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Quantifying coherence of quantum channels via trace distance

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Abstract
Quantum coherence is a fundamental resource in quantum information and computation. Deep insight can be gained into the nature of coherence by studying the quantum channels. Motivated by this, we propose a measure for coherence of a quantum channel utilizing the trace distance to quantify its commutativity with the completely dephasing operation. We further discuss the extremal property, monotonicity and convexity of the coherence measure of a quantum channel. Next, we calculate the coherence measure of Hadamard gate as 1/2 and show that the coherence measure of a composition of channels does not satisfy additivity. Finally, by calculating the coherence measure of noisy channels, it is pointed out that several typical quantum channels, such as the depolarizing quantum channel, the phase damping quantum channel and the amplitude damping quantum channel, are all incoherent. Moreover, we also give a sufficient condition for a given quantum channel to be coherent and compare with the existing result.

Keywords Coherence · Trace distance · Quantum channel

1 Introduction

Quantum coherence is the most fundamental feature of quantum mechanics that distinguishes the quantum from the classical world. It is the root of all the other intriguing quantum features such as entanglement [1, 2], quantum correlation [3, 4], quantum nonlocality [5], and so on. Recent studies have shown that coherence in a quantum system can be a useful resource in quantum algorithm [6, 7], quantum meteorology [8], quantum thermodynamics [9, 10], and quantum biology [11–13]. Therefore, the study of resource theory of quantum coherence is of great significance [14–17].

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In recent years, several kinds of coherence quantifiers such as the l_1-norm coherence, relative entropy coherence, skew information coherence [18] and the coherence measure based on entanglement [19] have been presented. Many researchers devote to study the properties of coherence of quantum states [20, 21]. There is also relevant work discussing coherence dynamics in decoherent environment [22, 23]. All this work on the coherence of quantum states has greatly enriched our understanding of quantum theory and has led to many applications.

Since most quantum information processes involve and depend on the properties of quantum channels, it is crucial to characterize all these channels and their impact on various physical resources [24, 25]. For instance, the authors demonstrated the revival and robustness of quantum dynamics under decoherence channels [26], quantum entanglement of a two-qubit state selectively undergoes independent nonidentical decoherence channels [27], and Bell correlation via quantum partially collapsing measurement [28]. In addition, these channels are also important for constructing coherence in resource theory. Therefore, many measures for coherence of quantum channels have been widely studied, such as the non-coherence-generating channel [29], the resource theory of coherence with respect to quantum operations [30], the coherence of quantum channels using the Choi-Jamiołkowski isomorphism [31], the detection-incoherent quantum operation and the creation incoherent quantum operation [32], and the framework for quantifying the coherence of quantum channels [33]. Many other works have been done for coherence measures of quantum channels, e.g., [34–36], which may provide inspiring evidence.

Although all the above measures are introduced from intuitive motivations and reasonable arguments, they are all notoriously difficult to calculate. In contrast to the relative entropic approach to the nonclassicality of operations [37], we will in this paper introduce an alternative measure for coherence of quantum channels in terms of the trace distance [24], which is motivated by the l_1-norm coherence of a quantum state, and has the advantage that it can be easier to evaluated.

The paper is organized as follows. In Sect. 2, we introduce some properties of the trace distance, define the measure for coherence of quantum channels in terms of the trace distance and discuss some fundamental properties of the coherence measure of channels, such as extremal property, monotonicity, convexity and non-additivity, by using a counterexample. Intuitively, a proper coherence measure should yield zero for noisy channels which does not generate coherence. In Sect. 3, taking several noise channels as examples, we show that the depolarizing channel, the phase damping quantum channel and the amplitude damping quantum channel are all incoherent by calculating the coherence measure, which perfectly matches this intuition. Moreover, a sufficient condition for a given quantum channel to be coherent is given. Finally, a summary is given in Sect. 4.

2 Quantum coherence of quantum channels

The trace distance between quantum states ρ and σ is defined in [24]:

$$D(\rho, \sigma) = \frac{1}{2}\mathrm{tr}|\rho - \sigma|,$$

where $|A| = \sqrt{A^\dagger A}$ is the positive square root of $A^\dagger A$.

The trace distance has many advantages and nice properties [24]:

(i) Contractibility of the trace distance. Suppose $\mathcal{E}$ is a trace-preserving quantum operation. Then,

$$D(\mathcal{E}(\rho), \mathcal{E}(\sigma)) \leq D(\rho, \sigma). \tag{1}$$

(ii) The trace distance is a metric. It is obviously that $D(\rho, \sigma) = 0$ if and only if $\rho = \sigma$; $D(\cdot, \cdot)$ is a symmetric function with respect to the input states, $D(\rho, \sigma) = D(\sigma, \rho)$; The triangle inequality holds, $D(\rho, \tau) \leq D(\rho, \sigma) + D(\sigma, \tau)$.

(iii) Joint convexity of the trace distance. Let $\{p_i\}$ be probability distribution, and ρ_i and σ_i be quantum states whose subscripts are taken from the same index set. Then,

$$D\left(\sum_i p_i \rho_i, \sum_i p_i \sigma_i\right) \leq \sum_i p_i D(\rho_i, \sigma_i). \tag{2}$$

For clarity, we first give some notation. Let H_A and H_B be two Hilbert spaces with dimensions m and n, and $\{|j\rangle\}_j$ and $\{|\alpha\rangle\}_\alpha$ be orthonormal bases of H_A and H_B, respectively. We always assume that the orthonormal bases are fixed, and adopt the tensor basis $\{|j\rangle|\alpha\rangle\}_{j\alpha}$ as the fixed basis when considering the multipartite system $H_{AB} = H_A \otimes H_B$. Let $\mathcal{D}(H_A)$ and $\mathcal{D}(H_B)$ be the set of all density operators on H_A and H_B, respectively, and $\mathcal{C}_{AB}$ denote the set of all channels from $\mathcal{D}(H_A)$ to $\mathcal{D}(H_B)$. A quantum channel $\Phi \in \mathcal{C}_{AB}$ is a completely positive trace-preserving operator [24].

The completely dephasing channel $\Delta^A \in \mathcal{C}_{AA}$ is defined as

$$\Delta^A(\rho^A) = \sum_j \langle j|\rho^A|j\rangle |j\rangle\langle j|, \quad \rho^A \in \mathcal{D}(H_A).$$

Recall that for quantum states, a state σ^A is called incoherent if

$$\Delta^A(\sigma^A) = \sigma^A,$$

or

$$\sigma^A = \sum_j \langle j|\sigma^A|j\rangle |j\rangle\langle j|.$$

That is to say, fixed a basis $\{|j\rangle\}$, σ^A is diagonalized on this basis. If a state is not incoherent, we say that it is coherent.

Definition 1 [33] A channel $\Phi \in \mathcal{C}_{AB}$ is called an incoherent channel, if

$$\Delta^B \circ \Phi \circ \Delta^A = \Phi.$$

If a channel is not incoherent, we say that it is coherent. Here, $\circ$ is the composition of operations.

Since $(\Delta^A)^2 = \Delta^A$, $(\Delta^B)^2 = \Delta^B$, incoherent channel Φ implies the commutation relation

$$\Delta^B \circ \Phi = \Delta^B \circ \Phi \circ \Delta^A = \Phi \circ \Delta^A. \tag{3}$$

As the authors mentioned in the Introduction in [32], it is both important to create and to detect coherence; therefore, one can define detection-incoherent operations and creation-incoherent operations, i.e., operations which cannot create coherence.

Definition 2 [32] A quantum operation $\Phi \in \mathcal{C}_{AB}$ is called detection-incoherent iff

$$\Delta^B \circ \Phi = \Delta^B \circ \Phi \circ \Delta^A.$$

A quantum operation $\Phi \in \mathcal{C}_{AB}$ is called creation-incoherent, if it cannot create coherence in system B if none were present in system A,

$$\Phi \circ \Delta^A = \Delta^B \circ \Phi \circ \Delta^A.$$

A quantum operation $\Phi \in \mathcal{C}_{AB}$ is called detection–creation-incoherent, if it can neither detect nor create coherence,

$$\Delta^B \circ \Phi = \Phi \circ \Delta^A.$$

Our contribution in this work is that we show how to quantify the abilities to create and detect coherence in a rigorous manner. The characterization of coherence quantifies the quantumness of the channels, i.e., it might be considered as the quantity which characterizes how much the channel is quantum. The explanation is similar to the quantumness of an operation Φ as [37]

$$W(\Phi) = \sup_{\rho} S\left(\Phi \circ \Delta^A(\rho) \parallel \Delta^B \circ \Phi(\rho)\right), \tag{4}$$

where $S(\rho \parallel \sigma) = \mathrm{tr}(\rho(\log\rho - \log\sigma))$ is the quantum relative entropy. In the following, we introduce an alternative measure for coherence of quantum channels in terms of the trace distance.

Definition 3 We define the coherence of a channel $\Phi \in \mathcal{C}_{AB}$ as

$$C(\Phi) = \sup_{\rho} D\left(\Phi \circ \Delta^A(\rho), \Delta^B \circ \Phi(\rho)\right), \tag{5}$$

where $D(\cdot, \cdot)$ is the trace distance, and the supremum in Eq. (5) is taken over all quantum states.

Remark 1 On the one hand, using the triangle inequality for the trace distance, we get

$$\begin{aligned} D\left(\Phi \circ \Delta^A(\rho), \Delta^B \circ \Phi(\rho)\right) \leq & \ D\left(\Delta^B \circ \Phi \circ \Delta^A(\rho), \Delta^B \circ \Phi(\rho)\right) \\ & + D\left(\Delta^B \circ \Phi \circ \Delta^A(\rho), \Phi \circ \Delta^A(\rho)\right). \end{aligned}$$

Then,

$$C(\Phi) \leq \sup_{\rho} D\left(\Delta^B \circ \Phi \circ \Delta^A(\rho), \Delta^B \circ \Phi(\rho)\right) + \sup_{\rho} D\left(\Delta^B \circ \Phi \circ \Delta^A(\rho), \Phi \circ \Delta^A(\rho)\right).$$

The first term on the right-hand side of the inequality, which we call the detecting power, characterizes how well a coherent channel Δ^B can detect between $\Phi \circ \Delta^A(\rho)$ and $\Phi(\rho)$. The second term, which we call the creating power, measures the ability of the map Φ to create a incoherent state out of a coherent input state.

Notice that the definition 2

$$\sup_{\rho} D\left(\Delta^B \circ \Phi \circ \Delta^A(\rho), \Delta^B \circ \Phi(\rho)\right)$$

can be used as a detection coherence measure, and

$$\sup_{\rho} D\left(\Delta^B \circ \Phi \circ \Delta^A(\rho), \Phi \circ \Delta^A(\rho)\right)$$

can be used as a creation coherence measure. Thus, the coherence of the channel Φ does not exceed the sum of its detection coherence and creation coherence.

On the other hand, notice that $(\Delta^B)^2 = \Delta^B$ and Δ^B is a trace-preserving quantum operation. Using the contractibility of the trace distance, we have

$$\begin{aligned} D\left(\Delta^B \circ \Phi \circ \Delta^A(\rho), \Delta^B \circ \Phi(\rho)\right) &= D\left(\Delta^B \circ \Phi \circ \Delta^A(\rho), \Delta^B \circ \Delta^B \circ \Phi(\rho)\right) \\ &\leq D\left(\Phi \circ \Delta^A(\rho), \Delta^B \circ \Phi(\rho)\right). \end{aligned}$$

So the coherence measure of the channel Φ is greater than or equal to its detection coherence measure. Therefore, the coherence of a channel Φ is between its detection coherence and the sum of the detection coherence and the creation coherence of the quantum channel Φ.

Here, we give the proof of property 1 that the maximum in the maximization for the coherence measure $C(\Phi)$ of a quantum channel Φ is always attained for a pure state.

Theorem 1 *Given* $\sup_{\rho} D(\Phi \circ \Delta(\rho), \Delta \circ \Phi(\rho))$, *there exists a pure state* $|\psi\rangle\langle\psi|$ *such that the supremum in Eq. (5) is attained when* $\rho = |\psi\rangle\langle\psi|$.

Proof For some mixed state ρ, we can spectrally decompose it as $\rho = \sum_i \mu_i |\psi_i\rangle\langle\psi_i|$, where $|\psi_i\rangle$ are its eigenstates. Since the trace distance is jointly convex, this implies that

$$
\begin{aligned}
&D(\Phi \circ \Delta(\rho), \Delta \circ \Phi(\rho)) \\
&= D\left(\Phi \circ \Delta\left(\sum_i \mu_i |\psi_i\rangle\langle\psi_i|\right), \Delta \circ \Phi\left(\sum_i \mu_i |\psi_i\rangle\langle\psi_i|\right)\right) \\
&= D\left(\sum_i \mu_i \Phi \circ \Delta\left(|\psi_i\rangle\langle\psi_i|\right), \sum_i \mu_i \Delta \circ \Phi\left(|\psi_i\rangle\langle\psi_i|\right)\right) \\
&\leq \sum_i \mu_i D(\Phi \circ \Delta(|\psi_i\rangle\langle\psi_i|), \Delta \circ \Phi(|\psi_i\rangle\langle\psi_i|)) \\
&\leq \sum_i \mu_i \sup_{|\psi\rangle} D(\Phi \circ \Delta(|\psi\rangle\langle\psi|), \Delta \circ \Phi(|\psi\rangle\langle\psi|)) \\
&= \sup_{|\psi\rangle} D(\Phi \circ \Delta(|\psi\rangle\langle\psi|), \Delta \circ \Phi(|\psi\rangle\langle\psi|)).
\end{aligned}
$$

Thus,

$$C(\Phi) \leq \sup_{|\psi\rangle} D(\Phi \circ \Delta(|\psi\rangle\langle\psi|), \Delta \circ \Phi(|\psi\rangle\langle\psi|)).$$

Therefore,

$$C(\Phi) = \sup_{|\psi\rangle} D(\Phi \circ \Delta(|\psi\rangle\langle\psi|), \Delta \circ \Phi(|\psi\rangle\langle\psi|)). \tag{6}$$

The proof is completed.

□

Next, we consider the property 2, stating that the measure $C(\Phi)$ is non-increasing under the composition with incoherent channels.

Theorem 2 *For any quantum channel* Φ, *if* Φ_0 *is a trace-preserving quantum operation and satisfies* $C(\Phi_0) = 0$, *then* $C(\Phi_0 \circ \Phi) \leq C(\Phi)$, *and* $C(\Phi \circ \Phi_0) \leq C(\Phi)$.

Proof By Theorem 1, we have

$$C(\Phi_0 \circ \Phi) = \sup_{|\psi\rangle} D(\Phi_0 \circ \Phi \circ \Delta(|\psi\rangle\langle\psi|), \Delta \circ \Phi_0 \circ \Phi(|\psi\rangle\langle\psi|)).$$

Since $C(\Phi_0) = 0$, Φ_0 is incoherent, which implies $\Phi_0 \circ \Delta = \Delta \circ \Phi_0$ by using Eq. (3). Consequently,

$$\begin{aligned}
&D(\Phi_0 \circ \Phi \circ \Delta(|\psi\rangle\langle\psi|), \Delta \circ \Phi_0 \circ \Phi(|\psi\rangle\langle\psi|)) \\
&\quad = D(\Phi_0 \circ \Phi \circ \Delta(|\psi\rangle\langle\psi|), \Phi_0 \circ \Delta \circ \Phi(|\psi\rangle\langle\psi|)) \\
&\quad \le D(\Phi \circ \Delta(|\psi\rangle\langle\psi|), \Delta \circ \Phi(|\psi\rangle\langle\psi|)).
\end{aligned}$$

The last inequality is due to the contractibility of the trace distance under trace-preserving quantum operation. Therefore,

$$C(\Phi_0 \circ \Phi) \le \sup_{|\psi\rangle} D(\Phi \circ \Delta(|\psi\rangle\langle\psi|), \Delta \circ \Phi(|\psi\rangle\langle\psi|)) = C(\Phi).$$

For the proof of the other inequality, we notice that

$$\begin{aligned}
&C(\Phi \circ \Phi_0) \\
&\quad = \sup_{\rho} D(\Phi \circ \Phi_0 \circ \Delta(\rho), \Delta \circ \Phi \circ \Phi_0(\rho)) \\
&\quad = \sup_{\rho} D(\Phi \circ \Delta \circ \Phi_0(\rho), \Delta \circ \Phi \circ \Phi_0(\rho)) \\
&\quad = \sup_{\sigma = \Phi_0(\rho)} D(\Phi \circ \Delta(\sigma), \Delta \circ \Phi(\sigma)) \\
&\quad \le \sup_{\rho} D(\Phi \circ \Delta(\rho), \Delta \circ \Phi(\rho)) \\
&\quad = C(\Phi).
\end{aligned}$$

Therefore, the conclusion holds. □

Corollary 1 *Let Δ^A, Δ^B be completely dephasing channels. Then,*

(i) $C(\Delta^A) = C(\Delta^B) = 0$*;*
(ii) For any quantum channel Φ, $C(\Delta^B \circ \Phi) \le C(\Phi)$, $C(\Phi \circ \Delta^A) \le C(\Phi)$.

Proof Notice that $\Delta^A\Delta^A\Delta^A = \Delta^A, \Delta^B\Delta^B\Delta^B = \Delta^B$. Thus, by the definition of incoherence of quantum channel, Δ^A, Δ^B are incoherent. Therefore, $C(\Delta^A) = 0, C(\Delta^B) = 0$. Conclusion (ii) is clearly true by use of Theorem 2. □

Finally, the third property is convexity of the coherence measure and is followed from the joint convexity of trace distance. It ensures that mixing does not create resources.

Theorem 3 *For some quantum channels Φ_i, and some positive real number λ_i, $\sum_i \lambda_i = 1$, we have*

$$C\left(\sum_i \lambda_i \Phi_i\right) \le \sum_i \lambda_i C(\Phi_i).$$

Proof By the joint convexity of trace distance, we have

$$\begin{aligned}
&C\left(\sum_i \lambda_i \Phi_i\right)\\
&= \sup_{|\psi\rangle} D\left(\Delta \circ \sum_i \lambda_i \Phi_i\,(|\psi\rangle\langle\psi|)\,, \sum_i \lambda_i \Phi_i \circ \Delta(|\psi\rangle\langle\psi|)\right)\\
&= \sup_{|\psi\rangle} D\left(\sum_i \lambda_i \Delta \circ \Phi_i(|\psi\rangle\langle\psi|), \sum_i \lambda_i \Phi_i \circ \Delta(|\psi\rangle\langle\psi|)\right)\\
&\leq \sum_i \lambda_i \sup_{|\psi\rangle} D(\Delta \circ \Phi_i(|\psi\rangle\langle\psi|), \Phi_i \circ \Delta(|\psi\rangle\langle\psi|))\\
&= \sum_i \lambda_i C(\Phi_i).
\end{aligned}$$

The proof is completed. □

Given two observers with incoherent channels Φ_i^A and Φ_i^B at their disposal, and shared source of randomness, they cannot create a coherent channel.

Corollary 2 *Let* $\Phi_i^A \in \mathcal{C}_{AA}, \Phi_i^B \in \mathcal{C}_{BB}, \forall i$. *If* $C(\Phi_i^A \otimes I^B) = 0, C(I^A \otimes \Phi_i^B) = 0,$ *then*

$$C\left(\sum_i p_i \Phi_i^A \otimes \Phi_i^B\right) = 0,$$

where $p_i \geq 0, \sum_i p_i = 1,$ I^A *and* I^B *are the identity operations on* H_A *and* H_B*, respectively.*

Proof By the convexity of the coherence measure, we have

$$C\left(\sum_i p_i \Phi_i^A \otimes \Phi_i^B\right) \leq \sum_i p_i C(\Phi_i^A \otimes \Phi_i^B).$$

Since $C(\Phi_i^A \otimes I^B) = 0, C(I^A \otimes \Phi_i^B) = 0$, we get

$$\Delta \circ (\Phi_i^A \otimes I^B) = (\Phi_i^A \otimes I^B) \circ \Delta,\ \Delta \circ (I^A \otimes \Phi_i^B) = (I^A \otimes \Phi_i^B) \circ \Delta.$$

Also

$$\Delta \circ (\Phi_i^A \otimes I^B) = (\Delta^A \otimes \Delta^B) \circ (\Phi_i^A \otimes I^B) = (\Delta^A \circ \Phi_i^A) \otimes \Delta^B,$$

$$(\Phi_i^A \otimes I^B) \circ \Delta = (\Phi_i^A \otimes I^B) \circ (\Delta^A \otimes \Delta^B) = (\Phi_i^A \circ \Delta^A) \otimes \Delta^B,$$

thus $\Delta^A \circ \Phi_i^A = \Phi_i^A \circ \Delta^A$. Similarly, we have $\Phi_i^B \circ \Delta^B = \Delta^B \circ \Phi_i^B$. Consequently,

$$\begin{aligned}
&\Delta \circ (\Phi_i^A \otimes \Phi_i^B) \\
&= (\Delta^A \otimes \Delta^B) \circ (\Phi_i^A \otimes \Phi_i^B) \\
&= (\Delta^A \circ \Phi_i^A) \otimes (\Delta^B \circ \Phi_i^B) \\
&= (\Phi_i^A \circ \Delta^A) \otimes (\Phi_i^B \circ \Delta^B) \\
&= (\Phi_i^A \otimes \Phi_i^B) \circ (\Delta^A \otimes \Delta^B) \\
&= (\Phi_i^A \otimes \Phi_i^B) \circ \Delta.
\end{aligned}$$

It follows that

$$C(\Phi_i^A \otimes \Phi_i^B) = 0.$$

Therefore,

$$C\left(\sum_i p_i \Phi_i^A \otimes \Phi_i^B\right) = 0.$$

The proof is completed. □

For any operator A, and any real number $p \geq 1$, one defines the Schatten p-norm of A in [38] as

$$\|A\|_p = \left(\operatorname{tr}\left((A^* A)^{\frac{p}{2}}\right)\right)^{\frac{1}{p}}.$$

In particular, one has the 1-norm of A as

$$\|A\|_1 = \operatorname{tr}\left((A^* A)^{\frac{1}{2}}\right) = \operatorname{tr}(|A|), \tag{7}$$

and

$$\|A\|_2 = \left(\operatorname{tr}(A^* A)\right)^{\frac{1}{2}}.$$

Now, we introduce a useful lemma.

Lemma 1 [38] *For every nonzero operator A, it holds that*

$$\|A\|_1 \leq \sqrt{\operatorname{rank}(A)}\|A\|_2. \tag{8}$$

Theorem 4 *For Hadamard gate H, we have $C(H) = \frac{1}{2}$.*

Proof For any pure state $|\psi\rangle$, it can be expressed as $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$, where $|\alpha|^2 + |\beta|^2 = 1$. We have

$$H \circ \Delta(|\psi\rangle\langle\psi|) = \frac{1}{2}|0\rangle\langle 0| + \frac{1}{2}|1\rangle\langle 1| + \frac{|\alpha|^2 - |\beta|^2}{2}(|0\rangle\langle 1| - |1\rangle\langle 0|),$$

and

$$\Delta \circ H(|\psi\rangle\langle\psi|) = \frac{1}{2}(1 + 2\mathrm{Re}(\alpha\bar{\beta}))|0\rangle\langle 0| + \frac{1}{2}(1 - 2\mathrm{Re}(\alpha\bar{\beta}))|1\rangle\langle 1|.$$

Hence, by direct calculation, we have

$$\begin{aligned}
\|\Delta \circ H(|\psi\rangle\langle\psi|) - H \circ \Delta(|\psi\rangle\langle\psi|)\|_2 &= \sqrt{2|\mathrm{Re}(\alpha\bar{\beta})|^2 + \frac{(|\alpha|^2 - |\beta|^2)^2}{2}} \\
&\le \sqrt{2|\alpha|^2|\beta|^2 + \frac{(|\alpha|^2 - |\beta|^2)^2}{2}} \\
&= \frac{1}{\sqrt{2}}.
\end{aligned}$$

In addition, by using Eq. (7), Lemma 1 and the above result, we get

$$\begin{aligned}
&D(\Delta \circ H(|\psi\rangle\langle\psi|), H \circ \Delta(|\psi\rangle\langle\psi|)) \\
&\quad = \frac{1}{2}\mathrm{tr}\,(|\Delta \circ H(|\psi\rangle\langle\psi|) - H \circ \Delta(|\psi\rangle\langle\psi|)|) \\
&\quad = \frac{1}{2}\|\Delta \circ H(|\psi\rangle\langle\psi|) - H \circ \Delta(|\psi\rangle\langle\psi|)\|_1 \\
&\quad \le \frac{\sqrt{2}}{2}\|\Delta \circ H(|\psi\rangle\langle\psi|) - H \circ \Delta(|\psi\rangle\langle\psi|)\|_2 \\
&\quad \le \frac{1}{2}.
\end{aligned}$$

It follows that $C(H) \le \frac{1}{2}$ in view of Theorem 1. On the other hand, it is straightforward to show that the maximization of $D(\Delta \circ H(\rho), H \circ \Delta(\rho))$ is therefore achieved for classical pure input states $|0\rangle$ or $|1\rangle$. Using $|0\rangle$, we compute directly

$$D(\Delta \circ H(|0\rangle\langle 0|), H \circ \Delta(|0\rangle\langle 0|)) = \frac{1}{2}.$$

Therefore, the conclusion holds, i.e., $C(H) = \frac{1}{2}$. □

Intuitively, a proper coherence measure should yield some nonzero constant for the Hadamard gate because the Hadamard gate can generate a maximally coherent state. So our result perfectly matches this intuition. In the following, we show that the coherence measure of a composition of operations does not satisfy additivity by using the coherence of the Hadamard gate.

Remark 2 Given a sequence of quantum channels, the coherence measure is not additive under the composition of channels so that

$$C(\Phi \circ \Psi \circ \Theta) \neq C(\Phi) + C(\Psi) + C(\Theta),$$

for three quantum channels Φ, Ψ and Θ. This is most easily demonstrated and explained by using a counterexample.

Consider $\Omega = \Delta \circ H \circ \Delta$, where H is Hadamard gate. Notice that $\Delta \circ \Omega = \Omega \circ \Delta$, so $C(\Omega) = 0$. In addition, since $C(\Delta) = 0$, $C(H) = \frac{1}{2}$, we have

$$C(\Delta \circ H \circ \Delta) \neq C(\Delta) + C(H) + C(\Delta).$$

3 Examples

In this section, we will employ several examples in order to compute the coherence measure of quantum channels. The depolarizing channel is an important type of quantum noise. Imagine we take a single qubit, with probability p it is replaced by the completely mixed state $\frac{I}{2}$, and with probability $1 - p$ the qubit is left untouched. We shall show that the depolarizing quantum channel is incoherent.

Example 1 Let $\mathcal{E}$ be the depolarizing channel. The state ρ of the quantum system after this noise is $\mathcal{E}(\rho) = \frac{p}{2} I + (1 - p)\rho$, then $C(\mathcal{E}) = 0$.

Proof For any pure state $|\psi\rangle$, it can be expressed as $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$, where $|\alpha|^2 + |\beta|^2 = 1$. We have

$$|\psi\rangle\langle\psi| = |\alpha^2|0\rangle\langle 0| + \alpha\bar{\beta}|0\rangle\langle 1| + \bar{\alpha}\beta|1\rangle\langle 0| + |\beta|^2|1\rangle\langle 1|.$$

Thus,

$$\begin{aligned}
\Delta \circ \mathcal{E}(|\psi\rangle\langle\psi|) &= \Delta\left(\frac{p}{2} I + (1 - p)|\psi\rangle\langle\psi|\right) \\
&= \frac{p}{2} I + (1 - p)\Delta(|\psi\rangle\langle\psi|) \\
&= \frac{p}{2} I + (1 - p)|\alpha|^2|0\rangle\langle 0| + (1 - p)|\beta|^2|1\rangle\langle 1|. \\
\mathcal{E} \circ \Delta(|\psi\rangle\langle\psi|) &= \mathcal{E}(|\alpha|^2|0\rangle\langle 0| + |\beta|^2|1\rangle\langle 1|) \\
&= \frac{p}{2} I + (1 - p)|\alpha|^2|0\rangle\langle 0| + (1 - p)|\beta|^2|1\rangle\langle 1| \\
&= \Delta \circ \mathcal{E}(|\psi\rangle\langle\psi|).
\end{aligned}$$

Therefore,

$$C(\mathcal{E}) = \sup_{|\psi\rangle} D(\Delta \circ \mathcal{E}(|\psi\rangle\langle\psi|), \mathcal{E} \circ \Delta(|\psi\rangle\langle\psi|)) = 0.$$

□

There is an important noise channel which describes the loss of quantum information without loss of energy in quantum mechanics. It is called the phase damping channel. In the following example, the coherence measure of the phase damping quantum channel is calculated to be zero, which indicates the channel is incoherent.

Example 2 Let $\mathcal{E}$ be the phase damping channel, $\mathcal{E}(\rho) = p\rho + (1-p)Z\rho Z$, where Z is the Pauli matrix. Then, $C(\mathcal{E}) = 0$.

For any pure state $|\psi\rangle$, it can be expressed as $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$, where $|\alpha|^2 + |\beta|^2 = 1$. We have $Z|\psi\rangle = \alpha|0\rangle - \beta|1\rangle$, which yields that

$$Z(|\psi\rangle\langle\psi|)Z = |\alpha|^2|0\rangle\langle 0| - \alpha\bar{\beta}|0\rangle\langle 1| - \beta\bar{\alpha}|1\rangle\langle 0| + |\beta|^2|1\rangle\langle 1|,$$

and then

$$\mathcal{E}(|\psi\rangle\langle\psi|) = |\alpha|^2|0\rangle\langle 0| + (2p-1)\alpha\bar{\beta}|0\rangle\langle 1| + (2p-1)\bar{\alpha}\beta|1\rangle\langle 0| + |\beta|^2|1\rangle\langle 1|.$$

Hence,

$$\Delta \circ \mathcal{E}(|\psi\rangle\langle\psi|) = |\alpha|^2|0\rangle\langle 0| + |\beta|^2|1\rangle\langle 1|.$$

Similarly, for any pure state $|\psi\rangle$, $\mathcal{E} \circ \Delta(|\psi\rangle\langle\psi|) = |\alpha|^2|0\rangle\langle 0| + |\beta|^2|1\rangle\langle 1| = \Delta \circ \mathcal{E}(|\psi\rangle\langle\psi|)$. Thus,

$$D(\Delta \circ \mathcal{E}(|\psi\rangle\langle\psi|), \mathcal{E} \circ \Delta(|\psi\rangle\langle\psi|)) = 0.$$

And so we have $C(\mathcal{E}) = 0$.

In the following, we shall show that the amplitude damping quantum channel is also incoherent.

Example 3 Let $\mathcal{E}$ be the amplitude damping channel, $\mathcal{E}(\rho) = E_0\rho E_0^\dagger + E_1\rho E_1^\dagger$, where $E_0 = \begin{bmatrix} 1 & 0 \\ 0 & \sqrt{1-r} \end{bmatrix}$, $E_1 = \begin{bmatrix} 0 & \sqrt{r} \\ 0 & 0 \end{bmatrix}$. Then, $C(\mathcal{E}) = 0$.

For any pure state $|\psi\rangle$, it can be expressed as $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$, where $|\alpha|^2 + |\beta|^2 = 1$. It is easy compute that $E_0|0\rangle = |0\rangle$, $E_0|1\rangle = \sqrt{1-r}|1\rangle$, $E_1|0\rangle = 0$, $E_1|1\rangle = \sqrt{r}|0\rangle$, and

$$\begin{aligned} E_0|\psi\rangle &= E_0(\alpha|0\rangle + \beta|1\rangle) = \alpha|0\rangle + \beta\sqrt{1-r}|1\rangle, \\ E_1|\psi\rangle &= E_1(\alpha|0\rangle + \beta|1\rangle) = \beta\sqrt{r}|0\rangle. \end{aligned}$$

Thus, $\Delta(|\psi\rangle\langle\psi|) = |\alpha|^2|0\rangle\langle 0| + |\beta|^2|1\rangle\langle 1|$, and

$$\begin{aligned} &\mathcal{E}(|\psi\rangle\langle\psi|) \\ &= E_0|\psi\rangle\langle\psi|E_0^\dagger + E_1|\psi\rangle\langle\psi|E_1^\dagger \\ &= (\alpha|0\rangle + \beta\sqrt{1-r}|1\rangle)(\bar{\alpha}\langle 0| + \bar{\beta}\sqrt{1-r}\langle 1|) + |\beta|^2 r|0\rangle\langle 0| \\ &= (|\alpha|^2 + |\beta|^2 r)|0\rangle\langle 0| + \alpha\bar{\beta}\sqrt{1-r}|0\rangle\langle 1| + \beta\bar{\alpha}\sqrt{1-r}|1\rangle\langle 0| + |\beta|^2(1-r)|1\rangle\langle 1|. \end{aligned}$$

And so we have

$$\begin{aligned}
&\Delta \circ \mathcal{E}(|\psi\rangle\langle\psi|) = (|\alpha|^2 + |\beta|^2 r)|0\rangle\langle 0| + |\beta|^2(1-r)|1\rangle\langle 1|,\\
&\mathcal{E} \circ \Delta(|\psi\rangle\langle\psi|)\\
&= E_0(|\alpha|^2|0\rangle\langle 0| + |\beta|^2|1\rangle\langle 1|)E_0^\dagger + E_1(|\alpha|^2|0\rangle\langle 0| + |\beta|^2|1\rangle\langle 1|)E_1^\dagger\\
&= (|\alpha|^2 + |\beta|^2 r)|0\rangle\langle 0| + |\beta|^2(1-r)|1\rangle\langle 1|\\
&= \Delta \circ \mathcal{E}(|\psi\rangle\langle\psi|).
\end{aligned}$$

Hence, for any pure state $|\psi\rangle$, we have $D(\Delta \circ \mathcal{E}(|\psi\rangle\langle\psi|), \mathcal{E} \circ \Delta(|\psi\rangle\langle\psi|) = 0$. Therefore, $C(\mathcal{E}) = 0$.

Above, we calculated that the coherence measure of several typical noisy channels are zero, which confirms that noisy channels can not generate coherence. In the following, we give an example for coherent channels.

Example 4 Let quantum channel Φ have the Kraus decomposition as $\Phi(\rho) = E_1(\rho)E_1^\dagger + E_2(\rho)E_2^\dagger$ with

$$E_1 = \begin{pmatrix} e^{i\eta}\cos\theta\cos\phi & 0 \\ -\sin\theta\sin\phi & e^{i\xi}\cos\phi \end{pmatrix},$$
$$E_2 = \begin{pmatrix} \sin\theta\cos\phi & e^{i\xi}\sin\phi \\ e^{-i\eta}\cos\theta\sin\phi & 0 \end{pmatrix}.$$

Here, θ, ϕ, ξ, and η are all real numbers.

For any pure state $|\psi\rangle$, it can be expressed as $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$, where $|\alpha|^2 + |\beta|^2 = 1$. It is easy compute that

$$\begin{aligned}
&\Delta \circ \Phi(|\psi\rangle\langle\psi|)\\
&= \begin{pmatrix} |\alpha|^2\cos^2\phi + |\beta|^2\sin^2\phi + a_{|\psi\rangle} & 0 \\ 0 & |\alpha|^2\sin^2\phi + |\beta|^2\cos^2\phi - a_{|\psi\rangle}, \end{pmatrix}
\end{aligned}$$

where $a_{|\psi\rangle} = 2\mathrm{Re}(\alpha\bar{\beta}e^{-i\xi})\sin\theta\sin\phi\cos\phi$, and

$$\begin{aligned}
&\Phi \circ \Delta(|\psi\rangle\langle\psi|)\\
&= \begin{pmatrix} |\alpha|^2\cos^2\phi + |\beta|^2\sin^2\phi & 0 \\ 0 & |\alpha|^2\sin^2\phi + |\beta|^2\cos^2\phi \end{pmatrix}.
\end{aligned}$$

The quantum channel Φ is incoherent if and only if $C(\Phi) = 0$, which is equivalent to

$$\Phi \circ \Delta(|\psi\rangle\langle\psi|) = \Delta \circ \Phi(|\psi\rangle\langle\psi|),$$

for any state $|\psi\rangle$, that is

$$\sin\theta\sin\phi\cos\phi = 0.$$

Therefore, Φ is not an incoherent channel unless $\sin\theta\sin\phi\cos\phi = 0$. It is concluded that Φ is a coherent channel when $\sin\theta\sin\phi\cos\phi \neq 0$. However, Hu obtained that Φ is not an incoherent channel when $\sin\theta\cos\theta\sin\phi\cos\phi \neq 0$ [29]. Clearly, the sufficient condition of Φ being a coherent channel in [29] is stronger than the sufficient condition in our work. In other words, we can find more coherent quantum channels using our measure.

Suppose $\mathcal{E}$ is a trace-preserving quantum operation for which there exists a density operator ρ_0 and a trace-preserving quantum operation $\mathcal{E}'$, such that

$$\mathcal{E}(\rho) = p\rho_0 + (1-p)\mathcal{E}'(\rho), \tag{9}$$

for some p, $0 < p \leq 1$. Physically, this means that with probability p the input state is thrown out and replaced with the fixed state ρ_0, while with probability $1-p$ the operation $\mathcal{E}'$ occurs.

Example 5 Suppose $\mathcal{E}$ is defined as Eq. (9). Then,

$$C(\mathcal{E}) \leq \frac{p}{2}C_{l_1}(\rho_0) + (1-p)C(\mathcal{E}'), \tag{10}$$

where $C_{l_1}(\rho_0) = \sum_{i\neq j}|a_{ij}|$ is the coherence measure of the quantum state $\rho_0 = \sum_{ij} a_{ij}|i\rangle\langle j|$ under l_1−norm.

Indeed, for any state ρ, by the joint convexity of the trace distance, we have

$$\begin{aligned}
&D(\mathcal{E}\circ\Delta(\rho), \Delta\circ\mathcal{E}(\rho))\\
&\quad = D(p\rho_0 + (1-p)\mathcal{E}'(\Delta(\rho)), \Delta(p\rho_0 + (1-p)\mathcal{E}'(\rho)))\\
&\quad = D(p\rho_0 + (1-p)\mathcal{E}'\circ\Delta(\rho), p\Delta(\rho_0) + (1-p)\Delta\circ\mathcal{E}'(\rho))\\
&\quad \leq pD(\rho_0, \Delta(\rho_0)) + (1-p)D(\mathcal{E}'\circ\Delta(\rho), \Delta\circ\mathcal{E}'(\rho)).
\end{aligned} \tag{11}$$

Thus, taking the supremum at both sides of Eq. (11) for all quantum states, we get

$$C(\mathcal{E}) \leq pD(\rho_0, \Delta(\rho_0)) + (1-p)C(\mathcal{E}').$$

Since $D(\rho_0, \Delta(\rho_0) = \frac{1}{2}C_{l_1}(\rho_0)$, the inequality (10) holds. We see that the total channel coherence $C(\mathcal{E})$ does not exceed two parts: $C(\mathcal{E}')$ accounts for the contribution of channel $\mathcal{E}'$ and $C_{l_1}(\rho_0)$ accounts for the contribution of state ρ_0.

In [33], under the coherence measure C_{l_1} which is defined by the Choi matrix, it holds that

$$C_{l_1}(\mathcal{E}) \leq pC_{l_1}(\rho_0) + (1-p)C_{l_1}(\mathcal{E}'). \tag{12}$$

Comparing Eqs. (10) and (12), we find that the upper bound on the coherence measure defined by the Choi matrix and trace distance has a similar form.

4 Conclusions

Since the quantum channel resource theory method has great practical value and the mathematical structure of quantum channels is more complex, the relevant resource theoretical framework may be very important and interesting from a mathematical point of view. In addition, the trace distance has many advantages and nice properties. Therefore, we proposed a coherence measure of the quantum channel utilizing the trace distance to quantify its commutativity with the completely dephasing operation. Basic properties of this coherence measure with respect to a quantum channel were discussed. The extremal property is that the maximum in the maximization for the coherence measure of a quantum channel is always attained for a pure state. The monotonicity states that the coherence measure is non-increasing under the composition with incoherent channels. The convexity of the coherence measure ensures that mixing does not create resources.

We calculated the coherence measure of Hadamard gate as 1/2 and showed that the coherence of a composition of operations does not have additivity. Finally, we found the amplitude damping quantum channel, the phase damping quantum channel and the depolarized quantum channel are all incoherent by calculation. The result perfectly matches intuitive understanding. Furthermore, we also gave a sufficient condition for a given quantum channel to be coherent and compared with the existing result.

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Data availability All data generated or analyzed during this study are included in this article.

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