary qubits for preparing the state (43), one qubit being distributed to each agent, one single-qubit Hadamard transformation and one single-qubit measurement being performed by each agent, and one-bit classical message being sent to the receiver by each agent.

In contrast, to implement the same task, the method in the Sect. 2 requires: $m(n+2)$ auxiliary qubits for preparing m copies of a $(n + 2)$-qubit GHZ state, m qubits being distributed to each agent, m single-qubit Hadamard transformation and m single-qubit measurements being performed by each agent, and m-bit classical message being sent to the receiver by each agent.

For the case of $m = 1$, the present method is not interesting since it requires one more auxiliary qubit than the method in the Sect. 2. However, the advantage of the present proposal appears when $m = 2$ and becomes apparent as m increases.

5 Controlled RIO of multiple partially unknown operations to many distant receivers

It is interesting to note that the method described above can be further extended to teleport multiple multiple partially unknown operation-strings to many distant receivers via the control of many agents in a network. Assume that Alice has s partially unknown operation strings labeled by $1, 2, \ldots, s$. The partially unknown operation string t contains m_t partially unknown operations, which is written as $\bigotimes_{j=1}^{m_t}\mathcal{U}_{d_{j,t}}$ $(t = 1, 2, \ldots, s)$. The total operation of the s partially unknown operation strings is given by $\bigotimes_{t=1}^{s}\bigotimes_{j=1}^{m_t}\mathcal{U}_{d_{j,t}}$. Suppose that Bob holds s qubit strings labeled by $1, 2, \ldots, s$. The qubit string t contains m_t message qubits, which is initially in the state $\bigotimes_{j=1}^{m_t}(\alpha_{j,t}|0\rangle_{j,t} + \beta_{j,t}|1\rangle_{j,t})$ $(t = 1, 2, \ldots, s)$, here $|\alpha_{j,t}|^2 + |\beta_{j,t}|^2 = 1$, $j = 1, 2, \ldots, m_t$ and $t = 1, 2, \ldots, s$. The state of the s qubit strings is given by $\bigotimes_{t=1}^{s}\bigotimes_{j=1}^{m_t}(\alpha_{j,t}|0\rangle_{j,t} + \beta_{j,t}|1\rangle_{j,t})$.

Now, Alice wishes to teleport the s partially unknown quantum operation strings to s distant receivers (the message carried by the partially unknown quantum operation string t is for the receiver t) via the control of n agents in a network, such that each receiver can fully recover the partially unknown quantum operation string acting on the corresponding qubit string, only if all the agents cooperate. The present task can be implemented with the following EPR-GHZ entangled state

$$\begin{aligned}&\bigotimes_{t=1}^{s}\bigotimes_{j=1}^{m_t}(|00\rangle_{j'j'',t}+|11\rangle_{j'j'',t})|GHZ\rangle_{+}\\&+\bigotimes_{t=1}^{s}\bigotimes_{j=1}^{m_t}(|00\rangle_{j'j'',t}-|11\rangle_{j'j'',t})|GHZ\rangle_{-},\end{aligned} \tag{54}$$

where $|GHZ\rangle_{\pm}$ are the GHZ states (39) and (40) of the $(n+1)$ GHZ qubits shared by the n agents and Alice; the m_t EPR qubits $(1'', 2'', \ldots, m_t'')$ for the set t belong to the receiver t, while the other m_t EPR qubits $(1', 2', \ldots, m_t')$ for the set t are kept by Alice. Here, the set t represents "the qubit string t and the m_t EPR group shared by Alice and the receiver t"$(t = 1, 2, \ldots, s)$. The state of whole system is given by

$$\begin{aligned}&\bigotimes_{t=1}^{s}\bigotimes_{j=1}^{m_t}[(\alpha_{j,t}|0\rangle_{j,t}+\beta_{j,t}|1\rangle_{j,t})(|00\rangle_{j'j'',t}+|11\rangle_{j'j'',t})]|GHZ\rangle_{+}\\&+\bigotimes_{t=1}^{s}\bigotimes_{j=1}^{m_t}[(\alpha_{j,t}|0\rangle_{j,t}+\beta_{j,t}|1\rangle_{j,t})(|00\rangle_{j'j'',t}-|11\rangle_{j'j'',t})]|GHZ\rangle_{-}.\end{aligned} \tag{55}$$

After doing the same as Step 2 and Step 3 of the Sect. 3, we obtain

$$\bigotimes_{t=1}^{s}|\zeta\rangle_t|GHZ\rangle_{+}+\bigotimes_{t=1}^{s}|\zeta'\rangle_t|GHZ\rangle_{-}, \tag{56}$$

where $|\zeta\rangle_t = \bigotimes_{j=1}^{m_t}|\zeta\rangle_{j,t}$ and $|\zeta'\rangle_t = \bigotimes_{j=1}^{m_t}|\zeta'\rangle_{j,t}$ are the states for m_t message qubits $(1, 2, \ldots, m_t)$ belonging to the receiver t. $|\zeta\rangle_{j,t}$ and $|\zeta'\rangle_{j,t}$ are the states of qubit j for the receiver t, which depend on the outcome of Alice's single-qubit measurements on the qubit j' for the set t and the message $d_{j,t}$ of matrix $\mathcal{U}_{d_{j,t}}$ used by Alice, and take the form of (31) and (32), respectively.

Not that the states $|\zeta\rangle_t$ and $|\zeta'\rangle_t$ have the same form as $|\zeta\rangle$ and $|\zeta'\rangle$ described in (30), respectively. Therefore, based on (56) and using the above procedure, it is straightforward to show that partially unknown quantum operation string acting on each qubit string can be recovered by the corresponding receiver, with the aid of all the agents.

In order to restore the partially unknown quantum operation string acting on the corresponding qubit string for each receiver, the following procedure can be followed: (i) Each agent and Alice need to perform a Hadamard transformation and then a measurement on their respective GHZ qubits. (ii) Each agent and Alice need to send each receiver their measurement results on their GHZ qubits. And (iii) Alice needs to send the receiver t the outcome of her single-qubit measurement on the EPR qubits $1', 2', \ldots, m_t'$ for the set t and the message $d_{j,t}$ of matrix $\mathcal{U}_{d_{j,t}}$ acted on EPR qubit j' for the set t and $j' \in \{1', 2' \ldots, m_t'\}$, so that the receiver t can recover the partially unknown quantum operation string acting on the corresponding qubit string t.

On the other hand, it can be shown from (56) that even if one agent does not collaborate, the density operator for each qubit belonging to each receiver takes the form of (38), i.e., no receiver can fully restore the partially unknown quantum operation string acting on the corresponding message qubit string without the cooperation of all the agents.

Remark 3 To realize the present task, the present method requires only: $3\sum_{t=1}^{s} m_t + n + 1$ auxiliary qubits for preparing the state (54), one quibt being assigned to each agent, one single-qubit Hadamard transformation and one single-qubit measurement being performed by each agent, and one-bit classical message being sent to each receiver by each agent.

In contrast, to implement the same task, the method in the Sect. 2 requires: $\sum_{t=1}^{s} m_t(n+2)$ auxiliary qubits for preparing $\sum_{t=1}^{s} m_t$ copies of a $(n+2)$ qubit GHZ state; $\sum_{t=1}^{s} m_t$ qubits being assigned to each agent, $\sum_{t=1}^{s} m_t$ single-qubit Hadamard transformations and $\sum_{t=1}^{s} m_t$ single-qubit measurements being performed by each agent, and m_t-bit classical message being sent to the receiver t by each agent.

One can see that even for $m_t = 1$ and $s = 2$, the present method is effective, since (i) the number of qubits distributed to each agent, the number of Hadamard transformations by each agent, or the number of measurement by each agent is 1, which is, however, 2 for the method in the Sect. 2; and (ii) the number of auxiliary qubits required is $n+5$, which is smaller than $2n+4$ needed in the method in the Sect. 2, when $n > 1$. More interestingly, with the increment of m_t, s, or n, the advantage of the present method becomes very apparent.

6 Discussion and conclusion

In the existing one-way RIO schemes, the transmitted quantum operation is only a unitary operation. The scheme in the Sect. 2 of this paper is one of this types, which has the simplest form of expression and is most understandable protocol with the control of many agents.

It is worth mentioning that in terms of the control efficiency of each agent on RIO, our schemes in the Sects. 3, 4 and 5 are identical to those in the Sects. 2, since the result (38) applied in our protocols in the Sects. 3, 4 and 5 are the same as those used in the Sect. 2. However, as mentioned above, our schemes in the Sects. 3, 4 and 5 are very simple and economical in the implementation of multiple partially unknown operations via the control of many agents in a network. What's more, in the protocol of the Step 3 of the third section, if the Eq. (29) is changed to $|\zeta\rangle(|0\rangle + |1\rangle)_C + |\zeta'\rangle(|0\rangle - |1\rangle)_C$, Bob also can recover the operation through the corresponding operation of the agent. But as seen in the Eqs. (17), (43), and (54), in our quantum channel Alice is both the sender and the controller. If she does not cooperate as a controller in the Step 4 of the protocol in Sects. 3, 4 and 5, then Bob cannot restore the corresponding quantum operation. These specific analysis process is similar to that of the non-cooperation with the agents in the Remark1.(i). This modification makes the sender has power to control whole protocol, which improves security of protocol again.

Our schemes in the Sects. 4 and 5 work essentially through having originally-non-entangled quantum informations, carried by two message qubits, to be entangled each other after Alice applies partially unknown quantum operations and then execute a series of single-qubit measurements. This can be seen from Eq. (48). For instance, let us consider the case of $m = 2$, i.e., applying $\mathcal{U}_{d_1} \bigotimes \mathcal{U}_{d_2}$ on the state $(\alpha_1|0\rangle_1 + \beta_1|1\rangle_1) \otimes (\alpha_2|0\rangle_2 + \beta_2|1\rangle_2)$ of the two message qubits $(1, 2)$ and teleporting it to Bob. Based on Eqs. (30)–(32), one can see that if Alice measures the qubits $1'$ and $2'$ in the basis $|0\rangle_{1'}\langle 0|$ and $|0\rangle_{2'}\langle 0|$, respectively, the state (48) for the remaining qubit system will be

$$
\begin{aligned}
&(\alpha_1 u_{d_1}|0\rangle_1 + \beta_1 u_{d_1\oplus 1}|1\rangle_1)(\alpha_2 u_{d_2}|0\rangle_2 + \beta_2 u_{d_2\oplus 1}|1\rangle_2)|\text{GHZ}\rangle_+ \\
&\quad + (\alpha_1 u_{d_1}|0\rangle_1 - \beta_1 u_{d_1\oplus 1}|1\rangle_1)(\alpha_2 u_{d_2}|0\rangle_2 - \beta_2 u_{d_2\oplus 1}|1\rangle_2)|\text{GHZ}\rangle_- .
\end{aligned}
\tag{57}
$$

The result (57) implies that if Bob measures the qubit 1 in the state $\alpha_1 u_{d_1}|0\rangle_1 + \beta_1 u_{d_1\oplus 1}|1\rangle_1$, he can predict that his qubit 2 must be in the state $\alpha_2 u_{d_2}|0\rangle_1 + \beta_2 u_{d_2\oplus 1}|1\rangle_2$. On the other hand, if Bob detects the qubit 1 in the state $\alpha_1 u_{d_1}|0\rangle_1 - \beta_1 u_{d_1\oplus 1}|1\rangle_1$, he knows that his qubit 2 must be in the state $\alpha_2 u_{d_2}|0\rangle_2 - \beta_2 u_{d_2\oplus 1}|1\rangle_2$. Hence, if Alice implements single-qubit measurements, the quantum informations carried by the two message qubis $(1, 2)$ via the action of $\mathcal{U}_{d_1} \bigotimes \mathcal{U}_{d_2}$ are not only transferred onto Bob's qubits $(1, 2)$ but also become entangled each other.

In quantum communication, the most common single-qubit measurement bases are $\{|0\rangle, |1\rangle\}$ and $\{|+\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle), |-\rangle = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)\}$. As mentioned above, the former is used in our schemes, because it is quite difficult to make a measurement in the basis $\{|+\rangle, |-\rangle\}$ for some kind of qubits such as superconducting charge and flux qubits, but straightforward in the basis $\{|0\rangle, |1\rangle\}$. In fact, based on $|0\rangle = \frac{1}{\sqrt{2}}(|+\rangle + |-\rangle)$ and $|1\rangle = \frac{1}{\sqrt{2}}(|+\rangle - |-\rangle)$, Hadamard transforms are not necessary in our protocols, because the same results can be got when each agent measures his/her qubit in the basis $\{|+\rangle, |-\rangle\}$, instead of a Hadamard transformation followed by a measurement in the basis $\{|0\rangle, |1\rangle\}$, and then sends his/her measurement outcome $|+\rangle$ or $|-\rangle$ by one-bit classical message to the receiver(s).

As shown above, Alice's single-qubit measurements on EPR qubits, Alice's single-qubit operation (Hadamard transformation/measurement) on GHZ qubit, and each agent's operation on GHZ qubit are independently executed on different qubits, so our protocols actually does not require the operating order among Alice's single-qubit measurements, Alice's single-qubit operation and each agent's operation.

Although we do not intend to conduct a comprehensive study on the security of our protocols here to prevent all possible forms of eavesdropping and/or deception, we believe that it may be quite secure for several reasons. First, the eavesdropping by entangling ancillary qubits with the receiver's qubits can be revealed by comparing a subset of the states the receiver received to ones Alice sent. Second, the qubits that Alice sends to Bob are basically useless without the classical information owned by Alice.

Hence, even if Eve was to intercept the qubits intended for the receiver, and replace them by fakes, and somehow eavesdropped on the (classical) communication channels through which all the agents disclose to the receiver their measurement outcomes, she would still not be able to recover the message qubits' original states without access to Alice's classical information (her measurement outcomes), given that Alice sends her classical information to the receiver using standard quantum cryptography [29]. It is conceivable that an eavesdropper might obtain partial information by entangling enough ancillary qubits with the qubits belonging to all the agents and the receiver, but presumably such entanglement could be detected by tests conducted on 'sample' EPR-GHZ entangled states initially shared by Alice and the other parties.

In summary, we have first proposed a RIO scheme of a partially unknown quantum operation via the control of n agents, which is a remote implementation protocol in the sense of current RIO. Inspired by the scheme, we put forward the RIO of multiple partially unknown quantum operations from a sender to a distant receiver via the control of one agent, and then have generalized it to RIO of multiple partially unknown quantum operations via the control of many agents in a network. Then, we proposed a series of protocols in Sects. 2 to 5. The difference of these protocols is the number of participants and transmitted operations. Obviously, the security is higher but the recover operations are more complex with increasing number of participants. Meanwhile, these different scenarios can solve different actual problems in the RIO network. A special feature of our entangling quantum information concept is to implement a control of multiple partially unknown quantum operations RIO. The present scheme needs to assign only one qubit to each agent, followed by every agent performing only one Hadamard transformation and one measurement, and the sending only one-bit classical message to the receiver. As a result, the required auxiliary qubit resources, the number of local operations, and the quantity of classical communication are greatly reduced in the present scheme. Then,to our knowledge, the present RIO protocols can be divided into three types in terms of the mode of transmission, such as unidirectional transmission, bidirectional transmission and circular transmission. In the first type, most schemes only contains the situation of one controller, one receiver or one sender, except for ref. [10], such transmission between multiple sender and receivers under the control of multiple controllers is still missing. Similarly, we also study the extension of many-party RIO (many controllers and many receivers) in this transmission type because incorporation of multiple participants further merits consideration towards the realization of versatile quantum networks. The difference of these two protocols is that the ref. [10] utilizes a product of GHZ state as channel, but our schemes adopt GHZ type states. This means that the former can be seen as a repetition of many simple schemes, and a unitary operation needs to be decomposed into the product of some restrictive operations in equation (1) before transmission. This unitary operation is too special. it is difficult to find such expressions for common unitary operations. Therefore, this greatly limits the application scope of the scheme. In our scheme, the restrictive operations are obtained directly, with the result that it is more convenient to implement. Moreover, our scheme can also be extended to multiple senders as ref. [10] easily by distributing each restrictive operation to each sender. In general, our scheme is not only easier to realize, but also offers a ideal of RIO network.

The method presented here can also be extended to implement a multiparty controlled RIO of multiple partially unknown quantum operation-strings to many distant receivers. We believe that our schemes are considerable interest, especially because of its relatively straightforward nature in realizing simultaneous control of multiple partially unknown quantum operations RIO in an efficient and simple manner.

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Data availability All data generated or analysed during this study are included in this published article.

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