Quantum Information Processing (2021) 20:340
https://doi.org/10.1007/s11128-021-03282-y

Perfect controlled joint remote state preparation of arbitrary multi-qubit states independent of entanglement degree of the quantum channel

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Received: 24 December 2020 / Accepted: 20 September 2021 / Published online: 13 October 2021

Abstract
We design a quantum circuit to generate a class of partially entangled quantum states. Using this kind of quantum state as quantum channel, we put forward deterministic schemes for controlled joint remote state preparation of arbitrary two- and three-qubit states and extend them to prepare arbitrary multi-qubit states. For each case, we give the concrete construction methods of multi-qubit measurement basis and unitary transformations to recover the initial original state. Unlike most previous works, where the parameters of the quantum channel are given to the receiver who can accomplish the task only probabilistically by consuming auxiliary resource, operation and measurement, here we give them to the supervisor. Thanks to the knowledge of quantum channel parameters, the supervisor can perform appropriate complete projection measurement. Combined with the feed-forward strategy adapted by the preparers, the measurement not only much simplifies the receiver's operation but also yields unit success probability. Amazingly, our protocols do not depend on the entanglement degree of the shared quantum channel, and they are within the reach realization of current quantum technologies.

Keywords Controlled joint remote state preparation · Partially entangled resource · Arbitrary multi-qubit state · Unit success probability · Feed-forward strategy

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1 Introduction

In the era of big data and cloud computing, data can bring much convenience to human life, but personal information leakage will be particularly prominent. Therefore, how to effectively ensure information security is of great significance.

Classical cryptography has always been a popular choice, but it will face the threat of breakthrough in polynomial time with the emergence of quantum algorithm [1,2]. Quantum cryptography has been proved to be unconditionally secure on insecure channels, and its security is guaranteed by quantum mechanics such as Heisenberg uncertainty principle and non-cloning theorem. Hence, more and more attention has been paid to quantum cryptography by industry and academia. In 1993, Bennett et al.[3] initiated the quantum teleportation (QT) protocol, which is to transfer an unknown state. Later, Lo [4] designed a simple protocol for transmitting a known state utilizing the same quantum resource as in QT but without Bell measurement and with lesser classical communication. Such protocol is called remote state preparation (RSP) [4,5]. The shortcomings of RSP are: (a) all information of the prepared state is disclosed to the preparer and (b) unit success probability cannot be reached in general. In 2007, Xia et al. [6] proposed a new protocol called joint remote state preparation (JRSP). In JRSP, there are several preparers, each of them is allowed to know only part of the information of the prepared state so that no subsets of them are able to infer the state, thus resolving the shortcoming (a). Moreover, by using feed-forward measurement [7] (i.e., measuring in sequence, and the future measurement basis depends on the previous measurement results), JRSP can always be successful [8–10], thus resolving the shortcoming (b).

In the realistic environment, quantum control of global task is often needed. This can be achieved by adding a supervisor who has the right to decide at the last minute to complete a task after careful consideration of all relevant situations. Controlled teleportation [11,12], controlled RSP [13–18], controlled quantum dialogue [19], controlled remote implementation of partially unknown quantum operation [20,21], etc. have been investigated in detail. In order to be able to control in a quantum way, the supervisor must share beforehand with the preparers as well as with the receiver a quantum resource served as a quantum channel which is generally considered to be maximally entangled for best performance. For instance, a maximally entangled quantum channel leads to unit success probability together with feed-forward measurement strategy [8–10]. However, due to the influence of noise, the decoherence effects will cause the maximally entangled states to evolve into partially entangled states. In addition, literature [22] proposed a possible solution to cope with an outside attack by using a partially entangled resource whose identifying parameters are kept confidential from any outsider. Based on the above reasons, some scholars use partially entangled quantum resources as quantum channels to design quantum information transmission protocols [23–25]. Usually, the parameters of the partially entangled resource are assumed to be known by the receiver [23,24], who can use this knowledge to recover the desired state from his/her collapsed state. The cost of recovery process is the mandatory requirement of auxiliary qubits, auxiliary two qubit gates as well as measurements on auxiliary qubits, and the total probability of success is always less than 1. If the knowledge of the parameters of the partially entangled quantum channel is transferred from the receiver to the supervisor who executes optimal POVM (positive operator-valued measure) measurements on his/her qubits to guide the receiver to recover the desired state without consuming any auxiliary resources, then there is always a finite probability of failure when an ambiguous measurement output is obtained [25]. Fortunately, in 2017, Peng et al. [26] proposed a perfect protocol for multi-hop controlled QT of arbitrary single-qubit state by using an appropriate class of partially entangled quantum resources as channels. In this protocol, instead of POVM measurement, each supervisor can reconstruct the desired state by performing a rotation operation and projection measurement instead of POVM measurement; thus, the defects are overcome mentioned above. Ref. [26] gives us an enlightenment: for an intended task, there may be appropriate resources via which the performance of the task would be the best. Are there any partially entangled quantum resources to be served as channel for perfect controlled JRSP of arbitrary multi-qubit states? In this paper, we will answer the above question, that is, how to generate such partially entangled quantum resources and how to use them to implement controlled JRSP perfectly. An added interesting feature is that the implementation is independent of the entanglement degree of the shared partially entangled quantum channels, which is different from all previous controlled JRSP protocols.

The rest of this paper is organized as follows. In Sect. 2, we construct a quantum circuit to prepare partially entangled state for our quantum tasks. In Sects. 3 and 4, we explore how to realize the controlled JRSP for arbitrary two- and three-qubit states, respectively. By parity of reasoning, we present the controlled JRSP protocol for arbitrary multi-qubit states and give the precise construction methods of multi-qubit measurement basis about real parameter and complex parameter in Sect. 5. Finally, some discussion and summary are given in Sect. 6.

2 The partially entangled quantum channel

In order to accomplish our quantum tasks, we employ the following four-qubit partially entangled state as quantum channel

$$|Q\rangle_{1234} = \frac{1}{\sqrt{2}}(|0000\rangle + \cos\theta|1110\rangle + \sin\theta|1111\rangle)_{1234}, \tag{1}$$

where $\theta \in [0, \pi/2]$. This state is characterized by the angle θ whose value is only known by the supervisor. The entanglement among qubit 1, qubit 2 and qubit 3 is always independent of parameter θ. However, parameter θ influences the entanglement between qubit 4 and the others. Obviously, if parameter θ satisfies its value of sine with 0, the qubit of the supervisor is totally disentangled from the other ones. Besides this case, the state (1) possesses some degree of entanglement between the qubit of the supervisor and the remaining qubit group, justifying him/her as "quantum" controller. In particular, for the value of the angle which the sine is equal to ± 1, the degree of entanglement is maximal. Of importance is the fact that the value of θ is consciously assigned only to the supervisor. This obviously enhances the security level. Even if the qubits of the supervisor and the receiver are unexpectedly captured by

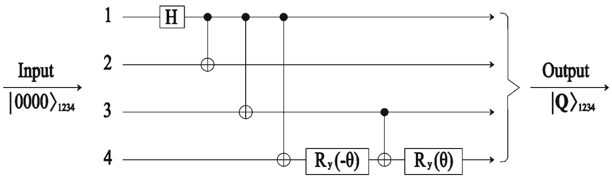

Fig. 1 Quantum circuit for preparing $|Q\rangle_{1234}$

the eavesdropper–a malicious enemy, the eavesdropper cannot determine the correct recovery actions because he/she has no information about the angle.

Now we proceed to constructing a quantum circuit that generates the state $|Q\rangle_{1234}$ from the initial separable state $|0000\rangle_{1234}$ as shown in Fig. 1.

In Fig. 1, the circuit is read from left to right and each single line denotes a qubit. This quantum circuit contains a single-qubit Hadamard gate $H = \frac{1}{\sqrt{2}}(|0\rangle\langle 0|+|1\rangle\langle 0|+|0\rangle\langle 1|-|1\rangle\langle 1|)$, the two-qubit controlled-Not gate (CNOT) $\mathscr{C}|i, j\rangle = |i, i \oplus j\rangle$ ($\oplus$ is an addition about modulo 2) and the rotation gate

$$R_y(\theta) = \begin{pmatrix} \cos(\theta/2) & -\sin(\theta/2) \\ \sin(\theta/2) & \cos(\theta/2) \end{pmatrix}$$

with θ the angle of rotation around y-axis. The Hadamard gate and the first three CNOTs transform the input state $|0000\rangle_{1234}$ to the well-known four-qubit GHZ state $|GHZ\rangle_{1234} = \frac{1}{\sqrt{2}}(|0000\rangle + |1111\rangle)_{1234}$. After the subsequent $R_y(-\theta)$, CNOT and $R_y(\theta)$ gates, $|GHZ\rangle_{1234}$ can be transformed to the desired four-qubit partially entangled state $|Q\rangle_{1234}$, expressed as (1). In this paper, we are interested in parameter $\theta \in (0, \pi/2)$ for which the state $|Q\rangle_{1234}$ is partially entangled with entanglement degree $|sin\theta|$ quantified by the concurrence.

3 Controlled JRSP for arbitrary two-qubit states

Suppose that the state to be prepared for a remote party, called the receiver Bob, has the form

$$|\phi\rangle_2 = a_0 e^{i\alpha_0}|00\rangle + a_1 e^{i\alpha_1}|01\rangle + a_2 e^{i\alpha_2}|10\rangle + a_3 e^{i\alpha_3}|11\rangle, \tag{2}$$

where real numbers $\alpha_j \in [0, 2\pi)$ $(j = 0, 1, 2, 3)$ and real numbers a_j $(j = 0, 1, 2, 3)$ satisfy $\sum_{j=0}^{3} a_j^2 = 1$. The values of a_0, a_1, a_2 and a_3 are given to Alice$_1$, while that of $\alpha_0, \alpha_1, \alpha_2$ and α_3 to Alice$_2$, who serve as the two preparers. Clearly, no one of the two preparers alone is able to infer $|\phi\rangle_2$. Let Charlie be the supervisor who, as Bob, knows nothing about $|\phi\rangle_2$. Quantum channel is composed of the following two partially entangled states

$$
\begin{aligned}
|Q\rangle_{A_1A_1'B_1C_1} &= \frac{1}{\sqrt{2}}(|0000\rangle + \cos\theta_1|1110\rangle + \sin\theta_1|1111\rangle)_{A_1A_1'B_1C_1},\\
|Q\rangle_{A_2A_2'B_2C_2} &= \frac{1}{\sqrt{2}}(|0000\rangle + \cos\theta_2|1110\rangle + \sin\theta_2|1111\rangle)_{A_2A_2'B_2C_2},
\end{aligned} \tag{3}
$$

where the real numbers $\theta_1, \theta_2 \in [0, \pi/2]$ whose values we let only the supervisor Charlie (not the receiver Bob) know. Without loss of generality, qubits (A_1, A_2) belong to Alice$_1$, while qubits (A_1', A_2'), (B_1, B_2) and (C_1, C_2) are hold by Alice$_2$, Bob and Charlie, respectively.

Step 1 For remotely preparing arbitrary two-qubit state (2), Alice$_1$ would perform the special measurement basis $\{|\xi_k\rangle | k = 0, 1, 2, 3\}$ on qubits (A_1, A_2), which are given by

$$
\begin{pmatrix} |\xi_0\rangle \\ |\xi_1\rangle \\ |\xi_2\rangle \\ |\xi_3\rangle \end{pmatrix} = \begin{pmatrix} a_0 & a_1 & a_2 & a_3 \\ a_1 & -a_0 & a_3 & -a_2 \\ a_2 & -a_3 & -a_0 & a_1 \\ a_3 & a_2 & -a_1 & -a_0 \end{pmatrix} \begin{pmatrix} |00\rangle \\ |01\rangle \\ |10\rangle \\ |11\rangle \end{pmatrix}. \tag{4}
$$

She but no one else can do that since a_0, a_1, a_2 and a_3 are known only to her. Under this basis, the whole quantum system $|Q\rangle_{A_1A_1'B_1C_1} \otimes |Q\rangle_{A_2A_2'B_2C_2}$ consisting of the eight qubits can be expressed as

$$
\begin{aligned}
&|Q\rangle_{A_1A_1'B_1C_1} \otimes |Q\rangle_{A_2A_2'B_2C_2}\\
&= \frac{1}{2}|\xi_0\rangle_{A_1A_2}\left(\sum_{i,j}|P_{\varepsilon_1^i}P_{\varepsilon_2^j}|\varepsilon_1^i\rangle_{C_1}|\varepsilon_2^j\rangle_{C_2}\right.\\
&\quad (a_0|0000\rangle + (-1)^j a_1|0011\rangle + (-1)^i a_2|1100\rangle + (-1)^{i+j} a_3|1111\rangle)_{A_1'B_1A_2'B_2}\\
&\quad + \frac{1}{2}|\xi_1\rangle_{A_1A_2}\left(\sum_{i,j}|P_{\varepsilon_1^i}P_{\varepsilon_2^j}|\varepsilon_1^i\rangle_{C_1}|\varepsilon_2^j\rangle_{C_2}\right.\\
&\quad (a_1|0000\rangle + (-1)^{j+1} a_0|0011\rangle + (-1)^i a_3|1100\rangle + (-1)^{i+j+1} a_2|1111\rangle)_{A_1'B_1A_2'B_2}\\
&\quad + \frac{1}{2}|\xi_2\rangle_{A_1A_2}\left(\sum_{i,j}|P_{\varepsilon_1^i}P_{\varepsilon_2^j}|\varepsilon_1^i\rangle_{C_1}|\varepsilon_2^j\rangle_{C_2}\right.\\
&\quad (a_2|0000\rangle + (-1)^{j+1} a_3|0011\rangle + (-1)^{i+1} a_0|1100\rangle + (-1)^{i+j} a_1|1111\rangle)_{A_1'B_1A_2'B_2}\\
&\quad + \frac{1}{2}|\xi_3\rangle_{A_1A_2}\left(\sum_{i,j}|P_{\varepsilon_1^i}P_{\varepsilon_2^j}|\varepsilon_1^i\rangle_{C_1}|\varepsilon_2^j\rangle_{C_2}\right.\\
&\quad (a_3|0000\rangle + (-1)^j a_2|0011\rangle + (-1)^{i+1} a_1|1100\rangle + (-1)^{i+j+1} a_0|1111\rangle)_{A_1'B_1A_2'B_2},
\end{aligned} \tag{5}
$$

where $i, j \in \{+, -\}$, $(-1)^+ = (-1)^0$, $(-1)^- = (-1)^1$. $P_{\varepsilon_1^\pm}$, $P_{\varepsilon_2^\pm}$, $|\varepsilon_1^\pm\rangle$ and $|\varepsilon_2^\pm\rangle$ are given by

$$P_{\varepsilon_j^{\pm}} = \sqrt{\frac{1}{2}(1 \pm \cos\theta_j)}, \tag{6}$$

and

$$|\varepsilon_j^{\pm}\rangle = \frac{1 \pm \cos\theta_j}{\sqrt{2(1 \pm \cos\theta_j)}}|0\rangle \pm \frac{\sin\theta_j}{\sqrt{2(1 \pm \cos\theta_j)}}|1\rangle,\ j \in \{1, 2\}. \tag{7}$$

As followed from Eq. (4), when performing the projective measurement $\{|\xi_k\rangle | k = 0, 1, 2, 3\}$ on qubits (A_1, A_2) Alice$_1$ obtains a state $|\xi_k\rangle_{A_1A_2}$ randomly with an equal probability of 1/4 and then informs Alice$_2$, Bob and Charlie of k which corresponds the measurement result $|\xi_k\rangle$ via classical channel.

Step 2 Just after announcement of Alice$_1$ about her outcome k, Alice$_2$ starts to measure her two qubits A_1' and A_2' in a delicately chosen basis which is important to achieve unit success probability without adding the local operations. That is to say, Alice$_2$ not only uses the set $\{\alpha_0, \alpha_1, \alpha_2, \alpha_3\}$, what was given to her a priori, but should also take into account Alice$_1$'s measurement outcome in terms of k. Specifically, the basis $\{|\eta_m^{(k)}\rangle | m = 0, 1, 2, 3\}$ $(k = 0, 1, 2, 3)$ for Alice$_2$'s measurement can be described as

$$\begin{pmatrix} |\eta_0^{(k)}\rangle \\ |\eta_1^{(k)}\rangle \\ |\eta_2^{(k)}\rangle \\ |\eta_3^{(k)}\rangle \end{pmatrix} = \frac{1}{2}\mathscr{G}^{(k)}(\alpha) \begin{pmatrix} |00\rangle \\ |01\rangle \\ |10\rangle \\ |11\rangle \end{pmatrix} \tag{8}$$

with

$$\begin{aligned}
\mathscr{G}^{(0)}(\alpha) &= \mathscr{G}(\alpha_0, \alpha_1, \alpha_2, \alpha_3) \\
&= \begin{pmatrix} e^{-i\alpha_0} & e^{-i\alpha_1} & e^{-i\alpha_2} & e^{-i\alpha_3} \\ e^{-i\alpha_0} & -e^{-i\alpha_1} & e^{-i\alpha_2} & -e^{-i\alpha_3} \\ e^{-i\alpha_0} & -e^{-i\alpha_1} & -e^{-i\alpha_2} & e^{-i\alpha_3} \\ e^{-i\alpha_0} & e^{-i\alpha_1} & -e^{-i\alpha_2} & -e^{-i\alpha_3} \end{pmatrix}, \\
\mathscr{G}^{(1)}(\alpha) &= \mathscr{G}(\alpha_1, \alpha_0, \alpha_3, \alpha_2), \\
\mathscr{G}^{(2)}(\alpha) &= \mathscr{G}(\alpha_2, \alpha_3, \alpha_0, \alpha_1), \\
\mathscr{G}^{(3)}(\alpha) &= \mathscr{G}(\alpha_3, \alpha_2, \alpha_1, \alpha_0).
\end{aligned}$$

Of course, Alice$_2$ can do such actions since only Alice$_2$ knows $\alpha_0, \alpha_1, \alpha_2$ and α_3. For each specific k, the states $\{|\eta_m^{(k)}\rangle | m = 0, 1, 2, 3\}$ comprise a complete set of orthonormal basis in a four-dimensional Hilbert space. The method of using this basis to measure is named feed-forward measurement strategy. Without loss of generality, if Alice$_1$'s measurement outcome is $|\xi_1\rangle_{A_1A_2}$, i.e., $k = 1$, then the corresponding collapsed state of qubits $(A_1', B_1, C_1, A_2', B_2, C_2)$ could be presented as follows

$$
\begin{aligned}
&{}_{A_1A_2}\langle\xi_1|Q\rangle_{A_1A_1'B_1C_1}\otimes|Q\rangle_{A_2A_2'B_2C_2}\\
&=\frac{1}{4}|\xi_1\rangle_{A_1A_2}(|\eta_0^{(1)}\rangle_{A_1'A_2'}|\gamma_0\rangle+|\eta_1^{(1)}\rangle_{A_1'A_2'}|\gamma_1\rangle+|\eta_2^{(1)}\rangle_{A_1'A_2'}|\gamma_2\rangle+|\eta_3^{(1)}\rangle_{A_1'A_2'}|\gamma_3\rangle),
\end{aligned}
\tag{9}
$$

here $i, j \in \{+, -\}$, $(-1)^i$, $(-1)^j$ defines as before,

$$
\begin{aligned}
|\gamma_0\rangle &= \sum_{i,j} P_{\varepsilon_1^i} P_{\varepsilon_2^j} |\varepsilon_1^i\rangle_{C_1}|\varepsilon_2^j\rangle_{C_2}(a_1e^{i\alpha_1}|00\rangle+(-1)^{j+1}a_0e^{i\alpha_0}|01\rangle\\
&\quad+(-1)^i a_3e^{i\alpha_3}|10\rangle+(-1)^{i+j+1}a_2e^{i\alpha_2}|11\rangle)_{B_1B_2},\\
|\gamma_1\rangle &= \sum_{i,j} P_{\varepsilon_1^i} P_{\varepsilon_2^j} |\varepsilon_1^i\rangle_{C_1}|\varepsilon_2^j\rangle_{C_2}(a_1e^{i\alpha_1}|00\rangle+(-1)^{j}a_0e^{i\alpha_0}|01\rangle\\
&\quad+(-1)^i a_3e^{i\alpha_3}|10\rangle+(-1)^{i+j}a_2e^{i\alpha_2}|11\rangle)_{B_1B_2},\\
|\gamma_2\rangle &= \sum_{i,j} P_{\varepsilon_1^i} P_{\varepsilon_2^j} |\varepsilon_1^i\rangle_{C_1}|\varepsilon_2^j\rangle_{C_2}(a_1e^{i\alpha_1}|00\rangle+(-1)^{j}a_0e^{i\alpha_0}|01\rangle\\
&\quad+(-1)^{i+1} a_3e^{i\alpha_3}|10\rangle+(-1)^{i+j+1}a_2e^{i\alpha_2}|11\rangle)_{B_1B_2},\\
|\gamma_3\rangle &= \sum_{i,j} P_{\varepsilon_1^i} P_{\varepsilon_2^j} |\varepsilon_1^i\rangle_{C_1}|\varepsilon_2^j\rangle_{C_2}(a_1e^{i\alpha_1}|00\rangle+(-1)^{j+1}a_0e^{i\alpha_0}|01\rangle\\
&\quad+(-1)^{i+1} a_3e^{i\alpha_3}|10\rangle+(-1)^{i+j}a_2e^{i\alpha_2}|11\rangle)_{B_1B_2}.
\end{aligned}
$$

Step 3 Alice$_2$ implements the projective measurement $\{|\eta_m^{(1)}\rangle | m = 0, 1, 2, 3\}$ on qubits (A_1', A_2') and informs Bob and Charlie of m which corresponds the measurement result $|\eta_m^{(1)}\rangle$ via classical channel. At this stage of the protocol, although having heard both the results k and m, Bob is not yet in the position to get the target state. The deciding role is now played by the supervisor Charlie, who should carefully review the overall situation concerning the real necessity of carrying out the task. If there are any adverse problems, she decides to stop or postpone the task and do nothing. Otherwise, if everything is favorable, she decides to proceed toward completion of the task by appropriately manipulating her qubits. After hearing the measurement information from Alice$_1$ and Alice$_2$, Charlie measures the qubits C_1 and C_2 via the basis $\{|\varepsilon_1^{\pm}\rangle\}$ and $\{|\varepsilon_2^{\pm}\rangle\}$, respectively, and then, he publicly broadcasts his measurement outcomes via classical channel. Note that here Charlie's measurement is a simple projective one, so it is a complete measurement, as opposed to incomplete POVM measurement [21]. It's also worth mentioning that Charlie is the only one who knows the values of θ_1 and θ_2 so no unauthorized parties are able to correctly manipulate qubits C_1 and C_2 even when they capture that qubits. This is a striking advantage of using partially entangled resources rather than the maximally entangled one (in the latter case $\theta = \pi/2$ which is known to everybody).

After receiving the measurement information, Bob just denotes $k = 2k_1 + k_2,\ m = 2m_1 + m_2$ $(k_1, k_2, m_1, m_2 \in \{0, 1\})$. Obviously, this is the binary representation of k

and m. Then, he reconstruct the target state expressed as Eq. (2) by using corresponding unitary operations as follows

$$(\sigma_x \otimes I)^{k_1}(I \otimes \sigma_x)^{k_2}(\sigma_z \otimes I)^{i\oplus m_1\oplus k_1}(I \otimes \sigma_z)^{j\oplus m_1\oplus m_2\oplus k_1\oplus k_2}, \tag{10}$$

here $I = |0\rangle\langle 0| + |1\rangle\langle 1|$, $\sigma_x = |0\rangle\langle 1| + |1\rangle\langle 0|$ and $\sigma_z = |0\rangle\langle 0| - |1\rangle\langle 1|$ are the Pauli operations, and $\sigma_z^+ = \sigma_z^0,\ \sigma_z^- = \sigma_z^1$.

From Eq. (4), there are four measurement results of Alice$_1$. For the other three measurement outputs of Alice$_1$, the desired quantum state can also be prepared by the same method.

Step 4 Now consider the probability of success of this scheme. From Eq. (5), there are four measurement results of Alice$_2$, Alice$_2$ obtains the outcome $|\eta_m^{(1)}\rangle_{A_1'A_2'}$ with an equal probability of 1/4. For each $|\eta_m^{(1)}\rangle_{A_1'A_2'}$, there are four Charlie's measurement outcomes $|\varepsilon_1^+\rangle_{C_1}|\varepsilon_2^+\rangle_{C_2}$, $|\varepsilon_1^+\rangle_{C_1}|\varepsilon_2^-\rangle_{C_2}$, $|\varepsilon_1^-\rangle_{C_1}|\varepsilon_2^+\rangle_{C_2}$ and $|\varepsilon_1^-\rangle_{C_1}|\varepsilon_2^-\rangle_{C_2}$ with corresponding probability $P_{\varepsilon_1^+}P_{\varepsilon_2^+}$, $P_{\varepsilon_1^+}P_{\varepsilon_2^-}$, $P_{\varepsilon_1^-}P_{\varepsilon_2^+}$ and $P_{\varepsilon_1^-}P_{\varepsilon_2^-}$, respectively. Obviously, for any joint measurement of Alice$_1$, Alice$_2$ and Charlie, Bob can always take corresponding transformations to recover the arbitrary two-qubit state. The successful probability is always viewed as an important factor for remote state preparation, and the total probability of this scheme can be calculated as

$$\begin{aligned}
p &= \left[\left(\left(\frac{P_{\varepsilon_1^+}P_{\varepsilon_2^+}}{4}\right)^2 + \left(\frac{P_{\varepsilon_1^+}P_{\varepsilon_2^-}}{4}\right)^2 + \left(\frac{P_{\varepsilon_1^-}P_{\varepsilon_2^+}}{4}\right)^2 + \left(\frac{P_{\varepsilon_1^-}P_{\varepsilon_2^-}}{4}\right)^2\right) \times 4\right] \times 4 \\
&= (P_{\varepsilon_1^+}P_{\varepsilon_2^+})^2 + (P_{\varepsilon_1^+}P_{\varepsilon_2^-})^2 + (P_{\varepsilon_1^-}P_{\varepsilon_2^+})^2 + (P_{\varepsilon_1^-}P_{\varepsilon_2^-})^2 \\
&= (P^2_{\varepsilon_1^+} + P^2_{\varepsilon_1^-})(P^2_{\varepsilon_2^+} + P^2_{\varepsilon_2^-}) \\
&= 1.
\end{aligned}$$

From the above discussions, unit successful probability of this controlled JRSP scheme is achieved for whatever entanglement degree of quantum channel, an amazing and obviously superior to all previous protocols. Here the deterministic feature is produced by three factors at the same time: (a) the feed-forward measurement strategy adapted by the preparers, (b) the knowledge of θ by the supervisor (not receiver) and (c) the use of the partially entangled states $|Q\rangle_{A_1A_1'B_1C_1}$ and $|Q\rangle_{A_2A_2'B_2C_2}$ as the quantum channel. Since not only $p = 1$ but also no additional resources/operations are required at all, our controlled JRSP is perfect.

4 Controlled JRSP for arbitrary three-qubit states

The arbitrary three-qubit states can be written as

$$\begin{aligned}
|\phi\rangle_3 = a_0e^{i\alpha_0}|000\rangle + a_1e^{i\alpha_1}|001\rangle + a_2e^{i\alpha_2}|010\rangle + a_3e^{i\alpha_3}|011\rangle \\
+ a_4e^{i\alpha_4}|100\rangle + a_5e^{i\alpha_5}|101\rangle + a_6e^{i\alpha_6}|110\rangle + a_7e^{i\alpha_7}|111\rangle,
\end{aligned} \tag{11}$$

where the real numbers $\alpha_j \in [0, 2\pi)$, a_j $(j = 0, 1, \cdots, 7)$ satisfy $\sum_{j=1}^{7} a_j^2 = 1$. The sender Alice$_1$ only knows a_j $(j = 0, 1, \cdots, 7)$, while the sender Alice$_2$ only knows $\alpha_j \in [0, 2\pi)$ $(j = 0, 1, \cdots, 7)$. Neither the receiver Bob nor the supervisor Charlie knows anything about $|\phi\rangle_3$. Clearly, no one of the two senders alone can help Bob to reconstruct the original state $|\phi\rangle_3$ in Eq. (11). Quantum channel is composed of the following three partially entangled states

$$\begin{aligned}
|Q\rangle_{A_1A_1'B_1C_1} &= \frac{1}{\sqrt{2}}(|0000\rangle + \cos\theta_1|1110\rangle + \sin\theta_1|1111\rangle)_{A_1A_1'B_1C_1},\\
|Q\rangle_{A_2A_2'B_2C_2} &= \frac{1}{\sqrt{2}}(|0000\rangle + \cos\theta_2|1110\rangle + \sin\theta_2|1111\rangle)_{A_2A_2'B_2C_2},\\
|Q\rangle_{A_3A_3'B_3C_3} &= \frac{1}{\sqrt{2}}(|0000\rangle + \cos\theta_3|1110\rangle + \sin\theta_3|1111\rangle)_{A_3A_3'B_3C_3}.
\end{aligned} \tag{12}$$

In the above equation, the real numbers $\theta_j \in [0, \pi/2]$ $(j = 1, 2, 3)$ are known only to the supervisor Charlie. The initial state of the total system including qubits $A_i,\ A_i',\ B_i,\ C_i$ $(i = 1, 2, 3)$ can be given by

$$|\mathscr{Q}\rangle = |Q\rangle_{A_1A_1'B_1C_1} \otimes |Q\rangle_{A_2A_2'B_2C_2} \otimes |Q\rangle_{A_3A_3'B_3C_3}. \tag{13}$$

In this case, Alice$_1$, Alice$_2$, Bob and Charlie have qubits (A_1, A_2, A_3), (A_1', A_2', A_3'), (B_1, B_2, B_3) and (C_1, C_2, C_3), respectively. The detailed execution steps are as follows.

Step 1 For performing controlled JRSP of arbitrary three-qubit states, Alice$_1$ needs to perform a projective measurement on her qubits (A_1, A_2, A_3) with the three-qubit mutually orthogonal measurement basis $\{|\xi_k\rangle | k = 0, 1, \cdots, 7\}$. The special measurement basis is given by

$$\begin{pmatrix} |\xi_0\rangle \\ |\xi_1\rangle \\ |\xi_2\rangle \\ |\xi_3\rangle \\ |\xi_4\rangle \\ |\xi_5\rangle \\ |\xi_6\rangle \\ |\xi_7\rangle \end{pmatrix} = \begin{pmatrix} a_0 & a_1 & a_2 & a_3 & a_4 & a_5 & a_6 & a_7 \\ a_1 & -a_0 & a_3 & -a_2 & a_5 & -a_4 & -a_7 & a_6 \\ a_2 & -a_3 & -a_0 & a_1 & a_6 & a_7 & -a_4 & -a_5 \\ a_3 & a_2 & -a_1 & -a_0 & a_7 & -a_6 & a_5 & -a_4 \\ a_4 & -a_5 & -a_6 & -a_7 & -a_0 & a_1 & a_2 & a_3 \\ a_5 & a_4 & -a_7 & a_6 & -a_1 & -a_0 & -a_3 & a_2 \\ a_6 & a_7 & a_4 & -a_5 & -a_2 & a_3 & -a_0 & -a_1 \\ a_7 & -a_6 & a_5 & a_4 & -a_3 & -a_2 & a_1 & -a_0 \end{pmatrix} ad \begin{pmatrix} |000\rangle \\ |001\rangle \\ |010\rangle \\ |011\rangle \\ |100\rangle \\ |101\rangle \\ |110\rangle \\ |111\rangle \end{pmatrix}. \tag{14}$$

There may be eight measurement outcomes on qubits (A_1, A_2, A_3). After Alice$_1$ measured the states of these qubits, the quantum channel $|\mathscr{Q}\rangle$ has collapsed to

$$|\mathscr{Q}^{(k)}\rangle_{A_1'B_1C_1A_2'B_2C_2A_3'B_3C_3} = {}_{A_1A_2A_3}\langle\xi_k|\mathscr{Q}\rangle \ \ (k = 0, 1, \cdots, 7).$$

Then, via classical channel, Alice$_1$ sends the message k to Alice$_2$, Bob and Charlie about her the projective measurement outcome $|\xi_k\rangle$.

Step 2 By using feed-forward measurement, Alice$_2$ utilizes Alice$_1$'s measurement message k and the set $\{\alpha_0, \alpha_1, \cdots, \alpha_7\}$ to construct the corresponding measurement basis $\{|\eta_m^{(k)}\rangle | m = 0, 1, \cdots, 7\}$ $(k = 0, 1, \cdots, 7)$, which is given by

$$\begin{pmatrix} |\eta_0^{(k)}\rangle \\ |\eta_1^{(k)}\rangle \\ \vdots \\ |\eta_7^{(k)}\rangle \end{pmatrix} = \frac{1}{2\sqrt{2}} \mathscr{W}^{(k)}(\alpha) \begin{pmatrix} |000\rangle \\ |001\rangle \\ \vdots \\ |111\rangle \end{pmatrix}, \tag{15}$$

where

$$\begin{aligned} \mathscr{W}^{(0)}(\alpha) &= \mathscr{W}(\alpha_0, \alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6, \alpha_7) \\ &= \begin{pmatrix} e^{-i\alpha_0} & e^{-i\alpha_1} & e^{-i\alpha_2} & e^{-i\alpha_3} & e^{-i\alpha_4} & e^{-i\alpha_5} & e^{-i\alpha_6} & e^{-i\alpha_7} \\ e^{-i\alpha_0} & -e^{-i\alpha_1} & e^{-i\alpha_2} & -e^{-i\alpha_3} & e^{-i\alpha_4} & -e^{-i\alpha_5} & -e^{-i\alpha_6} & e^{-i\alpha_7} \\ e^{-i\alpha_0} & -e^{-i\alpha_1} & -e^{-i\alpha_2} & e^{-i\alpha_3} & e^{-i\alpha_4} & e^{-i\alpha_5} & -e^{-i\alpha_6} & -e^{-i\alpha_7} \\ e^{-i\alpha_0} & e^{-i\alpha_1} & -e^{-i\alpha_2} & -e^{-i\alpha_3} & e^{-i\alpha_4} & -e^{-i\alpha_5} & e^{-i\alpha_6} & -e^{-i\alpha_7} \\ e^{-i\alpha_0} & -e^{-i\alpha_1} & -e^{-i\alpha_2} & -e^{-i\alpha_3} & -e^{-i\alpha_4} & e^{-i\alpha_5} & e^{-i\alpha_6} & e^{-i\alpha_7} \\ e^{-i\alpha_0} & e^{-i\alpha_1} & -e^{-i\alpha_2} & e^{-i\alpha_3} & -e^{-i\alpha_4} & -e^{-i\alpha_5} & -e^{-i\alpha_6} & e^{-i\alpha_7} \\ e^{-i\alpha_0} & e^{-i\alpha_1} & e^{-i\alpha_2} & -e^{-i\alpha_3} & -e^{-i\alpha_4} & e^{-i\alpha_5} & -e^{-i\alpha_6} & -e^{-i\alpha_7} \\ e^{-i\alpha_0} & -e^{-i\alpha_1} & e^{-i\alpha_2} & e^{-i\alpha_3} & -e^{-i\alpha_4} & -e^{-i\alpha_5} & e^{-i\alpha_6} & -e^{-i\alpha_7} \end{pmatrix}, \\ \mathscr{W}^{(1)}(\alpha) &= \mathscr{W}(\alpha_1, \alpha_0, \alpha_3, \alpha_2, \alpha_5, \alpha_4, \alpha_7, \alpha_6), \\ \mathscr{W}^{(2)}(\alpha) &= \mathscr{W}(\alpha_2, \alpha_3, \alpha_0, \alpha_1, \alpha_6, \alpha_7, \alpha_4, \alpha_5), \\ \mathscr{W}^{(3)}(\alpha) &= \mathscr{W}(\alpha_3, \alpha_2, \alpha_1, \alpha_0, \alpha_7, \alpha_6, \alpha_5, \alpha_4), \\ \mathscr{W}^{(4)}(\alpha) &= \mathscr{W}(\alpha_4, \alpha_5, \alpha_6, \alpha_7, \alpha_0, \alpha_1, \alpha_2, \alpha_3), \\ \mathscr{W}^{(5)}(\alpha) &= \mathscr{W}(\alpha_5, \alpha_4, \alpha_7, \alpha_6, \alpha_1, \alpha_0, \alpha_3, \alpha_2), \\ \mathscr{W}^{(6)}(\alpha) &= \mathscr{W}(\alpha_6, \alpha_7, \alpha_4, \alpha_5, \alpha_2, \alpha_3, \alpha_0, \alpha_1), \\ \mathscr{W}^{(7)}(\alpha) &= \mathscr{W}(\alpha_7, \alpha_6, \alpha_5, \alpha_4, \alpha_3, \alpha_2, \alpha_1, \alpha_0). \end{aligned}$$

After hearing Alice$_1$'s outcome k, Alice$_2$ performs a three-qubit measurement on qubits (A_1', A_2', A_3') with the basis $\{|\eta_m^{(k)}\rangle | m = 0, 1, \cdots, 7\}$ and informs Bob and Charlie of the measurement results. According to their priori agreement, a cbit message m corresponds to the measurement result $|\eta_m^{(k)}\rangle$. Then, the quantum state of the remaining six qubits B_1, B_2, B_3, C_1, C_2 and C_3 will collapse into the following state

$$|\mathscr{Q}_m^{(k)}\rangle_{B_1 C_1 B_2 C_2 B_3 C_3} = {}_{A_1' A_2' A_3'}\langle \eta_m^{(k)}|_{A_1 A_2 A_3}\langle \xi_k | \mathscr{Q}\rangle \ \ (m = 0, 1, \cdots, 7).$$

Step 3 After Charlie obtains the messages k and m, he needs to measure the qubits (C_1, C_2, C_3) in terms of the basis $\{|\varepsilon_j^{\pm}\rangle\}$, respectively. That is to say,

$$|\mathscr{Q}_m^{(k)}\rangle_{B_1C_1B_2C_2B_3C_3} = \frac{1}{8}\sum |\varepsilon_1^{\pm}\rangle_{C_1}|\varepsilon_2^{\pm}\rangle_{C_2}|\varepsilon_3^{\pm}\rangle_{C_3}|\mathscr{Q}^{\pm\pm\pm}\rangle_{B_1B_2B_3}, \tag{16}$$

where $|\mathscr{Q}^{\pm\pm\pm}\rangle_{B_1B_2B_3}$ is a combination of computable basis $\{|ijk\rangle : i, j, k = 0, 1, 2\}$ that the coefficients of measurement basis $|\varepsilon_j^{\pm}\rangle$ $(j = 1, 2, 3)$ in Eq. (7) depend on the messages k, m and the parameters $\cos\theta_j$, $\sin\theta_j$ of the jth quantum channel.

Without loss of generality, assume that Alice$_1$'s measurement result is $|\xi_2\rangle$ and Alice$_2$'s measurement outcome is $|\eta_5^{(2)}\rangle$, then the quantum state of the remaining qubits can be described as

$$\begin{aligned}
&|\mathscr{Q}_5^{(2)}\rangle_{B_1C_1B_2C_2B_3C_3} \\
&= {}_{A_1'A_2'A_3'}\langle\eta_5^{(2)}|_{A_1A_2A_3}\langle\xi_2|\mathscr{Q}\rangle \\
&= \frac{1}{8}\sum_{|\zeta_{C_j}\rangle=|\varepsilon_j^{\pm}\rangle_{C_j}} P_{\varepsilon_1}P_{\varepsilon_2}P_{\varepsilon_3}|\zeta_{C_1}\rangle\otimes|\zeta_{C_2}\rangle\otimes|\zeta_{C_3}\rangle \\
&\quad\otimes[a_2e^{i\alpha_2}|000\rangle - (-1)^{\langle\varepsilon_3^-|\zeta_{C_3}\rangle}a_3e^{i\alpha_3}|001\rangle + (-1)^{\langle\varepsilon_2^-|\zeta_{C_2}\rangle}a_0e^{i\alpha_0}|010\rangle \\
&\quad+(-1)^{\langle\varepsilon_2^-|\zeta_{C_2}\rangle}(-1)^{\langle\varepsilon_3^-|\zeta_{C_3}\rangle}a_1e^{i\alpha_1}|011\rangle - (-1)^{\langle\varepsilon_1^-|\zeta_{C_1}\rangle}a_6e^{i\alpha_6}|100\rangle \\
&\quad-(-1)^{\langle\varepsilon_1^-|\zeta_{C_1}\rangle}(-1)^{\langle\varepsilon_3^-|\zeta_{C_3}\rangle}a_7e^{i\alpha_7}|101\rangle + (-1)^{\langle\varepsilon_1^-|\zeta_{C_1}\rangle}(-1)^{\langle\varepsilon_2^-|\zeta_{C_2}\rangle}a_4e^{i\alpha_4}|110\rangle \\
&\quad-(-1)^{\langle\varepsilon_1^-|\zeta_{C_2}\rangle}(-1)^{\langle\varepsilon_2^-|\zeta_{C_2}\rangle}(-1)^{\langle\varepsilon_3^-|\zeta_{C_3}\rangle}a_5e^{i\alpha_5}|111\rangle]_{B_1B_2B_3},
\end{aligned} \tag{17}$$

Here, the state $|\zeta_{C_j}\rangle$ $(j = 1, 2, 3)$ represents the measurement result of the qubit C_j, and P_{ε_j} are given by

$$P_{\varepsilon_j} = \begin{cases} P_{\varepsilon_j^+}, \ |\zeta_{C_j}\rangle = |\varepsilon_j^+\rangle_{C_j}; \\ P_{\varepsilon_j^-}, \ |\zeta_{C_j}\rangle = |\varepsilon_j^-\rangle_{C_j}. \end{cases}$$

Charlie performs the single-qubit measurement on qubits C_1, C_2 and C_3, respectively, and announces the measurement results to Bob via classical channel.

Step 4 According to the classical information from Alice$_1$, Alice$_2$ and Charlie, Bob needs to take corresponding unitary transformations $U_m^{(k)}$ $(k, m \in \{0, 1, \cdots, 7\})$ to restore the initial state in Eq. (11). Suppose that the measurement results of qubits (A_1, A_2, A_3), (A_1', A_2', A_3') and (C_1, C_2, C_3) are $|\xi_2\rangle$, $|\eta_5^{(2)}\rangle$, $|\zeta_{C_1}\rangle$, $|\zeta_{C_2}\rangle$ and $|\zeta_{C_3}\rangle$, respectively, then the unitary transformation $U_5^{(2)}$ Bob needs to take is expressed as

$$
\begin{aligned}
U_5^{(2)} &= |010\rangle\langle 000| - (-1)^{\langle \varepsilon_3^- | \zeta_{C_3} \rangle} |011\rangle\langle 001| + (-1)^{\langle \varepsilon_2^- | \zeta_{C_2} \rangle} |000\rangle\langle 010| \\
&\quad + (-1)^{\langle \varepsilon_2^- | \zeta_{C_2} \rangle} (-1)^{\langle \varepsilon_3^- | \zeta_{C_3} \rangle} |001\rangle\langle 011| - (-1)^{\langle \varepsilon_1^- | \zeta_{C_1} \rangle} |110\rangle\langle 100| \\
&\quad - (-1)^{\langle \varepsilon_1^- | \zeta_{C_1} \rangle} (-1)^{\langle \varepsilon_3^- | \zeta_{C_3} \rangle} |111\rangle\langle 101| + (-1)^{\langle \varepsilon_1^- | \zeta_{C_1} \rangle} (-1)^{\langle \varepsilon_2^- | \zeta_{C_2} \rangle} |100\rangle\langle 110| \\
&\quad - (-1)^{\langle \varepsilon_1^- | \zeta_{C_1} \rangle} (-1)^{\langle \varepsilon_2^- | \zeta_{C_2} \rangle} (-1)^{\langle \varepsilon_3^- | \zeta_{C_3} \rangle} |101\rangle\langle 111|.
\end{aligned}
$$

As for the other measurement results of Alice$_1$, Alice$_2$ and Charlie, we can use the similar method to implement the controlled JRSP protocol for arbitrary three-qubit states.

From the above discussions, it can be seen that for any measurement outcome, Bob can always use the corresponding unitary transformation on qubits (B_1, B_2, B_3) to reconstruct the initial state. Therefore, our scheme can be used to determinately prepare arbitrary three qubit states, and the successful probability can reach up to 100%.

5 Controlled JRSP for arbitrary *n*-qubit states

For the convenience of discussion, we give a recursive method of constructing a new kind of matrix from a given matrix: let $\mathscr{W}(\beta_1, \beta_2, \cdots, \beta_{2^n})$ be a matrix composed of 2^n column vectors $\beta_1, \beta_2, \cdots, \beta_{2^n}$, we use recursive method to construct 2^{n-1} new matrices.

Recursive method of constructing matrices (RMCM)

(I) When $n = 1$, we exchange β_1 and β_2 in $\mathscr{W}^{(1)}(\beta_1, \beta_2) = \mathscr{W}(\beta_1, \beta_2)$ to obtain a new matrix $\mathscr{W}^{(2)}(\beta_2, \beta_1)$.

(II) When $n = 2$, we first divide the vectors $\beta_1, \beta_2, \beta_3$ and β_4 in $\mathscr{W}^{(1)}(\beta_1, \beta_2, \beta_3, \beta_4) = \mathscr{W}(\beta_1, \beta_2, \beta_3, \beta_4)$ into two groups and obtain the matrix $\mathscr{W}((\beta_1, \beta_2), (\beta_3, \beta_4))$. Secondly, we obtain matrix $\mathscr{W}((\beta_3, \beta_4), (\beta_1, \beta_2)) = \mathscr{W}^{(3)}(\beta_3, \beta_4, \beta_1, \beta_2)$ by exchanging (β_1, β_2) and (β_3, β_4). Finally, for each group in $\mathscr{W}((\beta_1, \beta_2), (\beta_3, \beta_4))$ and $\mathscr{W}((\beta_3, \beta_4), (\beta_1, \beta_2))$, we get $\mathscr{W}^{(2)}(\beta_2, \beta_1, \beta_4, \beta_3)$ and $\mathscr{W}^{(4)}(\beta_4, \beta_3, \beta_2, \beta_1)$ by using the method in (I). That is, we have

$$
\begin{aligned}
&\mathscr{W}^{(1)}(\beta_1, \beta_2, \beta_3, \beta_4), \mathscr{W}^{(2)}(\beta_2, \beta_1, \beta_4, \beta_3), \\
&\quad \mathscr{W}^{(3)}(\beta_3, \beta_4, \beta_1, \beta_2), \mathscr{W}^{(4)}(\beta_4, \beta_3, \beta_2, \beta_1),
\end{aligned}
$$

which is exactly the 2^2 matrices we need.

(III) when $n = k + 1$, we divide the vectors $\beta_1, \beta_2, \cdots, \beta_{2^{k+1}}$ in $\mathscr{W}^{(1)}(\beta_1, \beta_2, \cdots, \beta_{2^{k+1}}) = \mathscr{W}(\beta_1, \beta_2, \cdots, \beta_{2^{k+1}})$ into two groups and obtain the matrix $\mathscr{W}((\beta_1, \beta_2, \cdots, \beta_{2^k}), (\beta_{2^k+1}, \beta_{2^k+2}, \cdots, \beta_{2^{k+1}}))$, then exchange $(\beta_1, \beta_2, \cdots, \beta_{2^k})$ and $(\beta_{2^k+1}, \beta_{2^k+2}, \cdots, \beta_{2^{k+1}})$ to get the matrix

$$
\begin{aligned}
&\mathscr{W}^{(2^k+1)}(\beta_{2^k+1}, \beta_{2^k+2}, \cdots, \beta_{2^{k+1}}, \beta_1, \beta_2, \cdots, \beta_{2^k}) \\
&\quad = \mathscr{W}((\beta_{2^k+1}, \beta_{2^k+2}, \cdots, \beta_{2^{k+1}}), (\beta_1, \beta_2, \cdots, \beta_{2^k})).
\end{aligned}
$$

For each group in $\mathscr{W}((\beta_1, \beta_2, \cdots, \beta_{2^k}), (\beta_{2^k+1}, \beta_{2^k+2}, \cdots, \beta_{2^{k+1}}))$ and $\mathscr{W}((\beta_{2^k+1}, \beta_{2^k+2}, \cdots, \beta_{2^{k+1}}), (\beta_1, \beta_2, \cdots, \beta_{2^k}))$, we can obtain the 2^{k+1} desired matrices

$$\begin{aligned}&\mathscr{W}^{(1)}(\beta_1, \beta_2, \beta_3, \beta_4, \cdots, \beta_{2^{k+1}}),\\ &\quad\mathscr{W}^{(2)}(\beta_2, \beta_1, \beta_4, \beta_3, \cdots, \beta_{2^{k+1}}, \beta_{2^{k+1}-1}),\\ &\quad\cdots\\ &\quad\mathscr{W}^{(2^{k+1})}(\beta_{2^{k+1}}, \beta_{2^{k+1}-1}, \cdots, \beta_2, \beta_1)\end{aligned}$$

by using the technique in case $n = k$.

Now turn to our quantum task: we will extend the controlled JRSP protocols of preparing arbitrary two- and three-qubit states to arbitrary multi-qubit states. The n-qubit state can be written as

$$|\phi\rangle_n = \sum_{x=0}^{2^n-1} a_x e^{i\alpha_x} |d_n \cdots d_2 d_1\rangle, \quad d_j \in \{0, 1\}, x = \sum_{j=1}^{n} d_j \cdot 2^{j-1}, \tag{18}$$

here the real numbers a_x $(x = 0, 1, \cdots, 2^n - 1)$ satisfy the condition $\sum_{x=0}^{2^n-1} a_x^2 = 1$, and the real numbers $\alpha_x \in [0, 2\pi)$. The sender Alice$_1$ only knows a_x $(x = 0, 1, \cdots, 2^n - 1)$, while the sender Alice$_2$ only knows $\alpha_x \in [0, 2\pi)$ $(x = 0, 1, \cdots, 2^n - 1)$. Neither the receiver Bob nor the supervisor Charlie knows anything about $|\phi\rangle_n$.

In this case, for preparing arbitrary n-qubit states, we need to employ the following n partially entangled states

$$\begin{aligned}|\mathscr{Q}\rangle_{A_j A'_j B_j C_j} = \;& \frac{1}{\sqrt{2}}(|0000\rangle + \cos\theta_j |1110\rangle \\ &+ \sin\theta_j |1111\rangle)_{A_j A'_j B_j C_j}, \quad j = 1, 2, \cdots, n\end{aligned} \tag{19}$$

as quantum channel, where $\theta_j \in [0, \pi/2]$ $(j = 1, 2, \cdots, n)$ whose values are known only to the supervisor Charlie. The qubits A_j, A'_j, B_j and C_j belong to the sender Alice$_1$, Alice$_2$, the receiver Bob and the supervisor Charlie, respectively. The controlled JRSP procedure is composed of the following four steps.

Step 1 According to her own information a_x $(x = 0, 1, \cdots, 2^n - 1)$, Alice$_1$ needs to construct the special basis $\{|\xi_k\rangle | k = 0, 1, \cdots, 2^n - 1\}$ to measure her qubits $A_j (j = 0, 1, \cdots, 2^n - 1)$:

$$[|\xi_j\rangle]^T = [|\xi_0\rangle, |\xi_1\rangle, \cdots, |\xi_{2^n-1}\rangle]^T = U_n^n [|\lambda\rangle]^T, \tag{20}$$

where $[|\lambda\rangle] = [|0 \cdots 00\rangle, |0 \cdots 01\rangle, \cdots, |1 \cdots 11\rangle]$. In Ref.[23], the concrete construction processes of $2^n \times 2^n$ matrix U_n^n can be elaborated as follow: First, set the elements of 1th and $(2^{n-1} + 1)$th rows together with the 1th and $(2^{n-1} + 1)$th columns as (14); Second, if $n \geq 2$, set $U_n^n(i, j) = U_{n-1}^{n-1}(i, j)$, here $2 \leq i, j \leq 2^{n-1}$.

The $U_n^n(i,j)$ represents the element in the ith row and the jth column of the matrix U_n^n; Third, according to the special properties of the unitary matrix, set $\sum_{i=1}^{2^n} U_n^n(i,j)\cdot U_n^n(i,k)=0$, here $2\le j\le 2^{n-1}$, $k=1,2^{n-1}+1$. Then we can obtain the elements from 2nd column to 2^{n-1}th of the matrix U_n^n; Finally, based on the equation $\sum_{j=1}^{2^n} U_n^n(i,j)\cdot U_n^n(l,j)=0(2\le i\le 2^n, l=1,2^{n-1}+1, i\ne l)$, the elements from $(2^{n-1}+2)$th column to 2^nth column can be derived. Hence, from the above steps the whole elements of the matrix U_n^n can be obtained.

$$U_n^n=\begin{pmatrix} a_0 & a_1 & \cdots & a_{2^{n-1}-1} & a_{2^{n-1}} & a_{2^{n-1}+1} & \cdots & a_{2^n-1} \\ a_1 & \ddots & \cdots & \cdots & a_{2^{n-1}+1} & \cdots & \cdots & \cdots \\ \vdots & \vdots & \ddots & \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{2^{n-1}-1} & \vdots & \vdots & \ddots & a_{2^n-1} & \cdots & \cdots & \cdots \\ a_{2^{n-1}} & -a_{2^{n-1}+1} & \vdots & -a_{2^n-1} & -a_0 & a_1 & \cdots & a_{2^{n-1}-1} \\ a_{2^{n-1}+1} & \vdots & \vdots & \vdots & -a_1 & \ddots & \cdots & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \cdots \\ a_{2^n-1} & \vdots & \vdots & \vdots & -a_{2^{n-1}-1} & \vdots & \vdots & \ddots \end{pmatrix}. \tag{21}$$

Actually, when $n=1$, we can obtain

$$U_1^1=\begin{pmatrix} a_0 & a_1 \\ a_1 & -a_0 \end{pmatrix},$$

and when $n\ge 2$, we can use the above method to obtain the matrix U_n^n.

Step 2 By using information α_j $(j=0,1,\cdots,2^n-1)$ and Alice$_1$'s measurement outcome k, Alice$_2$ should construct a special basis $\{|\eta_m^{(k)}\rangle|m=0,1,\cdots,2^n-1\}$ to measure her particle A'_j $(j=0,1,\cdots,2^n-1)$.

$$[|\eta_m^{(k)}\rangle]^T=[|\eta_0^{(k)}\rangle,|\eta_1^{(k)}\rangle,\cdots,|\eta_{2^n-1}^{(k)}\rangle]^T=\frac{1}{\sqrt{2^n}}\mathscr{W}_{2^n}^{(k)}(\alpha)[|\lambda\rangle]^T, \tag{22}$$

where the construction process of the matrix $\mathscr{W}_{2^n}^{(1)}(\alpha)$ can described as follows by using the matrix U_n^n: replaces column 1, column 2, $\cdots$, column 2^n of the matrix U_n^n with $e^{-i\alpha_0}\Gamma$, $e^{-i\alpha_1}\Gamma$, $\cdots$, $e^{-i\alpha_{2^n-1}}\Gamma$, respectively, here Γ is a $2^n\times 1$ matrix $(1,1,\cdots,1)^T$, i.e., $\Gamma=(1,1,\cdots,1)^T$. And then the sign (positive or negative sign) of each element in $\mathscr{W}_{2^n}^{(1)}(\alpha)$ is exactly the same as that of the corresponding element in U_n^n. Obviously, $\mathscr{W}_{2^n}^{(1)}(\alpha)$ depends on $\alpha_0,\alpha_1,\cdots,\alpha_{2^n-1}$; thus, $\mathscr{W}_{2^n}^{(1)}(\alpha)$ can be denoted as $\mathscr{W}_{2^n}^{(1)}(\alpha_0,\alpha_1,\cdots,\alpha_{2^n-1})$. Based on $\mathscr{W}_{2^n}^{(1)}(\alpha_0,\alpha_1,\cdots,\alpha_{2^n-1})$, we can obtain $\mathscr{W}_{2^n}^{(2)}(\alpha)$, $\mathscr{W}_{2^n}^{(3)}(\alpha)$, $\cdots$, $\mathscr{W}_{2^n}^{(2^n-1)}(\alpha)$ by using the above RMCM.

It is worth mentioning that for a fixed k, $\{|\eta_m^{(k)}\rangle|m=0,1,\cdots,2^n-1\}$ is a set of completely orthogonal bases in Hilbert space C^{2^n}.

Step 3 After Charlie hears the information from Alice$_1$ and Alice$_2$, he needs to implement measurement basis $|\varepsilon_j^{\pm}\rangle$ in Eq. (7) on qubit C_j one by one and announces his measurement results to Bob in a classic channel subsequently. Thus, the quantum state of the whole system can be written as

$$
\begin{aligned}
|\mathscr{Q}\rangle_{\text{total}} &= |\mathscr{Q}\rangle_{A_1 A_1' B_1 C_1} \otimes |\mathscr{Q}\rangle_{A_2 A_2' B_2 C_2} \otimes \cdots \otimes |\mathscr{Q}\rangle_{A_n A_n' B_n C_n} \\
&= \sum_{|\zeta_{C_j}\rangle = |\varepsilon_j^{\pm}\rangle_{C_j}} \sum_{m,k=0,1,\cdots,2^n-1} \frac{1}{2^n} \prod_{j=1}^{n} P_{\varepsilon_j} |\xi_k\rangle_{A_1 A_2 \cdots A_n} |\eta_m^{(k)}\rangle_{A_1' A_2' \cdots A_n'} \\
&\quad \otimes |\zeta_{C_1}\rangle \otimes |\zeta_{C_2}\rangle \otimes \cdots \otimes |\zeta_{C_n}\rangle \otimes |T_m^{(k)}\rangle_{B_1 B_2 \cdots B_n},
\end{aligned}
\tag{23}
$$

where the state $|\zeta_{C_j}\rangle$ $(j = 1, 2, \cdots, n)$ represents the measurement result of the qubit C_j, and P_{ε_j} are given by

$$
P_{\varepsilon_j} = \begin{cases} P_{\varepsilon_j^+}, & |\zeta_{C_j}\rangle = |\varepsilon_j^+\rangle_{C_j}; \\ P_{\varepsilon_j^-}, & |\zeta_{C_j}\rangle = |\varepsilon_j^-\rangle_{C_j}, \end{cases}
$$

here $|\varepsilon_j^{\pm}\rangle$ is related to the parameters of the jth quantum channel. And the value of probability parameter P_{ε_j} depends on the measurement result of qubit C_j.

Step 4 After hearing all the measurement results from Alice$_1$, Alice$_2$ and Charlie, Bob needs to recover the initial state in Eq. (18) by adopting corresponding Pauli operations σ_x and σ_z that depends on the different value of n and measurement results. Definitely, the recover operator can be seen in Eq. (10) for $n = 2$.

It is worth emphasizing that our schemes are applicable to the controlled JRSP protocol for arbitrary multi-qubit states with the successful probability of 100%.

6 Discussion and conclusion

In Sects. 3–5, the total success probability for each scheme is 100% for whatever entanglement degree of the shared resource in terms of the quantum state $|\mathscr{Q}\rangle$, that is, our protocols are deterministic. It is commonly thought that the quality of a protocol scales with the degree of the shared entanglement. But, quite counter-intuitively, there exist kinds of information-theoretic tasks for which less entanglement turns out to be more useful [27–29]. Coming back to our problem, one may ask: "How if the receiver Bob (not the supervisor Charlie) knows θ_j ?" In this case Charlie measures his qubits in the basis $\{|\pm\rangle = (|0\rangle \pm |1\rangle)/\sqrt{2}\}$. It is not difficult to verify that then Bob can still recover the target state by sacrificing additional resource and operations, yet always succeed he cannot.

In the controlled JRSP scheme proposed in this paper, we mainly apply entangled resources, projective measurements and multi-qubit unitary transformations. From Sect. 2, we can see that the entanglement resources we employ can be generated by unitary operators. It has been proved that arbitrary multi-qubit unitary operations can be composed of single-qubit unitary transformations and two-qubit CNOT gates in quantum information [30]. Meanwhile, the single-qubit unitary operations and two-qubit CNOT gates can be realized in different physical experiments, such as the linear optics [31,32], ion traps [33,34] and cavity QED system [35]. Consequently, the entangled resources and multi-qubit unitary operations used in our schemes might be implemented in physical experiments. Moreover, it can be seen that single-qubit projective measurements and two-qubit projective measurements can be realized experimentally in practice [36,37]. Additionally, the multi-qubit projective measurement basis, similar to Eqs. (4, 13, 19) in this paper, is available for the previous RSP proposals [38–40], and the measurement basis in Eqs. (6), (15) and (21) are special cases in Eqs. (4), (13) and (20), respectively. Hence, our protocols might be realizable physically in near future.

In summary, we first construct the quantum circuit to output the state $|Q\rangle$ of Eq. (1). Next we have forward perfect schemes for controlled JRSP of arbitrary two- and three-qubit states and extended them to prepare arbitrary multi-qubit states via the quantum channel in terms of a suitable chosen non-maximally entangled resource $|Q\rangle$, whose entangled degree is determined by parameters θ_j. For each controlled JRSP scheme, we have given the specific measurement basis performed by the two senders and the supervisor, and also the unitary transformations needed to restore the initial states by the receiver. Traditionally, the receiver is allowed to know the values of some θ_j and he/she can recover his/her qubits to be in the desired state after hearing all the measurement outcomes. However, the receiver needs to pay for additional quantum resources, quantum operations and quantum measurements, yet the performance can only be probabilistic with the total success probability depending sensitively on some θ_j. In our schemes, we let the supervisor (instead of the receiver) know these θ_j. If so, the supervisor is able to do projective (not POVM) measurements on his/her qubits in the right basis determined by θ_j, so that the receiver needs a proper unitary transformation to faithfully obtain the target state without consuming anything else. Another crucial merit is that, combined with feed-forward measurements by the two preparers, the total success probability of each of our protocols is always 1, independent of the entanglement degree of the quantum channel. Although entanglement is necessary, any amount (even tiny) of it does equally well in our protocols. This feature is very interesting and amazing in exploring entanglement, especially partial entanglement, to complete global quantum tasks through local operation and classical communication.

Acknowledgements This work is supported partially by Natural Science Foundation of China (Grant Nos. 11071178, 11671284), Sichuan Provincial Natural Science Foundation of China (Grant. 2017JY0197) and Sichuan Province Education Department Scientific Research Innovation Team Foundation (No. 15TD0027).

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