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Average distillated coherence without complete waste of resources

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Abstract

To distill quantum coherence, in [Phys. Rev. A 99, 012321 (2019)], authors proposed a strictly incoherent operation that produces one of a family of maximally coherent states of variable dimension from any pure quantum state. For a d-dimensional pure state, an incoherent state may be obtained with a nonzero probability, which results in a complete waste of resource, namely the probability of transforming a given pure state to the incoherent state is not zero. Here, we give a specific method to avoid a complete waste of resource with the maximal probability to transform the pure state to a d-dimensional maximally coherent state and study the range of average coherence of the corresponding output state in this case. We also give a method to transform a mixed state to a maximally coherent state with probability.

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1 Introduction

The resource theory of quantum coherence [1–8] plays an important role in characterizing the intrinsic feature of quantum mechanics. Over the past decades, the properties of a resource have been extensively investigated under a suitable set of allowed free operations [9–14]. In this setting, quantum coherence is regarded as a useful quantum resource allowing us to do many tasks more efficiently. The free states, i.e., the incoherent states are defined as the diagonal density matrices under a fixed reference basis. Free operations are the ones that transform the free states to free states. To single out a unique class of free operations under which the operational features of coherence should be investigated, many definitions of free operations are proposed, such as the maximally incoherent operations (MIOs) [1], the incoherent operations (IOs) [2], the dephasing-covariant incoherent operations (DIOs) [9,15], and the strictly incoherent operations (SIOs) [16–21].

One of the most fundamental operational problems within a resource theory is how to manipulate the resource by using free operations. Among such manipulations, the coherence distillation stands out as one of the most important problems in the resource theory of quantum coherence [19,22–27]. It focuses on the conversion of copies of a given d-dimensional state ρ into the canonical unit resource $|\Phi_q\rangle$ (q-dimensional maximally coherent state). In a realistic setting, only finite supply of states is available. Thus, it is of significance to consider the one-shot version of coherence distillation [29–32].

In Ref. [26], the authors proposed a method to transform a pure state to a maximally coherent state with certain probability via SIOs. They presented an optimal probabilistic protocol to distill quantum coherence with reduced waste of resources. The protocol was expanded to possibly avert a complete waste of resources by exploiting an additional transformation into a suitable intermediate state.

In this paper, we give specific conditions when a d-dimensional pure state can be transformed to a maximally coherent state $|\Psi_d\rangle = \frac{1}{\sqrt{d}}\sum_i |i\rangle$ without complete waste of resource with the maximal probability for a given protocol proposed in Ref. [26]. We also give a protocol to transform a mixed state ρ to a maximally coherent state via a series of SIOs with probability based on the pure state decomposition of ρ. Moreover, we show that the probability to get a maximally coherent state is independent of the pure state decomposition.

2 Preliminary

For a given d-dimensional Hilbert space $\mathcal{H}$ with an orthonormal basis $\{|i\rangle\}_{i=1}^{d}$, the incoherent states are defined as: $\sigma = \sum_{i=1}^{d} p_i|i\rangle\langle i|$, with $p_i \geq 0$, and $\sum_{i=1}^{d} p_i = 1$. The set of all the incoherent states is denoted as $\mathcal{I}$. Without loss of generality, any

d-dimensional pure coherent state can be expressed as

$$|\psi\rangle = \sum_{i=1}^{d} \psi_i |i\rangle, \tag{1}$$

where $\{\psi_i\}$ are nonnegative real numbers in nonincreasing ordering, i.e., $\psi_i \geq \psi_{i+1} \geq 0$, and $\sum_{i=1}^{d} \psi_i^2 = 1$. Here, one should note that not every pure state can be expressed in this form, but it can be brought into this form by using an SIO unitary in a reversible manner.

The one-shot coherence distillation is the process of transforming a given state ρ into a maximally coherent pure state under different classes of free operations [28,29]. For a given state ρ and $\epsilon \geq 0$, the one-shot coherence distillation rate with error ϵ under free operation $\mathcal{O}$ is defined as

$$C_{d,\mathcal{O}}^{(1),\epsilon}(\rho) := \log \max\{m \in \mathbb{N}, F_{\mathcal{O}}(\rho, \psi_m) \geq 1 - \epsilon\},$$

where $F_{\mathcal{O}}(\rho, \psi_m) = \max_{\Lambda \in \mathcal{O}} \langle \Lambda(\rho), \psi_m \rangle$ is the fidelity. Such one-shot distillable rates can be quantified by the relative entropy of coherence [28].

A completely positive trace-preserving map Φ is called a strictly incoherent operation (SIO) if $\Phi(\rho) = \sum_i K_i \rho K_i^\dagger$, with the Kraus operators $\{K_i\}$ satisfying $K_i \mathcal{I} K_i^\dagger \subseteq \mathcal{I}$, $K_i^\dagger \mathcal{I} K_i \subseteq \mathcal{I}$, and $\sum_i K_i^\dagger K_i = I$. In this paper, by using a class of particular SIOs, we focus on transforming a single copy of the input state ρ given in Eq. (1) to a maximally coherent state with certain probability under incoherent operations. A class of SIOs Φ for the coherence distillation from $|\psi\rangle$ defined in Eq. (1) has been presented in [26],

$$\Phi(\rho) = \sum_{q=1}^{d} K_q \rho K_q^\dagger, \tag{2}$$

with the Kraus operators K_q satisfying

$$K_q |\psi\rangle = \sqrt{p_q} |\Psi_q\rangle, \tag{3}$$

where

$$K_q = \sqrt{p_q} \left(\frac{1}{\sqrt{q}} \sum_{i=1}^{q} \frac{|i\rangle\langle i|}{\psi_i} \right), \tag{4}$$

$$p_d = d\psi_d^2, \quad p_q = q(\psi_q^2 - \psi_{q+1}^2), \quad q = 1, 2, \ldots, d-1, \tag{5}$$

and $|\Psi_q\rangle = \frac{1}{\sqrt{q}} \sum_{j=1}^{q} |i\rangle$ $(1 \leq q \leq d)$ is the q-dimensional maximally coherent state. From Eqs. (3), (4), and (5), one can see that the expression of SIO depends on the pure state from which we want to distillate the maximally coherent states.

As the coherence cannot be increased under SIOs, one may want to know how much coherence is lost on average during the protocol. Here, we adopt the l_1-norm as the measure of coherence [2]. The l_1-norm of coherence is defined as

$$\mathcal{C}_{l_1}(\rho)=\sum_{i\neq j}|\rho_{ij}|,$$

where $\rho_{ij} = \langle i|\rho|j\rangle$. Thus, the average coherence for the output ensemble $\{p_q,|\Psi_q\rangle\}_{q=1,2,\ldots,d}$ of $|\psi\rangle_{out}$ can be given by [26]

$$\bar{\mathcal{C}}_{l_1}(|\psi\rangle_{out})=\sum_{j=2}^{d}(2j-2)\psi_j^2. \tag{6}$$

3 The extreme value of $\bar{\mathcal{C}}_{l_1}(|\psi\rangle_{out})$ without complete waste of resources

In this section, for a given initial pure state $|\psi\rangle$, we give a specific method to avoid a complete waste of resources with the maximal probability to get $|\Psi_d\rangle$, and calculate the range of the average coherence of $|\psi\rangle_{out}$ via SIOs defined in Eq. (2).

We first recall the concept of majorization introduced in [33,34]. For two d-dimensional real vectors $x = (x_1, x_2, \ldots, x_d)^t$ and $(y_1, y_2, \ldots, y_d)^t$, we say that x is majorized by y if $\sum_{j=1}^{k} x_j^{\downarrow} \le \sum_{j=1}^{k} y_j^{\downarrow}$, where $x_j^{\downarrow}$ $(y_j^{\downarrow})$ denotes the components of x (y) arranged in decreasing order. We denote $x \prec y$ if x is majorized by y.

Without a complete waste of resources means that the probability of transforming the given pure state $|\psi\rangle$ to the incoherent state $|\Psi_1\rangle$ is zero via SIO defined in Eq. (2), namely $p_1 = 0$ in Eq. (3) [26]. By different IOs, a pure state $|\psi\rangle$ can be transformed into different pure states $|\phi\rangle$, $|\psi\rangle \xrightarrow{IO} |\phi\rangle$ iff $|\psi\rangle \prec |\phi\rangle$ [35]. Thus, for the case $p_1 \neq 0$, there may exit many IOs to make $p_1 = 0$ while keep p_d unchanged if we first transform $|\psi\rangle$ to $|\phi\rangle$ via a proper IO. Specifically, when we say that $|\Psi_d\rangle$ can be distilled from a pure state $|\psi\rangle$ given in Eq. (1) with maximal probability via SIOs defined in Eq. (2) and avoid a complete waste of resources, we mean the following two cases: if $\psi_1 = \psi_2$, then $p_d = d\psi_d^2$ is the maximal probability to transform $|\psi\rangle$ to $|\Psi_d\rangle$, and $p_1 = 0$ via the SIOs defined in Eq. (2); if $\psi_1 > \psi_2$, and there exists at least one incoherent operation such that $|\psi\rangle \xrightarrow{IO} |\phi\rangle = \sum_{i=1}^{d} \phi_i |i\rangle$, where $\{\phi_i\}$ are nonnegative real numbers in nonincreasing ordering with $\phi_1 = \phi_2$ and $\phi_d = \psi_d$, one gets that $p_1' = 0$, and $p_d' = d\psi_d^2$ is the maximal probability, with p_1' and p_d' similarly defined in Eq. (5). One can see that if such IOs exist, $|\phi\rangle$ can be always transformed into a maximally coherent one via SIOs defined in Eq. (2) with $p_1' = 0$, and $p_d' = d\psi_d^2$, although p_1 may be larger than 0 for $|\psi\rangle$.

For the case $d = 2$, if $\psi_1 > \psi_2$, there are no IOs to change the value of ψ_1 while keep ψ_2 unchanged. Thus, $|\Psi_2\rangle$ cannot be distilled from a pure state $|\psi\rangle$ given in Eq. (1) for the case $\psi_1 > \psi_2$, with maximal probability via SIOs defined in Eq. (2) and avoid a complete waste of resources. For the case $d = 3$, if $\psi_1 > \psi_2$, there are no

IOs that transform $|\psi\rangle$ to $|\phi\rangle$ such that $\phi_1 = \phi_2 \geq \psi_1$ and $\phi_3 = \psi_3$. In other words, $|\Psi_3\rangle$ cannot be distilled from a pure state $|\psi\rangle$ given in Eq. (1) for the case $\psi_1 > \psi_2$, with maximal probability via SIOs defined in Eq. (2) and avoid a complete waste of resources. Nevertheless, for the cases $d \geq 4$, we have

Theorem 1 *For a pure state $|\psi\rangle$ given in (1), we can obtain the maximally coherent state $|\Psi_d\rangle$ with the maximal probability via SIOs defined in Eq. (2), and avoid a complete waste of resources at the same time if and only if $\sum_{j=2}^{d-1}\psi_j^2 \geq \psi_1^2 + (d-3)\psi_d^2$.*

Proof Suppose $\sum_{j=2}^{d-1}\psi_j^2 \geq \psi_1^2 + (d-3)\psi_d^2$. Let $|\phi\rangle = \sum_j \phi_j|j\rangle$ with $\phi_1 = \phi_2 = \sqrt{\frac{1-(d-2)\psi_d^2}{2}}$, and $\phi_j = \psi_d,\ j = 3, 4, \ldots, d$. Obviously, $\{\phi_j\}$ is in a nonincreasing order. We prove now there indeed exists an incoherent operation which transforms $|\psi\rangle$ to $|\phi\rangle$. In other words, we need to prove $\sum_{j=1}^{s}\psi_j^2 \leq \sum_{j=1}^{s}\phi_j^2$ for $s = 1, 2, \ldots, d$. From $\sum_{j=2}^{d-1}\psi_j^2 \geq \psi_1^2 + (d-3)\psi_d^2$, we see that $2\psi_1^2 + (d-2)\psi_d^2 \leq \sum_{j=2}^{d-1}\psi_j^2 + \psi_1^2 + \psi_d^2 = 1$. Thus, $\psi_1^2 \leq \frac{1-(d-2)\psi_d^2}{2} = \phi_1^2$. Then, one can also find that $\psi_1^2 + \psi_2^2 \leq 2\psi_1^2 \leq 2\phi_1^2 = \phi_1^2 + \phi_2^2$. For any $s \geq 3$, we have $\sum_{j=1}^{s}\psi_j^2 = 1 - \sum_{j=s+1}^{d}\psi_j^2 \leq 1 - (d-s)\psi_d^2 = \sum_{j=1}^{s}\phi_i^2$.

Now suppose that one wants to obtain the maximally coherent state $|\Psi_d\rangle$ with the maximal probability, and avoid a complete waste of resources at the same time via SIOs defined in Eq. (2), i.e., $p_1 = 0$ and $p_d = d\psi_d^2$. Assume there exists an incoherent operation which transforms $|\psi\rangle$ to $|\phi\rangle = \sum_i \phi_i|i\rangle$. Then, $\phi_1 = \phi_2 \geq \psi_1$, $\phi_d = \psi_d$, and $\phi_i \geq \phi_{i+1}$ for $i = 1, 2, \ldots, d-1$. Thus,

$$\begin{aligned}\sum_{j=2}^{d-1}\psi_j^2 &= 1 - (\psi_1^2 + \psi_d^2)\\ &= \sum_{j=1}^{d}\phi_d^2 - (\psi_1^2 + \psi_d^2)\\ &\geq 2\psi_1^2 + (d-2)\psi_d^2 - (\psi_1^2 + \psi_d^2)\\ &= \psi_1^2 + (d-3)\psi_d^2.\end{aligned} \tag{7}$$

□

In fact, for a given $|\psi\rangle$, there may exist many IOs satisfying $|\psi\rangle \xrightarrow{IO} |\phi\rangle$, such that one obtains $|\Psi_d\rangle$ with maximal probability and avoid a complete waste of resource at the same time. Different $|\phi\rangle$ given in $|\psi\rangle \xrightarrow{IO} |\phi\rangle$ gives rise to different values of $\overline{C}_{l_1}(|\phi\rangle_{out})$ via the corresponding SIOs defined in Eq. (2). Then, for a given pure state $|\psi\rangle$, we can get a range of $\overline{C}_{l_1}(|\phi\rangle_{out})$ over all $|\phi\rangle$ satisfying $\phi_1 = \phi_2$ and $\phi_d = \psi_d$. In the following, $\overline{C}_{l_1}(|\psi\rangle_{out})$ also stands for $\overline{C}_{l_1}(|\phi\rangle_{out})$ with $|\psi\rangle \xrightarrow{IO} |\phi\rangle$ satisfying $\phi_1 = \phi_2$, $\phi_d = \psi_d$ and $\psi_1 \geq \psi_2$.

Next, we study the lower and upper bounds of $\overline{C}_{l_1}(|\psi\rangle_{out})$ at the conditions of obtaining the maximally coherent state $|\Psi_d\rangle$ with maximal probability and avoiding a complete waste of resource at the same time via SIOs defined in Eq. (2).

Our approach can be summarized as follows. For remain the condition $p_1 = 0$ and $p_d = d\psi_d^2$, we should always fix $\phi_1 = \phi_2$ and $\phi_d = \psi_d$ while we transform $|\psi\rangle$ into

$|\phi\rangle$ under IOs. From Eq. (6), one can see that the value of $\bar{C}_{l_1}(|\psi\rangle_{out})$ depends on the values of $\{\psi_j\}$. Specifically, the value of ψ_j is related to its coefficient $2j-2$ and $\sum_{j=1}^{d}\psi_j^2=1$. Thus, for an arbitrary pure state $|\psi\rangle$ defined in Eq. (1), if we want to get the maximal value $\overline{C}_{l_1}(|\psi\rangle_{out})_{\max}$ of $\overline{C}_{l_1}(|\psi\rangle_{out})$, from Eq. (6), we should keep ϕ_i as large as possible under the condition $|\psi\rangle \xrightarrow{IO} |\phi\rangle$ when i increases. Similarly, if we want to get the minimal value $\overline{C}_{l_1}(|\psi\rangle_{out})_{\min}$ of $\overline{C}_{l_1}(|\psi\rangle_{out})$, we should keep ϕ_i as small as possible under the condition $|\psi\rangle \xrightarrow{IO} |\phi\rangle$ when i increases.

For $d=4$, from Theorem 1 we obtain the maximal probability of getting $|\Psi_4\rangle$ and avoid a complete waste of resource at the same time if and only if $\psi_1^2+\psi_4^2 \le \psi_2^2+\psi_3^2$. Hence, to get the $\overline{C}_{l_1}(|\psi\rangle_{out})_{\max}$, one should keep $\phi_1=\phi_2$ as small as possible under the condition $|\psi\rangle \xrightarrow{IO} |\phi\rangle$. Thus, $\phi_1=\phi_2=\psi_1$. In this case, $\phi_3=\sqrt{\psi_2^2+\psi_3^2-\psi_1^2}$ and $\overline{C}_{l_1}(|\psi\rangle_{out})_{\max}=4+2\psi_4^2-6\psi_1^2$. To get $\overline{C}_{l_1}(|\psi\rangle_{out})_{\min}$, one should keep $\phi_1=\phi_2$ as large as possible and ϕ_3 as small as possible under the condition $|\psi\rangle \xrightarrow{IO} |\phi\rangle$. Thus, $\phi_3=\phi_4=\psi_4$, $\phi_1=\phi_2=\sqrt{\frac{1-2\psi_4^2}{2}}$, and $\overline{C}_{l_1}(|\psi\rangle_{out})_{\min}=1+8\psi_4^2$. Easily, one can see that for both cases, $|\psi\rangle \prec |\phi\rangle$. Therefore, we have

Theorem 2 *For $d=4$ and $|\psi\rangle$ given in (1) with $\psi_1^2+\psi_4^2 \le \psi_2^2+\psi_3^2$, we have*

$$1+8\psi_4^2 \le \bar{C}_{l_1}(|\psi\rangle_{out}) \le 4+2\psi_4^2-6\psi_1^2, \tag{8}$$

if the maximally coherent state $|\Psi_4\rangle$ is obtained with the maximal probability via SIOs defined in Eq. (2) without a complete waste of resources. Moreover, the left equality holds if $\phi_1=\phi_2=\sqrt{\frac{1-2\psi_4^2}{2}}$ and $\phi_3=\phi_4=\psi_4$. The right equality holds if $\phi_1=\phi_2=\psi_1$, $\phi_3=\sqrt{\psi_2^2+\psi_3^2-\psi_1^2}$ and $\phi_4=\psi_4$.

To illustrate Theorem 2, we show the lower and upper bounds of $\bar{C}_{l_1}(|\psi\rangle_{out})$ in Fig. 1, where the intersecting line of the green and blue surfaces stands for the case $\psi_1^2+\psi_4^2=\frac{1}{2}$, and from the nonincreasing order of $\{\psi_j\}_{j=1}^4$ and $\sum_{j=1}^4\psi_j^2=1$, we have $0\le\psi_4\le\frac{1}{2}$ and $\sqrt{\frac{1-\psi_4^2}{3}} \le \psi_1 \le \sqrt{\frac{1}{2}-\psi_4^2}$. Specially, for $\psi_4=0$, see Fig. 2.

For $d=5$, from (6) one gets $\overline{C}_{l_1}(|\psi\rangle_{out})_{\max}$ when $\phi_1=\phi_2=\psi_1$, $\phi_3=\sqrt{\psi_2^2+\psi_3^2-\psi_1^2}$, $\phi_4=\psi_4$ and $\phi_5=\psi_5$ under the condition $|\psi\rangle \xrightarrow{IO} |\phi\rangle$ when $\psi_1^2+\psi_4^2 \le \psi_2^2+\psi_3^2$ via SIOs defined in Eq. (2). In this case, $\overline{C}_{l_1}(|\psi\rangle_{out})_{\max}=4\psi_2^2+4\psi_3^2+6\psi_4^2+8\psi_5^2-2\psi_1^2$. Similar to the discussions on the case of $d=4$, we have $\overline{C}_{l_1}(|\psi\rangle_{out})_{\min}=1+15\psi_5^2$. However, for $\psi_1^2+\psi_4^2>\psi_2^2+\psi_3^2$, there exist no incoherent operations such that $|\psi\rangle \xrightarrow{IO} |\phi\rangle$ with $\phi_1=\phi_2=\psi_1$, $\phi_3=\sqrt{\psi_2^2+\psi_3^2-\psi_1^2}$, $\phi_4=\psi_4$ and $\phi_5=\psi_5$, since one can find that $\phi_3<\phi_4$. Thus, one needs to change ϕ_3 and ϕ_4 while keeps $\phi_1=\phi_2=\psi_1$. From Eq. (6), we have $\phi_3=\phi_4=\sqrt{\frac{\psi_2^2+\psi_3^2+\psi_4^2-\psi_1^2}{2}}$. However, the condition $|\psi\rangle \xrightarrow{IO} |\phi\rangle$ requires

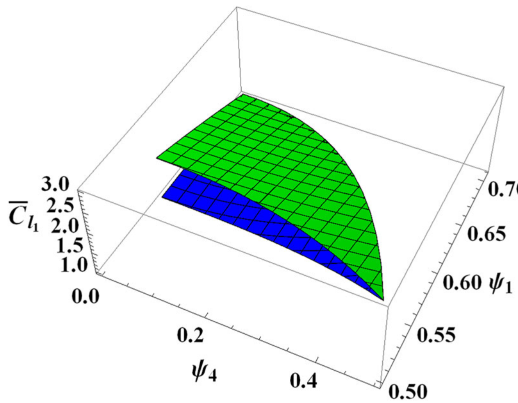

Fig. 1 The green surface corresponds to the maximal value of $\bar{C}_{l_1}(|\psi\rangle_{out})$, and the blue one corresponds to the minimal value of $\bar{C}_{l_1}(|\psi\rangle_{out})$. From the relations $\sum_{i=1}^{4}\psi_i^2 = 1, \psi_1 \geq \psi_2 \geq \psi_3 \geq \psi_4$, and $\psi_1^2+\psi_4^2 \leq \psi_2^2+\psi_3^2$, one can see $0 \leq \psi_4 \leq \frac{1}{2}$, and $\sqrt{\frac{1-\psi_4^2}{3}} \leq \psi_1 \leq \sqrt{\frac{1-2\psi_4^2}{2}}$. Particularly, from Fig. 1, one can see that for the case $\psi_1 = \psi_4 = \frac{1}{2}$, $\bar{C}_{l_1}(|\psi\rangle_{out})_{\max} = \bar{C}_{l_1}(|\psi\rangle_{out})_{\min}$. This is in consistent with the fact that for the case $|\psi\rangle = \frac{1}{2}\sum_{j=1}^{4}|j\rangle$ and $p_4 = 1$, $\bar{C}_{l_1}(|\psi\rangle_{out})_{\max} = \bar{C}_{l_1}(|\psi\rangle_{out})_{\min} = 3$

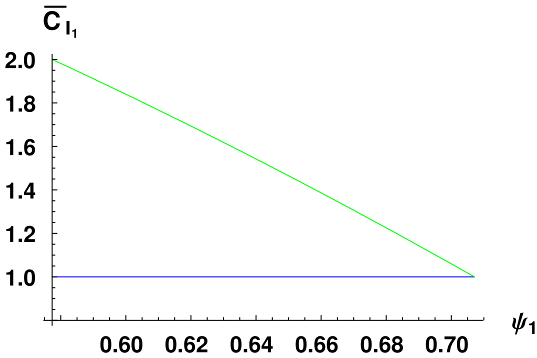

Fig. 2 The case of $\psi_4 = 0$. Blue line and green line correspond to the minimal and maximal value of $\bar{C}_{l_1}(|\psi\rangle_{out})$, respectively

that $\phi_4 \geq \phi_5$, i.e., $\psi_2^2+\psi_3^2+\psi_4^2 \geq \psi_1^2+2\psi_5^2$. In this case, $\overline{C}_{l_1}(|\psi\rangle_{out})_{\max} = 5\psi_2^2+5\psi_3^2+5\psi_4^2+8\psi_5^2-3\psi_1^2$. Similarly, $\overline{C}_{l_1}(|\psi\rangle_{out})_{\min} = 1+15\psi_5^2$. By Theorem 1, one cannot obtain the maximal probability with respect to $|\Psi_5\rangle$ without a complete waste of resource when $\psi_2^2+\psi_3^2+\psi_4^2 < \psi_1^2+2\psi_5^2$ via SIOs defined in Eq. (2).

Following the discussions above, for a general d-dimension pure state $|\psi\rangle$ with $d \geq 5$, one can deal with the problem via SIOs defined in Eq. (2) according to the following $d-3$ cases.

Case 1: $\psi_2^2+\psi_3^2 \geq \psi_1^2+\psi_4^2$

From Eq. (6), if $\phi_1 = \phi_2 = \psi_1$, $\phi_3 = \sqrt{\psi_2^2+\psi_3^2-\psi_1^2}$ and $\phi_j = \psi_j,\ j = 4, 5, \ldots, d$, we have the maximal value, $\bar{C}_{l_1}(|\psi\rangle_{out})_{\max} = 4\sum_{j=2}^{3}\psi_j^2 + 2\sum_{j=4}^{d}(j-1)\psi_j^2 - 2\psi_1^2$.

Conversely, we can get the minimal value of $\bar{C}_{l_1}(|\psi\rangle_{out})$ if $\phi_1=\phi_2=\sqrt{\frac{1-(d-2)\psi_d^2}{2}}$, and $\phi_j=\psi_d,\ j=3,4,\ldots,d$, $\bar{C}_{l_1}(|\psi\rangle_{out})_{\min}=1+d(d-2)\psi_d^2$.

Case 2: $\psi_2^2+\psi_3^2<\psi_1^2+\psi_4^2$ and $\psi_2^2+\psi_3^2+\psi_4^2\geq\psi_1^2+2\psi_5^2$

From Eq. (6), if $\phi_1=\phi_2=\psi_1, \phi_3=\phi_4=\sqrt{\frac{\psi_2^2+\psi_3^2+\psi_4^2-\psi_1^2}{2}}$ and $\phi_j=\psi_j,\ j=5,6,\ldots,d$, we get the maximal value $\bar{C}_{l_1}(|\psi\rangle_{out})_{\max}=5\sum_{j=2}^{4}\psi_j^2+2\sum_{j=5}^{d}(j-1)\psi_j^2-3\psi_1^2$. If $\phi_1=\phi_2=\sqrt{\frac{1-(d-2)\psi_d^2}{2}}$ and $\phi_j=\psi_d,\ j=3,4,\ldots,d$, we get the minimal value of $\bar{C}_{l_1}(|\psi\rangle_{out})$, $\bar{C}_{l_1}(|\psi\rangle_{out})_{\min}=1+d(d-2)\psi_d^2$.

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Case k: $\sum_{j=2}^{k+1}\psi_j^2<\psi_1^2+(k-1)\psi_{2+k}^2$ and $\sum_{j=2}^{k+2}\psi_j^2\geq\psi_1^2+k\psi_{3+k}^2$

From Eq. (6), if $\phi_1=\phi_2=\psi_1, \phi_3=\phi_4=\cdots=\phi_{2+k}=\sqrt{\frac{\Sigma_{j=2}^{k+2}\psi_j^2-\psi_1^2}{k}}$ and $\phi_j=\psi_j,\ j=3+k,4+k,\ldots,d$, we get the maximal value

$$\bar{C}_{l_1}(|\psi\rangle_{out})_{\max}=(k+3)\sum_{j=2}^{k+2}\psi_j^2+2\sum_{j=k+3}^{d}(j-1)\psi_j^2-(k+1)\psi_1^2.$$

If $\phi_1=\phi_2=\sqrt{\frac{1-(d-2)\psi_d^2}{2}}$ and $\phi_j=\psi_d,\ j=3,4,\ldots,d$, we obtain the minimal value of $\bar{C}_{l_1}(|\psi\rangle_{out})$, $\bar{C}_{l_1}(|\psi\rangle_{out})_{\min}=1+d(d-2)\psi_d^2$.

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Case $d-3$: $\sum_{j=2}^{d-2}\psi_j^2<\psi_1^2+(d-4)\psi_{d-1}^2$ and $\sum_{j=2}^{d-1}\psi_j^2\geq\psi_1^2+(d-3)\psi_d^2$

From Eq. (6), if $\phi_1=\phi_2=\psi_1, \phi_3=\phi_4=\cdots=\phi_{d-1}=\sqrt{\frac{\Sigma_{j=2}^{d-1}\psi_j^2-\psi_1^2}{d-3}}$ and $\phi_d=\psi_d$, we have the maximal value,

$$\bar{C}_{l_1}(|\psi\rangle_{out})_{\max}=d\sum_{j=2}^{d-1}\psi_j^2+2(d-1)\psi_d^2-(d-2)\psi_1^2.$$

If $\phi_1=\phi_2=\sqrt{\frac{1-(d-2)\psi_d^2}{2}}$ and $\phi_j=\psi_d,\ j=3,4,\ldots,d$, we get the minimal value of $\bar{C}_{l_1}(|\psi\rangle_{out})$, $\bar{C}_{l_1}(|\psi\rangle_{out})_{\min}=1+d(d-2)\psi_d^2$.

One can see that the $d-3$ cases together are just the condition $\sum_{j=2}^{d-1}\psi_j^2\geq\psi_1^2+(d-3)\psi_d^2$ in Theorem 1 to realize the one-shot coherence distillation via SIOs defined in Eq. (2). Thus, we can just discuss the $d-3$ cases above based on the SIOs defined in Eq. (2). Here, in the discussions of the $d-3$ cases above, we have assumed that $|\psi\rangle\xrightarrow{IO}|\phi\rangle$, i.e., $\sum_{m=1}^{j}\psi_m^2\leq\sum_{m=1}^{j}\phi_m^2$, and $\phi_m\geq\phi_{m+1}$ for $m=1,2,\ldots,d-1$ and $j=1,2,\ldots,d$, which needs to be proved further.

For case k $(1\leq k\leq d-3)$, concerning the maximal value of $\bar{C}_{l_1}(|\psi\rangle_{out})$, as $\phi_1=\phi_2=\psi_1$ and $\phi_j=\psi_j$ for $j=3+k,4+k,\ldots,d$, for the proof of $\sum_{m=1}^{j}\psi_m^2\leq$

$\sum_{m=1}^{j}\phi_m^2$, we only need to prove $\sum_{m=1}^{j}\psi_m^2 \leq \sum_{m=1}^{j}\phi_m^2$ for $j = 3, 4, \ldots, k+2$. We have

$$\begin{aligned}\sum_{m=1}^{j}\phi_m^2 - \sum_{m=1}^{j}\psi_m^2 &= \frac{(k+2-j)\psi_1^2+k(\psi_{j+1}^2+\cdots+\psi_{k+1}^2)+(j-2)\psi_{k+2}^2-(k+2-j)(\psi_2^2+\cdots+\psi_{k+1}^2)}{k}\\ &\geq \frac{(k+2-j)\psi_1^2+[k(k-j+1)+j-2]\psi_{k+2}^2-(k+2-j)(\psi_2^2+\cdots+\psi_{k+1}^2)}{k}\\ &= \frac{(k+2-j)[\psi_1^2+(k-1)\psi_{k+2}^2-(\psi_2^2+\cdots+\psi_{k+1}^2)}{k}]\\ &\geq 0\end{aligned} \tag{9}$$

for $j = 3, 4, \ldots, k+2$. The last inequality is due to the condition $\sum_{j=2}^{k+1}\psi_j^2 < \psi_1^2 + (k-1)\psi_{2+k}^2$.

For the proof of $\phi_m \geq \phi_{m+1}$ for $m = 1, 2, \ldots, d-1$, we only need to prove $\phi_2 \geq \phi_3$ and $\phi_{2+k} \geq \phi_{3+k}$. Obviously,

$$\phi_2 \geq \phi_3 \Leftrightarrow \phi_2^2 \geq \phi_3^2 \Leftrightarrow \psi_1^2 \geq \frac{\sum_{j=2}^{k+2}\psi_j^2-\psi_1^2}{k} \Leftrightarrow (k+1)\psi_1^2 \geq \sum_{j=2}^{k+2}\psi_j^2, \tag{10}$$

and

$$\phi_{2+k} \geq \phi_{k+3} \Leftrightarrow \phi_{2+k}^2 \geq \phi_{k+3}^2 \Leftrightarrow \frac{\sum_{j=2}^{k+2}-\psi_1^2}{k} \geq \phi_{3+k}^2 \Leftrightarrow \sum_{j=2}^{k+2}\psi_j^2 \geq \psi_1^2 + k\psi_{3+k}^2. \tag{11}$$

Thus, we get $\phi_m \geq \phi_{m+1}$ for $m = 1, 2, \ldots, d-1$. Concerning the minimal value of $\bar{C}_{l_1}(|\psi\rangle_{out})$, similarly, one can also prove that $|\psi\rangle \xrightarrow{IO} |\phi\rangle$. Altogether, we have the following theorem for the general Case k.

Theorem 3 *For the pure state $|\psi\rangle$ given in Eq. (1) with $\sum_{j=2}^{k+1}\psi_j^2 < \psi_1^2+(k-1)\psi_{2+k}^2$ and $\sum_{j=2}^{k+2}\psi_j^2 \geq \psi_1^2 + k\psi_{3+k}^2$ $(1 \leq k \leq d-3)$, $\bar{C}_{l_1}(|\psi\rangle_{out})$ satisfies the following relations,*

$$1 + d(d-2)\psi_d^2 \leq \bar{C}_{l_1}(|\psi\rangle_{out}) \leq (k+3)\sum_{j=2}^{k+2}\psi_j^2 + 2\sum_{j=k+3}^{d}(j-1)\psi_j^2 - (k+1)\psi_1^2, \tag{12}$$

if the maximal probability of getting $|\Psi_d\rangle$ is attained without a complete waste of resources via SIOs defined in Eq. (2). The left equality holds when $\phi_1 = \phi_2 = \sqrt{\frac{1-(d-2)\psi_d^2}{2}}$ and $\phi_j = \psi_d,\ j = 3, 4, \ldots, d$. The right equality holds when $\phi_1 = \phi_2 = \psi_1$, $\phi_3 = \phi_4 = \cdots = \phi_{2+k} = \sqrt{\frac{\sum_{j=2}^{k+2}\psi_j^2-\psi_1^2}{k}}$ and $\phi_j = \psi_j,\ j = 3+k, 4+k, \ldots, d$.

From Theorem 3, we know that the average coherence of output ensemble depends on the dimensional d. To illustrate this, we show the lower bound of $\bar{C}_{l_1}(|\psi\rangle_{out})$ in Fig. 3.

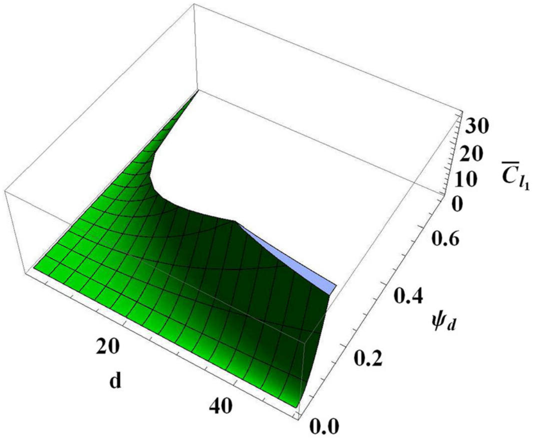

Fig. 3 The dependence of the average coherence $\bar{C}_{l_1}$ on the dimension d as well as on the value of ψ_d

From Theorems 2 and 3, for a given initial pure state, one can judge which method one can use to get $|\Psi_d\rangle$ with the maximal probability, and avoid a complete waste of resources at the same time via SIOs defined in Eq. (2). One can also obtain the range of the corresponding average loss of coherence. From the inequality (12), one can see that $\bar{C}_{l_1}(|\psi\rangle_{out})_{\min}$ only depends on the value of ψ_d, while $\bar{C}_{l_1}(|\psi\rangle_{out})_{\max}$ depends on $\{\psi_i\}$. Since the lower and upper bounds of $\bar{C}_{l_1}(|\psi\rangle_{out})$ depend on the initially given pure states, one can only estimate $\bar{C}_{l_1}(|\psi\rangle_{out})_{\max}$ or $\bar{C}_{l_1}(|\psi\rangle_{out})_{\min}$ for given pure states. In other words, if one does not know the specific expressions of two initial pure states $|\psi\rangle$ and $|\widetilde{\psi}\rangle$ which belong to different cases, one cannot say which one of $\bar{C}_{l_1}(|\psi\rangle_{out})_{\min}$ and $\bar{C}_{l_1}(|\widetilde{\psi}\rangle_{out})_{\min}$ is larger, as one cannot assure which one of ψ_d and $\widetilde{\psi}_d$ is larger. The same is for case of $\bar{C}_{l_1}(|\psi\rangle_{out})_{\max}$. What is more, if we want to get the maximal value of $\bar{C}_{l_1}(|\psi\rangle_{out})_{\max}$ over all pure states, ψ_1 should be as small as possible. Thus, the maximal value of $\bar{C}_{l_1}(|\psi\rangle_{out})_{\max}$ over all pure states can be attained at $\psi_1 = \frac{1}{\sqrt{d}}$, i.e., $|\psi\rangle = |\Psi_d\rangle$.

To illustrate our results, let us consider an example for the case $d = 4$.

Example 1 Let

$$|\psi\rangle = \sqrt{0.28}|1\rangle + \sqrt{0.25}|2\rangle + \sqrt{0.22}|3\rangle + \sqrt{0.15}|4\rangle + \sqrt{0.1}|5\rangle,$$
$$|\widetilde{\psi}\rangle = \sqrt{0.305}|1\rangle + \sqrt{0.25}|2\rangle + \sqrt{0.2}|3\rangle + \sqrt{0.13}|4\rangle + \sqrt{0.115}|5\rangle$$

and

$$|\psi'\rangle = \sqrt{0.3}|1\rangle + \sqrt{0.25}|2\rangle + \sqrt{0.2}|3\rangle + \sqrt{0.14}|4\rangle + \sqrt{0.11}|5\rangle.$$

For the pure state $|\psi\rangle$, one finds that $\psi_1^2 + \psi_4^2 = 0.43 \leq \psi_2^2 + \psi_3^2 = 0.47$. Thus, $|\psi\rangle$ is a pure state of the Case 1. By Theorem 3, we have

$$\bar{C}_{l_1}(|\psi\rangle_{out})_{\max} = \bar{C}_{l_1}((|\phi\rangle_{\max})_{out}) = 3.02,$$

with $|\psi\rangle \xrightarrow{IO} |\phi\rangle_{\max} = \sqrt{0.28}|1\rangle + \sqrt{0.28}|2\rangle + \sqrt{0.19}|3\rangle + \sqrt{0.15}|4\rangle + \sqrt{0.1}|5\rangle$, and

$$\bar{C}_{l_1}(|\psi\rangle_{out})_{\min} = \bar{C}_{l_1}((|\phi\rangle_{\min})_{out}) = 2.5$$

with $|\psi\rangle \xrightarrow{IO} |\phi\rangle_{\min} = \sqrt{0.35}|1\rangle + \sqrt{0.35}|2\rangle + \sqrt{0.1}|3\rangle + \sqrt{0.1}|4\rangle + \sqrt{0.1}|5\rangle$.

Let $|\phi\rangle_{\max}$, $|\widetilde{\phi}\rangle_{\max}$ and $|\phi'\rangle_{\max}$ be the corresponding pure states satisfying $|\psi\rangle \xrightarrow{IO} |\phi\rangle_{\max}$, $|\widetilde{\psi}\rangle \xrightarrow{IO} |\widetilde{\phi}\rangle_{\max}$ and $|\psi'\rangle \xrightarrow{IO} |\phi'\rangle_{\max}$, respectively. Similarly, one finds that $|\widetilde{\psi}\rangle$ belongs to the Case 1 with $\bar{C}_{l_1}(|\widetilde{\psi}\rangle_{out})_{\max} = 2.89$ and $\bar{C}_{l_1}(|\psi\rangle_{out})_{\min} = 2.725$. $|\psi'\rangle$ belongs to the Case 2 with $\bar{C}_{l_1}(|\psi'\rangle_{out})_{\max} = 2.93$ and $\bar{C}_{l_1}(|\psi\rangle_{out})_{\min} = 2.65$.

4 Coherence loss related to mixed states

In Ref. [27], Lami *et.al* showed that one may have a very limited coherence distillation power via SIOs when mixed input states are concerned. In Ref. [36], the authors showed that no mixed qubit state can be distilled to a maximally coherent qubit state with nonzero probability using a SIO. In fact, we can generally prove the following.

Lemma 1 *No d-dimensional mixed states ρ can be distilled to any d-dimensional pure states via a SIO with nonzero probability.*

Proof First, a ρ is pure if and only if $\rho_{jj}\rho_{kk} = |\rho_{jk}|^2$, since

$$\begin{aligned} \operatorname{tr}\rho^2 &= \sum_{jk} |\rho_{jk}|^2 \\ &= \sum_{j} |\rho_{jj}|^2 + 2\sum_{j<k} |\rho_{jk}|^2 \\ &= \sum_{j} |\rho_{jj}|^2 + 2\sum_{j<k} \rho_{jj}\rho_{kk} \\ &= \left(\sum_{j}\rho_{jj}\right)^2 \\ &= 1. \end{aligned} \tag{13}$$

Let Φ be a SIO. Then, the Kraus operators $\{K_n\}$ of Φ can be the form of $K_n = \sum_j K_{nj}|f(j)\rangle\langle j|$ with f a bijection. One has

$$K_n\rho K_n^\dagger = \sum_{jk} K_{nj}K_{nk}^*\rho_{jk}|f(j)\rangle\langle f(k)|. \tag{14}$$

If ρ can be distilled to a d-dimensional pure state via Φ with nonzero probability, then $|K_{nj}|^2|K_{nk}|^2\rho_{jj}\rho_{kk} = |K_{nj}K_{nk}^*|^2|\rho_{jk}|^2$, which give rise to $\rho_{jj}\rho_{kk} = |\rho_{jk}|^2$, i.e., ρ is a pure state, which completes the proof. □

Therefore, to find a method to transform a mixed state into a maximally coherent state via a series of SIOs is necessary, which also generalizes the results of coherence loss for pure states [26]. In the following, we present a way to transform a mixed state into maximally coherent states with probability via a series of SIOs on the pure states of the ensemble of the mixed state.

For a mixed state ρ, without loss of generality, one can assume $\rho_{ii} \geq \rho_{i+1,i+1}$ for $i = 1, 2, \ldots, d-1$. Let $\rho = \sum_i q_i |\varphi_i\rangle\langle\varphi_i|$ be a pure state decomposition of ρ, where $|\varphi_i\rangle = \sum_{j=1}^{d} \varphi_{ij}|j\rangle$ with $\varphi_{ij} \geq \varphi_{i,j+1} \geq 0$ for $j = 1, \ldots, d-1$. Here, it should be noted that the coefficients of the pure states may be not in non-increasing order. Nevertheless, one can always apply unitary SIOs to let the corresponding pure states satisfy this condition. Define Φ_i to be the SIO corresponding to $|\varphi_i\rangle$,

$$\Phi_i(|\varphi_i\rangle) = \sum_{j=1}^{d} K_{ij}|\varphi_i\rangle\langle\varphi_i|K_{ij}^{\dagger}, \tag{15}$$

where

$$K_{ij} = \sqrt{p_{ij}}\left(\frac{1}{\sqrt{j}}\sum_{q=1}^{j}\frac{|q\rangle\langle q|}{\varphi_{iq}}\right), \tag{16}$$

with

$$p_{id} = d\varphi_{id}^2, \quad p_{iq} = q(\varphi_{iq}^2 - \varphi_{i,q+1}^2), \quad q = 1, 2, \ldots, d-1. \tag{17}$$

First, we prepare a series of SIOs $\{\Phi_i\}$ on $|\varphi_i\rangle$, with $\{p_i, |\varphi_i\rangle\}$ the pure state decomposition of ρ and Φ_i defined in Eq. (15). Define Φ as

$$\Phi(\rho) = \sum_i p_i \Phi_i(|\varphi_i\rangle). \tag{18}$$

Theorem 4 *Let Φ defined in Eq. (18) with Φ_i defined in Eq. (15). Φ can transform ρ into a maximally coherent state $|\Psi_q\rangle$ with the probability $p_q = q(\rho_{qq} - \rho_{q+1,q+1})$, $q = 1, 2, \ldots, d-1$, and $p_d = d\rho_{dd}$. Moreover, Φ is independent of the pure decomposition of ρ.*

Proof From Eq. (15) and the expression of $|\varphi_i\rangle$, Φ_i can transform $|\varphi_i\rangle$ into a maximally coherent state $|\Psi_q\rangle$ with the probability p_{iq}. Then, Φ can transform ρ to $|\Psi_q\rangle$ with probability $p_q = \sum_i q_i p_{iq}$. Then, the conclusion follows from Eq. (17). □

Corollary 1 *For an arbitrary mixed state ρ, we have*

$$\bar{C}_{l_1}(\rho_{out}) = 2\sum_{q=2}^{d}(q-1)\rho_{qq} \tag{19}$$

with $\rho_{out} = \Phi(\rho)$ and Φ defined in Eq. (18).

Similar to the pure state case, one can also get the conditions related to the no complete waste of resource for mixed states ρ.

Theorem 5 *For a given d-dimensional mixed state ρ with $d \geq 4$, if one can avoid a complete waste of resource while $p_d = d\rho_{dd}$ via Φ defined in Eq. (18), then $\sum_{j=2}^{d-1} \rho_{jj} \geq \rho_{11} + (d-3)\rho_{dd}$, where p_q are defined in Theorem 4 for $q = 1, 2, \ldots, d$.*

Proof Let $\rho = \sum_i q_i |\varphi_i\rangle\langle\varphi_i|$ with $|\varphi_i\rangle = \sum_j \varphi_{ij}|j\rangle$, $\varphi_{ij} \geq \varphi_{i,j+1} \geq 0$ for $j = 1, \ldots, d-1$. Assuming $p_1 = 0$ we have $p_{i1} = 0$, where p_{i1} is defined in Eq. (17). Thus, for each pure state $|\varphi_i\rangle$, one has $\sum_{j=2}^{d-1} \varphi_{ij}^2 \geq \varphi_{i1}^2 + (d-3)\varphi_{id}^2$. Therefore, $\sum_{j=2}^{d-1} \rho_{jj} \geq \rho_{11} + (d-3)\rho_{dd}$. □

From Theorem 5, one sees that via operations defined in Eq. (18), there is always possibility of a complete waste of resource while p_d remains nonzero if $\sum_{j=2}^{d-1} \rho_{jj} < \rho_{11} + (d-3)\rho_{dd}$.

5 Conclusion

We presented a protocol of one-shot coherence distillation with the maximal probability to transform a d-dimensional pure state $|\psi\rangle$ into the maximally coherent state $|\Psi_d\rangle$ without complete waste of resource. In this process, an incoherent operation may be used to transform $|\psi\rangle$ to $|\phi\rangle$. For mixed states ρ, we also proposed a method to transform ρ into a maximally coherent state without complete waste of resource. Our method is independent of the pure decompositions of ρ. These results may highlight further investigations on the theory of coherence manipulations.

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