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Revisiting unambiguous discrimination

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Abstract

The unambiguous discrimination is a standard strategy of detecting linearly independent quantum states and has many applications in quantum information processing. In the standard unambiguous discrimination, the measuring operators contain two types of measuring operators: the success operators and an inconclusive operator. It is conventionally regarded that an inconclusive operator gives no more information about the input states. In this paper, we propose a new unambiguous discrimination by replacing the inconclusive operator by the correct operators, and thus, more information about input states can be obtained. We take three examples to demonstrate the efficiency of our detection scheme. Our scenario improves the standard unambiguous discrimination.

Keywords State discrimination · Unambiguous discrimination · Optimal minimum probability · Linearly independent states

1 Introduction

The quantum information encoded in a set of quantum states by a sender can be delivered to a receiver in a distance, who performs a suitable measurement to extract this information. A proper measurement strategy is required when the receiver wants to obtain information from nonorthogonal quantum states because those states cannot be perfectly discriminated [1–3]. Quantum measurement strategies may be classified by the constraints on the conclusive or inconclusive results. In quantum state discrimination (QSD), an inconclusive result indicates that the given quantum states cannot

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be definitely discriminated, unless quantum states are mutually orthogonal. The standard minimum-error discrimination (SMD) [4–12] is able to minimize the average error of conclusive results without an inconclusive result. The standard unambiguous discrimination (SUD) [13–24] and maximum confidence discrimination (MCD) [25] strategies permit an inconclusive result and minimize individual errors associated with the conclusive results. A general state discrimination strategy with a fixed rate of inconclusive results (FRIR) was studied [26–32].

The SUD strategy was first proposed by Ivanovic, Dieks and Peres, i.e., the IDP measurement [13–15]. The IDP measurement is constructed by two types of measuring operators: the success operators and an inconclusive operator. If the success operators are measured, it gives deterministic result about the input states without errors. If an inconclusive result is obtained, however, the detection is failure, or in other words, an inconclusive result gives no more information about the input states.

The SUD scheme was first designed only in detecting a pair of two pure states [13–16], and afterward, as a conventional scheme, it is applied to the case of $N \geq 3$ quantum states. Up to date, its optimality is never proved. For the SUD scheme, it is usually regarded that an inconclusive operator cannot give any information about input states. Nevertheless, in investigating the performance of the SUD strategy on the detection of three linearly independent states in Ref. [33], Peres, who proposed the theory of positive-operator-valued measures (POVM) [34] and designed the SUD scheme [15], perceptively found that the inconclusive answers still carry some information. This implies that the SUD is optimal only for the detection of special, highly symmetric quantum states, but not for general quantum states. In this paper, we design a new unambiguous discrimination (NUD), which is expected to improve the SUD. In the NUD, we replace an inconclusive operator by the correct operators and apply to discriminating the two pure states, the three linearly independent symmetric states [35], and the three real quantum states with the real overlaps among them by using a POVM [34]. Our result shows that in detecting two states, the two correct operators are actually reduced to the two inconclusive operators, and hence, the SUD is optimal for discriminating two quantum states. But in detecting three linearly independent symmetric state and three real quantum states, except for the success operator, the correct operators can also give some information about the input states. In particular, when applied to third example, the NUD demonstrates a great improvement to the SUD. Thus, we present a new strategy for the unambiguous discrimination. The IDP measurement has many applications, such as quantum cryptography [36], entanglement concentration [17], conclusive quantum teleportation [37], entanglement swapping [38] via nonmaximally entangled channels and quantum tomography [39]. Therefore, our strategy proposed may find its applications in quantum information processing.

This paper is organized as follows. In Sect. 2, we briefly review the SME strategy for discriminating the two pure states and the SUD strategy applied to the three linearly independent symmetric states. In Sect. 3, we introduce our NUD and demonstrate three examples. The paper ends with a summary.

2 SMD and SUD

The problem of quantum state discrimination can be briefly posed as follows: detecting one of quantum states from a set of N known pure quantum states $\{\eta_i; |\psi_i\rangle\}_{i=1}^N$, described by the density operators $\hat{\rho}_i = |\psi_i\rangle\langle\psi_i|$ and a prior probabilities η_i for $\sum_{i=1}^N \eta_i = 1$. The original quantum ensemble may then be expressed as $\hat{\rho} = \sum_{i=1}^N \eta_i \hat{\rho}_i$. Generally, this problem can be solved by exploiting a POVM [34], where the POVM element $\left\{\hat{\Pi}_i\right\}_{i=1}^N$ must satisfy the following conditions:

$$\hat{\Pi}_i \geq 0, \quad \sum_{i=1}^N \hat{\Pi}_i = \hat{I}, \tag{1}$$

where $\hat{I}$ is the identity operator.

For the SME strategy, it is sufficient to define only one type of the measuring operators $\left\{\hat{\Pi}_i\right\}_{i=1}^N$ of correctly identifying the input states $\hat{\rho}_i$. The individual correct and error probabilities are given according to

$$p_{ci} = \mathrm{Tr}\left(\hat{\rho}_i \hat{\Pi}_i\right), \quad e_i = \sum_{\substack{j=1 \\ j\neq i}}^N e_{ij} = \sum_{\substack{j=1 \\ j\neq i}}^N \mathrm{Tr}\left(\hat{\rho}_i \hat{\Pi}_j\right). \tag{2}$$

The individual probabilities have explicitly physical meanings. When the input state is $\hat{\rho}_i$, if $\hat{\Pi}_i$ is measured one can correctly known that the input state is $\hat{\rho}_i$ with the correct probability p_{ci}. When the input state is $\hat{\rho}_i$ and if $\hat{\Pi}_j$ is measured, however, one then erroneously regards the input state $\hat{\rho}_i$ is $\hat{\rho}_j$ with the probability of error $e_{ij} = \mathrm{Tr}\left(\hat{\rho}_i \hat{\Pi}_j\right)$. So the individual error probability of erroneously detecting the state $\hat{\rho}_i$ is $e_i = \sum_{\substack{j=1 \\ j\neq i}}^N e_{ij}$. It is obvious that $p_{ci} + e_i = 1$ because of $\mathrm{Tr}\left(\hat{\rho}_i \sum_{j=1}^N \hat{\Pi}_j\right) = \mathrm{Tr}\left(\hat{\rho}_i \hat{\Pi}_i\right) + \mathrm{Tr}\left(\sum_{\substack{j=1 \\ i\neq j}}^N \hat{\rho}_i \hat{\Pi}_j\right) = 1$. Associated with a prior probabilities η_i for $\sum_{i=1}^N \eta_i = 1$, the average correct and error probabilities are conventionally defined as

$$P_c = \sum_{i=1}^N \eta_i p_{ci}, \quad P_e = \sum_{i=1}^N \eta_i e_i, \tag{3}$$

and they have the relation $P_c + P_e = 1$. For the detection of nonorthogonal quantum states, it is oblivious that $P_c < 1$ and $P_e > 0$, or it will lead to the result of the perfect detection of nonorthogonal quantum states. The task of the SME scheme is to find the minimum of the average error probability P_e (or to maximize the value of P_c).

To unambiguously discriminate a set of N linearly independent states $\{|\psi_i\rangle\}_{i=1}^N$, the POVM elements of the SUD strategy consist of two types of measuring operators: the N success operators $\left\{\hat{M}_i\right\}_{i=1}^N$ and one inconclusive operator $\hat{O}$, satisfying with the condition $\sum_{i=1}^N \hat{M}_i + \hat{O} = \hat{I}$. The individual success and inconclusive probabilities are given according to

$$\mathrm{Tr}\left(\hat{\rho}_i \hat{M}_j\right) = p_{si}\delta_{ij}, \quad \mathrm{Tr}\left(\hat{\rho}_i \hat{O}\right) = q_i. \tag{4}$$

If the success operator $\hat{M}_i$ is measured with the probability p_{si}, then one can unambiguously determine that the input state is $\hat{\rho}_i$ without error. If an inconclusive result occurs with the probability q_i, however, the SUD fails, that is, one does not learn which of the input states is given. In other words, the inconclusive result can give no more information about the input states. These properties of the measuring operators imply that the SUD is an error-free strategy, but sometimes will fail in the case of an inconclusive result occurs. It is obvious that $p_{si} + q_i = 1$ because of $\mathrm{Tr}\left(\hat{\rho}_i\left(\sum_{j=1}^N \hat{M}_j + \hat{O}\right)\right) = \mathrm{Tr}\left(\hat{\rho}_i \hat{M}_i\right) + \mathrm{Tr}\left(\hat{\rho}_i \hat{O}\right) = 1$. The average success probability P_s and the average inconclusive probability Q are conventionally defined as

$$P_s = \sum_{i=1}^N \eta_i p_{si}, \quad Q = \sum_{i=1}^N \eta_i q_i, \tag{5}$$

and the two probabilities have the relation $P_s + Q = 1$. In the SUD strategy, the task is to minimize the value of Q (or to maximize the value of P_s).

We next show the performances of the SME and the SUD on the state detection. In detecting the two pure states $\{\eta_i; |\psi_i\rangle\}_{i=1}^2$ with the overlap among them $\langle\psi_1 \mid \psi_2\rangle = s \in [0, 1]$, by using the SMD strategy, the average correct probability, the well-known Helstrom bound [4], is derived as

$$P_c = \frac{1}{2}\left(1 + \sqrt{1 - 4\eta_1\eta_2 s^2}\right). \tag{6}$$

If using the SUD strategy, the well-known IDP measurement [13–15], the average success probability is derived as [16]

$$P_s = 1 - 2\sqrt{\eta_1\eta_2}s. \tag{7}$$

Consider the three linearly independent symmetric states in the form [35]

$$\begin{aligned}
|\zeta_0\rangle &= c_0|0\rangle + c_1|1\rangle + c_2|2\rangle, \\
|\zeta_1\rangle &= c_0|0\rangle + c_1 e^{i\frac{2}{3}\pi}|1\rangle + c_2 e^{i\frac{4}{3}\pi}|2\rangle, \\
|\zeta_2\rangle &= c_0|0\rangle + c_1 e^{i\frac{4}{3}\pi}|1\rangle + c_2 e^{i\frac{2}{3}\pi}|2\rangle,
\end{aligned} \tag{8}$$

with the normalization condition $\sum_{i=0}^{2}|c_i|^2=1$, and the overlaps among them are $\langle\zeta_0 \mid \zeta_1\rangle=\langle\zeta_1 \mid \zeta_2\rangle=\langle\zeta_2 \mid \zeta_0\rangle=\sum_{j=0}^{2}\left|c_j\right|^2 e^{2\pi ij/3}=se^{i\alpha}$. When prior probabilities are equal, $\eta_i=1/3$, by using the SUD scheme, the upper bound of the average success probability of detecting them reaches [33]

$$P_s \leq 3\times \min|c_i|^2. \tag{9}$$

Thus, an average inconclusive probability is $Q=1-P_s\geq 1-3\times\min|c_i|^2$, conventionally regarded to give no more information about the input states. In the next section, we show that an inconclusive result may also extract some information about the input states.

3 NUD

In the SUD scenario, an inconclusive result given by an inconclusive operator is conventionally viewed as a completely useless one from which no more information about the input states can be attained. In Ref. [33], Peres definitely pointed out that the SUD is optimal only for the detection of some special, highly symmetric quantum states, but not for general quantum states. In other words, in discriminating general quantum states, an inconclusive operator is not suitable to be introduced. This implies a new UD scheme is needed. In this section, we replace an inconclusive operator by the correct operators and investigate the performance of our scheme in unambiguously discriminating linearly independent quantum states.

The measuring operators of our new strategy consist of the success operators $\hat{M}_i$ and the correct operators $\hat{\Pi}_i$, satisfying the following conditions:

$$\hat{M}_i,\hat{\Pi}_i\geq 0,\quad \sum_{i=1}^{N}\left(\hat{M}_i+\hat{\Pi}_i\right)=\hat{I}. \tag{10}$$

In the SUD, once the success operators $\left\{\hat{M}_i\right\}_{i=1}^{N}$ are defined, then an inconclusive operator is determined as $\hat{O}=\hat{I}-\sum_{i=1}^{N}\hat{M}_i$ correspondingly. Traditionally, the measurement of an inclusive operator gives no more information about the input states. But it is not the case [33]. We replace an inconclusive operator by some correct operators. Or in other words, in doing so mathematically, an inconclusive operator that can give no more information about the input states is decomposed as the correct operators that can give some information about the input states, that is, $\hat{O}=\sum_{i=1}^{N}\hat{\Pi}_i=\hat{I}-\sum_{i=1}^{N}\hat{M}_i$. Therefore, our strategy is a general SUD strategy.

For our purpose, the measuring operators must be satisfied with the following constrains.

$$\mathrm{Tr}\left(\hat{\rho}_i\hat{M}_j\right)=p_{si}\delta_{ij},\quad p_{ci}=\mathrm{Tr}\left(\hat{\rho}_i\hat{\Pi}_i\right),\quad e_i=\sum_{\substack{j=1\\ j\neq i}}^{N}e_{ij}=\sum_{\substack{j=1\\ j\neq i}}^{N}\mathrm{Tr}\left(\hat{\rho}_i\hat{\Pi}_j\right). \tag{11}$$

The corresponding probabilities as defined in Sect. 2, such as the average success probability $P_s = \sum_{i=1}^{N} \eta_i p_{si}$, the average correct probability $P_c = \sum_{i=1}^{N} \eta_i p_{ci}$ and the average error probability $P_e = \sum_{i=1}^{N} \eta_i e_i$, obviously have the two relations:

$$p_{si} + p_{ci} + e_i = 1, \quad P_s + P_c + P_e = 1. \tag{12}$$

In order to quantify the efficiency of the average correct probability of the NUD, we introduce the definition of the relative average correct probability.

$$P_c^{(R)} = \frac{P_c}{P_c + P_e} = \frac{P_c}{1 - P_s}. \tag{13}$$

Where we have use the condition $P_s + P_c + P_e = 1$. We here give some remarks on the NUD proposed. To unambiguously discriminate some sets of linearly independent quantum states $\{\eta_i; |\psi_i\rangle\}_{i=1}^{N}$ and in the case of the maximal average inconclusive probabilities of the SUD and NUD strategies being equal, if the relative average correct probability is $P_c^{(R)} > 1\big/N$, then the NUD performs better than the SUD. If the relative average correct probability is $P_c^{(R)} = 1\big/N$, however, the NUD is essentially equivalent to the SUD (in this case, the correct operators are actually reduced to the inconclusive operators).

We first consider the case of detecting two quantum states. For the two pure states $\{\eta_i; |\psi_i\rangle\}_{i=1}^{2}$ with the overlap among them $\langle\psi_1 \mid \psi_2\rangle = s \in [0, 1]$, we define the success operators $\hat{M}_i = |m_i\rangle\langle m_i|$ and the correct operators $\hat{\Pi}_i = |\pi_i\rangle\langle\pi_i|$, where the corresponding states are defined as

$$|m_1\rangle = \frac{\sqrt{p_{s1}}}{1 - s^2}(|\psi_1\rangle - s|\psi_2\rangle), \quad |m_2\rangle = \frac{\sqrt{p_{s2}}}{1 - s^2}(|\psi_2\rangle - s|\psi_1\rangle), \tag{14}$$

$$\begin{aligned} |\pi_1\rangle &= \frac{\sqrt{p_{c1}} - s\sqrt{e_2}}{1 - s^2}|\psi_1\rangle + \frac{\sqrt{e_2} - s\sqrt{p_{c1}}}{1 - s^2}|\psi_2\rangle, \\ |\pi_2\rangle &= \frac{\sqrt{p_{c2}} - s\sqrt{e_1}}{1 - s^2}|\psi_1\rangle + \frac{\sqrt{e_1} - s\sqrt{p_{c2}}}{1 - s^2}|\psi_2\rangle. \end{aligned} \tag{15}$$

For $i, j = 1, 2$ and $i \neq j$, it is ready to verify the properties of the individual success probability $p_{si} = \mathrm{Tr}\left(\hat{\rho}_i \hat{M}_i\right)$ and $\mathrm{Tr}\left(\hat{\rho}_i \hat{M}_j\right) = 0$ from Eq. (14), and the individual correct probability $p_{ci} = \mathrm{Tr}\left(\hat{\rho}_i \hat{\Pi}_i\right)$ and the individual error probability $e_i = e_{ij} = \mathrm{Tr}\left(\hat{\rho}_i \hat{\Pi}_j\right)$ from Eq. (15). It can also verify that the sum of the measuring operators is the identity operator, regardless of the explicit expressions of the input states. Note that the probabilities are independent variables. So, the forms of the measuring operators proposed are the most general ones.

Let us show the performance of our strategy on the two-state discrimination. For the fixed values of the individual success probabilities p_{ci}, our task is then to find the maximum of the average correct probabilities $P_c = \sum_{i=1}^{2} \eta_i p_{ci}$. Multiplying $\langle\psi_k|$ in the left-hand and $|\psi_l\rangle$ in the right-hand on both sides of the equation

$\sum_{i=1}^{2}\left(\hat{M}_i+\hat{\Pi}_i\right)=\hat{I}$, we get some constraints

$$p_{s1}+p_{c1}+e_1=1,\quad p_{s2}+p_{c2}+e_2=1, \tag{16}$$

$$\sqrt{p_{c1}e_2}+\sqrt{e_1 p_{c2}}=s. \tag{17}$$

Our next task is then to find the maximal value of the average correct probabilities $P_c=\sum_{i=1}^{2}\eta_i p_{ci}$ under Eqs. (16), (17). By using the method of Lagrange multipliers, when the individual correct probability is

$$\begin{aligned} p_{c1} &= \frac{a(a\eta_1+b\eta_2)\left(a\eta_1+b\eta_2+\sqrt{\Delta}\right)-2s^2\eta_2\sqrt{\Delta}-4as^2\eta_1\eta_2}{2\Delta}, \\ p_{c2} &= \frac{b\sqrt{\Delta}\left(a\eta_1+b\eta_2+\sqrt{\Delta}\right)-2s^2\eta_1\sqrt{\Delta}}{2\Delta}, \end{aligned} \tag{18}$$

where

$$\begin{aligned} &a=1-p_{s1},\quad b=1-p_{s2}, \\ &\Delta=(1-\eta_1 p_{s1}-\eta_2 p_{s2})^2-4\eta_1\eta_2 s^2, \end{aligned} \tag{19}$$

the maximum of the average correct probability is obtained as

$$\begin{aligned} P_c &= \frac{1}{2}\left[1-\eta_1 p_{s1}-\eta_2 p_{s2}+\sqrt{(1-\eta_1 p_{s1}-\eta_2 p_{s2})^2-4\eta_1\eta_2 s^2}\right] \\ &= \frac{1}{2}\left[1-P_s+\sqrt{(1-P_s)^2-4\eta_1\eta_2 s^2}\right], \end{aligned} \tag{20}$$

as a function of the average success probability $P_s=\sum_{i=1}^{2}\eta_i p_{si}$. Thus, the relative average correct probability is given as

$$P_c^{(R)}=\frac{1}{2}\left[1+\sqrt{1-\left(\frac{2\sqrt{\eta_1\eta_2}s}{1-P_s}\right)^2}\right]. \tag{21}$$

If taking $P_s=0$, i.e., $p_{si}=0$, our strategy can be reduced to the SMD strategy [4–12] with the average correct probability $P_c=P_c^{(R)}=\left(1+\sqrt{1-4\eta_1\eta_2 s^2}\right)\big/2$ given by Eq. (6). When the average success probability reaches the maximum value $P_s=1-2\sqrt{\eta_1\eta_2}s$ given by Eq. (7), the relative average correct probability is $P_c^{(R)}=1\big/2$, which gives no more information about the input states. In this case, the correct operators in fact turn to be the two inconclusive operators, and our scheme is reduced to the SUD. Therefore, it is indeed that in the case of detecting two states, the correct operators are unnecessary to be introduced. For discriminating the three linearly independent quantum states, however, the situation is much different.

We now consider the detection of the three linearly independent symmetric states given by Eq. (8). For convenience in the following calculations, we assume $\eta_i = \eta_j = 1/3$. Due to the symmetry of the three symmetric states, it will have $p_{si} = p_{sj} = p_s = P_s$ and $p_{ci} = p_{cj} = p_c = P_c$. In order to detect the three symmetric states, we will need three success operators $\hat{M}_i = |m_i\rangle\langle m_i|$ and three correct operators $\hat{\Pi}_i = |\pi_i\rangle\langle\pi_i|$, where the corresponding states are defined as

$$\begin{aligned}
|m_0\rangle &= \frac{\sqrt{P_s}}{3}\left(\frac{1}{c_0}|0\rangle + \frac{1}{c_1}|1\rangle + \frac{1}{c_2}|2\rangle\right),\\
|m_1\rangle &= \frac{\sqrt{P_s}}{3}\left(\frac{1}{c_0}|0\rangle + \frac{1}{c_1}e^{i\frac{2\pi}{3}}|1\rangle + \frac{1}{c_2}e^{i\frac{4\pi}{3}}|2\rangle\right),\\
|m_2\rangle &= \frac{\sqrt{P_s}}{3}\left(\frac{1}{c_0}|0\rangle + \frac{1}{c_1}e^{i\frac{4\pi}{3}}|1\rangle + \frac{1}{c_1}e^{i\frac{2\pi}{3}}|2\rangle\right),
\end{aligned} \tag{22}$$

$$\begin{aligned}
|\pi_0\rangle &= \frac{\sqrt{c_0^2 - P_s/3}}{\sqrt{3}c_0}|0\rangle + \frac{\sqrt{c_1^2 - P_s/3}}{\sqrt{3}c_1}|1\rangle + \frac{\sqrt{c_2^2 - P_s/3}}{\sqrt{3}c_2}|2\rangle,\\
|\pi_1\rangle &= \frac{\sqrt{c_0^2 - P_s/3}}{\sqrt{3}c_0}|0\rangle + \frac{\sqrt{c_1^2 - P_s/3}}{\sqrt{3}c_1}e^{i\frac{2\pi}{3}}|1\rangle + \frac{\sqrt{c_2^2 - P_s/3}}{\sqrt{3}c_2}e^{i\frac{4\pi}{3}}|2\rangle,\\
|\pi_2\rangle &= \frac{\sqrt{c_0^2 - P_s/3}}{\sqrt{3}c_0}|0\rangle + \frac{\sqrt{c_1^2 - P_s/3}}{\sqrt{3}c_1}e^{i\frac{4\pi}{3}}|1\rangle + \frac{\sqrt{c_2^2 - P_s/3}}{\sqrt{3}c_2}e^{i\frac{2\pi}{3}}|2\rangle.
\end{aligned} \tag{23}$$

With the help of the explicit expressions of the measuring operators above, it is ready to verify the set of the measuring operators fulfill the completeness condition, $\sum_{i=0}^{2}\left(\hat{M}_i + \hat{\Pi}_i\right) = \hat{I}$. It is also easily to calculate the corresponding individual probabilities, $P_s = \mathrm{Tr}\left(\hat{\rho}_i\hat{M}_j\right) = P_s\delta_{ij}$ and $P_c = \mathrm{Tr}\left(\hat{\rho}_i\hat{\Pi}_i\right) = \left(\sum_{j=0}^{2}\sqrt{c_j^2 - P_s/3}\right)^2$. Thus, the relative average correct probability is easily obtained.

$$P_c^{(R)} = \frac{1}{3}\frac{\sum_{i=0}^{2}\left[\eta_i\left(\sum_{j=0}^{2}\sqrt{c_j^2 - P_s/3}\right)^2\right]}{1 - \sum_{i=0}^{2}\eta_i P_s}. \tag{24}$$

Therefore, Eq. (24) is reduced to

$$P_c^{(R)} = \frac{1}{3}\frac{\left(\sum_{i=0}^{2}\sqrt{c_i^2 - P_s/3}\right)^2}{1 - P_s}. \tag{25}$$

Due to that the overlaps among the three linearly independent symmetric states are all equal, $\langle\zeta_0 \mid \zeta_1\rangle = \sum_{j=0}^{2}\left|c_j\right|^2 e^{2\pi ij/3} = se^{i\alpha}$, $s \in [0, 1]$ and $\alpha \in [0, 2\pi)$, without loss of generality, we assume that $1 > c_{\max} = c_0 \geq c_1 \geq c_2 = c_{\min} > 0$. Therefore, it is immediately to obtain the upper bound of the average success probability $P_s \leq 3c_2^2$, given by Eq. (9), from the inequality $\sqrt{c_i^2 - P_s/3} \geq 0$. This result agrees with the previous contribution [35].

For the SUD strategy, the maximal value of the average success probability reaches $P_s = 3c_2^2$, so that the minimal value of the average inconclusive probability is $Q = 1 - P_s = 1 - 3c_2^2 = \left(c_0^2 - c_2^2\right) + \left(c_1^2 - c_2^2\right)$ (we have used the relation $\sum_{i=0}^{2} c_i^2 = 1$), which is conventionality regarded to give no more information about the three lineally independent symmetric states. Let us look the efficiency of our scheme. When $P_s = 3c_2^2$, the relative average correct probability is given as

$$P_c^{(R)} = \frac{1}{3}\frac{\left(\sum_{i=0}^{2}\sqrt{c_i^2 - P_s/3}\right)^2}{1 - P_s} = \frac{1}{3}\frac{\left(\sqrt{c_0^2 - c_2^2} + \sqrt{c_1^2 - c_2^2}\right)^2}{\left(c_0^2 - c_2^2\right) + \left(c_1^2 - c_2^2\right)} \in \left[\frac{1}{3}, \frac{2}{3}\right]. \quad (26)$$

By exploiting the explicit expression (26), some analysis can be given. When $c_0 = c_1 > c_2$, the relative correct probability is maximal $p_c^{(R)} = 2/3$. When $c_0 > c_1 > c_2$, the relative correct probability is maximal $p_c^{(R)} \in \left(1/3, 2/3\right)$. When $c_0 > c_1 = c_2$, it has $p_c^{(R)} = 1/3$, giving no more information about the three symmetric states. In the case of $p_c^{(R)} = 1/3$, the correct operators turn to be the three inconclusive operators.

Let us find the so-called special, highly symmetric quantum states in 3 dimensions, conjectured in Ref. [33]. Exploiting the conditions of $c_0 > c_1 = c_2$ and $\sum_{i=0}^{2} c_i^2 = 1$ can immediately derive the value of the overlap among three symmetric quantum states $se^{i\alpha} = \sum_{j=0}^{2} \left|c_j\right|^2 e^{2\pi ij/3} = 1 - 3c_1^2 \in (0, 1)$, which implies $\alpha = 0$. When $\alpha \neq 0$, it always has $p_c^{(R)} > 1/3$, implying that the performance of the NUD is better than that of the SUD.

We next consider the third example. Suppose the three equiprobable quantum states takes the following forms:

$$|\xi_1\rangle = |0\rangle, \quad |\xi_2\rangle = \frac{1}{3}(|0\rangle + 2|1\rangle + 2|2\rangle), \quad |\xi_3\rangle = \frac{1}{3}(|0\rangle + 2|1\rangle - 2|2\rangle), \quad (27)$$

with the real overlaps among them

$$s_{23} = s_1 = \frac{1}{9}, \quad s_{12} = s_3 = \frac{1}{3}, \quad s_{31} = s_2 = \frac{1}{3}. \quad (28)$$

By using Eqs. (9)–(11) given in the contributions in Ref. [21] and taking the indices as $i = 2$, $j = 3$ and $k = 1$, the average probabilities of the SUD are derived as $Q = 10/27$ and $P_s = 1 - Q = 17/27$. For our NUD, we design the measuring elements of the POVM as follows:

$$\hat{m}_1^\dagger = \sqrt{\frac{27}{81}}|0\rangle - \frac{1}{2\sqrt{3}}|1\rangle, \quad \hat{m}_2^\dagger = \frac{\sqrt{7}}{4}|1\rangle + \frac{\sqrt{7}}{4}|2\rangle, \quad \hat{m}_3^\dagger = \frac{\sqrt{7}}{4}|1\rangle - \frac{\sqrt{7}}{4}|2\rangle,$$

$$\hat{\pi}_1^\dagger = \sqrt{\frac{52}{81}}|0\rangle + \frac{\sqrt{13}}{18}|1\rangle, \quad \hat{\pi}_2^\dagger = \sqrt{\frac{1}{81}}|0\rangle + \frac{1}{36}|1\rangle + \frac{1}{4}|2\rangle \quad \hat{\pi}_3^\dagger = \sqrt{\frac{1}{81}}|0\rangle + \frac{1}{36}|1\rangle - \frac{1}{4}|2\rangle. \tag{29}$$

It can be verified without any difficulty that $\sum_{i=1}^{3}\left(\hat{m}_i^\dagger \hat{m}_i + \hat{\pi}_i^\dagger \hat{\pi}_i\right) = \hat{I}$ and $\mathrm{Tr}\left(\hat{\rho}_i \hat{m}_j^\dagger \hat{m}_j\right) = 0$ for $i, j = 1, 2, 3$ and $i \neq j$. The average success and correct probabilities are obtained as

$$P_s = \frac{1}{3}\mathrm{Tr}\left(\sum\nolimits_{i=1}^{3}\left(\hat{\rho}_i \hat{m}_i^\dagger \hat{m}_i\right)\right) = \frac{17}{27}, \quad P_c = \frac{1}{3}\mathrm{Tr}\sum\nolimits_{i=1}^{3}\left(\hat{\rho}_i \hat{\pi}_i^\dagger \hat{\pi}_i\right) = \frac{20}{81}. \tag{30}$$

Thus, the relative average correct probability is calculated as

$$P_c^{(R)} = \frac{P_s}{1 - P_s} = \frac{2}{3}. \tag{31}$$

There is a reasonable conjecture that the upper bound of the probability of correctly detecting pure quantum states from a set of N equiprobable D-dimensional quantum pure states is D/N (i.e., the upper bound can reach unit for linearly independent pure states). By using the SUD, the inconclusive probability gives no more information about the input states; but if exploiting our NUD this part of the probability still contributes the relative average correct probability to. In comparison with the SUD, therefore, the third example illustrates the best efficiency of the NUD. It may be conjectured that in detecting a set of equiprobable linearly independent quantum pure states, when the average success probabilities are maximal (equal to that of the SUD), the maximum of the relative average correct probability may reach. This is another type of special, equiprobable highly symmetric linearly independent quantum states.

Finally, we would like to give some remarks. Our NUD obviously improves the SUD. It is an interesting topic in quantum information processing that how to extract maximal information in detecting a set of linearly independent quantum states. Here, the optimality of the detection strategy means to obtain the maximal information accessible allowed by quantum mechanics. The optimality of the IDP measurement is never proved. In this paper, we are also unable to prove the optimality of our strategy, but it is indeed superior to the SUD strategy. Maybe, more efficient scenario than our NUD may be proposed in future. The IDP measurement has many applications, and therefore, our strategy proposed may find its applications in quantum information processing.

4 Summaries

We have investigated the detection of a set of linearly independent quantum states. A new scheme of the unambiguous discrimination is proposed. The measuring operators consist of the success and correct operators. Our strategy can extract more information than the SUD strategy. We take three examples to show the efficiency of our strategy.

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