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Tighter monogamy relations in multi-qubit systems

Yudie Gu[1] · Yanmin Yang[1] · Jialing Zhang[1] · Wei Chen[2]

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Abstract
In this paper, we present some monogamy relations of multiqubit quantum entanglement in terms of the βth power of concurrence, entanglement of formation and convex-roof extended negativity. These monogamy relations are proved to be tighter than the existing ones, together with detailed examples showing the tightness.

Keywords Monogamy relations · Concurrence · Entanglement of formation · Convex-roof extended negativity

1 Introduction

Quantum entanglement is widely used as a very important resource in quantum information processing [1–4]. With the emergence of quantum information theory, quantum entanglement plays a very important role in quantum cryptography, quantum teleportation and measurement-based quantum computing. An important issue related to the entanglement metric is the limited shareability of the two-part entanglement in a multipartite entangled qubit system, that is, the single duality of entanglement [5]. Monogamy of entanglement (MoE) plays a very important role in many quantum information and communication processing tasks, such as security proof of quantum cryptography schemes and security analysis of quantum key distribution [6, 7].

For a tripartite quantum state ρ_{ABC}, MoE can be described as $E(\rho_{A|BC}) \geq E(\rho_{AB}) + E(\rho_{AC})$, where $\rho_{AB} = \mathrm{tr}_C(\rho_{ABC})$, $\rho_{AC} = \mathrm{tr}_B(\rho_{ABC})$, $E(\rho_{A|BC})$ denotes the entanglement between systems A and BC. A remarkable result was established by Coffman, Kundu and Wootters (CKW) [8] for three qubits that was the simultaneous squares satisfy monogamy inequality. Then, the so-called CKW inequality was generalized to any N-qubit system [9]. Interestingly, it is further proved that similar

✉ Yanmin Yang
ym.yang@kust.edu.cn

1 Faculty of Science, Kunming University of Science and Technology, Kunming 650500, China

2 School of Computer Science and Technology, Dongguan University of Technology, Dongguan 523808, China

inequalities of polyqubit monogamy can be established for negativity and convex-roof extended negativity (CREN) [10–12], the entanglement of formation (EoF) [13, 14], Rényi-α entanglement [15, 16] and Tsallis-q entanglement [17].

Our paper is organized as follows. In Sect. 2, we present and prove two monogamy inequalities for the βth ($\beta \geq 2$) power of concurrence in N-qubit system. In Sect. 3, we give a tighter monogamy relation for the βth ($\beta \geq \sqrt{2}$) power of EoF in $2\otimes 2\otimes 2^{N-2}$ system. Then, we extend the result to N-qubit system. In Sect. 4, the monogamy relation for the βth ($\beta \geq 2$) power of CREN in N-qubit system is discussed. In addition, detailed examples are given to illustrate the tightness. In Sect. 5, we summarize our results.

2 Tighter monogamy relations using concurrence

Given a bipartite pure state $|\phi\rangle_{AB}$ on Hilbert space $H_A \otimes H_B$, the concurrence is given by [18–20]

$$C(|\phi\rangle_{AB}) = \sqrt{2(1-\mathrm{Tr}(\rho_A^2))}, \tag{1}$$

where ρ_A is the reduced density matrix by tracing over the subsystem B, $\rho_A = \mathrm{Tr}_B(|\phi\rangle_{AB}\langle\phi|)$. For a bipartite mixed state ρ_{AB}, the concurrence is defined by the convex-roof,

$$C(\rho_{AB}) = \min_{\{p_i,|\phi_i\rangle\}} \sum_i p_i C(|\phi_i\rangle_{AB}), \tag{2}$$

where the minimum is taken over all possible pure state decompositions of $\rho_{AB} = \sum_i p_i|\phi_i\rangle\langle\phi_i|$, with $\sum_i p_i = 1$ and $p_i \geq 0$.

For any N-qubit mixed state $\rho_{AB_1\cdots B_{N-1}}$, the concurrence $C(\rho_{A|B_1\cdots B_{N-1}})$ of the state $\rho_{AB_1\cdots B_{N-1}}$ under bipartite partition A and $B_1\cdots B_{N-1}$ satisfies [21]

$$C^\beta(\rho_{A|B_1\cdots B_{N-1}}) \;\geq C^\beta(\rho_{AB_1}) + C^\beta(\rho_{AB_2}) + \cdots + C^\beta(\rho_{AB_{N-1}}), \tag{3}$$

for $\beta \geq 2$. Furthermore, for an N-qubit mixed state, if $C_{AB_i} \geq C_{A|B_{i+1}\cdots B_{N-1}}$ for $i = 1, 2, \ldots, m$, and $C_{AB_j} \leq C_{A|B_{j+1}\cdots B_{N-1}}$ for $j = m+1, \ldots, N-2$, a generalized monogamy relation for $\beta \geq 2$ was presented as [22]:

$$\begin{aligned} &C^\beta(\rho_{A|B_1\cdots B_{N-1}}) \\ &\geq C^\beta(\rho_{AB_1}) + (2^{\frac{\beta}{2}}-1)C^\beta(\rho_{AB_2}) + \cdots + \left(2^{\frac{\beta}{2}}-1\right)^{m-1}C^\beta(\rho_{AB_m}) \\ &\quad +\left(2^{\frac{\beta}{2}}-1\right)^{m+1}[C^\beta(\rho_{AB_{m+1}}) + \cdots + C^\beta(\rho_{AB_{N-2}})] + \left(2^{\frac{\beta}{2}}-1\right)^{m}C^\beta(\rho_{AB_{N-1}}), \end{aligned} \tag{4}$$

where $1 \leq m \leq N-3$, $N \geq 4$.

In the following, we will show that these monogamy relations for concurrence can be further tightened under some conditions. Before that, we first introduce two lemmas as follows.

Lemma 1 *For any $x \in [0, 1]$ and $t \geq 1$, we have*

$$(1+x)^t \geq 1 + \frac{t}{2}x + \frac{(t-1)^2}{4}x^2 + (2^t - \frac{t}{2} + \frac{(t-1)^2}{4} - 1)x^t - \frac{(t-1)^2}{2}x^{t+1}$$
$$\geq 1 + \frac{t}{2}x + (2^t - \frac{t}{2} - 1)x^t \geq 1 + (2^t - 1)x^t. \tag{5}$$

Proof Let us consider the function $f(t,x) = \frac{(1+x)^t - 1 - \frac{t}{2}x - \frac{(t-1)^2}{4}x^2 + \frac{(t-1)^2}{2}x^{t+1}}{x^t}$. Then, $\frac{\partial f(t,x)}{\partial x} = \frac{tx^{t-1}[1 + \frac{t-1}{2}x + \frac{(t-1)^2(t-2)}{4t}x^2 + \frac{(t-1)^2}{2t}x^{t+1} - (1+x)^{t-1}]}{x^{2t}}$. Next, we will prove that

$$1 + \frac{t-1}{2}x + \frac{(t-1)^2(t-2)}{4t}x^2 + \frac{(t-1)^2}{2t}x^{t+1} \leq (1+x)^{t-1}, \tag{6}$$

thus $\frac{\partial f(t,x)}{\partial x} \leq 0$, $f(t,x)$ is a decreasing function of x, i.e., $f(t,x) \geq f(t,1) = 2^t - \frac{t}{2} + \frac{(t-1)^2}{4} - 1$. It follows that $(1+x)^t \geq 1 + \frac{t}{2}x + \frac{(t-1)^2}{4}x^2 + (2^t - \frac{t}{2} + \frac{(t-1)^2}{4} - 1)x^t - \frac{(t-1)^2}{2}x^{t+1}$.

For the case $1 \leq t \leq 2$, it is obvious that $(1+x)^{t-1} \geq 1 + (t-1)x + \frac{(t-1)(t-2)}{2}x^2$. Besides, we have

$$\begin{aligned}\frac{(t-1)(t-2)}{2}x^2 &= \frac{t-1}{4t}2t(t-2)x^2\\ &= \frac{t-1}{4t}[(t-1)(t-2)x^2 + (t^2+t-2)x^2 - 2tx^2]\\ &\geq \frac{t-1}{4t}[(t-1)(t-2)x^2 + (2t-2)x^{t+1} - 2tx].\end{aligned}$$

Thus, Eq. (6) is hold.

For the case $t \geq 2$, it is obvious that $(1+x)^{t-1} \geq 1 + (t-1)x + \frac{(t-1)(t-2)}{4}x^2$. Besides, we have

$$\begin{aligned}\frac{(t-1)(t-2)}{4}x^2 &= \frac{t-1}{4t}t(t-2)x^2 = \frac{t-1}{4t}[(t-1)(t-2)x^2 + (2t-2)x^2 - tx^2]\\ &\geq \frac{t-1}{4t}[(t-1)(t-2)x^2 + 2(t-1)x^{t+1} - 2tx].\end{aligned}$$

Thus, Eq. (6) is hold.

On the other hand, since $x^2 - 2x^{t+1} + x^t \geq 0$ and $\frac{(t-1)^2}{4} \geq 0$, for $t \geq 1$ and $x \in [0, 1]$, we can get $(1+x)^t \geq 1 + \frac{t}{2}x + \frac{(t-1)^2}{4}x^2 + (2^t - \frac{t}{2} + \frac{(t-1)^2}{4} - 1)x^t - \frac{(t-1)^2}{2}x^{t+1} \geq 1 + \frac{t}{2}x + (2^t - \frac{t}{2} - 1)x^t \geq 1 + (2^t - 1)x^t$. □

Lemma 2 *For any mixed state ρ_{ABC} in a $2 \otimes 2 \otimes 2^{N-2}$ system, suppose that $C_{AB} \geq C_{AC}$, we have*

$$C^\beta_{A|BC} \geq C^\beta_{AB} + hC^\beta_{AC} + \frac{\beta}{4}C^2_{AC}\left(C^{\beta-2}_{AB} - C^{\beta-2}_{AC}\right)$$

$$+\frac{(\beta-2)^2}{16}C_{AC}^4\left(C_{AB}^{\beta-4}+C_{AC}^{\beta-4}-2C_{AC}^{\beta-2}C_{AB}^{-2}\right), \tag{7}$$

for all $\beta \geq 2$*, where* $h = 2^{\frac{\beta}{2}} - 1$, $C_{A|BC} = C(\rho_{A|BC})$*, analogously for* C_{AB} *and* C_{AC}.

Proof Since $C_{AB} \geq C_{AC}$, we obtain

$$\begin{aligned}
C_{A|BC}^{\beta} &\geq (C_{AB}^2 + C_{AC}^2)^{\frac{\beta}{2}} = C_{AB}^{\beta}\left(1+\frac{C_{AC}^2}{C_{AB}^2}\right)^{\frac{\beta}{2}} \\
&\geq C_{AB}^{\beta}\left[1+\frac{\beta}{4}\frac{C_{AC}^2}{C_{AB}^2}+\frac{(\beta-2)^2}{16}\frac{C_{AC}^4}{C_{AB}^4}+\left(2^{\frac{\beta}{2}}-\frac{\beta}{4}+\frac{(\beta-2)^2}{16}-1\right)\frac{C_{AC}^{\beta}}{C_{AB}^{\beta}}\right. \\
&\quad \left.-\frac{(\beta-2)^2}{8}\frac{C_{AC}^{\beta+2}}{C_{AB}^{\beta+2}}\right] \\
&= C_{AB}^{\beta} + hC_{AC}^{\beta} + \frac{\beta}{4}C_{AC}^2\left(C_{AB}^{\beta-2}-C_{AC}^{\beta-2}\right) \\
&\quad +\frac{(\beta-2)^2}{16}C_{AC}^4\left(C_{AB}^{\beta-4}+C_{AC}^{\beta-4}-2C_{AC}^{\beta-2}C_{AB}^{-2}\right),
\end{aligned} \tag{8}$$

where the first inequality is due to the fact that $C_{A|BC}^2 \geq C_{AB}^2 + C_{AC}^2$ for any $2 \otimes 2 \otimes 2^{N-2}$ tripartite state $\rho_{A|BC}$ [9, 23] and the second inequality is due to Lemma 1. □

Theorem 1 *For any N-qubit mixed state* $\rho_{AB_1\cdots B_{N-1}}$*, if* $C_{AB_i} \geq C_{A|B_{i+1}\cdots B_{N-1}}$*, for* $i = 1, 2, \ldots, N-2$*, we have*

$$C_{A|B_1\cdots B_{N-1}}^{\beta} \geq \sum_{i=1}^{N-2} h^{i-1}\left(C_{AB_i}^{\beta} + P_{AB_i}\right) + h^{N-2}C_{AB_{N-1}}^{\beta}, \tag{9}$$

for all $N \geq 3$, $\beta \geq 2$*, where* $h = 2^{\frac{\beta}{2}} - 1$, $P_{AB_i} = \frac{\beta}{4}C_{A|B_{i+1}\cdots B_{N-1}}^2(C_{AB_i}^{\beta-2} - C_{A|B_{i+1}\cdots B_{N-1}}^{\beta-2}) + \frac{(\beta-2)^2}{16}C_{A|B_{i+1}\cdots B_{N-1}}^4(C_{AB_i}^{\beta-4} + C_{A|B_{i+1}\cdots B_{N-1}}^{\beta-4} - 2C_{A|B_{i+1}\cdots B_{N-1}}^{\beta-2}C_{AB_i}^{-2})$.

Proof Due to Eq. (7), we obtain

$$\begin{aligned}
&C_{A|B_1\cdots B_{N-1}}^{\beta} \\
&\geq C_{AB_1}^{\beta} + hC_{A|B_2\cdots B_{N-1}}^{\beta} + \frac{\beta}{4}C_{A|B_2\cdots B_{N-1}}^2\left(C_{AB_1}^{\beta-2} - C_{A|B_2\cdots B_{N-1}}^{\beta-2}\right) \\
&\quad +\frac{(\beta-2)^2}{16}C_{A|B_2\cdots B_{N-1}}^4\left(C_{AB_1}^{\beta-4} + C_{A|B_2\cdots B_{N-1}}^{\beta-4} - 2C_{A|B_2\cdots B_{N-1}}^{\beta-2}C_{AB_1}^{-2}\right) \\
&\geq C_{AB_1}^{\beta} + h\left[C_{AB_2}^{\beta} + hC_{A|B_3\cdots B_{N-1}}^{\beta} + \frac{\beta}{4}C_{A|B_3\cdots B_{N-1}}^2(C_{AB_2}^{\beta-2} - C_{A|B_3\cdots B_{N-1}}^{\beta-2})\right. \\
&\quad \left.+\frac{(\beta-2)^2}{16}C_{A|B_3\cdots B_{N-1}}^4\left(C_{AB_2}^{\beta-4} + C_{A|B_3\cdots B_{N-1}}^{\beta-4} - 2C_{A|B_3\cdots B_{N-1}}^{\beta-2}C_{AB_2}^{-2}\right)\right]
\end{aligned}$$

$$
\begin{aligned}
&+\frac{\beta}{4}C^{2}_{A|B_2\cdots B_{N-1}}\left(C^{\beta-2}_{AB_1}-C^{\beta-2}_{A|B_2\cdots B_{N-1}}\right)\\
&+\frac{(\beta-2)^2}{16}C^{4}_{A|B_2\cdots B_{N-1}}\left(C^{\beta-4}_{AB_1}+C^{\beta-4}_{A|B_2\cdots B_{N-1}}-2C^{\beta-2}_{A|B_2\cdots B_{N-1}}C^{-2}_{AB_1}\right)\\
\geq&\cdots\\
\geq&\, C^{\beta}_{AB_1}+hC^{\beta}_{AB_2}+\cdots+h^{N-2}C^{\beta}_{AB_{N-1}}+h^{N-3}\left[\frac{\beta}{4}C^{2}_{AB_{N-1}}\left(C^{\beta-2}_{AB_{N-2}}-C^{\beta-2}_{AB_{N-1}}\right)\right.\\
&\left.+\frac{(\beta-2)^2}{16}C^{4}_{AB_{N-1}}\left(C^{\beta-4}_{AB_{N-2}}+C^{\beta-4}_{AB_{N-1}}-2C^{\beta-2}_{AB_{N-1}}C^{-2}_{AB_{N-2}}\right)\right]\\
&+\cdots+h\left[\frac{\beta}{4}C^{2}_{A|B_3\cdots B_{N-1}}\left(C^{\beta-2}_{AB_2}-C^{\beta-2}_{A|B_3\cdots B_{N-1}}\right)\right.\\
&\left.+\frac{(\beta-2)^2}{16}C^{4}_{A|B_3\cdots B_{N-1}}\left(C^{\beta-4}_{AB_2}+C^{\beta-4}_{A|B_3\cdots B_{N-1}}-2C^{\beta-2}_{A|B_3\cdots B_{N-1}}C^{-2}_{AB_2}\right)\right]\\
&+\frac{\beta}{4}C^{2}_{A|B_2\cdots B_{N-1}}\left(C^{\beta-2}_{AB_1}-C^{\beta-2}_{A|B_2\cdots B_{N-1}}\right)\\
&+\frac{(\beta-2)^2}{16}C^{4}_{A|B_2\cdots B_{N-1}}\left(C^{\beta-4}_{AB_1}+C^{\beta-4}_{A|B_2\cdots B_{N-1}}-2C^{\beta-2}_{A|B_2\cdots B_{N-1}}C^{-2}_{AB_1}\right). \qquad (10)
\end{aligned}
$$

By the denotation of P_{AB_i}, we complete the proof. □

Theorem 2 *For any N-qubit mixed state $\rho_{AB_1\cdots B_{N-1}}$, if $C_{AB_i}\geq C_{A|B_{i+1}\cdots B_{N-1}}$ for $i=1,2,\ldots,m$, and $C_{AB_j}\leq C_{A|B_{j+1}\cdots B_{N-1}}$ for $j=m+1,\ldots,N-2$, $\forall\, 1\leq m\leq N-3$, we have*

$$
\begin{aligned}
C^{\beta}_{A|B_1\cdots B_{N-1}}\geq&\sum_{i=1}^{m}h^{i-1}\left(C^{\beta}_{AB_i}+P_{AB_i}\right)\\
&+h^{m}\sum_{j=m+1}^{N-2}\left(hC^{\beta}_{AB_j}+P^{1}_{AB_j}\right)+h^{m}C^{\beta}_{AB_{N-1}}, \qquad (11)
\end{aligned}
$$

for all $N\geq 4$, $\beta\geq 2$, where $h=2^{\frac{\beta}{2}}-1$, $P_{AB_i}=\frac{\beta}{4}C^{2}_{A|B_{i+1}\cdots B_{N-1}}(C^{\beta-2}_{AB_i}-C^{\beta-2}_{A|B_{i+1}\cdots B_{N-1}})+\frac{(\beta-2)^2}{16}C^{4}_{A|B_{i+1}\cdots B_{N-1}}(C^{\beta-4}_{AB_i}+C^{\beta-4}_{A|B_{i+1}\cdots B_{N-1}}-2C^{\beta-2}_{A|B_{i+1}\cdots B_{N-1}}C^{-2}_{AB_i})$, $P^{1}_{AB_j}=\frac{\beta}{4}C^{2}_{AB_j}(C^{\beta-2}_{A|B_{j+1}\cdots B_{N-1}}-C^{\beta-2}_{AB_j})+\frac{(\beta-2)^2}{16}C^{4}_{AB_j}(C^{\beta-4}_{A|B_{j+1}\cdots B_{N-1}}+C^{\beta-4}_{AB_j}-2C^{\beta-2}_{AB_j}C^{-2}_{A|B_{j+1}\cdots B_{N-1}})$.

Proof Due to the proof process of Theorem 1, we can get that

$$
C^{\beta}_{A|B_1\cdots B_{N-1}}\geq\sum_{i=1}^{m}h^{i-1}\left(C^{\beta}_{AB_i}+P_{AB_i}\right)+h^{m}C^{\beta}_{A|B_{m+1}\cdots B_{N-1}}. \qquad (12)
$$

In addition, since $C_{AB_j}\leq C_{A|B_{j+1}\cdots B_{N-1}}$ for $j=m+1,\ldots,N-2$, hence

$$
C^{\beta}_{A|B_{m+1}\cdots B_{N-1}}
$$

$$\geq C^{\beta}_{A|B_{m+2}\cdots B_{N-1}} + hC^{\beta}_{AB_{m+1}} + \frac{\beta}{4}C^{2}_{AB_{m+1}}\left(C^{\beta-2}_{A|B_{m+2}\cdots B_{N-1}} - C^{\beta-2}_{AB_{m+1}}\right)$$
$$+\frac{(\beta-2)^2}{16}C^{4}_{AB_{m+1}}\left(C^{\beta-4}_{A|B_{m+2}\cdots B_{N-1}} + C^{\beta-4}_{AB_{m+1}} - 2C^{\beta-2}_{AB_{m+1}}C^{-2}_{A|B_{m+2}\cdots B_{N-1}}\right)$$
$$\geq \sum_{j=m+1}^{N-2}\left(hC^{\beta}_{AB_j} + P^{1}_{AB_j}\right) + C^{\beta}_{AB_{N-1}}. \tag{13}$$

Combing Eqs. (12) and (13), we can get the inequality (11). □

Example 1 Consider the three-qubit state $|\psi\rangle_{ABC}$ in generalized Schmidt decomposition form [25, 26]:

$$|\psi\rangle_{ABC} = \lambda_0|000\rangle + \lambda_1 e^{i\varphi}|100\rangle + \lambda_2|101\rangle + \lambda_3|110\rangle + \lambda_4|111\rangle, \tag{14}$$

where $\lambda_i \geq 0$, $i = 0, 1, 2, 3, 4$, and $\sum_{i=0}^{4}\lambda_i^2 = 1$. A direct calculation shows that $C_{A|BC} = 2\lambda_0\sqrt{\lambda_2^2 + \lambda_3^2 + \lambda_4^2}$, $C_{AB} = 2\lambda_0\lambda_2$ and $C_{AC} = 2\lambda_0\lambda_3$.

Set $\lambda_0 = \frac{\sqrt{2}}{3}, \lambda_1 = 0, \lambda_2 = \frac{\sqrt{5}}{3}, \lambda_3 = \frac{\sqrt{2}}{3}, \lambda_4 = 0$. We have $C_{A|BC} = \frac{2\sqrt{14}}{9}$, $C_{AB} = \frac{2\sqrt{10}}{9}$ and $C_{AC} = \frac{4}{9}$. Then, $C^{\beta}_{A|BC} = (\frac{2\sqrt{14}}{9})^{\beta} \geq C^{\beta}_{AB} + hC^{\beta}_{AC} + \frac{\beta}{4}C^2_{AC}(C^{\beta-2}_{AB} - C^{\beta-2}_{AC}) + \frac{(\beta-2)^2}{16}C^4_{AC}(C^{\beta-4}_{AB} + C^{\beta-4}_{AC} - 2C^{\beta-2}_{AC}C^{-2}_{AB}) = (\frac{2\sqrt{10}}{9})^{\beta} + h(\frac{4}{9})^{\beta} + \frac{\beta}{4}(\frac{4}{9})^2\Big[(\frac{2\sqrt{10}}{9})^{\beta-2} - (\frac{4}{9})^{\beta-2}\Big] + \frac{(\beta-2)^2}{16}(\frac{4}{9})^4\Big[(\frac{2\sqrt{10}}{9})^{\beta-4} + (\frac{4}{9})^{\beta-4} - 2(\frac{4}{9})^{\beta-2}(\frac{2\sqrt{10}}{9})^{-2}\Big]$. However, the result in [24] is $C^{\beta}_{AB} + hC^{\beta}_{AC} + \frac{\beta}{4}C^2_{AC}(C^{\beta-2}_{AB} - C^{\beta-2}_{AC}) = (\frac{2\sqrt{10}}{9})^{\beta} + h(\frac{4}{9})^{\beta} + \frac{\beta}{4}(\frac{4}{9})^2\Big[(\frac{2\sqrt{10}}{9})^{\beta-2} - (\frac{4}{9})^{\beta-2}\Big]$. We can see that our results are better than the ones in [24] for $\beta \geq 2$, see Fig. 1.

3 Tighter monogamy relations using EoF

Let H_A and H_B be two Hilbert spaces with dimension m and n $(m \leq n)$. The entanglement of formation (EoF) of a pure state $|\phi\rangle_{AB}$ on Hilbert space $H_A \otimes H_B$, is defined as [27, 28]

$$E(|\phi\rangle_{AB}) = S(\rho_A) = -\mathrm{Tr}(\rho_A \log_2 \rho_A), \tag{15}$$

where $S(\rho) = -\mathrm{Tr}(\rho \log_2 \rho)$ and $\rho_A = \mathrm{Tr}_B(|\phi\rangle_{AB}\langle\phi|)$. For a bipartite mixed state $|\phi\rangle_{AB}$ on Hilbert space $H_A \otimes H_B$, the EoF is given by

$$E(\rho_{AB}) = \inf_{\{p_i, |\phi_i\rangle\}} \sum_i p_i E(|\phi_i\rangle), \tag{16}$$

where the infimum is taken over all possible pure state decompositions of ρ_{AB}.

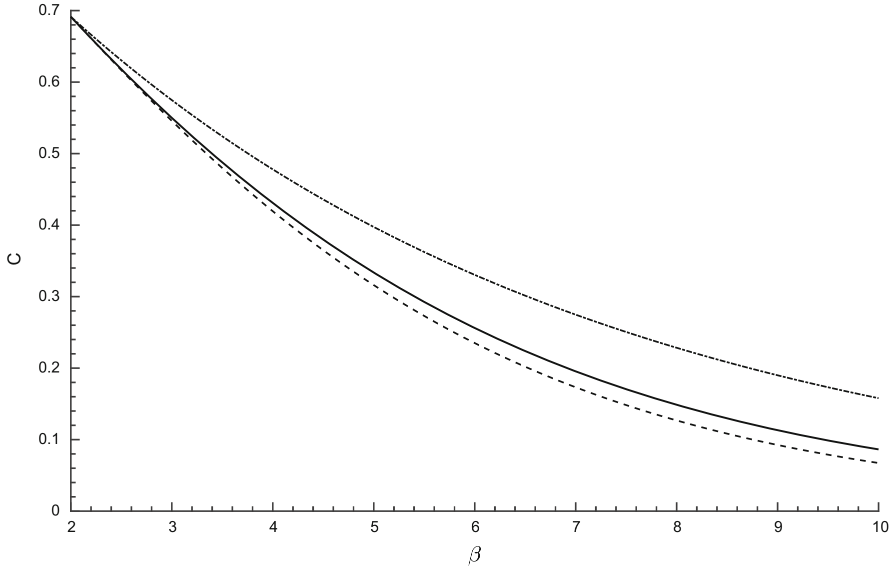

Fig. 1 Dash dotted line, $C^{\beta}_{A|BC}$ as a function of β ($2 \leq \beta \leq 10$); solid line, the lower bound of $C^{\beta}_{A|BC}$ as a function of β ($2 \leq \beta \leq 10$) in Eq. (11); dash line, the lower bound of $C^{\beta}_{A|BC}$ as a function of β ($2 \leq \beta \leq 10$) in [24]

Let $g(x) = H\left(\frac{1+\sqrt{1-x}}{2}\right)$ and $H(x) = -x\log_2 x - (1-x)\log_2(1-x)$, it is obvious that $g(x)$ is a monotonically increasing function for $0 \leq x \leq 1$, and satisfies

$$g^{\sqrt{2}}(x^2 + y^2) \geq g^{\sqrt{2}}(x^2) + g^{\sqrt{2}}(y^2), \tag{17}$$

where $g^{\sqrt{2}}(x^2 + y^2) = [g(x^2 + y^2)]^{\sqrt{2}}$.

From Eqs. (15) and (16), we have $E(|\phi\rangle) = g(\mathcal{C}^2(|\phi\rangle))$ for $2 \otimes d$ ($d \geq 2$) pure state $|\phi\rangle$. And $E(\rho) = g(\mathcal{C}^2(\rho))$ for arbitrary two-qubit mixed state ρ [29].

Wootters [8] shows that the EoF does not satisfy the monogamy inequality $E_{AB} + E_{AC} \leq E_{A|BC}$. In [30], the authors shows that EoF is a monotonic function satisfying $E^2(C^2_{A|B_1B_2\cdots B_{N-1}}) \geq E^2 \sum_{i=1}^{N-1}(C^2_{AB_i})$. For N-qubit systems, one has [21]

$$E^{\beta}_{A|B_1B_2\cdots B_{N-1}} \geq E^{\beta}_{AB_1} + E^{\beta}_{AB_2} + \cdots + E^{\beta}_{AB_{N-1}}, \tag{18}$$

for $\beta \geq \sqrt{2}$, where $E_{A|B_1B_2\cdots B_{N-1}}$ is the EoF of ρ under bipartite partition $A|B_1B_2\cdots B_{N-1}$, E_{AB_i} is the EoF of the mixed state $\rho_{AB_i} = \mathrm{Tr}_{B_1\cdots B_{i-1},B_{i+1}\cdots B_{N-1}}(\rho)$ for $i = 1, 2, \ldots, N-1$.

Lemma 3 *For any mixed state ρ_{ABC} in a $2\otimes2\otimes2^{N-2}$ system, $\beta \geq \sqrt{2}$, if $C_{AB} \geq C_{AC}$, then we have*

$$E^{\beta}_{A|BC} \geq E^{\beta}_{AB} + hE^{\beta}_{AC} + \frac{t}{2}E^{\sqrt{2}}_{AC}\left(E^{\beta-\sqrt{2}}_{AB} - E^{\beta-\sqrt{2}}_{AC}\right)$$

$$+\frac{(t-1)^2}{4}E_{AC}^{2\sqrt{2}}\left(E_{AB}^{\beta-2\sqrt{2}}+E_{AC}^{\beta-2\sqrt{2}}-2E_{AC}^{\beta-\sqrt{2}}E_{AB}^{-\sqrt{2}}\right), \tag{19}$$

where $t=\frac{\beta}{\sqrt{2}}$, $h=2^t-1$.

Proof The proof is similar to the proof of Lemma 2. □

In fact, the result can be generalized to N-qubit mixed state $\rho_{AB_1\cdots B_{N-1}}$. The following theorem holds for $\rho_{AB_1\cdots B_{N-1}}$.

Theorem 3 *For any* N*-qubit mixed state* $\rho_{AB_1\cdots B_{N-1}}$*, if* $C_{AB_i}\geq C_{A|B_{i+1}\cdots B_{N-1}}$ *for* $i=1,2,\ldots,N-2$*, we have*

$$E_{A|B_1\cdots B_{N-1}}^{\beta}\geq\sum_{i=1}^{N-2}h^{i-1}\left(E_{AB_i}^{\beta}+Q_{AB_i}\right)+h^{N-2}E_{AB_{N-1}}^{\beta}, \tag{20}$$

for $\beta\geq\sqrt{2}$*, where* $h=2^t-1$, $t=\frac{\beta}{\sqrt{2}}$, $Q_{AB_i}=\frac{t}{2}(E_{AB_{i+1}}^{\sqrt{2}}+\cdots+E_{AB_{N-1}}^{\sqrt{2}})(E_{AB_i}^{\beta-\sqrt{2}}-E_{A|B_{i+1}\cdots B_{N-1}}^{\beta-\sqrt{2}})+\frac{(t-1)^2}{4}(E_{AB_{i+1}}^{2\sqrt{2}}+\cdots+E_{AB_{N-1}}^{2\sqrt{2}})[E_{AB_i}^{\beta-2\sqrt{2}}+\cdots+E_{AB_{N-1}}^{\beta-2\sqrt{2}}-2(E_{A|B_{i+1}\cdots B_{N-1}}^{\beta-\sqrt{2}})E_{AB_i}^{-\sqrt{2}}]$.

Proof Let $\rho=\sum_i p_i|\psi_i\rangle\langle\psi_i|\in H_A\otimes H_{B_1}\otimes\cdots H_{B_{N-1}}$ be the optimal decomposition of $E_{A|B_1B_2\cdots B_{N-1}}(\rho)$ for the N-qubit mixed state ρ, we have [22]

$$E_{A|B_1B_2\cdots B_{N-1}}\geq g\left(C_{A|B_1B_2\cdots B_{N-1}}^2\right). \tag{21}$$

In addition, for $\beta\geq\sqrt{2}$, we have

$$\begin{aligned}g^{\beta}(x^2+y^2)&=\left[g^{\sqrt{2}}(x^2+y^2)\right]^t\geq\left[g^{\sqrt{2}}(x^2)+g^{\sqrt{2}}(y^2)\right]^t\\&\geq g^{\beta}(x^2)+(2^t-1)g^{\beta}(y^2)+\frac{t}{2}g^{\sqrt{2}}(y^2)\\&\quad\left[g^{\beta-\sqrt{2}}(x^2)-g^{\beta-\sqrt{2}}(y^2)\right]+\frac{(t-1)^2}{4}g^{2\sqrt{2}}(y^2)\\&\quad\left[g^{\beta-2\sqrt{2}}(x^2)+g^{\beta-2\sqrt{2}}(y^2)-2g^{\beta-\sqrt{2}}(y^2)g^{-\sqrt{2}}(x^2)\right],\end{aligned} \tag{22}$$

where the first inequality is due to Eq. (17), and without loss of generality, we can assume $x^2\geq y^2$, then the second inequality is obtained from the monotonicity of $g(x)$ and Eq. (5).

Thus, combining Eqs. (21) and (22), we obtain

$$\begin{aligned}&E_{A|B_1B_2\cdots B_{N-1}}^{\beta}\\&\geq g^{\beta}\left(C_{AB_1}^2+C_{AB_2}^2+\cdots+C_{AB_{N-1}}^2\right)\end{aligned}$$

$$
\begin{aligned}
&\geq g^{\beta}\left(C_{AB_1}^2\right)+hg^{\beta}\left(C_{AB_2}^2+\cdots+C_{AB_{N-1}}^2\right)+\frac{t}{2}g^{\sqrt{2}}\left(C_{AB_2}^2+\cdots+C_{AB_{N-1}}^2\right)\\
&\quad\left[g^{\beta-\sqrt{2}}\left(C_{AB_1}^2\right)-g^{\beta-\sqrt{2}}\left(C_{AB_2}^2+\cdots+C_{AB_{N-1}}^2\right)\right]\\
&\quad+\frac{(t-1)^2}{4}g^{2\sqrt{2}}\left(C_{AB_2}^2+\cdots+C_{AB_{N-1}}^2\right)\\
&\quad\left[g^{\beta-2\sqrt{2}}\left(C_{AB_1}^2\right)+g^{\beta-2\sqrt{2}}\left(C_{AB_2}^2+\cdots+C_{AB_{N-1}}^2\right)\right.\\
&\quad\left.-2g^{\beta-\sqrt{2}}\left(C_{AB_2}^2+\cdots+C_{AB_{N-1}}^2\right)g^{-\sqrt{2}}(C_{AB_1}^2)\right]\\
&\geq g^{\beta}\left(C_{AB_1}^2\right)+hg^{\beta}\left(C_{AB_2}^2+\cdots+C_{AB_{N-1}}^2\right)\\
&\quad+\frac{t}{2}\left[g^{\sqrt{2}}\left(C_{AB_2}^2\right)+\cdots+g^{\sqrt{2}}\left(C_{AB_{N-1}}^2\right)\right]\\
&\quad\cdot\left[g^{\beta-\sqrt{2}}\left(C_{AB_1}^2\right)-g^{\beta-\sqrt{2}}\left(C_{A|B_2\cdots B_{N-1}}^2\right)\right]\\
&\quad+\frac{(t-1)^2}{4}\left[g^{2\sqrt{2}}(C_{AB_2}^2)+\cdots+g^{2\sqrt{2}}\left(C_{AB_{N-1}}^2\right)\right]\\
&\quad\cdot\left[g^{\beta-2\sqrt{2}}(C_{AB_1}^2)+\cdots+g^{\beta-2\sqrt{2}}\left(C_{AB_{N-1}}^2\right)\right.\\
&\quad\left.-2g^{\beta-\sqrt{2}}(C_{A|B_2\cdots B_{N-1}}^2)g^{-\sqrt{2}}\left(C_{AB_1}^2\right)\right]\\
&\geq g^{\beta}\left(C_{AB_1}^2\right)+hg^{\beta}\left(C_{AB_2}^2\right)+\cdots+h^{N-2}g^{\beta}\left(C_{AB_{N-1}}^2\right)\\
&\quad+h^{N-3}\cdot\frac{t}{2}\cdot g^{\sqrt{2}}\left(C_{AB_{N-1}}^2\right)\left[g^{\beta-\sqrt{2}}\left(C_{AB_{N-2}}^2\right)-g^{\beta-\sqrt{2}}\left(C_{AB_{N-1}}^2\right)\right]+\cdots\\
&\quad+\frac{t}{2}\left[g^{\sqrt{2}}\left(C_{AB_2}^2\right)+\cdots+g^{\sqrt{2}}\left(C_{AB_{N-1}}^2\right)\right]\\
&\quad\cdot\left[g^{\beta-\sqrt{2}}\left(C_{AB_1}^2\right)-g^{\beta-\sqrt{2}}\left(C_{A|B_2\cdots B_{N-1}}^2\right)\right]\\
&\quad+h^{N-3}\cdot\frac{(t-1)^2}{4}\cdot g^{2\sqrt{2}}\left(C_{AB_{N-1}}^2\right)\left[g^{\beta-2\sqrt{2}}\left(C_{AB_{N-2}}^2\right)+g^{\beta-2\sqrt{2}}\left(C_{AB_{N-1}}^2\right)\right.\\
&\quad\left.-2g^{\beta-\sqrt{2}}\left(C_{AB_{N-1}}^2\right)g^{-\sqrt{2}}\left(C_{AB_{N-2}}^2\right)\right]+\cdots\\
&\quad+\frac{(t-1)^2}{4}\left[g^{2\sqrt{2}}\left(C_{AB_2}^2\right)+\cdots+g^{2\sqrt{2}}\left(C_{AB_{N-1}}^2\right)\right]\\
&\quad\cdot\left[g^{\beta-2\sqrt{2}}\left(C_{AB_1}^2\right)+\cdots+g^{\beta-2\sqrt{2}}\left(C_{AB_{N-1}}^2\right)\right.\\
&\quad\left.-2g^{\beta-\sqrt{2}}\left(C_{A|B_2\cdots B_{N-1}}^2\right)g^{-\sqrt{2}}\left(C_{AB_1}^2\right)\right],
\end{aligned}
\tag{23}
$$

where we have utilized Eq. (3) and the monotonicity of $g(x)$ to obtain the first inequality, the third and the forth inequalities are due to Eq. (17) and the monotonicity of the function $g(x)$.

According to Eq. (21) and the fact that $g(\mathcal{C}^2(\rho)) = E(\rho)$ for arbitrary two-qubit mixed state ρ, we obtain Eq. (20). □

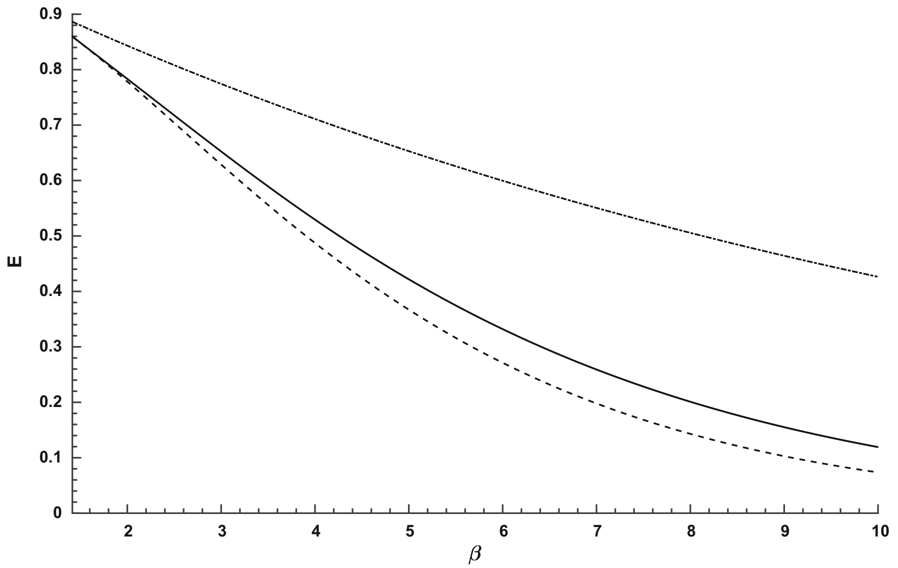

Fig. 2 Dash dotted line, $E^{\beta}_{A|BC}$ as a function of β ($\sqrt{2} \le \beta \le 10$); solid line, the lower bound of $E^{\beta}_{A|BC}$ as a function of β ($\sqrt{2} \le \beta \le 10$) in Eq. (20); dash line, the lower bound of $E^{\beta}_{A|BC}$ as a function of β ($\sqrt{2} \le \beta \le 10$) in [24]

Example 2 Let us consider the state in (14) given in Example 1. Set $\lambda_0 = \frac{\sqrt{6}}{3}, \lambda_1 = 0, \lambda_2 = \frac{\sqrt{2}}{3}, \lambda_3 = \frac{1}{3}, \lambda_4 = 0$, we have $E_{A|BC} = 0.91829$, $E_{AB} = 0.68193$, $E_{AC} = 0.40416$. Then, $E^{\beta}_{A|BC} = (0.91829)^{\beta} \ge E^{\beta}_{AB} + hE^{\beta}_{AC} + \frac{\beta}{2\sqrt{2}} E^{\sqrt{2}}_{AC}(E^{\beta-\sqrt{2}}_{AB} - E^{\beta-\sqrt{2}}_{AC}) + \frac{(\beta-\sqrt{2})^2}{8} E^{2\sqrt{2}}_{AC}(E^{\beta-2\sqrt{2}}_{AB} + E^{\beta-2\sqrt{2}}_{AC} - 2E^{\beta-\sqrt{2}}_{AC} E^{-\sqrt{2}}_{AB}) = (0.68193)^{\beta} + h(0.40416)^{\beta} + \frac{\beta}{2\sqrt{2}}(0.40416)^{\sqrt{2}}\Big[(0.68193)^{\beta-\sqrt{2}} - (0.40416)^{\beta-\sqrt{2}}\Big] + \frac{(\beta-\sqrt{2})^2}{8}(0.40416)^{2\sqrt{2}}\Big[(0.68193)^{\beta-2\sqrt{2}} + (0.40416)^{\beta-2\sqrt{2}} - 2(0.40416)^{\beta-\sqrt{2}} (0.68193)^{-\sqrt{2}}\Big]$. While the result in [24] is $E^{\beta}_{AB} + hE^{\beta}_{AC} + \frac{\beta}{2\sqrt{2}} E^{\sqrt{2}}_{AC}(E^{\beta-\sqrt{2}}_{AB} - E^{\beta-\sqrt{2}}_{AC}) = (0.68193)^{\beta} + h(0.40416)^{\beta} + \frac{\beta}{2\sqrt{2}}(0.40416)^{\sqrt{2}}\Big[(0.68193)^{\beta-\sqrt{2}} - (0.40416)^{\beta-\sqrt{2}}\Big]$. We can see that our results are better than the ones in [24], see Fig. 2.

4 Tighter monogamy relations using negativity

The negativity is a well-known quantifier of bipartite entanglement. Given a bipartite state ρ_{AB} in Hilbert space $H_A \otimes H_B$, the negativity is defined as [31]:

$$\mathcal{N}(\rho_{AB}) = \frac{\|\rho^{T_A}_{AB}\| - 1}{2}, \tag{24}$$

where $\rho_{AB}^{T_A}$ is the partial transposed matrix of ρ_{AB} with respect to the subsystem A and $\|X\|$ denotes the trace norm of X, i.e., $\|X\| = \mathrm{Tr}\sqrt{XX^\dagger}$. For convenience, we use the definition of negativity as: $\|\rho_{AB}^{T_A}\| - 1$ [10].

If a bipartite pure state $|\phi\rangle_{AB}$ with the Schmidt decomposition, $|\phi\rangle_{AB} = \sum_i \sqrt{\lambda_i}|ii\rangle$, $\lambda_i \geq 0$, $\sum_i \lambda_i = 1$, then [10]

$$\mathcal{N}(|\phi\rangle_{AB}) = 2\sum_{i<j}\sqrt{\lambda_i\lambda_j}. \tag{25}$$

From the definition of concurrence (1), we have

$$C(|\phi\rangle_{AB}) = 2\sqrt{\sum_{i<j}\lambda_i\lambda_j}. \tag{26}$$

As a consequence, for any bipartite pure state $|\phi\rangle_{AB}$ with Schmidt rank 2, one has $\mathcal{N}(|\phi\rangle_{AB}) = C(|\phi\rangle_{AB})$.

For a mixed state ρ_{AB}, the convex-roof extended negativity (CREN) is given by

$$\mathcal{N}_c(\rho_{AB}) = \min_{\{p_i,|\phi_i\rangle\}}\sum_i p_i\mathcal{N}(|\phi_i\rangle), \tag{27}$$

where the minimum is taken over all possible pure state decomposition of ρ_{AB}. CREN gives a perfect discrimination between PPT bound entangled states and separable states in any bipartite quantum system [32]. It follows that for any $2\otimes d$ ($d \geq 2$) mixed state ρ_{AB}, we have

$$\mathcal{N}_c(\rho_{AB}) = \min_{\{p_i,|\phi_i\rangle\}}\sum_i p_i\mathcal{N}(|\phi_i\rangle) = \min_{\{p_i,|\phi_i\rangle\}}\sum_i p_i C(|\phi_i\rangle) = C(\rho_{AB}). \tag{28}$$

According to the relation between CREN and concurrence, we have the following results for the lower bound of $\mathcal{N}_{cA|B_1\cdots B_{N-1}}^{\beta}$.

Theorem 4 *For any N-qubit mixed state $\rho_{AB_1\cdots B_{N-1}}$, if $\mathcal{N}_{cAB_i} \geq \mathcal{N}_{cAB_{i+1}\cdots B_{N-1}}$ for $i = 1, 2, \ldots, m$, and $\mathcal{N}_{cAB_j} \leq \mathcal{N}_{cA|B_{j+1}\cdots B_{N-1}}$ for $j = m+1, \ldots, N-2$, $\forall 1 \leq m \leq N-3$, then we have*

$$\begin{aligned}\mathcal{N}_{cA|B_1\cdots B_{N-1}}^{\beta} \geq& \sum_{i=1}^{m} h^{i-1}(\mathcal{N}_{cAB_i}^{\beta} + R_{AB_i})\\ &+h^m\sum_{j=m+1}^{N-2}(h\mathcal{N}_{cAB_j}^{\beta} + R_{AB_j}^1) + h^m\mathcal{N}_{cAB_{N-1}}^{\beta},\end{aligned} \tag{29}$$

for all $N \geq 4$, $\beta \geq 2$, where $h = 2^{\frac{\beta}{2}} - 1$, $R_{AB_i} = \frac{\beta}{4}\mathcal{N}_{cA|B_{i+1}\cdots B_{N-1}}^2(\mathcal{N}_{cAB_i}^{\beta-2} - \mathcal{N}_{cA|B_{i+1}\cdots B_{N-1}}^{\beta-2}) + \frac{(\beta-2)^2}{16}\mathcal{N}_{cA|B_{i+1}\cdots B_{N-1}}^4(\mathcal{N}_{cAB_i}^{\beta-4} + \mathcal{N}_{cA|B_{i+1}\cdots B_{N-1}}^{\beta-4} - 2\mathcal{N}_{cA|B_{i+1}\cdots B_{N-1}}^{\beta-2}$

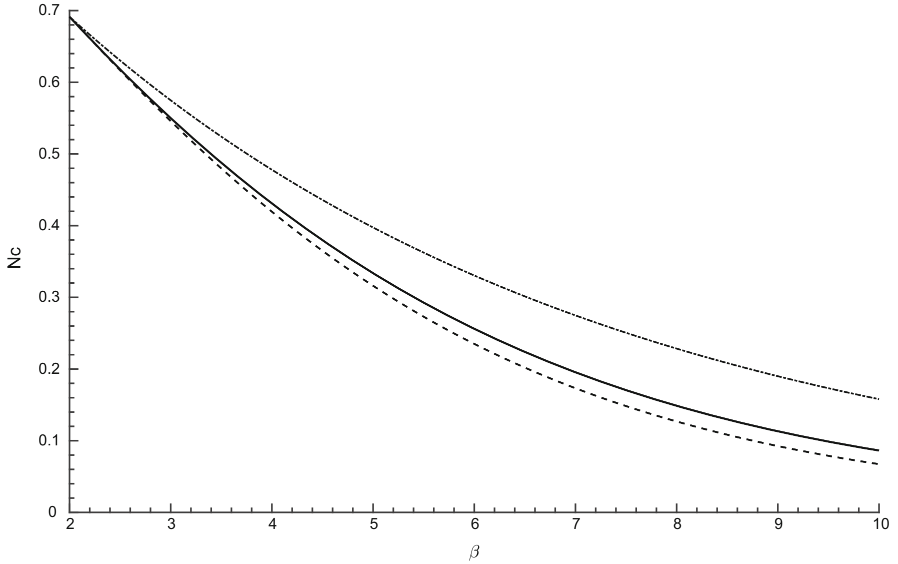

Fig. 3 Dash dotted line, $\mathcal{N}^{\beta}_{cA|BC}$ as a function of β ($2 \le \beta \le 10$); solid line, the lower bound of $\mathcal{N}^{\beta}_{cA|BC}$ as a function of β ($2 \le \beta \le 10$) in Eq. (30); dash line, the lower bound of $\mathcal{N}^{\beta}_{cA|BC}$ as a function of β ($2 \le \beta \le 10$) in [24]

$\mathcal{N}^{-2}_{cAB_i})$, $R^1_{AB_j} = \frac{\beta}{4}\mathcal{N}^2_{cAB_j}(\mathcal{N}^{\beta-2}_{cA|B_{j+1}\cdots B_{N-1}} - \mathcal{N}^{\beta-2}_{cAB_j}) + \frac{(\beta-2)^2}{16}\mathcal{N}^4_{cAB_j}(\mathcal{N}^{\beta-4}_{cA|B_{j+1}\cdots B_{N-1}} + \mathcal{N}^{\beta-4}_{cAB_j} - 2\mathcal{N}^{\beta-2}_{cAB_j}\mathcal{N}^{-2}_{cA|B_{j+1}\cdots B_{N-1}})$.

Theorem 5 *For any N-qubit mixed state $\rho_{AB_1\cdots B_{N-1}}$, if $\mathcal{N}_{cAB_i} \ge \mathcal{N}_{cA|B_{i+1}\cdots B_{N-1}}$ for $i = 1, 2, \ldots, N-2$, then we can obtain*

$$\mathcal{N}^{\beta}_{cA|B_1\cdots B_{N-1}} \ge \sum_{i=1}^{N-2} h^{i-1}\left(\mathcal{N}^{\beta}_{cAB_i} + R_{AB_i}\right) + h^{N-2}\mathcal{N}^{\beta}_{cAB_{N-1}}, \tag{30}$$

for all $N \ge 3$, $\beta \ge 2$, where $h = 2^{\frac{\beta}{2}} - 1$, $R_{AB_i} = \frac{\beta}{4}\mathcal{N}^2_{cA|B_{i+1}\cdots B_{N-1}}(\mathcal{N}^{\beta-2}_{cAB_i} - \mathcal{N}^{\beta-2}_{cA|B_{i+1}\cdots B_{N-1}}) + \frac{(\beta-2)^2}{16}\mathcal{N}^4_{cA|B_{i+1}\cdots B_{N-1}}(N^{\beta-4}_{cAB_i} + \mathcal{N}^{\beta-4}_{cA|B_{i+1}\cdots B_{N-1}} - 2\mathcal{N}^{\beta-2}_{cA|B_{i+1}\cdots B_{N-1}}\mathcal{N}^{-2}_{cAB_i})$.

Example 3 Let us consider the state in (14) given in Example 1. We have $\mathcal{N}_{cA|BC} = 2\lambda_0\sqrt{\lambda_2^2 + \lambda_3^2 + \lambda_4^2}$, $\mathcal{N}_{cAB} = 2\lambda_0\lambda_2$ and $\mathcal{N}_{cAC} = 2\lambda_0\lambda_3$. Set $\lambda_0 = \frac{\sqrt{2}}{3}$, $\lambda_1 = 0$, $\lambda_2 = \frac{\sqrt{5}}{3}$, $\lambda_3 = \frac{\sqrt{2}}{3}$, $\lambda_4 = 0$. We have $\mathcal{N}^{\beta}_{cA|BC} \ge \mathcal{N}^{\beta}_{cAB} + h\mathcal{N}^{\beta}_{cAC} + \frac{\beta}{4}\mathcal{N}^2_{cAC}(\mathcal{N}^{\beta-2}_{cAB} - \mathcal{N}^{\beta-2}_{cAC}) + \frac{(\beta-2)^2}{16}\mathcal{N}^4_{cAC}(\mathcal{N}^{\beta-4}_{cAB} + \mathcal{N}^{\beta-4}_{cAC} - 2\mathcal{N}^{\beta-2}_{cAC}\mathcal{N}^{-2}_{cAB}) = (\frac{2\sqrt{10}}{9})^{\beta} + h(\frac{4}{9})^{\beta} + \frac{\beta}{4}(\frac{4}{9})^2\left[(\frac{2\sqrt{10}}{9})^{\beta-2} - (\frac{4}{9})^{\beta-2}\right] + \frac{(\beta-2)^2}{16}(\frac{4}{9})^4\left[(\frac{2\sqrt{10}}{9})^{\beta-4} + (\frac{4}{9})^{\beta-4} - 2(\frac{4}{9})^{\beta-2}(\frac{2\sqrt{10}}{9})^{-2}\right)$. While the result in[24] is $\mathcal{N}^{\beta}_{cAB} + h\mathcal{N}^{\beta}_{cAC} + \frac{\beta}{4}\mathcal{N}^2_{cAC}(\mathcal{N}^{\beta-2}_{cAB} - \mathcal{N}^{\beta-2}_{cAC}) = (\frac{2\sqrt{10}}{9})^{\beta} +$

$h(\frac{4}{9})^\beta + \frac{\beta}{4}(\frac{4}{9})^2\Big[(\frac{2\sqrt{10}}{9})^{\beta-2} - (\frac{4}{9})^{\beta-2}\Big]$. We can see that our result is better than the one in [24] for $\beta \geq 2$, see Fig. 3.

5 Conclusion

Entanglement monogamy relations are fundamental properties of multipartite entangled states. In this paper, we have provided the multipartite entanglement based on the monogamy relations for βth power of concurrence $C^{\beta}_{A|B_1\cdots B_{N-1}}$ $(\beta \geq 2)$, entanglement of formation $E^{\beta}_{A|B_1\cdots B_{N-1}}$ $(\beta \geq \sqrt{2})$ and convex-roof extended negativity $\mathcal{N}^{\beta}_{cA|B_1\cdots B_{N-1}}$ $(\beta \geq 2)$. Our monogamy relations have larger lower bounds and are tighter than the existing results [24]. These tighter monogamy inequalities can also provide a finer description of the entanglement distribution. In multi-qubit system, our research results provide a rich reference for future research on multi-party quantum entanglement. Our method can also be applied to the study of other properties of monogamy related to quantum correlations.

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Data Availability All data generated or analyzed during this study are included in this submitted article.

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