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New monogamy relations for multiqubit systems

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Abstract
Recently, a new class of monogamy relations (actually, exponentially many) was provided by Christopher Eltschka et al. in terms of squared concurrence. Their approach is restricted to the distribution of bipartite entanglement shared between different subsystems of a global state. We have critically analysed those monogamy relations in three as well as in four-qubit pure states using squared negativity. We have been able to prove that in the case of pure three-qubit states those relations are always true in terms of squared negativity. However, if we consider the pure four-qubit states, the results are not always true. Rather, we find opposite behaviour in some particular classes of four-qubit pure states where some of the monogamy relations are violated. We have provided analytical and numerical evidences in support of our claim.

Keywords Entanglement · Monogamy · Negativity

1 Introduction

Entanglement is one of the most important ideas in quantum information theory and it is in fact the main form of quantum correlation which shows clear advantages over several aspects of classical theory. Classification and characterization of entanglement have

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always been a challenging field of research. One important feature of entanglement is that it could be used as a resource that allows one to perform certain quantum information tasks, e.g. dense coding [1], teleportation [2], quantum computation [3,4], etc. Now, as far as the number of parties is concerned, bipartite entanglement is well understood at least for two-qubit system, whereas for multipartite systems only few ideas are available.

Monogamy is one of the most important properties of entanglement that provide us the information about the distribution of entanglement in a multipartite system [5]. Monogamy was possibly first studied by Coffman et al. [6] in terms of squared concurrence. Concurrence is defined as a bipartite measure of entanglement. For a two-qubit state ρ_{AB}, concurrence is defined by, $C(\rho_{AB}) = max\{0, \lambda_1 - \lambda_2 - \lambda_3 - \lambda_4\}$ where $\lambda_1, \lambda_2, \lambda_3, \lambda_4$ are the square root of the eigenvalues of the matrix $\rho_{AB}((\sigma_y \otimes \sigma_y)\rho^*_{AB}(\sigma_y \otimes \sigma_y))$ in decreasing order, σ_y is the Pauli spin matrix and ρ^*_{AB} is conjugate of ρ_{AB}. For pure bipartite states, concurrence can be computed through $C(\rho_{AB}) = 2\sqrt{det\rho_A}$ where ρ_A is obtained from ρ_{AB} by taking partial trace over the subsystem B. We will use the notation C_{AB} instead of $C(\rho_{AB})$ for any state ρ_{AB}. The **CKW** (Coffman, Kundu, Wootters) inequality [6] is given by,

$$C^2_{A|BC} \geq C^2_{AB} + C^2_{AC} \tag{1}$$

where C denotes the measure of concurrence for a bipartite state. The meaning of the above CKW inequality could be stated as: sum of the amount of entanglement (measured in terms of square of the concurrence) shared between parties A, B and the amount of entanglement shared between the parties A, C cannot exceed the amount of entanglement between the parties A and BC. They had also conjectured that the extension of their monogamy relation for n-qubit states would be as follows:

$$C^2_{A_1|A_2A_3\ldots A_n} \geq C^2_{A_1A_2} + C^2_{A_1A_3} + \cdots . + C^2_{A_1A_n}. \tag{2}$$

This conjecture later proved by Osborne et al. [7]. Since the introduction of CKW inequality, several works had been done on monogamy where **CKW** inequality is modified, generalized and also replaced by other entanglement measures [8–12]. All such investigations enable us to understand the entanglement behaviour of composite quantum systems more profoundly. In [13,14], the authors tried to describe monogamy property without using **CKW** type inequality [6]. Recently, C. Eltschka et. al. [15] provided a new kind of monogamy relation for multipartite (say, N number of parties) d dimensional pure states. They adopt the methodology that any functional relation between measures of entanglement in different subsets of parties could be considered as a monogamy relation because the free distribution of entanglement between different parties has been constrained by it. The monogamy relations in the compact form are [15] given by,

$$\sum_{\Phi \neq S \subset \{1,2,\ldots,N\}} (-1)^{|S \cap T|+1} C^2_{S|S^c} \geq 0 \tag{3}$$

where $\Phi \neq T \subseteq \{1, 2, \ldots, N\}$. There are actually $2^N - 1$ number of monogamy relations where we find one inequality for each T, and when $|T|$ (the cardinality of T) is odd, we shall get only the trivial inequality $0 \geq 0$. Inspired by their results, we have studied in this paper three-qubit and four-qubit systems through another quantity, the squared negativity.

Negativity is an important measure of entanglement [16]. It is an entanglement monotone and invariant under local unitary operations. The negativity is a rare bipartite entanglement measure which is easy to compute for pure as well for mixed bipartite states. From Peres criterion [17], it is known that for a separable state partial transpose of its density matrix will also be a density matrix. Partial transpose in general preserves hermiticity but not positivity. Thus, after taking partial transpose on a density matrix representing a bipartite state, if we obtain at least one negative eigenvalue, then we could certainly say that the state is an entangled state. The definition of negativity for a bipartite state ρ_{AB} (pure or mixed) is given by,

$$N(\rho_{AB}) = \frac{\|\rho_{AB}^{t_A}\|_1 - 1}{2} \tag{4}$$

where $\|X\|_1 = tr\sqrt{XX^\dagger}$ and partial transposition is taken with respect to subsystem A. In other words, the negativity is the absolute sum of negative eigenvalues of $\rho_{AB}^{t_A}$ and it measures how much $\rho_{AB}^{t_A}$ fails to be a positive definite matrix. We will use the notation N_{AB} instead of $N(\rho_{AB})$.

We have organized our paper as follows: In Sect. 2, we will discuss motivation of our work. In Sects. 3 and 4, we will discuss monogamy relations for three-qubit and four-qubit pure states, respectively. Section 5 ended with conclusion.

2 Motivation

The generalized T inversion map [15] is,

$$\mathcal{I}_T(\rho) = \sum_{S\subseteq\{1,2,\ldots,N\}} (-)^{|S\cap T|}(Tr_{S^c}\rho) \otimes I_{S^c} \tag{5}$$

where T is any subset of $\{1, 2, \ldots, N\}$. Using positivity property of $\mathcal{I}_T(.)$, for two semi definite positive operator M_1 and M_2 one has

$$Tr_S[M_1\mathcal{I}_T(M_2)] \geq 0. \tag{6}$$

As $Tr_S[(M_1)Tr_{S^c}(M_2)] = Tr_S[Tr_{S^c}(M_1)Tr_{S^c}(M_2)]$ putting Eq. (5) in (6), one will get

$$\sum_{S\subseteq\{1,2,\ldots,N\}} (-)^{|S\cap T|}Tr_S[Tr_{S^c}(M_1)Tr_{S^c}(M_2)] \geq 0 \tag{7}$$

where T is any subset of $\{1, 2, \ldots, N\}$. This inequality is called shadow inequality [18,19].

Now, if one consider $M_1 = M_2 = |\psi_{N,D}\rangle$ an N partite D dimensional pure state, then one can directly get the monogamy inequalities,

$$\sum_{\Phi \neq S \subset \{1,2,\ldots,N\}} (-1)^{|S \cap T|+1} C^2_{S|S^c} \geq 0 \tag{8}$$

where $\Phi \neq T \subseteq \{1, 2, \ldots, N\}$ and here $C_{S|S^c}$ is concurrence of the pure state along the bipartition. So, the relation (8) is direct consequences of shadow inequality or rather the algebraic property of generalized T inverter.

Again, the shadow enumerator polynomial [18] is,

$$S_{M_1 M_2}(x, y) = \sum_{j=0}^{N} S_j(M_1 M_2) x^{N-j} y^j \tag{9}$$

where the coefficient is defined as follows

$$S_j(M_1 M_2) = \sum_{|T|=j} \sum_{S \subseteq \{1,\ldots,N\}} (-1)^{|S \cap T^c|} \mathcal{A}^{/}_S(M_1, M_2) \tag{10}$$

(the first sum is over all subset of size j) and $\mathcal{A}^{/}_S(M_1, M_2) = Tr_S[Tr_{S^c}(M_1) Tr_{S^c}(M_2)]$. If in particular $M_1 = M_2 = |\psi_{N,D}\rangle$, then the inequalities (8) will imply that $S_j(|\psi_{N,D}\rangle) \geq 0$.

Further, $S_j(M_1 M_2)$ can be written in terms of coefficient of Shor–Laflamme enumerator [20] which is

$$S_j(M_1 M_2) = \sum_{l=0}^{N} K_{N-j}(l; N) A^{/}_l(M_1, M_2) \tag{11}$$

where $K_{N-j}(l; N)$ is the Krawtchouk polynomial

$$K_m(l; N) = \sum_{\alpha} (-1)^{\alpha} \binom{n-l}{m-\alpha} \binom{l}{\alpha}.$$

Now, when $M_1 = M_2 = |\psi_{N,D}\rangle$, then $A^{/}_l(M_1, M_2) = \binom{N}{l} D^{-min(l,N-l)}$.

Therefore,

$$S_j(|\Psi_{N,D}\rangle) = \sum_{l=0}^{N} K_{N-j}(l; N) \binom{N}{l} D^{-min(l,N-l)}. \tag{12}$$

If for a pure state $|\psi_{N,D}\rangle$, $S_j(|\psi_{N,D}\rangle)$ becomes negative, then an Absolute Maximally Entangled (AME) [20] state on N parties having D dimension cannot exist as it will contradict $S_j(|\psi_{N,D}\rangle) \geq 0$.

A particular example is $|\psi_{4,2}\rangle$, where $S_0(|\psi_{4,2}\rangle) = \sum_{l=0}^{4}(-1)^l \binom{4}{l} 2^{-min(l,4-l)} = -\frac{1}{2} < 0$. Therefore, there does not exist a 4 partite 2 local dimensional AME state [20].

The inequalities (8) are very important class of monogamy inequalities, as because in one hand, it is derived from an algebraic property of generalized T inverter and on the other hand, it helps one in excluding the existence of AME states in N partite D local dimensions. A simple question that arises from their work is whether this type of monogamy holds for other entanglement measures or not. In our work, we have examined the above set of monogamy relations using negativity as an entanglement measure for three and four-qubit pure states.

3 Monogamy relations for three-qubit pure states

We start this section with a relation between negativity and concurrence.

Theorem 1 *[10] For an N partite pure state $|\psi_{A_1A_2\ldots A_N}\rangle$ in a $2 \otimes 2 \otimes \ldots \otimes 2$($N$ times) system, the negativity of bipartition $A_1|A_2\ldots A_N$ is half of its concurrence, i.e. $N_{A_1|A_2\ldots A_N} = \frac{1}{2}C_{A_1|A_2\ldots A_N}$.*

Proof is given in "Appendix 3".

We will use the above theorem to form monogamy relations for three- and four-qubit systems from relation (3) with respect to squared negativity. For a three-qubit pure state, from monogamy relation (3), we have

$$\sum_{\phi\neq S\subset\{1,2,3\}} (-1)^{|S\cap T|+1} C^2_{S|S^c} \geq 0 \tag{13}$$

where we will get one inequality for each $\Phi \neq T \subseteq \{1,2,3\}$, i.e. total $2^3 - 1 = 7$ monogamy relations. When $|T|$ is odd we shall obtain trivial inequality $0 \geq 0$. Expanding (13) for $T = \{1,2\}$, $T = \{1,3\}$, $T = \{2,3\}$, we get, respectively

$$C^2_{1|23} + C^2_{2|13} \geq C^2_{3|12} \tag{14}$$

$$C^2_{1|23} + C^2_{3|12} \geq C^2_{2|13} \tag{15}$$

$$C^2_{2|13} + C^2_{3|12} \geq C^2_{1|23}. \tag{16}$$

Now, using Theorem 1 for $2\otimes2\otimes2$ dimensional pure states, we have $C_{i|jk} = 2 \times N_{i|jk}$ and thus from (14), (15), (16) we can write,

$$N^2_{1|23} + N^2_{2|13} \geq N^2_{3|12} \tag{17}$$

$$N^2_{1|23} + N^2_{3|12} \geq N^2_{2|13} \tag{18}$$

$$N^2_{2|13} + N^2_{3|12} \geq N^2_{1|23}. \tag{19}$$

The above three monogamy inequalities can also be written compactly as

$$\sum_{\phi\neq S\subset\{1,2,3\}}(-1)^{|S\cap T|+1}N^2_{S|S^c}\geq 0 \tag{20}$$

where one inequality is associated for each $\Phi\neq T\subset\{1,2,3\}$, i.e. total $(2^3-2)=6$ inequalities. When $|T|$ is odd we shall get only the trivial inequality $0\geq 0$. Thus, Theorem 1 completely determines the monogamy relations in terms of squared negativity from the relation (13). Next, we will consider pure four-qubit states and observe whether it is similar to that of three-qubit case or not.

4 Monogamy relations for four-qubit pure states

For a four-qubit pure state, relation (3) looks like

$$\sum_{\phi\neq S\subset\{1,2,3,4\}}(-1)^{|S\cap T|+1}C^2_{S|S^c}\geq 0 \tag{21}$$

where one inequality is associated for each $\Phi\neq T\subseteq\{1,2,3,4\}$, i.e. total $2^4-1=15$ monogamy relations, out of which eight are trivial inequalities $0\geq 0$ when $|T|$ is an odd number. The inequalities (21) are given in details in "Appendix 1". We now state another relation between concurrence and negativity in the following theorem.

Theorem 2 *For an N partite pure state $|\psi_{A_1A_2\ldots A_N}\rangle$ in a $d_1\otimes d_2\otimes\ldots\otimes d_N$ dimensional system where each $d_i>2\ \forall i=1,2,\ldots,N$, $N_{A_1|A_2\ldots A_N}>\frac{1}{2}C_{A_1|A_2\ldots A_N}$.*

Proof is given in "Appendix 3".

As stated in Theorem 2, the replacement of concurrence by negativity in the relation (21) is not always possible like in the three-qubit case, since in some expressions, the focus party is of dimension 4, hence Theorem 1 will not be applicable to such cases.

We now denote δ_i, $\forall i=1,2,\ldots,15$ as follows,

$$\delta_i=\sum_{\phi\neq S\subset\{1,2,3,4\}}(-1)^{|S\cap T|+1}N^2_{S|S^c} \tag{22}$$

where we obtain, for each $\Phi\neq T\subseteq\{1,2,3,4\}$, total $2^4-1=15$ expression. When $|T|$ is odd we shall get zero in the right hand side of (22). We take the nonzero expressions as $\delta_1,\delta_2,\ldots\delta_7$ and $\delta_8=\delta_9=\ldots=\delta_{15}=0$. Expansion of expressions (22) are given in "Appendix 1".

Whenever $\delta_i\geq 0$, $\forall i=1,2,\ldots,7$, we have the relations (30)–(36), given in "Appendix 1", are true. As there exist infinitely many SLOCC inequivalent classes for four-qubit pure states, we will consider the four-qubit generic class [21] and other important four-qubit classes to check the sign of δ_i's $\forall i=1,2,\ldots,7$.

4.1 Monogamy relations in some particular classes of four-qubit pure states

Generic Class: The generic class of pure states is dense under SLOCC in four-qubit state space. It even contains uncountable SLOCC inequivalent subclasses [22]. We denote this class by $\mathcal{A}$ and is defined as

$$\mathcal{A} = \{au_1 + bu_2 + cu_3 + du_4 \quad | \quad a, b, c, d \in \mathbb{C} \\ \text{and} \quad |a|^2 + |b|^2 + |c|^2 + |d|^2 = 1\}$$

where $u_1 \equiv |\Phi^+\rangle|\Phi^+\rangle$, $u_2 \equiv |\Phi^-\rangle|\Phi^-\rangle$, $u_3 \equiv |\Psi^+\rangle|\Psi^+\rangle$, $u_4 \equiv |\Psi^-\rangle|\Psi^-\rangle$, $|\Phi^\pm\rangle = \frac{|00\rangle \pm |11\rangle}{\sqrt{2}}$ and $|\Psi^\pm\rangle = \frac{|01\rangle \pm |10\rangle}{\sqrt{2}}$ We now consider two special subclasses of generic class [22] of four-qubit pure states

$$\mathcal{B} = \{au_1 + au_2 + cu_3 + cu_4 \quad | \quad a, c \in \mathbb{C} \\ \text{and} \quad 2(|a|^2 + |c|^2) = 1\}$$

and

$$\mathcal{D} = \{au_1 + bu_2 + cu_3 + du_4 \quad | \quad a, b, c, d \in \mathbb{R} \\ \text{and} \quad |a|^2 + |b|^2 + |c|^2 + |d|^2 = 1\}$$

For states in subclass $\mathcal{B}$, we have $N_{1|234} = N_{2|134} = N_{3|124} = N_{4|123} = \frac{1}{2}$, $\quad N_{12|34} = N_{14|23} = |a|^2 + |c|^2 + 4|ac|$ and $N_{13|24} = |a^2 - c^2|$. So,

$\delta_1 = \delta_2 = \delta_5 = \delta_6 = |a^2 - c^2|^2 \geq 0$,

$\delta_3 = \delta_4 = |a|^4 + |c|^4 + 16|ac|[|a|^2 + |c|^2] + 2[18|ac|^2 + Re(a^2 c^{*2})] \geq 0$, as $Re(a^2 c^{*2}) \leq |a^2 c^{*2}| = |a^2 c^2|$.

Due to the difficulties in finding the sign of δ_7, numerical simulation (Fig. 1) has been performed with 10^5 random pure states from class $\mathcal{B}$, which clearly shows that $\delta_7 < 0$ for most of the cases.

In particular, if we take a and c as real numbers, then we have obtained the graph of δ_7 versus a (Fig. 2).

For the states in subclass $\mathcal{D}$ (see details in "Appendix 2") due to the difficulty in computation of sign of δ_i, $\forall i = 1, 2, \ldots, 7$, we present numerical evidences using 10^5 random pure states from class $\mathcal{D}$ which shows $\delta_1 = \delta_2 \geq 0$ (Fig. 3),

Also, $\delta_3 = \delta_4 \geq 0$ & $\delta_5 = \delta_6 \geq 0$ (Figs. 8, 9 in "Appendix 2") in all cases. But, numerical evidences for δ_7 (Fig. 4) show that it is negative for most of the cases except for a small number.

Cluster States: Cluster states are used in quantum nonlocality test [23], quantum error correction code [24], etc. Four-qubit cluster states [25] can be written as

$|\psi\rangle = a|0000\rangle + b|0011\rangle + c|1100\rangle - d|1111\rangle$

where $a, b, c, d \in \mathcal{C}$ and $|a|^2 + |b|^2 + |c|^2 + |d|^2 = 1$. Calculating negativity for this state, we observe that $\delta_i \geq 0 \; \forall i = 1, 2, \ldots, 6$ (see "Appendix 2"). For δ_7, numerical simulation with 10^5 random states from this class has been performed (Fig. 5), which shows that for most of the cases $\delta_7 < 0$.

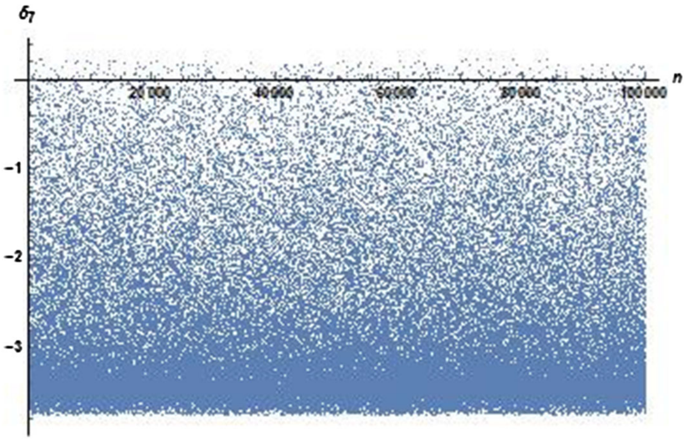

Fig. 1 δ_7 for states in $\mathcal{B}$

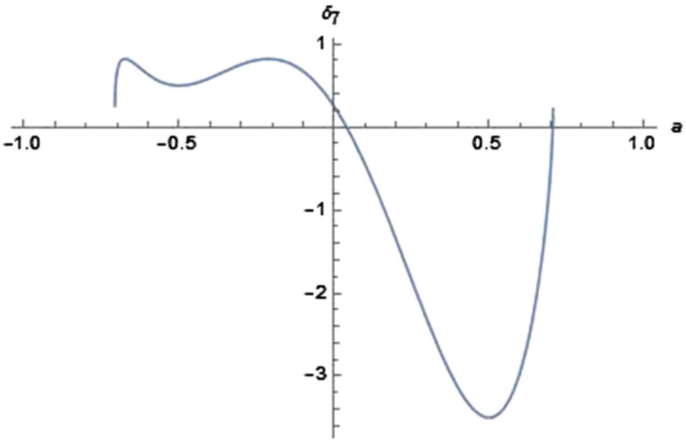

Fig. 2 a vs δ_7 for state in $\mathcal{B}$

Dicke States: A four-qubit Dicke state [26] is given by,

$$|S(4,k)\rangle = \sqrt{\frac{k!(4-k)!}{4!}} \sum_{permutation} |0\rangle^{\otimes(4-k)}|1\rangle^{\otimes k}$$

where the summation is over all possible permutations of the product state having $k(\leq 4)$ qubit in excited state $|1\rangle$ and remaining $(4-k)$ qubits are in ground state. $|S(4,0)\rangle = |0000\rangle$ and $|S(4,4)\rangle = |1111\rangle$, are separable states. $|S(4,1)\rangle = |W\rangle$ and $|S(4,3)\rangle = |\tilde{W}\rangle$. For $|W\rangle$ and $|\tilde{W}\rangle$, we get, $\delta_i = \frac{1}{4}$ $\forall i = 1, 2, \ldots, 6$ and $\delta_7 = 0$ (see "Appendix 2"). When $k = 2$, we get $|S(4,2)\rangle = (|0011\rangle + |1100\rangle + |0110\rangle + |1001\rangle + |1010\rangle + |0101\rangle)/\sqrt{6}$. For this state, we have $\delta_i = \frac{25}{36} > 0, \forall i = 1, 2, \ldots, 6$ and this time, $\delta_7 = -\frac{13}{12} < 0$ (see "Appendix 2").

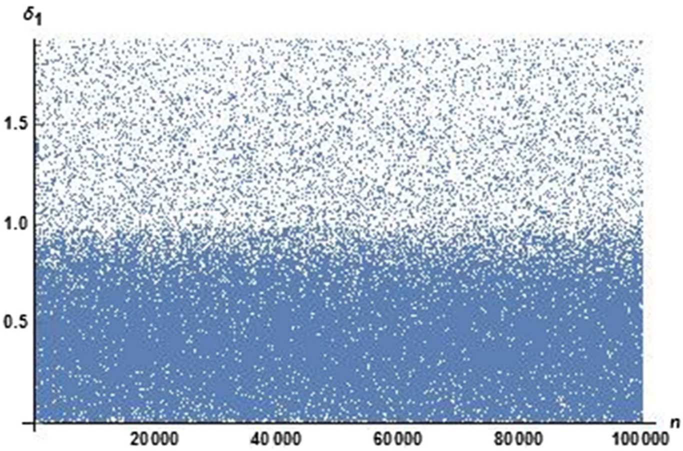

Fig. 3 δ_1 for states in $\mathcal{D}$

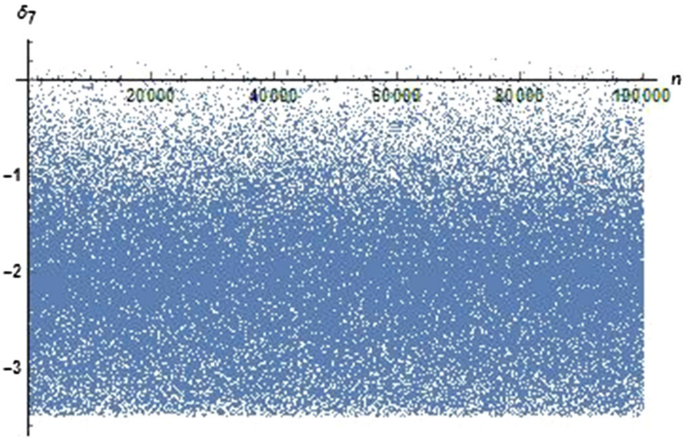

Fig. 4 δ_7 for states in $\mathcal{D}$

Generalized GHZ State: Four-qubit generalized GHZ state is $|GGHZ\rangle = a|0000\rangle + b|1111\rangle$ where $a, b \in \mathbb{C}$ and $|a|^2 + |b|^2 = 1$. Simple calculations have yielded that $N_{1|234} = N_{2|134} = N_{3|124} = N_{4|123} = N_{12|34} = N_{13|24} = N_{14|23} = |ab|$. Hence $\delta_i = |ab|^2 > 0$, $\forall i = 1, 2, \ldots, 7$.

Generalized W State: Four-qubit generalized W state is given by, $|GW\rangle = a|0001\rangle + b|0010\rangle + c|0100\rangle + d|1000\rangle$ where $a, b, c, d \in \mathbb{C}$ and $|a|^2 + |b|^2 + |c|^2 + |d|^2 = 1$. Simple calculations (see "Appendix 2") have revealed that $\delta_i \geq 0$ $\forall i = 1, 2, \ldots, 6$ and $\delta_7 = 0$. Obviously, the results for W state can be derived directly from the generalized W state.

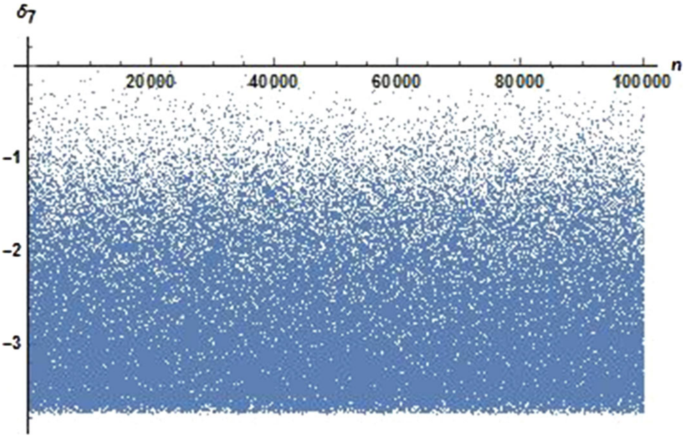

Fig. 5 δ_7 for cluster states

4.2 Monogamy relations in superposition of some pure states

Superposition of $|W\rangle$ *and* $|\tilde{W}\rangle$ *states:* Consider the superposition of $|W\rangle \& |\tilde{W}\rangle$ as $|\psi\rangle = a|\tilde{W}\rangle + be^{i\theta}|W\rangle$ where $a, b \in (0, 1)$, $a^2 + b^2 = 1 \& \ \theta \in [0, 2\pi)$. Here, $N_{1|234} = N_{2|134} = N_{3|124} = N_{4|123} = \frac{1}{4}\sqrt{3 + 4a^2b^2}$ and $N_{12|34} = N_{13|24} = N_{14|23} = \frac{1}{2}$. Therefore, we have $\delta_i = \frac{1}{4} > 0 \ \forall i = 1, 2, \ldots, 6$ and $\delta_7 = a^2b^2 > 0$.

Superposition of $|GW\rangle$ *and* $|0000\rangle$: Suppose, $|\psi\rangle = \sqrt{p}|GW\rangle + \sqrt{1-p}|0000\rangle$ where $0 < p < 1$, $|GW\rangle = a|0001\rangle + b|0010\rangle + c|0100\rangle + d|1000\rangle$, $a, b, c, d \in \mathbb{C}$ such that $|a|^2 + |b|^2 + |c|^2 + |d|^2 = 1$. For this case, we have $\delta_i \geq 0, \ \forall i = 1, 2, \ldots, 6$ and $\delta_7 = 0$ (see "Appendix 2").

Superposition of $|GGHZ\rangle$ *and* $|W\rangle$: Consider, $|\psi\rangle = c_1(a_1|0000\rangle + b_1|1111\rangle) + c_2(|0001\rangle + |0010\rangle + |0100\rangle + |1000\rangle)/2$, $a_1, b_1, c_1, c_2 \in \mathbb{C}$ such that $|a_1|^2 + |b_1|^2 = 1$ and $|c_1|^2 + |c_2|^2 = 1$. Considering c_1a_1, c_1b_1, c_2 as a, b, c, respectively $|\psi\rangle = a|0000\rangle + b|1111\rangle + \frac{c}{2}(|0001\rangle + |0010\rangle + |0100\rangle + |1000\rangle)$ where $a, b, c \in \mathbb{C}$ such that $|a|^2 + |b|^2 + |c|^2 = 1$. For this case we have $\delta_i \geq 0, \ \forall i = 1, 2, \ldots, 6$ (see "Appendix 2"). Due to the difficulty in computation of sign of δ_7, numerical evidence (Fig. 6) is presented using 10^5 random pure states from this class. Figure 6 clearly explains that δ_7 can be positive, negative or even zero for states in this superposed class (Fig. 6).

Particularly assuming, $a = b = \sqrt{p/2}$ and $c = \sqrt{1-p}$ where $p \in (0, 1)$ and we have obtained p vs δ_7 graph (Fig. 7).

For the four-qubit case, we consider different physically important pure states and some subclasses of generic class. It is observed that the relations (30)–(35) are well satisfied for all the mentioned classes and states in this paper, but peculiar behaviour of the relation (36) have been noticed here. We have proved that the relation (36) holds for generalized GHZ state, generalized W state, superposition of $|W\rangle$ and $|\tilde{W}\rangle$ state, superposition of generalized W and ground state $|0000\rangle$, whereas violation is observed in subclasses $\mathcal{B}$, $\mathcal{D}$ of four-qubit pure generic class, Dicke $|S(4, 2)\rangle$ and by cluster state. The most counter-intuitive result has been noticed through the superposition of W state

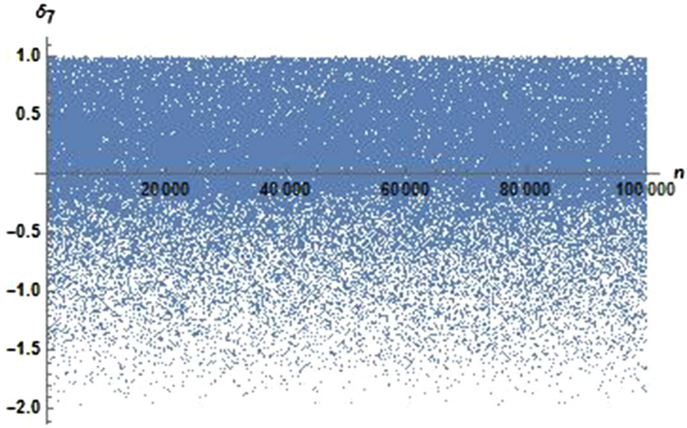

Fig. 6 δ_7 for superposition states of $|GGHZ\rangle$ and $|W\rangle$

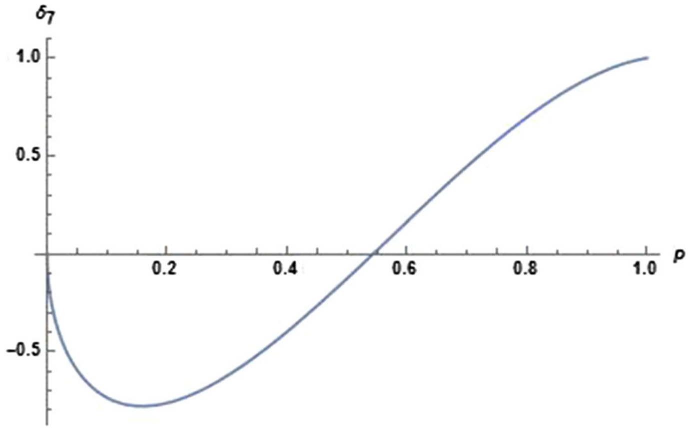

Fig. 7 p vs δ_7 for superposition of $|GHZ\rangle$ and $|W\rangle$

and generalized GHZ state where we see (36) has been violated as well as satisfied for large number of random states. Another important observation of our work enlightens the fact that superposition of states also plays a crucial role on status of (36), contrary to (30)–(35). $\delta_7 = 0$ for $|W\rangle$ and $|\tilde{W}\rangle$ but for their superposition $\delta_7 > 0$. Similar, peculiar behaviour of (36) has been observed for superposition of $|GGHZ\rangle$ and $|W\rangle$, where δ_7 changes sign near $p = 0.55$ (approx.) (Fig. 7), i.e. in this case, (36) violated and satisfied depending on the value of p.

5 Conclusion

In conclusion, we have analysed a new set of monogamy relations in terms of squared negativity for three-qubit and four-qubit pure states. With the help of theorem 1, we have proved three monogamy relations (17)–(19) analytically and compactly. We can write them as

$$\sum_{\phi \neq S \subset \{1,2,3\}} (-1)^{|S \cap T|+1} N^2_{S|S^c} \geq 0$$

where we will get one inequality for each $\Phi \neq T \subset \{1, 2, 3\}$. In four-qubit case for squared negativity, we see that the six relations (30)–(35) plus eight trivial inequalities $(0 \geq 0)$, i.e. total fourteen monogamy relations of type

$$\sum_{\phi \neq S \subset \{1,2,3,4\}} (-1)^{|S \cap T|+1} N^2_{S|S^c} \geq 0$$

where we will get one inequality for each $\Phi \neq T \subset \{1, 2, 3, 4\}$ are always true in all the considered cases of this paper. We have observed that for three-qubit case when $T = \{1, 2, 3\}$, we get a trivial inequality $0 \geq 0$ and in four-qubit case when $T = \{1, 2, 3, 4\}$, the corresponding inequities (36) show different behaviours for different classes. That is why we have excluded the case when T is the set of all parties. We conjecture that for N-qubit pure states the monogamy relations are

$$\sum_{\phi \neq S \subset \{1,2,\ldots,N\}} (-1)^{|S \cap T|+1} N^2_{S|S^c} \geq 0$$

where we will get one inequality for each $\Phi \neq T \subset \{1, 2, \ldots, N\}$, i.e. total $(2^N - 2)$ inequalities, and when $|T|$ is odd, we will get the trivial inequality $0 \geq 0$. We hope our result will provide further insight into entanglement distribution of multipartite systems and could be applied on possible areas of quantum key distributions and quantum cryptography.

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Appendix 1

$$C^2_{1|234} + C^2_{2|134} + C^2_{13|24} + C^2_{14|23} \geq C^2_{3|124} + C^2_{4|123} + C^2_{12|34} \quad for \quad T = \{1, 2\} \tag{23}$$

$$C^2_{3|124} + C^2_{4|123} + C^2_{13|24} + C^2_{14|23} \geq C^2_{1|234} + C^2_{2|134} + C^2_{12|34} \quad for \quad T = \{3, 4\} \tag{24}$$

$$C^2_{1|234} + C^2_{3|124} + C^2_{12|34} + C^2_{14|23} \geq C^2_{4|123} + C^2_{2|134} + C^2_{13|24} \quad for \quad T = \{1, 3\} \tag{25}$$

$$C^2_{4|123} + C^2_{2|134} + C^2_{12|34} + C^2_{14|23} \geq C^2_{1|234} + C^2_{3|124} + C^2_{13|24} \quad for \quad T = \{2, 4\} \tag{26}$$

$$C^2_{1|234} + C^2_{4|123} + C^2_{12|34} + C^2_{13|24} \geq C^2_{2|134} + C^2_{3|124} + C^2_{14|23} \quad for \quad T = \{1, 4\} \tag{27}$$

$$C^2_{2|134} + C^2_{3|124} + C^2_{12|34} + C^2_{13|24} \geq C^2_{1|234} + C^2_{4|123} + C^2_{14|23} \quad for \quad T = \{2, 3\} \tag{28}$$

$$C^2_{1|234} + C^2_{2|134} + C^2_{3|124} + C^2_{4|123} \geq C^2_{12|34} + C^2_{13|24} + C^2_{14|23} \quad for \quad T = \{1, 2, 3, 4\} \tag{29}$$

$$\delta_1 = N^2_{1|234} + N^2_{2|134} + N^2_{13|24} + N^2_{14|23} - N^2_{3|124} - N^2_{4|123} - N^2_{12|34} \quad for \quad T = \{1, 2\}$$

$$
\begin{aligned}
&\delta_2 = N^2_{3|124} + N^2_{4|123} + N^2_{13|24} + N^2_{14|23} - N^2_{1|234} - N^2_{2|134} - N^2_{12|34} \quad for \quad T = \{3, 4\}\\
&\delta_3 = N^2_{1|234} + N^2_{3|124} + N^2_{12|34} + N^2_{14|23} - N^2_{4|123} - N^2_{2|134} - N^2_{13|24} \quad for \quad T = \{1, 3\}\\
&\delta_4 = N^2_{4|123} + N^2_{2|134} + N^2_{12|34} + N^2_{14|23} - N^2_{1|234} - N^2_{3|124} - N^2_{13|24} \quad for \quad T = \{2, 4\}\\
&\delta_5 = N^2_{1|234} + N^2_{4|123} + N^2_{12|34} + N^2_{13|24} - N^2_{2|134} - N^2_{3|124} - N^2_{14|23} \quad for \quad T = \{1, 4\}\\
&\delta_6 = N^2_{2|134} + N^2_{3|124} + N^2_{12|34} + N^2_{13|24} - N^2_{1|234} - N^2_{4|123} - N^2_{14|23} \quad for \quad T = \{2, 3\}\\
&\delta_7 = N^2_{1|234} + N^2_{2|134} + N^2_{3|124} + N^2_{4|123} - N^2_{12|34} - N^2_{13|24} - N^2_{14|23} \quad for \quad T = \{1, 2, 3, 4\}\\
&\delta_8 = \delta_9 = \ldots = \delta_{15} = 0 \; when \; |T| \; is \; odd \; number.
\end{aligned}
$$

$$N^2_{1|234} + N^2_{2|134} + N^2_{13|24} + N^2_{14|23} \geq N^2_{3|124} + N^2_{4|123} + N^2_{12|34} \quad for \quad T = \{1, 2\} \tag{30}$$

$$N^2_{3|124} + N^2_{4|123} + N^2_{13|24} + N^2_{14|23} \geq N^2_{1|234} + N^2_{2|134} + N^2_{12|34} \quad for \quad T = \{3, 4\} \tag{31}$$

$$N^2_{1|234} + N^2_{3|124} + N^2_{12|34} + N^2_{14|23} \geq N^2_{4|123} + N^2_{2|134} + N^2_{13|24} \quad for \quad T = \{1, 3\} \tag{32}$$

$$N^2_{4|123} + N^2_{2|134} + N^2_{12|34} + N^2_{14|23} \geq N^2_{1|234} + N^2_{3|124} + N^2_{13|24} \quad for \quad T = \{2, 4\} \tag{33}$$

$$N^2_{1|234} + N^2_{4|123} + N^2_{12|34} + N^2_{13|24} \geq N^2_{2|134} + N^2_{3|124} + N^2_{14|23} \quad for \quad T = \{1, 4\} \tag{34}$$

$$N^2_{2|134} + N^2_{3|124} + N^2_{12|34} + N^2_{13|24} \geq N^2_{1|234} + N^2_{4|123} + N^2_{14|23} \quad for \quad T = \{2, 3\} \tag{35}$$

$$N^2_{1|234} + N^2_{2|134} + N^2_{3|124} + N^2_{4|123} \geq N^2_{12|34} + N^2_{13|24} + N^2_{14|23} \quad for \quad T = \{1, 2, 3, 4\} \tag{36}$$

Appendix 2

The subclass of four-qubit pure generic state $\mathcal{D}$ is $\mathcal{D} = \{au_1 + bu_2 + cu_3 + du_4 \mid a, b, c, d \in \mathbb{R} \text{ and } |a|^2 + |b|^2 + |c|^2 + |d|^2 = 1\}$

For the states in subclass $\mathcal{D}$ we have

$$
\begin{aligned}
N_{1|234} &= N_{2|134} = N_{3|124} = N_{4|123} = \frac{1}{2},\\
N_{13|24} &= \{|(a+b)^2 - (c+d)^2| + |(a-b)^2 - (c-d)^2| + |(a+c)^2 - (b+d)^2|\\
&\quad +|(a-c)^2 + (b-d)^2| + |(a+d)^2 - (b+c)^2| + |(a-d)^2 - (b-c)^2|\}/4,\\
N_{14|23} &= \{|(a+b)^2 - (c-d)^2| + |(a-b)^2 - (c+d)^2| + |(a+c)^2\\
&\quad -(b-d)^2| + |(a-c)^2 + (b+d)^2| + |(a+d)^2\\
&\quad -(b-c)^2| + |(a-d)^2 - (b+c)^2|\}/4,\\
N_{12|34} &= |ab| + |ac| + |ad| + |bc| + |bd| + |cd|.\\
\delta_1 &= \delta_2 = N^2_{13|24} + N^2_{14|23} - N^2_{12|34},\\
\delta_3 &= \delta_4 = N^2_{12|34} + N^2_{14|23} - N^2_{13|24},\\
\delta_5 &= \delta_6 = N^2_{12|34} + N^2_{13|24} - N^2_{14|23},\\
\delta_7 &= 1 - N^2_{12|34} - N^2_{13|24} - N^2_{14|23}.
\end{aligned}
$$

The numerical simulations using 10^5 pure random states from class $\mathcal{D}$ shows that $\delta_3 = \delta_4 \geq 0$ (Fig. 8) and $\delta_5 = \delta_6 \geq 0$ (Fig. 9).

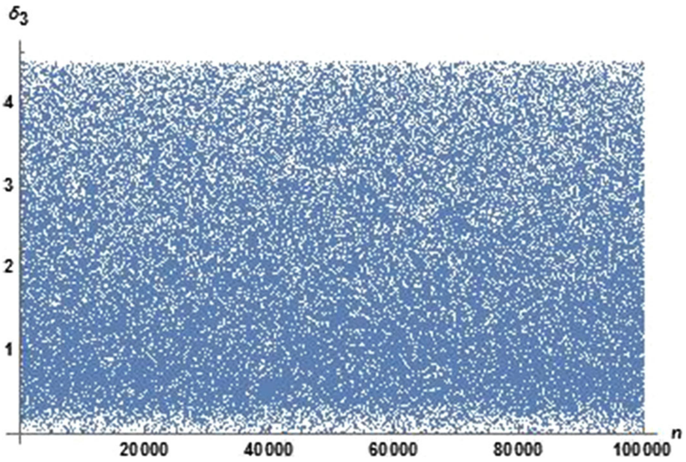

Fig. 8 δ_3 for state in subclass $\mathcal{D}$

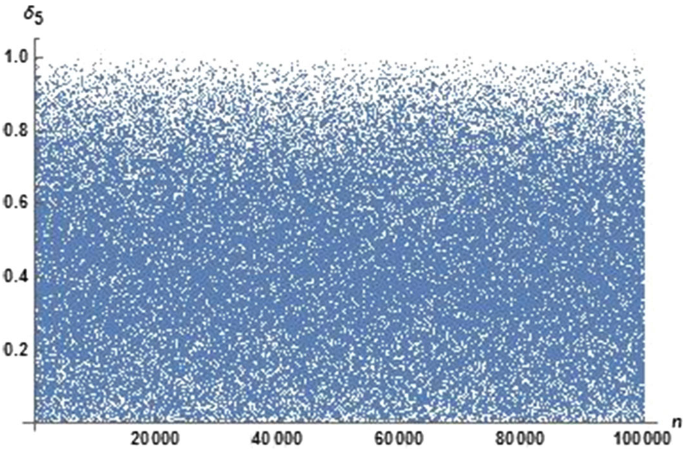

Fig. 9 δ_5 for state in subclass $\mathcal{D}$

Four-qubit cluster state is $|\psi\rangle = a|0000\rangle + b|0011\rangle + c|1100\rangle - d|1111\rangle$ where $a, b, c, d \in \mathcal{C}$ and $|a|^2 + |b|^2 + |c|^2 + |d|^2 = 1$. Negativities of cluster state are $N_{12|34} = |bc + ad|$,

$$\begin{aligned}
N_{13|24} &= N_{14|23} = |ab| + |ac| + |ad| + |bc| + |bd| + |cd|, \\
N_{1|234} &= N_{2|134} = \sqrt{(|a|^2 + |b|^2)(|c|^2 + |d|^2)}, \\
N_{3|124} &= N_{4|123} = \sqrt{(|a|^2 + |c|^2)(|b|^2 + |d|^2)}. \\
\delta_3 &= \delta_4 = N^2_{12|34} + N^2_{14|23} - N^2_{13|24} = |bc + ad|^2 \geq 0, \\
\delta_5 &= \delta_6 = N^2_{12|34} + N^2_{13|23} - N^2_{14|24} = |bc + ad|^2 \geq 0, \\
\delta_1 &= 4(|ac|^2 + |bd|^2) + (|bc|^2 + |ad|^2) + 2(|bcad| - Re(bca^*d^*)) + 2L \geq 0,
\end{aligned}$$

$$\delta_2 = 4(|ab|^2 + |cd|^2) + (|bc|^2 + |ad|^2) + 2(|bcad| - Re(bca^*d^*)) + 2L \geq 0$$
$$[\because |bc||ad| \geq Re(bca^*d^*)],$$

where L is sum of product of $\{|ab|, |ac|, |ad|, |bc|, |bd|, |cd|\}$ taken two at a time except the product $|bc||ad|$.

The $|W\rangle$ and $|\tilde{W}\rangle$ states are

$$|W\rangle = \frac{1}{2}(|0001\rangle + |0010\rangle + |0100\rangle + |1000\rangle)$$
$$|\tilde{W}\rangle = \frac{1}{2}(|1110\rangle + |1101\rangle + |1011\rangle + |0111\rangle)$$

Negativities of $|W\rangle$ and $\tilde{W}$ states are $N_{1|234} = N_{2|134} = N_{3|124} = N_{4|123} = \frac{\sqrt{3}}{4}$ and $N_{12|34} = N_{13|24} = N_{14|23} = \frac{1}{2}$. Hence, $\delta_i = \frac{1}{4} > 0\ \forall i = 1, 2, \ldots, 6$, but $\delta_7 = 0$. The negativities of $|S(4, 2)\rangle$ among different bipartition are $N_{1|234} = N_{2|134} = N_{3|124} = N_{4|123} = \frac{1}{2}$ and $N_{12|34} = N_{13|24} = N_{14|23} = \frac{5}{6}$. Thus, $\delta_i = \frac{25}{36} > 0, \forall i = 1, 2, \ldots, 6$ and $\delta_7 = -\frac{13}{12} < 0$.

Generalized W state is

$|GW\rangle = a|0001\rangle + b|0010\rangle + c|0100\rangle + d|1000\rangle$ where $a, b, c, d \in \mathbb{C}$ and $|a|^2 + |b|^2 + |c|^2 + |d|^2 = 1$.

The negativities are

$$N_{1|234} = |d|\sqrt{|a|^2 + |b|^2 + |c|^2},$$
$$N_{2|134} = |c|\sqrt{|a|^2 + |b|^2 + |d|^2},$$
$$N_{3|124} = |b|\sqrt{|a|^2 + |d|^2 + |c|^2},$$
$$N_{4|123} = |a|\sqrt{|b|^2 + |c|^2 + |d|^2},$$
$$N_{12|34} = \sqrt{(|a|^2 + |b|^2)(|c|^2 + |d|^2)},$$
$$N_{13|24} = \sqrt{(|a|^2 + |c|^2)(|b|^2 + |d|^2)},$$
$$N_{14|23} = \sqrt{(|b|^2 + |c|^2)(|a|^2 + |d|^2)}.$$

$\delta_1 = 4|c|^2|d|^2$, $\delta_2 = 4|a|^2|b|^2$, $\delta_3 = 4|b|^2|d|^2$, $\delta_4 = 4|a|^2|c|^2$, $\delta_5 = 4|a|^2|d|^2$, $\delta_6 = 4|b|^2|c|^2$ and $\delta_7 = 0$. So $\delta_i \geq 0\ \forall i = 1, 2, \ldots, 6$.

Superposition of $|GW\rangle$ and $|0000\rangle$ is $|\psi\rangle = \sqrt{p}|GW\rangle + \sqrt{1-p}|0000\rangle$ where $0 < p < 1$,$|GW\rangle = a|0001\rangle + b|0010\rangle + c|0100\rangle + d|1000\rangle$, $a, b, c, d \in \mathbb{C}$ s.t. $|a|^2 + |b|^2 + |c|^2 + |d|^2 = 1$. The Negativities are, $N_{1|234} = p|d|\sqrt{|a|^2 + |b|^2 + |c|^2}$,

$$N_{2|134} = p|c|\sqrt{|a|^2 + |b|^2 + |d|^2},$$
$$N_{3|124} = p|b|\sqrt{|a|^2 + |d|^2 + |c|^2},$$
$$N_{4|123} = p|a|\sqrt{|b|^2 + |c|^2 + |d|^2},$$

$$N_{12|34} = p\sqrt{(|a|^2+|b|^2)(|c|^2+|d|^2)},$$
$$N_{13|24} = p\sqrt{(|a|^2+|c|^2)(|b|^2+|d|^2)},$$
$$N_{14|23} = p\sqrt{(|b|^2+|c|^2)(|a|^2+|d|^2)}.$$

$\delta_1 = 4p^2|c|^2|d|^2$, $\delta_2 = 4p^2|a|^2|b|^2$, $\delta_3 = 4p^2|b|^2|d|^2$, $\delta_4 = 4p^2|a|^2|c|^2$, $\delta_5 = 4p^2|a|^2|d|^2$, $\delta_6 = 4p^2|b|^2|c|^2$. So $\delta_i \geq 0\ \forall i = 1, 2, \ldots, 6$.

Superposition of $|GGHZ\rangle$ and $|W\rangle$ state is

$|\psi\rangle = a|0000\rangle + b|1111\rangle + \frac{c}{2}(|0001\rangle + |0010\rangle + |0100\rangle + |1000\rangle)$ where $a, b, c \in \mathbb{C}$ s.t. $|a|^2 + |b|^2 + |c|^2 = 1$.

$N_{1|234} = N_{2|134} = \sqrt{16|a|^2|b|^2 + 12|b|^2|c|^2 + 3|c|^4}/4 = N_{3|124} = N_{4|123}$, $N_{12|34} = \frac{|c|^2}{2} + \sqrt{2|a|^2|b|^2 + 2|b|^2|c|^2 - 2\sqrt{|a|^2|b|^4(|a|^2 + 2|c|^2)}} = N_{13|24} = N_{14|23}$.

Since $N_{1|234} = N_{2|134} = N_{3|124} = N_{4|123}$ and $N_{12|34} = N_{13|24} = N_{14|23}$ we have $\delta_i = N_{12|34}^2 \geq 0 \forall i = 1, 2, \ldots, 6$.

Appendix 3

Theorem 1 *For an N partite pure state $|\psi_{A_1A_2\ldots A_N}\rangle$ in a $2\otimes2\otimes\ldots\otimes2$(N times) system the negativity of bipartition $A_1|A_2\ldots A_N$ is half of its concurrence, i.e. $N_{A_1|A_2\ldots A_N} = \frac{1}{2}C_{A_1|A_2\ldots A_N}$ [10].*

Proof For simplicity we write, $A_1 = A$ and $A_2A_3\ldots A_N = B$. By Schmidt decomposition, any bipartite state can be written as $|\psi_{A|B}\rangle = \sum_i \sqrt{\lambda_i}|\phi_A^i\rangle \otimes |\phi_B^i\rangle$ where λ_i are Schmidt coefficients and $\{|\phi_A^i\rangle\}$, $\{|\phi_B^i\rangle\}$ are orthogonal basis for the subsystems A and B.

Now, $\rho_{AB} = \sum_{i,j} \sqrt{\lambda_i\lambda_j}|\phi_A^i\rangle\left\langle\phi_A^j\right| \otimes |\phi_B^i\rangle\left\langle\phi_B^j\right|$

$\implies \rho_{AB}^{t_A} = \sum_{i,j} \sqrt{\lambda_i\lambda_j}|\phi_A^{j'}\rangle\left\langle\phi_A^{i'}\right| \otimes |\phi_B^i\rangle\left\langle\phi_B^j\right|$

So, we have

$$\begin{aligned}
N_{AB} &= \frac{\|\rho_{AB}^{t_A}\|_1 - 1}{2}\\
&= \frac{1}{2}\{\|\sum_{i,j}\sqrt{\lambda_i\lambda_j}|\phi_A^{j'}\rangle\left\langle\phi_A^{i'}\right| \otimes |\phi_B^i\rangle\left\langle\phi_B^j\right|\|_1 - 1\}\\
&= \frac{1}{2}\{\|\sum_{i,j}\sqrt{\lambda_i\lambda_j}|\phi_A^{j'}\rangle\left\langle\phi_B^{j}\right| \otimes |\phi_B^i\rangle\left\langle\phi_A^{i'}\right|\|_1 - 1\}\\
&= \frac{1}{2}\{\|\sum_{j}\sqrt{\lambda_j}|\phi_A^{j'}\rangle\left\langle\phi_B^{j}\right| \otimes \sum_i\sqrt{\lambda_i}|\phi_B^i\rangle\left\langle\phi_A^{i'}\right|\|_1 - 1\}\\
&= \frac{1}{2}\{\|Z \otimes Z^\dagger\|_1 - 1\} \quad [Z = \sum_{j=1}^{2}\sqrt{\lambda_j}|\phi_A^{j'}\rangle\left\langle\phi_B^{j}\right|]
\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{2}\{\|Z\|_1^2 - 1\} \quad [\|A \otimes B\| = \|A\|\|B\|] \\
&= \frac{1}{2}\{(\sqrt{\lambda_1} + \sqrt{\lambda_2})^2 - 1\} \\
&= \frac{1}{2} \times 2\sqrt{\lambda_1\lambda_2} \quad [\sum_{i=1}^{2} \lambda_i = 1] \\
&= \frac{1}{2} \times 2\sqrt{det(\rho_A)} \\
&= \frac{1}{2} C_{AB}
\end{aligned}
$$

Hence, $N_{A_1|A_2 \ldots A_N} = \frac{1}{2} C_{A_1|A_2 \ldots A_N}$ (proved). □

Theorem 2 *For an N partite pure state* $|\psi_{A_1 A_2 \ldots A_N}\rangle$ *in a* $d_1 \otimes d_2 \otimes \ldots \otimes d_N$ *dimensional system where* $d_i > 2\ \forall i = 1, 2, \ldots, N$, $N_{A_1|A_2 \ldots A_N} \geq \frac{1}{2} C_{A_1|A_2 \ldots A_N}$.

Proof For simplicity we write $A_1 = A$ & $A_2 \otimes A_3 \otimes \ldots \otimes A_N = B$. Suppose, $d \leq min\{d_1, d_2.d_3 \ldots d_N\}$, then by Schmidt decomposition for any bipartite state, we write, $|\Psi_{A|B}\rangle = \sum_{i=1}^{d} \sqrt{\lambda_i}|\phi_A^i\rangle \otimes |\phi_B^i\rangle$ where λ_i are Schmidt coefficients and $\{|\phi_A^i\rangle\}, \{|\phi_B^i\rangle\}$ are orthogonal basis for the subsystems A and B, respectively. By the similar calculations from theorem 1 we can say that □

$$
\begin{aligned}
N_{AB} &= \frac{1}{2}\{\|Z\|_1^2 - 1\} = \frac{1}{2}\{[\sum_{i=1}^{d} \sqrt{\lambda_i}]^2 - 1\} \\
&= \frac{1}{2}(2 \sum_{i \neq j = 1}^{d} \sqrt{\lambda_i \lambda_j}) \geq \frac{1}{2} \times 2 \times \binom{d}{2} \sqrt{\prod_{i=1}^{d} \lambda_i} \geq \frac{1}{2} \times 2 \sqrt{\prod_{i=1}^{d} \lambda_i} \\
&\implies N_{AB} \geq \frac{1}{2} \times 2\sqrt{\lambda_1 \lambda_2 \ldots \lambda_d} \\
&\implies N_{AB} \geq \frac{1}{2} \times 2\sqrt{det(\rho_A)} \\
&\implies N_{AB} \geq \frac{1}{2} C_{AB}
\end{aligned}
$$

where $Z = \sum_{i=1}^{d} \sqrt{\lambda_i}|\phi_A^i\rangle\langle\phi_B^i|$, $\|A \otimes B\| = \|A\|\|B\|$ and $\sum_{i=1}^{d} \lambda_i = 1$
Hence, $N_{A_1|A_2 \ldots A_N} \geq \frac{1}{2} C_{A_1|A_2 \ldots A_N}$ (proved).

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