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Strong subadditivity lower bound and quantum channels

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Abstract
We derive the strong subadditivity of the von Neumann entropy with a strict lower bound, dependent on the distribution of quantum correlation in the system. We investigate the structure of states saturating the bounded subadditivity and examine its consequences for the quantum data processing inequality. The quantum data processing achieves a lower bound associated with the locally inaccessible information.

Keywords Quantum entanglement · Quantum correlation · Strong subadditivity of the von Neumann entropy · Quantum data processing

1 Introduction

In information theory, the central constraints on how information can be distributed among parties are given by the subadditivity inequalities [1,2]. Given two random variables $X : \{x\}$ and $Y : \{y\}$, assuming the values x and y with probabilities p_x and p_y, respectively, the weak subadditivity is given by

$$H(X, Y) \leq H(X) + H(Y), \tag{1}$$

in terms of the Shannon entropies $H(X) = -\sum_{X:\{x\}} p_x \ln p_x$, $H(Y) = -\sum_{Y:\{y\}} p_y \ln p_y$, and the joint entropy $H(X, Y) = -\sum_{X:\{x\},Y:\{y\}} p_{x,y} \ln p_{x,y}$. In essence, this inequality states the positivity of the mutual information, $I(X : Y) \equiv H(X) + H(Y) - H(X, Y) \geq 0$, which bounds the correlations between X and Y. On the other hand, the strong subadditivity

$$H(X, Y, Z) + H(Y) \leq H(X, Y) + H(Y, Z), \tag{2}$$

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imposes the positivity of the conditional mutual information defined by $I(X:Z|Y) \equiv H(X,Y)+H(Y,Z)-H(X,Y,Z)-H(Y) \geq 0$. As simple as they look, these two positivity bounds govern what one can or cannot do in communication, since they are related to the communication channel capacity and to all other relevant inequalities in information theory [1].

Quantum information theory is concerned with information processing and tasks performing in the quantum regime, and the main quantity for that is the von Neumann entropy

$$S(\rho) = -\mathrm{Tr}\left(\rho \log \rho\right), \tag{3}$$

in terms of the system state ρ. Similarly to classical information theory, the subadditivity for the von Neumann entropy is enormously relevant. The weak subadditivity states as

$$S(A,B) \leq S(A) + S(B), \tag{4}$$

being $S(A) \equiv S(\rho_A)$, and $S(A,B) \equiv S(\rho_{AB})$. We may also define the mutual information as

$$I(A:B) = S(A) + S(B) - S(A,B), \tag{5}$$

being always positive. Moreover, the strong subadditivity (SSA) proved by Lieb and Ruskai [3] gives that

$$S(A,B,C) + S(B) \leq S(A,B) + S(B,C), \tag{6}$$

being $S(ABC) \equiv S(\rho_{ABC})$. This inequality holds for a tripartite system with density matrix ρ_{ABC} living in $\mathcal{H}_{ABC} = \mathcal{H}_A \otimes \mathcal{H}_B \otimes \mathcal{H}_C$, and each of the reduced density matrices was taken by a partial trace of the density matrix of the whole system, such that $\mathrm{Tr}_{BC}(\rho_{ABC}) = \rho_A$. The SSA is fairly well studied and very important, since it bounds the most relevant results inside quantum information theory, such as the limiting bounds in the information transmitted in quantum channels [4]. Similarly to the classical instance, the SSA of the von Neumann entropy reduces to the positivity of the conditional mutual information in the form

$$I(A:C|B) = S(A,B) + S(B,C) - S(B) - S(A,B,C) \geq 0. \tag{7}$$

Its relevance is so extensive that it applies, e.g., to the search for lower bounds for the free energy in many-body physics [5]. In fact, all other inequalities, but one, in quantum information theory are equivalent to the SSA [6].

There are some implicit problems in the distinction of genuinely quantum from classical correlation when the mutual information for quantum systems is given by (5). This is because the mutual information can be more precisely defined as

$$I(A:B) = S(A) - S(A|B), \tag{8}$$

where $S(A|B)$ is the conditional information on ρ_A given the knowledge of state ρ_B. The form $S(A|B) = S(A, B) - S(B)$ is an extension of the classical information theory definition, and contrarily to the latter, it can be negative, indicating the presence of entanglement. Strictly speaking, in quantum information, to obtain knowledge about the quantum state, a measurement must be performed and therefore a more appropriate form of conditional entropy must be employed [7]. It turns out that when such an approach is considered some intriguing relations between entanglement and quantum correlation in general emerge [6], leading to a weak monotonicity relation of the von Neumann entropy [8] with additional restrictions on the balance of quantum correlations in the system as measured by the entanglement of formation (Eof)[9] and the quantum discord (QD) [7]. Since the standard weak monotonicity is equivalent to the SSA, it would be interesting to extend the discussion to understand the existence of possible new bounds.

In this work, we further investigate this problem by deriving a strong subadditivity relation from the bounded weak monotonicity. We show that it intrinsically involves new bounds which allows distinction of genuine quantum correlations. In [10] it was shown that the structure of states saturating the SSA (6) is given by

$$\rho_{ABC} = \bigoplus_j q_j \rho_{Ab_j^L} \otimes \rho_{b_j^R C}, \tag{9}$$

where $\rho_{Ab_j^L} \in \mathcal{H}_A \otimes \mathcal{H}_{b_j^L}$ and $\rho_{b_j^R C} \in \mathcal{H}_{b_j^R} \otimes \mathcal{H}_C$, with probability distribution $\{q_j\}$, such that the Hilbert space be decomposable as $\mathcal{H}_B = \bigoplus_j \mathcal{H}_{b_j^L} \otimes \mathcal{H}_{b_j^R}$. This structure was enormously relevant for the derivation of a hierarchy of independent inequalities for the von Neumann entropy [11]. Here we extend this analysis to understand the structure of states saturating the bounded SSA. To show the extension of this bound, we apply it to the quantum data processing inequality. The paper is organized as follows. In Sect. 2, we develop the new bounds on the SSA and discuss its implications in several instances. In Sect. 3, we investigate the structure of states saturating the bounded-SSA. Finally, in Sect. 4, we apply the bounded-SSA to understand the imposed restrictions on the quantum data processing inequality [4]. In Sect. 5 a conclusion encloses the paper.

2 Bounded strong subadditivity

The mutual information (MI) in terms of the von Neumann entropy measures the amount of information shared by two quantum systems A and B. In other words, it quantifies the total amount of correlations (quantum and classical) of a bipartite quantum state ρ_{AB}. It is given by Eq. (8). Assuming the classical form for the conditional entropy to the quantum case as being $S(A|B) = S(A, B) - S(B)$, we end up with Eq. (5). In contrast, by taking into account the fact that in quantum systems for the description of the conditional entropy, $S(A|B)$, the prior knowledge about ρ_B is achieved by a type of measurement one obtains

$$J^{\leftarrow}_{A|B} = \max_{\{\Pi_k\}} \left[S(A) - \sum_k p_k S(A|k) \right], \tag{10}$$

where $S(A|k) \equiv S(\rho_{A|k})$ is the conditional entropy after a measurement in B, where $\rho_{A|k} = \mathrm{Tr}_B(\Pi_k \rho_{AB} \Pi_k)/\mathrm{Tr}_{AB}(\Pi_k \rho_{AB} \Pi_k)$ is the reduced post-selected state of A after obtaining the outcome k in B. $\{\Pi_k\}$ is a complete set of positive operator valued measurement (POVM) resulting in the outcome k with probability $p_k = \mathrm{Tr}_{AB}(\Pi_k \rho_{AB} \Pi_k)$. In this case, since a measurement might give different results depending on the basis choice, a maximization is required. Thus, $J_{A|B}$ measures the amount of mutual information accessible by local measurement in B only. Due to that distinction in its definition, one can quantify the amount of information not accessible by local measurements in B by

$$\delta^{\leftarrow}_{AB} = I(A:B) - J^{\leftarrow}_{A|B}, \tag{11}$$

the so-called quantum discord [7].

For an arbitrarily mixed tripartite system state, ρ_{ABC}, there exists an important relation known as the Koashi-Winter inequality [6,8] given by

$$E_{AB} \leq \delta^{\leftarrow}_{AC} + S_{A|C}, \tag{12}$$

where E_{AB} quantifies the entanglement of formation (Eof) between A and B

$$E_{AB} = \min_{\{p_i, |\psi_i\rangle\}} \sum_i p_i S(\rho^i_A), \tag{13}$$

where the minimization is over all ensembles of pure states $\{p_i, |\psi_i\rangle_{AB}\}$, and $\delta^{\leftarrow}_{AC}$ is the quantum discord (QD) between A and C (given measurements in C). In fact it is possible to show [8] that in general

$$S(B) + S(C) + \Delta \leq S(A, B) + S(A, C), \tag{14}$$

where Δ is the balance of correlations in a tripartite system, and is given by $\Delta = E_{AB} + E_{AC} - \delta^{\leftarrow}_{AB} - \delta^{\leftarrow}_{AC}$. Equation (14) imposes an additional term in the weak monotonicity of Entropy, which surely will affect the SSA. Pure quantum states, ρ_{ABC}, necessarily satisfy $S(B) = S(A, C)$ and $S(C) = S(A, B)$, and saturate (14) in a way that $\Delta = 0$, or

$$E_{AB} + E_{AC} = \delta^{\leftarrow}_{AB} + \delta^{\leftarrow}_{AC}. \tag{15}$$

The balance of quantum correlations above can be viewed as a conservation relation, indicating that the entanglement of formation of a bipartite system is going to be increased or decreased by the same amount that the quantum discord of the same bipartite system in relation to a part of the pure tripartite global state.

We start by adding an ancilla R, which allow us to write Eq. (14) as

$$S(B) + S(R) + \widetilde{\Delta} \leq S(A, B) + S(A, R) \tag{16}$$

and obtain a global pure state, ρ_{ABCR}, where now

$$\widetilde{\Delta} = E_{AB} + E_{AR} - \delta^{\leftarrow}_{AB} - \delta^{\leftarrow}_{AR}. \tag{17}$$

Therefore, using that $S(R) = S(A, B, C)$ and $S(A, R) = S(B, C)$ we get

$$S(B) + S(A, B, C) + \widetilde{\Delta} \leq S(A, B) + S(B, C). \tag{18}$$

To write $\widetilde{\Delta}$ we divide the system into a pure tripartite state $\rho_{A(BC)R}$ and use the conservation relation (15) for that partition

$$E_{A(BC)} + E_{AR} = \delta^{\leftarrow}_{A(BC)} + \delta^{\leftarrow}_{AR}, \tag{19}$$

then,

$$\begin{aligned}\widetilde{\Delta} &= E_{AB} - E_{A(BC)} + \delta^{\leftarrow}_{A(BC)} + \delta^{\leftarrow}_{AR} - \delta^{\leftarrow}_{AB} - \delta^{\leftarrow}_{AR} \\ &= E_{AB} - E_{A(BC)} - \left(\delta^{\leftarrow}_{AB} - \delta^{\leftarrow}_{A(BC)}\right).\end{aligned} \tag{20}$$

Therefore, we end up with

$$S(A, B, C) + S(B) + \widetilde{\Delta} \leq S(A, B) + S(B, C). \tag{21}$$

The inequality in Eq. (21) is similar to the SSA, but for the additional term $\widetilde{\Delta}$. Since $\widetilde{\Delta}$ can take both positive and negative values it can be a stronger or weaker bound to the SSA, and therefore we call it as the bounded SSA, or b-SSA for short. Since the positivity of the conditional mutual information $I(A : C|B)$ (7) was independently proved by Lieb and Ruskai [3], in fact in (21) effectively one has to take it as

$$I(A : C|B) \geq \max\{0, \widetilde{\Delta}\}. \tag{22}$$

It is important to note how $\widetilde{\Delta}$ now involves measurement over the partition BC. In transitioning from (14) to (21), there is a change in sign in the balance of quantum correlations. So that the difference of the positivity of the Δ to $\widetilde{\Delta}$ does not depend only on the difference between the QD and the Eof of the same bipartitions, becoming more complex to evaluate in general. Remark that the entanglement of formation is a measure that will be null if the system state is separable, while the quantum discord, $\delta^{\leftarrow}_{AB}$ is null only for states of the form [12] $\rho = \sum_j p_j \rho^A_j \otimes |\psi_j\rangle\langle\psi_j|^B$. What we can precisely say about $\widetilde{\Delta}$ is that it carries information on the residual entanglement and correlation of the partition A with BC, and with B. Therefore whenever $E_{AB} - E_{A(BC)} \geq \delta^{\leftarrow}_{AB} - \delta^{\leftarrow}_{A(BC)}$, then $\widetilde{\Delta} \geq 0$.

3 Structure of states saturating the b-SSA

The structure of states that saturate the b-SSA (21) can be obtained by using a theorem due to Petz [13], regarding situations when the quantum relative entropy remains unchanged after the action of a certain map. That is possible because the conditional mutual information, as well as the other measures of quantum correlation in Eq. (21) can all be rephrased in terms of the quantum relative entropy. To begin this analysis, let us remind that the quantum discord is defined as the difference between the total correlations that a system shares and the classical correlations in the system, as in Eq. (11). The mutual information is given by the relative entropy as

$$I(A:B) = S(\rho_{AB}||\rho_A \otimes \rho_B), \tag{23}$$

while the classical correlation is given by Eq. (18). Here we will make use of non-selective von Neumann measurements, following the previous section, if the state is ρ_{AB}, the measurement will be on the B part of the state and if the state is ρ_{ABC} it will be on the BC part. Hence, we obtain

$$\begin{aligned}\Phi_B(\rho_{AB}) &= \sum_i \left(\mathbb{1}_A \otimes \Pi_B^i\right) \rho_{AB} \left(\mathbb{1}_A \otimes \Pi_B^i\right) \\ &= \sum_i p_i \rho_A^i \otimes |\psi_i\rangle_B \langle\psi_i|,\end{aligned} \tag{24}$$

and similarly $\Phi_B(\rho_B) = \sum_i p_i |\psi_i\rangle_B \langle\psi_i|$. Enabling us to rewrite the classical correlation as

$$J_{AB}^{\leftarrow} = S(\Phi(\rho_{AB})||\rho_A \otimes \Phi(\rho_B)). \tag{25}$$

Therefore, from (11) we have that

$$\delta_{AB}^{\leftarrow} = \min_{\{\Pi_B^i\}} \left[S(\rho_{AB}||\rho_A \otimes \rho_B) - S(\Phi_B(\rho_{AB})||\rho_A \otimes \Phi_B(\rho_B))\right]. \tag{26}$$

In a similar fashion, for a tripartite system given by ρ_{ABC},

$$\delta_{A(BC)}^{\leftarrow} = \min_{\{\Pi_{BC}^i\}} \left[S(\rho_{ABC}||\rho_A \otimes \rho_{BC}) - S(\Phi_{BC}(\rho_{ABC})||\rho_A \otimes \Phi_{BC}(\rho_{BC}))\right], \tag{27}$$

with $\Phi_{BC}(\rho_{ABC}) = \sum_i p_i \rho_A^i \otimes |\psi_i\rangle_{BC} \langle\psi_i|$ and $\Phi_{BC}(\rho_{BC}) = \sum_i p_i |\psi_i\rangle_{BC} \langle\psi_i|$.

The minimization over the projectors is maintained since it is necessary to get the optimized value for the QD. Further on, the minimization is not going to be carried for the lack of necessity, i.e., the main calculations work for any minimization performed, so that the result is valid, and the minimization can be taken afterward.

Since $S(A|B) = -S(\rho_{AB}||\mathbf{1}_A \otimes \rho_B)$, we obtain

$$E_{AB} = \min_{\{\Pi^i_B\}} S(\Phi_B(\rho_{AB})||\mathbf{1}_A \otimes \Phi_B(\rho_B)), \tag{28}$$

and through a similar derivation,

$$E_{A(BC)} = \min_{\{\Pi^i_{BC}\}} S(\Phi_{BC}(\rho_{ABC})||\mathbf{1}_A \otimes \Phi_{BC}(\rho_{BC})). \tag{29}$$

By substituting Eqs. (26)–(29) in the b-SSA (21), we get

$$\begin{aligned} I(A:C|B) \geq & \min_{\{\Pi^i_B\}} S(\Phi_B(\rho_{AB})||\mathbf{1}_A \otimes \Phi_B(\rho_B)) \\ & - \min_{\{\Pi^i_{BC}\}} S(\Phi_{BC}(\rho_{ABC})||\mathbf{1}_A \otimes \Phi_{BC}(\rho_{BC})) \\ & + \min_{\{\Pi^i_{BC}\}} [S(\rho_{ABC}||\rho_A \otimes \rho_{BC}) - S(\Phi_{BC}(\rho_{ABC})||\rho_A \otimes \Phi_{BC}(\rho_{BC}))] \\ & - \min_{\{\Pi^i_B\}} [S(\rho_{AB}||\rho_A \otimes \rho_B) - S(\Phi_B(\rho_{AB})||\rho_A \otimes \Phi_B(\rho_B))] . \end{aligned} \tag{30}$$

We can check that the mutual information, $I(A;C|B)$, is also present in the RHS, so it is easy to see that the condition for saturation of the b-SSA rests in the equality condition of the remaining terms of Eq. (30), i.e., necessarily

$$S(\Phi_{BC}(\rho_{ABC})||\rho_A \otimes \Phi_{BC}(\rho_{BC})) = S(\Phi_B(\rho_{AB})||\rho_A \otimes \Phi_B(\rho_B)), \tag{31}$$

and

$$S(\Phi_{BC}(\rho_{ABC})||\mathbf{1}_A \otimes \Phi_{BC}(\rho_{BC})) = S(\Phi_B(\rho_{AB})||\mathbf{1}_A \otimes \Phi_B(\rho_B)). \tag{32}$$

As it turns out, these necessary equalities reduce the action of the maps to be relevant only over a single party B. It is good to observe that in the relations written above, we are not taking into account the minimizations that are taken on the POVM's. This is due to the fact that the same optimization is enacted in both terms for each equation, so for a given optimum value the best basis is found, and we can proceed with our analysis getting to the equations above. We know that the equality condition for the monotonicity of the relative entropy is guaranteed if there exists a quantum operation $\hat{T}$ that maps $T\rho$ to ρ. So assuming that there exists a quantum operation that takes B to BC in the form of a recovery map $R_{B\to BC}$ [13], we actuate over the states $\Phi_{BC}(\rho_{ABC})$ and $\Phi_B(\rho_{AB})$ so that

$$R_{B\to BC}(\Phi_B(\rho_{AB})) = R_{B\to BC}\left(\sum_i p_i \rho^i_A \otimes |\psi_i\rangle_B \langle\psi_i|\right), \tag{33}$$

and the structure of states that saturate the b-SSA will take the form

$$\Phi_{BC}(\rho_{ABC}) = \bigoplus_j q_j \sum_i p_{j|i} \rho_A^i \otimes |\tilde{\psi}_j\rangle_{b_i^L} \langle\tilde{\psi}_j| \otimes \omega_{b_i^R C}, \tag{34}$$

where $\omega_{b_i^R C} \in H_{b_i^R} \otimes H_C$, $|\tilde{\psi}_j\rangle_{b_i^L} \in H_{b_i^L}$ and $H_B = \bigoplus_i H_{b_i^L} \otimes H_{b_i^R}$. It is clear that the possibility of recovering the state belonging to H_{ABC} from the state belonging to H_{AB}, makes a sufficient condition for us to call this kind of states as short Quantum Markov chains, similarly to the states that saturate the SSA [10]. Equations (31) and (32) demand that $J^{\leftarrow}_{A(BC)} = J^{\leftarrow}_{AB}$, which does not present much relevancy, but it also demands that $E_{A(BC)} = E_{AB}$, that is, the entanglement of formation must be monogamous [14] for those states. This can be very interesting in a quantum cryptographic setting, where we are trying to minimize the access of third parties in a two-part protocol.

4 Quantum data processing

Now we are going to apply the findings about the b-SSA in a well-known inequality, the quantum data processing inequality. The quantum data processing inequality was first introduced by Schumacher and Nielsen [4], where they also introduce a measure of entanglement—the coherent information, which obeys the data processing in the quantum regime. The coherent information is defined as

$$I_c(A\rangle B) \equiv S(B) - S(A|B), \tag{35}$$

, i.e., the negative of the conditional entropy, $S(A|B)$ (which itself is negative when the system AB is entangled). For the scheme in Fig. 1, the data processing is

$$I_c(A\rangle B_1) \geq I_c(A\rangle B_2), \tag{36}$$

where there are two parties Alice and Bob, and they share a bipartite state. Bob is the one that operates in his part of the state in Fig. 1, and there are two stages corresponding to two operations in Bob's part. The first stage can be understood as the action of encoding information and produces B_1, the second stage could be some error correction to extract the information and yields B_2. Both environments start in a pure state and each interaction is unitary, guaranteeing the purity of the global state in all stages of the process. Also, there is a change in the global state at those different stages—in the first the global system is AB_1E_1, and in the second part it is $AB_2E_1E_2$.

The standard quantum data processing inequality (36) says that in processing a quantum state we can only decrease the quantum correlations between two parts, in agreement with its classical version. However, similarly to the SSA, the limiting bounds might change if quantum correlations are properly taken into account as we show now. By writing $I_c(A\rangle B_1)$ and $I_c(A\rangle B_1)$ explicitly in terms of the definition (35), and noticing that the quantum conditional mutual information, after the second

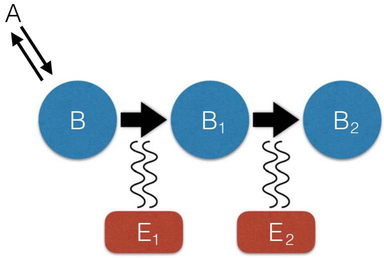


Fig. 1 An illustration of a two stage noisy quantum channel. Alice (A) and Bob (B) share a bipartite state. Bob locally operates his state, evolving as $B \to B_1 \to B_2$ through the interaction with environments E_1 and E_2, respectively

process, is

$$I(A : E_2|E_1) = S(A, E_1) + S(E_1, E_2) - S(E_1) - S(A, E_1, E_2), \tag{37}$$

we see that

$$I_c(A\rangle B_1) - I_c(A\rangle B_2) = I(A : E_2|E_1). \tag{38}$$

By Eq. (14), and recalling that the global system (AB_1E_1, and $AB_2E_1E_2$) is pure, such that $E_{AE_1} - E_{A(E_1E_2)} + \delta^{\leftarrow}_{A(E_1E_2)} - \delta^{\leftarrow}_{AE_1} = E_{AB_2} - E_{AB_1} - \delta^{\leftarrow}_{AB_2} + \delta^{\leftarrow}_{AB_1}$, we can write

$$\widetilde{\Delta} = (E_{AB_2} - \delta^{\leftarrow}_{AB_2}) - (E_{AB_1} - \delta^{\leftarrow}_{AB_1}), \tag{39}$$

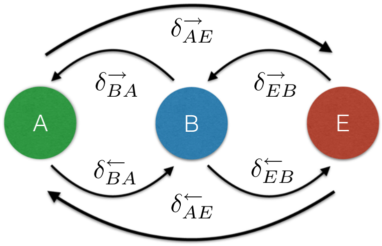
in terms of A, B_1 and B_2 to obtain

$$I_c(A\rangle B_1) \geq I_c(A\rangle B_2) + \widetilde{\Delta}. \tag{40}$$

This is the quantum data processing inequality when quantum correlations captured in Δ are appropriately included. Different situations of the inequality can be analyzed in terms of the quantum correlation shared between the subsystem A with B_1 and B_2.

We now analyze the balance of entanglement and QD in each of the stages between A and B_1, and A and B_2. We can see that if the entanglement distributed in the system balances the quantum correlations (besides entanglement) we get the standard quantum data processing inequality. Otherwise, the lower bound could be weaker or stronger. Therefore, if the Eof is equal to the QD in each stage, $\widetilde{\Delta} = 0$, and we recover the standard inequality. That happens when both the systems AB_1 and AB_2 are pure, possible only when the environments E_1 and E_2 are uncorrelated (not even classically) from B_1 and B_2, respectively. Therefore, the evolution $AB \to AB_1 \to AB_2$ is unitary. Of course, this is not the only situation where $\widetilde{\Delta} = 0$—it might happen that the exceeding correlation (entanglement) in one stage cancels out the exceeding correlation (entanglement) at the other stage. There are many situations when this is possible whenever the states of AB_1 and AB_2 alone are mixed. Therefore, we assume that the standard result of the quantum data processing applies specifically in these cases. The situation when $\widetilde{\Delta} > 0$ is fascinating as it imposes a stronger lower bound on the data processing inequality. Rephrasing its meaning, the quantum data processing

Fig. 2 A schematic representation of the cycle of the Locally Inaccessible Information (LII), measured by Quantum Discord



inequality (40) says that in processing a quantum state we can only decrease the quantum correlations between two parts, and the amount of this decreasing is bounded by the balance of quantum correlations in the process. In contrast, if $\widetilde{\Delta} < 0$, in principle the processing could be improved. However, since this lower bound is weaker than the standard quantum data processing, it is not a relevant case, but it means that the correlations at the final stage could be larger than initially, as if the environments were contributing to the processing with an extra amount of quantum correlations making the processing better. This last case can be considered non-physical, since there are different proofs attesting the non-negativity of the quantum data processing inequality, and the bound being smaller than zero would violate the standard inequality.

Lastly, an intuition can be given by a different lower bound for the data processing in terms of the flow of locally inaccessible information[15], as in Fig. 2. Noting that the difference between the Eof and the QD can be written as

$$E_{AB_2} - \delta^{\leftarrow}_{AB_2} \equiv \frac{1}{2}(\mathcal{L}_{E_1E_2 \rightarrow A \rightarrow B} - \mathcal{L}_{B \rightarrow A \rightarrow E_1E_2}). \tag{41}$$

The relation on the right-hand side of (41) represents the net flow of locally inaccessible information (LII)

$$\mathcal{L}_{R\{E_1E_2\}} \equiv \frac{1}{2}(\mathcal{L}_{E_1E_2 \rightarrow A \rightarrow B} - \mathcal{L}_{B \rightarrow A \rightarrow E_1E_2}), \tag{42}$$

from $\{E_1E_2\}$ to A to B and from B to A to $\{E_1E_2\}$. The notation $R\{E_1E_2\}$ is there to specify that the net flow of LII is in respect with both environments in each stage of the processing, while $R\{E_1\}$ implies a net flow in respect to the environment in the first stage only. By Eq. (30) we establish a lower bound for the quantum data processing inequality based on the difference of the net flow of LII in an out of the environments E_1 and E_2 as

$$I_c(A\rangle B_1) - I_c(A\rangle B_2) \geq \mathcal{L}_{R\{E_1E_2\}} - \mathcal{L}_{R\{E_1\}}. \tag{43}$$

The flow of LII is based on measurements of the quantum discord viewed in tripartite systems, where those measurements are taken from bipartite sides in both directions, capturing only the quantum correlations. By those contributions, while most of the exchange of LII is happening from B to the environments some exchange is happening from the system A and the environments, even though there is no operation in $A's$ part of the state.

The intuition is that the locally inaccessible information in respect to the whole processing $\mathcal{L}_{R\{E_1E_2\}}$ should be greater than the locally inaccessible information on the first stage $\mathcal{L}_{R\{E_1\}}$ since it should be harder to disturb the system after being processed twice, and the bound should be greater than zero. However, noticing what the standard quantum data processing inequality tells, we believe that while the bound is greater than zero, the locally inaccessible information acquired after both stages is not usable nor by Alice or by Bob, since it is not locally accessible. This is different from saying that the LII is destroyed during the processing, because it is only not accessible by both parts. The question then would be if there is a way to harness the extra LII in order to strengthen the standard inequality.

5 Conclusion

Starting from the bounded weak monotonicity, we derived an equivalent inequality that we called bounded strong subadditivity (SSA) of the von Neumann entropy. We showed that the lower bound obtained for the SSA can take a range of values, positive, negative, or null. Depending on the mixture of states utilized, it can give a stronger bound than usual on the quantum conditional information, since the lower bound is written as a balance of quantum correlations in the system described by the entanglement of formation and quantum discord. Both measures, Eof and QD were rewritten as relative entropies in order to use Petz's theorem [13] and the Hayden et al. [10] result to obtain the structure of states that would saturate the bounded quantum conditional entropy. The resulting states exhibit the form of short quantum Markov chains similarly to the states that saturate the standard strong subadditivity, in the aspect that we can recover the global state from a reduced form. This structure also demands the entanglement of formation to respect a monogamous relation, which can make those states useful in cryptography protocols. In addition, we examined the consequences of the bounded SSA in the quantum data processing inequality, although it is not clear why the data processing should not be stricter than usual given the bound in terms of the Eof and QD, a lower bound in terms of the difference in net flow of locally inaccessible information was achieved given additional insight. Even though the bound is greater than zero, it is possible that the locally inaccessible information seen by Bob is not extractable or useful in the processing, being only possible to use the locally accessible part. Whether it is conceivable a protocol where we can use the LII in the system and the quantum data processing is violated remains for further investigation.

After completion of this work, we became aware of an interesting alternative derivation of the SSA, taking the role of quantum correlations[16]. While in that work the authors consider the implications of the weak monotonicity obtained in [8] when the system is extended to a purified global state ρ_{ABCE}, here we are concerned with the implication for the states saturating the b-SSA for arbitrarily mixed tripartite states.

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