Quantum Information Processing (2020) 19:256
https://doi.org/10.1007/s11128-020-02761-y

Stronger uncertainty relations of mixed states

Yajing Fan[1] · Huaixin Cao[2] · Liang Chen[3] · Huixian Meng[4]

Received: 1 February 2020 / Accepted: 11 July 2020 / Published online: 23 July 2020

Abstract
The Heisenberg–Robertson uncertainty relation bounds the product of the variances in the two possible measurement outcomes in terms of the expectation of the commutator of the observables. Notably, it does not capture the concept of incompatible observables because it can be trivial, i.e., the lower bound can be null even for two noncompatible observables. Here, we give two stronger uncertainty relations, relating to the sum of variances with respect to density matrix, whose lower bounds are guaranteed to be nontrivial whenever the two observables are incompatible on the state of the system; moreover, two stronger uncertainty relations in terms of the product of the variances of two observables are established. Also, several stronger uncertainty relations for three observables are established, relating to the sum and product of variances with respect to density matrix, respectively.

Keywords Uncertainty relation · Variance · Observable

1 Introduction

Uncertainty relations are fundamental in quantum mechanics, underlying many conceptual differences between classical and quantum theories. The Heisenberg–Robertson uncertainty relations [1] are expressed in terms of the product $V_\rho(A)V_\rho(B)$ of the variances of the measurement results of the observables A and B, and the product can be null even when one of the two variances is different from zero. Here, we provide a different uncertainty relation, based on the sum $V_\rho(A)+V_\rho(B)$, that is guaranteed to

✉ Yajing Fan
fanyajing@snnu.edu.cn

1 School of Mathematics and Information Science, Ningxia Key Laboratory of Intelligent Information and Big Data Processing, North Minzu University, Yinchuan 750021, China

2 School of Mathematics and Information Science, Shaanxi Normal University, Xi'an 710062, China

3 Department of Mathematics, Changji College, Changji 831100, China

4 School of Mathematics and Physics, North China Electric Power University, Beijing 102206, China

be nontrivial whenever the observables are incompatible on the state. Previous uncertainty relations that provide a bound to the sum of the variances comprise a lower bound in terms of the variance of the sum of observables [2], a lower bound based on the entropic uncertainty relations [3], a sum uncertainty relation for angular momentum observables [4], sum uncertainty relations for arbitrary N observables [5], a series of uncertainty inequalities in the qubit system and a state-independent bound for the sum of variances [6], a unified and exact framework for the variance-based uncertainty relations [7], a lower bound based on the Wigner–Yanase skew information or Wigner–Yanase–Dyson skew information uncertainty relations [8–12]. Uncertainty relations are useful in many areas related or even unrelated to quantum mechanics: entanglement detection [13,14], quantum cryptography [15], signal processing [16], etc. Owing back to entanglement measure, Zidan's model for quantum computing [17–20] was discovered. This model was used to solve an extended version of the Deutsch–Jozsa algorithm. This extension was intractable for more than 27 years using the quantum circuit model [18]. So, developing uncertainty relations could be helpful to produce new quantum technologies.

The quantum mechanical uncertainties associated with observables A and B in the state $|\psi\rangle$ are defined via $\Delta A^2 = \langle\psi|A^2|\psi\rangle - \langle\psi|A|\psi\rangle^2$ and $\Delta B^2 = \langle\psi|B^2|\psi\rangle - \langle\psi|B|\psi\rangle^2$. Similarly, we can define the uncertainty in the sum of two observables as $\Delta(A+B)^2 = \langle\psi|(A+B)^2|\psi\rangle - \langle\psi|(A+B)|\psi\rangle^2$. The expectation value of A is defined as $\langle A\rangle = \langle\psi|A|\psi\rangle$. The Heisenberg–Robertson uncertainty relation [1] bounds the product of the variances through the expectation value of the commutator

$$\Delta A^2 \Delta B^2 \geq \left|\frac{1}{2}\langle[A, B]\rangle\right|^2. \tag{1}$$

It was strengthened by Schrödinger [21], obtaining

$$\Delta A^2 \Delta B^2 \geq \left|\frac{1}{2}\langle[A, B]\rangle\right|^2 + \left|\frac{1}{2}\langle\{A, B\}\rangle - \langle A\rangle\langle B\rangle\right|^2. \tag{2}$$

In [22], Maccone and Pati gave two stronger uncertainty relations, relating to the sum of variances with respect to pure state, whose lower bound is guaranteed to be nontrivial whenever the two observables are incompatible on the state of the system. The following two inequalities have lower bounds which are nontrivial. The first inequality is

$$\Delta A^2 + \Delta B^2 \geq \pm \mathrm{i}\langle[A, B]\rangle + \left|\langle\psi|A \pm \mathrm{i}B|\psi^{\perp}\rangle\right|^2, \tag{3}$$

which is valid for arbitrary states $|\psi^{\perp}\rangle$ orthogonal to the state of the system $|\psi\rangle$.

A second inequality with nontrivial bound even if $|\psi\rangle$ is an eigenstate either of A or of B is

$$\Delta A^2 + \Delta B^2 \geq \frac{1}{2}\left|\langle\psi^{\perp}_{A+B}|A+B|\psi\rangle\right|^2, \tag{4}$$

where $|\psi^{\perp}_{A+B}\rangle = (A + B - \langle A + B\rangle)|\psi\rangle/N$ is a state orthogonal to the state of the system $|\psi\rangle$. One can also obtain an amended Heisenberg–Robertson inequality

$$\Delta A \Delta B \geq \pm \mathrm{i}\langle [A, B]\rangle \Bigg/ \left(1 - \frac{1}{2}\left|\left\langle \psi \left| \frac{A}{\Delta A} \pm \mathrm{i}\frac{B}{\Delta B}\right| \psi^{\perp}\right\rangle\right|^2\right), \tag{5}$$

which reduces to (1) when minimizing the lower bound over $|\psi^{\perp}\rangle$ and becomes an equality when maximizing it.

In [25], the authors prove that the following stronger inequality exists:

$$\begin{aligned}\Delta A^2 + \Delta B^2 + \Delta C^2 \geq \frac{1}{3}\Delta(A + B + C)^2 \pm \mathrm{i}\frac{\sqrt{3}}{3}\langle [A, B, C]\rangle \\ + \frac{2}{3}\left|\left\langle \psi \left| \left(A + e^{\pm \mathrm{i}\frac{2\pi}{3}} B + e^{\pm \mathrm{i}\frac{4\pi}{3}} C\right)\right| \psi^{\perp}\right\rangle\right|^2,\end{aligned} \tag{6}$$

which is valid for arbitrary states $\left|\psi^{\perp}\right\rangle$ orthogonal to the state of the system $|\psi\rangle$, where $\langle [A, B, C]\rangle = \langle [A, B]\rangle + \langle [B, C]\rangle + \langle [C, A]\rangle$ and the sign should be chosen so that $\pm\mathrm{i}\frac{\sqrt{3}}{3}\langle [A, B, C]\rangle$ (a real quantity) is positive.

For a quantum state ρ and observables A and B, the Heisenberg uncertainty relation was expressed as follows:

$$V_\rho(A)V_\rho(B) \geq \frac{1}{4} \mid \mathrm{Tr}(\rho[A, B]) \mid^2 . \tag{7}$$

The further stronger result was given by Schrödinger:

$$V_\rho(A)V_\rho(B) - |\mathrm{Re}\{\mathrm{Cov}_\rho(A, B)\}|^2 \geq \frac{1}{4} \mid \mathrm{Tr}(\rho[A, B]) \mid^2, \tag{8}$$

where the covariance is defined by $\mathrm{Cov}_\rho(A, B) = \mathrm{Tr}\big[\rho(A - (\mathrm{Tr}\rho A)I)(B - (\mathrm{Tr}\rho B)I)\big]$, and $V_\rho(A) = \mathrm{Cov}_\rho(A, A)$.

In this paper, we will give two stronger uncertainty relations, relating to the sum of variances with respect to density matrix, whose lower bound is guaranteed to be nontrivial whenever the two observables are incompatible on the state of the system; moreover, two stronger uncertainty relations in terms of the product of the variances of two observables will be established in Sect. 2. Also, several stronger uncertainty relations for three observables will be established in Sect. 3.

2 Stronger uncertainty relations for two observables

Let H be a separable complex Hilbert space with the inner product $(\cdot, \cdot)$, and $B(H)$ the algebra of all bounded linear operators on H. The set of all trace-class operators on H is denoted by $L^1(H)$. Recall that an operator $A \in B(H)$ is said to be a *Hilbert–Schmidt* operator if

$$\|A\|_2 := \left(\sum_{n\in I}\langle e_n|A^*A|e_n\rangle\right)^{1/2} < \infty$$

for some orthonormal basis $\{|e_n\rangle\}_{n\in I}$ for H. The set of all Hilbert–Schmidt operators on H is denoted by $L^2(H)$. It is well known that both $L^1(H)$ and $L^2(H)$ are self-ideals of the C^*-algebra $B(H)$, and the product of two Hilbert–Schmidt operators on H is a trace class on H. The Hilbert–Schmidt inner product $\langle A, B\rangle := \mathrm{Tr}A^*B$.

In what follows, for an operator $A \in B(H)$, the adjoint of A is denoted by A^*. An operator $A \in B(H)$ is said to be self-adjoint if $A = A^*$. The set of all self-adjoint operators on H is denoted by $S(H)$. A state is given by a positive operator ρ of trace 1, called a density operator. The set of all states is denoted by $D(H)$.

For a mixed state ρ with the spectral decomposition $\rho = \sum_i \lambda_i|\psi_i\rangle\langle\psi_i|$, denote $\Delta A_i^2 = \langle\psi_i|A^2|\psi_i\rangle - \langle\psi_i|A|\psi_i\rangle^2$, $\Delta B_i^2 = \langle\psi_i|B^2|\psi_i\rangle - \langle\psi_i|B|\psi_i\rangle^2$. As is well known, the variance $V_\rho(A)$ is a concave function with respect to ρ, which is the following lemma.

Lemma 1 [23] *Let* $\rho_i \in D(H), \forall i$, $A \in S(H)$. *Then,*

$$V_{\sum_i p_i\rho_i}(A) \geq \sum_i p_i V_{\rho_i}(A),$$

where $\sum_i p_i = 1$, $p_i \geq 0, \forall i$.

Theorem 1 *Let* $\rho \in D(H)$, *and* $\rho = \sum_i \lambda_i|\psi_i\rangle\langle\psi_i|$. *If there exists* $|\psi^\perp\rangle$ *such that* $\langle\psi^\perp|\psi_i\rangle = 0, \forall i$, *then*

$$V_\rho(A) + V_\rho(B) \geq \pm\mathrm{i}\mathrm{Tr}(\rho[A, B]) + \left\|\rho^{\frac{1}{2}}(A \pm \mathrm{i}B)|\psi^\perp\rangle\right\|_2^2.$$

Proof We see from (3) that

$$\begin{aligned}\Delta A_i^2 + \Delta B_i^2 &\geq \pm\mathrm{i}\langle[A, B]\rangle + \left|\langle\psi_i|A \pm \mathrm{i}B|\psi^\perp\rangle\right|^2 \\ &= \pm\mathrm{i}\mathrm{Tr}([A, B]|\psi_i\rangle\langle\psi_i|) + \mathrm{Tr}\left((A \pm \mathrm{i}B)|\psi^\perp\rangle\langle\psi^\perp|(A \mp \mathrm{i}B)|\psi_i\rangle\langle\psi_i|\right).\end{aligned}$$

By multiplying both members by λ_i and summing over i, we obtain the mixed-state extension of (3):

$$\sum_i \lambda_i\Delta A_i^2 + \sum_i \lambda_i\Delta B_i^2 \geq \pm\mathrm{i}\mathrm{Tr}([A, B]\rho) + \mathrm{Tr}\left((A \pm \mathrm{i}B)|\psi^\perp\rangle\langle\psi^\perp|(A \mp \mathrm{i}B)\rho\right).$$

By the concavity of the variance, we get $V_\rho(A) \geq \sum_i \lambda_i\Delta A_i^2$, $V_\rho(B) \geq \sum_i \lambda_i\Delta B_i^2$. Hence, we obtain the conclusion. □

Write $O(H) = \{\rho \in D(H) | \exists \rho_\perp \in D(H), \text{s.t. } \rho\rho_\perp = 0\}$.

Theorem 2 *Let* $\rho \in O(H)$, $A, B \in S(H)$, *and* $\exists\ \rho_\perp \in D(H)$, s.t. $\rho\rho_\perp = 0$. *Then,*

$$V_\rho(A) + V_\rho(B) \geq \pm \mathrm{iTr}(\rho[A, B]) + \left\| \rho^{\frac{1}{2}} (A \pm \mathrm{i}B) \rho_\perp^{\frac{1}{2}} \right\|_2^2 .$$

Proof Denote $C = A - \mathrm{Tr}(\rho A)$, $D = B - \mathrm{Tr}(\rho B)$. So $\|C\rho^{\frac{1}{2}}\|_2^2 = \mathrm{Tr}(\rho C^2) = V_\rho(A)$, and $\|\mathrm{i}D\rho^{\frac{1}{2}}\|_2^2 = V_\rho(B)$. Thus,

$$\begin{aligned}
\left\| (C \mp \mathrm{i}D)\rho^{\frac{1}{2}} \right\|_2^2 &= \left\langle C\rho^{\frac{1}{2}} \mp \mathrm{i}D\rho^{\frac{1}{2}}, C\rho^{\frac{1}{2}} \mp \mathrm{i}D\rho^{\frac{1}{2}} \right\rangle \\
&= \left\| C\rho^{\frac{1}{2}} \right\|_2^2 + \left\| \mathrm{i}D\rho^{\frac{1}{2}} \right\|_2^2 \mp \mathrm{i} \left\langle C\rho^{\frac{1}{2}}, D\rho^{\frac{1}{2}} \right\rangle \pm \mathrm{i} \left\langle D\rho^{\frac{1}{2}}, C\rho^{\frac{1}{2}} \right\rangle \\
&= \left\| C\rho^{\frac{1}{2}} \right\|_2^2 + \left\| \mathrm{i}D\rho^{\frac{1}{2}} \right\|_2^2 \mp \mathrm{iTr}(\rho[C, D]) \\
&= V_\rho(A) + V_\rho(B) \mp \mathrm{iTr}(\rho[A, B]).
\end{aligned}$$

On the other hand, since $\rho\rho_\perp = 0$,

$$\begin{aligned}
\left\| \rho^{\frac{1}{2}} (A \pm \mathrm{i}B) \rho_\perp^{\frac{1}{2}} \right\|_2^2 &= \left\| \rho^{\frac{1}{2}} (C \pm \mathrm{i}D) \rho_\perp^{\frac{1}{2}} \right\|_2^2 \\
&\leq \left\| \rho^{\frac{1}{2}} (C \pm \mathrm{i}D) \right\|_2^2 \cdot \left\| \rho_\perp^{\frac{1}{2}} \right\|^2 \\
&\leq \left\| \rho^{\frac{1}{2}} (C \pm \mathrm{i}D) \right\|_2^2 = \left\| (C \mp \mathrm{i}D)\rho^{\frac{1}{2}} \right\|_2^2 .
\end{aligned}$$

Therefore, we obtain the conclusion. □

It is easy to compute that $V_\rho(A) + V_\rho(B) \mp \mathrm{iTr}(\rho[A, B]) = \mathrm{Cov}_\rho(A \pm \mathrm{i}B, A \mp \mathrm{i}B)$. Thus, by the use of Theorem 2, we can get the following corollary.

Corollary 1 *Let* $\rho \in O(H)$, $A, B \in S(H)$, *and* $\exists\ \rho_\perp \in D(H)$, s.t. $\rho\rho_\perp = 0$. *Then,*

$$\mathrm{Cov}_\rho(A \pm \mathrm{i}B, A \mp \mathrm{i}B) \geq \left\| \rho^{\frac{1}{2}} (A \pm \mathrm{i}B) \rho_\perp^{\frac{1}{2}} \right\|_2^2 .$$

Remark 1 If we take $\rho = |\psi\rangle\langle\psi|$, $\rho_\perp = \left|\psi^\perp\right\rangle\left\langle\psi^\perp\right|$, then the inequality in Theorem 2 can degenerate to (3).

Theorem 3 *Let* $\rho \in O(H)$, $A, B \in S(H)$, *and* $\exists\ \rho_\perp \in D(H)$, s.t. $\rho\rho_\perp = 0$. *Then,*

$$V_\rho(A) + V_\rho(B) \geq \frac{1}{2} \left\| \rho^{\frac{1}{2}} (A + B) \rho_\perp^{\frac{1}{2}} \right\|_2^2 .$$

Proof Denote $C = A - \mathrm{Tr}(\rho A)$, $D = B - \mathrm{Tr}(\rho B)$. So $\left\| C\rho^{\frac{1}{2}} \right\|_2^2 = V_\rho(A)$, and $\left\| D\rho^{\frac{1}{2}} \right\|_2^2 = V_\rho(B)$. Using the parallelogram law in Hilbert space, we have

$$\left\| C\rho^{\frac{1}{2}} \right\|_2^2 + \left\| D\rho^{\frac{1}{2}} \right\|_2^2 = \frac{1}{2}\left[\left\| (C+D)\rho^{\frac{1}{2}} \right\|_2^2 + \left\| (C-D)\rho^{\frac{1}{2}} \right\|_2^2 \right] \geq \frac{1}{2}\left\| (C+D)\rho^{\frac{1}{2}} \right\|_2^2.$$

On the other hand, since $\rho\rho_\perp = 0$,

$$\begin{aligned} \frac{1}{2}\left\| \rho^{\frac{1}{2}}(A+B)\rho_\perp^{\frac{1}{2}} \right\|_2^2 &= \frac{1}{2}\left\| \rho^{\frac{1}{2}}(C+D)\rho_\perp^{\frac{1}{2}} \right\|_2^2 \\ &\leq \frac{1}{2}\left\| \rho^{\frac{1}{2}}(C+D) \right\|_2^2 \cdot \left\| \rho_\perp^{\frac{1}{2}} \right\|^2 \\ &\leq \frac{1}{2}\left\| \rho^{\frac{1}{2}}(C+D) \right\|_2^2. \end{aligned}$$

Therefore, we obtain the conclusion. □

Corollary 2 *Let* $|\psi\rangle$, $\left|\psi^\perp\right\rangle$ *be orthogonal unit vectors on* H, $A, B \in S(H)$. *Then,*

$$\Delta A^2 + \Delta B^2 \geq \frac{1}{2}\left| \langle \psi^\perp | (A+B) | \psi \rangle \right|^2. \tag{9}$$

Proof Take $\rho = |\psi\rangle\langle\psi|$, $\rho_\perp = |\psi^\perp\rangle\langle\psi^\perp|$ in Theorem 3, then $V_\rho(A) = \Delta A^2$, $V_\rho(B) = \Delta B^2$, and

$$\frac{1}{2}\left\| \rho^{\frac{1}{2}}(A+B)\rho_\perp^{\frac{1}{2}} \right\|_2^2 = \frac{1}{2}\left| \langle \psi^\perp | (A+B) | \psi \rangle \right|^2.$$

Therefore, using Theorem 3, we get the conclusion. □

In Corollary 2, if we choose $|\psi^\perp\rangle = \left|\psi_{A+B}^\perp\right\rangle$, then the inequality (9) degenerates to (4).

Corollary 3 *Let* $\rho \in D(H)$, *and* $\rho = \sum_i \lambda_i |\psi_i\rangle\langle\psi_i|$. *If there exists* $|\psi^\perp\rangle$ *such that* $\langle\psi^\perp|\psi_i\rangle = 0, \forall i$, *then*

$$V_\rho(A) + V_\rho(B) \geq \frac{1}{2}\left\| \rho^{\frac{1}{2}}(A+B)|\psi^\perp\rangle \right\|_2^2.$$

Proof Put $\rho_\perp = |\psi^\perp\rangle\langle\psi^\perp|$, then $\rho\rho_\perp = 0$. By the use of Theorem 3, we have

$$V_\rho(A) + V_\rho(B) \geq \frac{1}{2}\left\| \rho^{\frac{1}{2}}(A+B)\rho_\perp^{\frac{1}{2}} \right\|_2^2 = \frac{1}{2}\left\| \rho^{\frac{1}{2}}(A+B)|\psi^\perp\rangle \right\|_2^2.$$

This completes the proof. □

Proposition 1 *Let* $\rho \in D(H)$, $A, B \in S(H)$. *Then,*

$$\sqrt{V_\rho(A+B)} \le \sqrt{V_\rho(A)} + \sqrt{V_\rho(B)}.$$

Proof Denote $C = A - \mathrm{Tr}(\rho A)$, $D = B - \mathrm{Tr}(\rho B)$. So $\left\| C\rho^{\frac{1}{2}} \right\|_2^2 = V_\rho(A)$, and $\left\| D\rho^{\frac{1}{2}} \right\|_2^2 = V_\rho(B)$. Thus,

$$\begin{aligned} V_\rho(A+B) = \left\| (C+D)\rho^{\frac{1}{2}} \right\|_2^2 &= \left\| C\rho^{\frac{1}{2}} \right\|_2^2 + \left\| D\rho^{\frac{1}{2}} \right\|_2^2 + 2\mathrm{Re}\left\langle C\rho^{\frac{1}{2}}, D\rho^{\frac{1}{2}} \right\rangle \\ &\le \left\| C\rho^{\frac{1}{2}} \right\|_2^2 + \left\| D\rho^{\frac{1}{2}} \right\|_2^2 + 2\left| \left\langle C\rho^{\frac{1}{2}}, D\rho^{\frac{1}{2}} \right\rangle \right|. \end{aligned}$$

Further using Schwarz inequality, we have

$$\begin{aligned} V_\rho(A+B) &\le V_\rho(A) + V_\rho(B) + 2\left\| C\rho^{\frac{1}{2}} \right\|_2^{\frac{1}{2}} \cdot \left\| D\rho^{\frac{1}{2}} \right\|_2^{\frac{1}{2}} \\ &= V_\rho(A) + V_\rho(B) + 2\sqrt{V_\rho(A)}\sqrt{V_\rho(B)} \\ &= \left(\sqrt{V_\rho(A)} + \sqrt{V_\rho(B)} \right)^2, \end{aligned}$$

which yields the conclusion. □

In general, for N observables $A_1, A_2, \cdots, A_N$ and mixed state ρ, we have

$$\sqrt{V_\rho\left(\sum_{i=1}^N A_i\right)} \le \sum_{i=1}^N \sqrt{V_\rho(A_i)}.$$

Holevo derived the following useful relation [24]:

$$\sqrt{V_\rho(A)} + \sqrt{V_{\rho'}(A)} \ge \left| \left(E_\rho(A) - E_{\rho'}(A)\right) \mathrm{Tr}\left(\rho^{\frac{1}{2}}\rho'^{\frac{1}{2}}\right) \right| \Big/ \sqrt{2\left(1 - \mathrm{Tr}\left(\rho^{\frac{1}{2}}\rho'^{\frac{1}{2}}\right)\right)}, \tag{10}$$

where $E_\rho(A) = \mathrm{Tr}(\rho A)$, $E_{\rho'}(A) = \mathrm{Tr}(\rho' A)$ are the expectation values of A on the states ρ and ρ', respectively. By using the square-modulus inequality and following a procedure analogous to the one employed by Holevo to derive the inequality (10), we can get the following relation.

Lemma 2 *Let* $\rho \in D(H)$, $\varphi \in (-\frac{\pi}{2}, \frac{\pi}{2})$, *and* λ *be any real number. If* $\sigma \in L^2(H)$ *satisfies* $\mathrm{Tr}\left(\rho^{\frac{1}{2}}\sigma\right) = 0$, $\|\sigma\|_2 = 1$, *and put*

$$A_\varphi = \cos\varphi \cdot \rho^{\frac{1}{2}} + e^{\mathrm{i}\lambda} \sin\varphi \cdot \sigma, \ \rho' = A_\varphi A_\varphi^*, \tag{11}$$

then

(i) ρ' *is a state.*
(ii) $\mathrm{Tr}\left(A_\varphi^* A_\varphi\right) = 1.$
(iii) $\mathrm{Tr}\left(\rho^{\frac{1}{2}} A_\varphi\right) = \cos\varphi.$
(iv) *In the sense of* $\|\cdot\|_2$, *we have* $\lim_{\varphi\to 0} A_\varphi = \rho^{\frac{1}{2}}$, $\lim_{\varphi\to 0} A_\varphi^* = \rho^{\frac{1}{2}}$, *and* $\lim_{\varphi\to 0} \rho' = \rho$.

Theorem 4 *Let* $\rho \in D(H)$, $A, B \in S(H)$. *If* $\sigma \in L^2(H)$ *satisfies* $\mathrm{Tr}\left(\rho^{\frac{1}{2}}\sigma\right) = 0$ *and* $\|\sigma\|_2 = 1$, *then*

$$\sqrt{V_\rho(A)V_\rho(B)} \geq \pm\frac{\mathrm{i}}{2}\mathrm{Tr}(\rho[A,B]) \Big/ \left[1 - \frac{1}{2}\left|\mathrm{Tr}\left(\rho^{\frac{1}{2}}\left(\frac{A}{\sqrt{V_\rho(A)}} \pm \mathrm{i}\frac{B}{\sqrt{V_\rho(B)}}\right)\sigma\right)\right|^2\right]. \tag{12}$$

Proof Let Eq. (11) be valid. Start from the following inequality

$$\left\| c_A(A-a)\rho^{\frac{1}{2}} \pm \mathrm{i}c_B(B-b')A_\varphi + c\left(\rho^{\frac{1}{2}} - A_\varphi\right)\right\|_2^2 \geq 0, \tag{13}$$

where $b' = \mathrm{Tr}(\rho' B)$, $a = \mathrm{Tr}(\rho A)$, c_A, c_B, c real constants. Calculating the square modulus, we find

$$c_A^2 V_\rho(A) + c_B^2 V_{\rho'}(B) \geq -c^2 r - c_A c_B c\delta \pm \mathrm{i}c_A c_B k, \tag{14}$$

with $V_\rho(A)$ and $V_{\rho'}(B)$ the variance of A and B on ρ and ρ', respectively, and where

$$r = 2(1-\cos\varphi), \quad k = 2\mathrm{iIm}\left\{\mathrm{Tr}\left(A_\varphi^*(B-b')(A-a)\rho^{\frac{1}{2}}\right)\right\},$$
$$\delta = 2\mathrm{Re}\left\{\mathrm{Tr}\left(\rho^{\frac{1}{2}}\left(\frac{a-A}{c_B} \pm \mathrm{i}\frac{B-b'}{c_A}\right)A_\varphi\right)\right\}.$$

Now choose the value of c that maximizes the right-hand side of (14), namely $c = -\frac{c_A c_B \delta}{2r}$, then (14) becomes

$$c_A^2 V_\rho(A) + c_B^2 V_{\rho'}(B) \geq \frac{(c_A c_B \delta)^2}{4r} \pm \mathrm{i}c_A c_B k. \tag{15}$$

Put $c_A = \sqrt{V_{\rho'}(B)}$, $c_B = -\sqrt{V_\rho(A)}$, then

$$\sqrt{V_\rho(A)V_{\rho'}(B)} \geq \frac{\sqrt{V_\rho(A)V_{\rho'}(B)}\delta^2}{8r} \mp \frac{\mathrm{i}}{2}k. \tag{16}$$

Denote $b = \mathrm{Tr}(\rho B)$, and by the use of Lemma 2, we have

$$\lim_{\varphi\to 0} k = \lim_{\varphi\to 0} 2\mathrm{iIm}\left\{\mathrm{Tr}\left(A_\varphi^*(B-b')(A-a)\rho^{\frac{1}{2}}\right)\right\}$$
$$= 2\mathrm{iIm}\left\{\mathrm{Tr}\left(\rho^{\frac{1}{2}}(B-b)(A-a)\rho^{\frac{1}{2}}\right)\right\}$$

$$
\begin{aligned}
&= 2\mathrm{iIm}\left\{\mathrm{Tr}\left(\rho(B-b)(A-a)\right)\right\} \\
&= 2\mathrm{iIm}\{\mathrm{Cov}_\rho(B,A)\} \\
&= \mathrm{Cov}_\rho(B,A) - \mathrm{Cov}_\rho(A,B) \\
&= -\mathrm{Tr}(\rho[A,B]),
\end{aligned}
$$

and $\lim\limits_{\varphi\to 0} V_{\rho'}(B) = V_\rho(B)$. In the following, we compute the limit of $\frac{\delta^2}{8r}$.

$$
\begin{aligned}
\lim_{\varphi\to 0}\frac{\delta^2}{8r} &= \lim_{\varphi\to 0}\frac{\left(\mathrm{Re}\left\{\mathrm{Tr}\left(\rho^{\frac{1}{2}}\left(\frac{A-a}{\sqrt{V_\rho(A)}} \pm \mathrm{i}\frac{B-b'}{\sqrt{V_{\rho'}(B)}}\right)A_\varphi\right)\right\}\right)^2}{4\left(1-\cos\varphi\right)} \\
&= \lim_{\varphi\to 0}\frac{\left(\mathrm{Re}\left\{\mathrm{Tr}\left(\rho^{\frac{1}{2}}\left(\frac{A-a}{\sqrt{V_\rho(A)}} \pm \mathrm{i}\frac{B-b}{\sqrt{V_{\rho}(B)}}\right)A_\varphi\right)\right\}\right)^2}{4\left(1-\cos\varphi\right)}.
\end{aligned}
$$

Denote $D = \frac{A-a}{\sqrt{V_\rho(A)}} \pm \mathrm{i}\frac{B-b}{\sqrt{V_\rho(B)}}$, then

$$
\frac{\left(\mathrm{Re}\left\{\mathrm{Tr}\left(\rho^{\frac{1}{2}}\left(\frac{A-a}{\sqrt{V_\rho(A)}} \pm \mathrm{i}\frac{B-b}{\sqrt{V_{\rho}(B)}}\right)A_\varphi\right)\right\}\right)^2}{4\left(1-\cos\varphi\right)} = \frac{\left(\mathrm{Re}\left\{\mathrm{Tr}\left(\rho^{\frac{1}{2}}DA_\varphi\right)\right\}\right)^2}{4\left(1-\cos\varphi\right)}.
$$

Note that

$$
\mathrm{Tr}\left(\rho^{\frac{1}{2}}DA_\varphi\right) = \mathrm{Tr}\left(\rho^{\frac{1}{2}}D(\cos\varphi\cdot\rho^{\frac{1}{2}} + e^{\mathrm{i}\lambda}\sin\varphi\cdot\sigma)\right) = e^{\mathrm{i}\lambda}\sin\varphi\mathrm{Tr}\left(\rho^{\frac{1}{2}}D\sigma\right).
$$

We can choose appropriate λ so that the term $\mathrm{Tr}\left(\rho^{\frac{1}{2}}DA_\varphi\right)$ is real. Therefore, $\left(\mathrm{Re}\left\{\mathrm{Tr}\left(\rho^{\frac{1}{2}}DA_\varphi\right)\right\}\right)^2 = \sin^2\varphi\left|\mathrm{Tr}\left(\rho^{\frac{1}{2}}D\sigma\right)\right|^2$, which yields

$$
\frac{\left(\mathrm{Re}\left\{\mathrm{Tr}\left(\rho^{\frac{1}{2}}DA_\varphi\right)\right\}\right)^2}{4\left(1-\cos\varphi\right)} = \frac{\sin^2\varphi\cdot\left|\mathrm{Tr}\left(\rho^{\frac{1}{2}}D\sigma\right)\right|^2}{4(1-\cos\varphi)},
$$

and so

$$
\begin{aligned}
\lim_{\varphi\to 0}\frac{\delta^2}{8r} &= \frac{\left|\mathrm{Tr}\left(\rho^{\frac{1}{2}}D\sigma\right)\right|^2}{2} = \frac{\left|\mathrm{Tr}\left(\rho^{\frac{1}{2}}\left(\frac{A-a}{\sqrt{V_\rho(A)}} \pm \mathrm{i}\frac{B-b}{\sqrt{V_\rho(B)}}\right)\sigma\right)\right|^2}{2} \\
&= \frac{\left|\mathrm{Tr}\left(\rho^{\frac{1}{2}}\left(\frac{A}{\sqrt{V_\rho(A)}} \pm \mathrm{i}\frac{B}{\sqrt{V_\rho(B)}}\right)\sigma\right)\right|^2}{2}.
\end{aligned}
$$

The last equality holds because of $\mathrm{Tr}\left(\rho^{\frac{1}{2}}\sigma\right)=0$. Hence, the inequality (16) becomes

$$\sqrt{V_\rho(A)V_\rho(B)} \geq \sqrt{V_\rho(A)V_\rho(B)} \cdot \frac{\left|\mathrm{Tr}\left(\rho^{\frac{1}{2}}\left(\frac{A}{\sqrt{V_\rho(A)}} \pm \mathrm{i}\frac{B}{\sqrt{V_\rho(B)}}\right)\sigma\right)\right|^2}{2} \pm \frac{\mathrm{i}}{2}\mathrm{Tr}(\rho[A,B]),$$

which is equivalent to the conclusion. □

Remark 2 If we take $\rho = |\psi\rangle\langle\psi|, \sigma = |\psi^\perp\rangle\langle\psi|$, where $\langle\psi|\psi^\perp\rangle = 0$, then $\mathrm{Tr}\left(\rho^{\frac{1}{2}}\sigma\right) = 0$. It is easy to compute $V_\rho(A) = \Delta A^2$, $V_\rho(B) = \Delta B^2$, and $\mathrm{Tr}\left(\rho^{\frac{1}{2}}\left(\frac{A}{\sqrt{V_\rho(A)}} \pm \mathrm{i}\frac{B}{\sqrt{V_\rho(B)}}\right)\sigma\right) = \left\langle\psi\right|\left(\frac{A}{\Delta A} \pm \mathrm{i}\frac{B}{\Delta B}\right)\left|\psi^\perp\right\rangle$, which recover to the inequality (5).

Remark 3 Let

$$\rho = \begin{pmatrix} \frac{1}{4} & 0 \\ 0 & \frac{3}{4} \end{pmatrix}, A = \begin{pmatrix} 0 & \mathrm{i} \\ -\mathrm{i} & 0 \end{pmatrix}, B = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \sigma = \begin{pmatrix} \frac{\sqrt{3}}{2} & 0 \\ 0 & -\frac{1}{2} \end{pmatrix}.$$

Then, $\mathrm{Tr}\left(\rho^{\frac{1}{2}}\sigma\right) = 0$ and $\|\sigma\|_2 = 1$. It is easy to compute that $\sqrt{V_\rho(A)V_\rho(B)} = 1$, and the right of (12)

$$\pm\frac{\mathrm{i}}{2}\mathrm{Tr}(\rho[A,B]) \Bigg/ \left[1 - \frac{1}{2}\left|\mathrm{Tr}\left(\rho^{\frac{1}{2}}\left(\frac{A}{\sqrt{V_\rho(A)}} \pm \mathrm{i}\frac{B}{\sqrt{V_\rho(B)}}\right)\sigma\right)\right|^2\right] = \pm\frac{16}{29}.$$

Thus, the inequality (12) holds.

In the following, we will give the other improved Schrödinger uncertainty relation, by choosing an arbitrary phase factor $e^{\mathrm{i}\tau}$ in place of the imaginary constant i in (13).

Theorem 5 *Let* $\rho \in D(H)$, $A, B \in S(H)$. *If* $\sigma \in L^2(H)$ *satisfies* $\mathrm{Tr}\left(\rho^{\frac{1}{2}}\sigma\right) = 0$ *and* $\|\sigma\|_2 = 1$, *then*

$$V_\rho(A)V_\rho(B) \geq \left|\mathrm{Cov}_\rho(A,B)\right|^2 \Bigg/ \left[1 - \frac{1}{2}\left|\mathrm{Tr}\left(\rho^{\frac{1}{2}}\left(\frac{A}{\sqrt{V_\rho(A)}} + e^{\mathrm{i}\alpha}\frac{B}{\sqrt{V_\rho(B)}}\right)\sigma\right)\right|^2\right]^2, \tag{17}$$

where α *satisfies* $e^{\mathrm{i}\alpha}\mathrm{Cov}_\rho(A,B) = |\mathrm{Cov}_\rho(A,B)|$.

Proof Let Eq. (11) be valid. Start from the following inequality

$$\left\|c_A(A-a)\rho^{\frac{1}{2}} - c_B e^{\mathrm{i}\tau}(B-b')A_\varphi + c\left(\rho^{\frac{1}{2}} - A_\varphi\right)\right\|_2^2 \geq 0, \tag{18}$$

where $b' = \mathrm{Tr}(\rho' B)$, $a = \mathrm{Tr}(\rho A)$ and c_A, c_B, c real constants. Calculating the square modulus, we find

$$c_A^2 V_\rho(A) + c_B^2 V_{\rho'}(B) \geq -c^2 r + c c_A c_B \delta + c_A c_B k, \tag{19}$$

with $V_\rho(A)$ and $V_{\rho'}(B)$ the variance of A and B on ρ and ρ', respectively, and where

$$
\begin{aligned}
r &= 2\,(1-\cos\varphi)\,,\\
k &= 2\mathrm{Re}\left\{e^{\mathrm{i}\tau}\mathrm{Tr}\left(\rho^{\frac{1}{2}}(A-a)(B-b')A_\varphi\right)\right\},\\
\delta &= 2\mathrm{Re}\left\{\mathrm{Tr}\left(\rho^{\frac{1}{2}}\left(\frac{A-a}{c_B}+e^{\mathrm{i}\tau}\frac{B-b'}{c_A}\right)A_\varphi\right)\right\}.
\end{aligned}
$$

Now choose the value of c that maximizes the right-hand side of (19), namely $c = \frac{c_A c_B \delta}{2r}$, then (19) becomes

$$c_A^2 V_\rho(A) + c_B^2 V_{\rho'}(B) \geq \frac{(c_A c_B \delta)^2}{4r} + c_A c_B k. \tag{20}$$

Put $c_A = \sqrt{V_{\rho'}(B)}$, $c_B = \sqrt{V_\rho(A)}$, then

$$\sqrt{V_\rho(A)V_{\rho'}(B)} \geq \frac{\sqrt{V_\rho(A)V_{\rho'}(B)}\,\delta^2}{8r} + \frac{k}{2}. \tag{21}$$

Denote $b = \mathrm{Tr}(\rho B)$, and by the use of Lemma 2, we get

$$
\begin{aligned}
\lim_{\varphi\to 0}\frac{k}{2} &= \lim_{\varphi\to 0}\mathrm{Re}\left\{e^{\mathrm{i}\tau}\mathrm{Tr}\left(\rho^{\frac{1}{2}}(A-a)(B-b')A_\varphi\right)\right\}\\
&= \mathrm{Re}\left\{e^{\mathrm{i}\tau}\mathrm{Cov}_\rho(A,B)\right\},
\end{aligned}
$$

and $\lim_{\varphi\to 0} V_{\rho'}(B) = V_\rho(B)$. In a similar way to the proof of Theorem 4, it is easy to obtain that

$$\lim_{\varphi\to 0}\frac{\delta^2}{8r} = \frac{\left|\mathrm{Tr}\left(\rho^{\frac{1}{2}}\left(\frac{A}{\sqrt{V_\rho(A)}}+e^{\mathrm{i}\tau}\frac{B}{\sqrt{V_\rho(B)}}\right)\sigma\right)\right|^2}{2}.$$

We can choose appropriate $\tau = \alpha$, so that $e^{\mathrm{i}\tau}\mathrm{Cov}_\rho(A,B)$ is real and can be written as $|\mathrm{Cov}_\rho(A,B)|$. Thus, (21) can be rewritten as

$$\sqrt{V_\rho(A)V_\rho(B)} \geq \frac{|\mathrm{Cov}_\rho(A,B)|}{1-\frac{1}{2}\left|\mathrm{Tr}\left(\rho^{\frac{1}{2}}\left(\frac{A}{\sqrt{V_\rho(A)}}+e^{\mathrm{i}\alpha}\frac{B}{\sqrt{V_\rho(B)}}\right)\sigma\right)\right|^2}.$$

This completes the proof. □

Note that

$$\mathrm{Cov}_\rho(A,B) = \mathrm{Tr}(\rho AB) - \mathrm{Tr}(\rho A)\mathrm{Tr}(\rho B)$$

$$= \frac{1}{2}\mathrm{Tr}(\rho[A, B]) + \frac{1}{2}\mathrm{Tr}(\rho\{A, B\}) - \mathrm{Tr}(\rho A)\mathrm{Tr}(\rho B).$$

Therefore, the inequality (17) can be represented as

$$\sqrt{V_\rho(A)V_\rho(B)} \geq \frac{\left|\frac{1}{2}\mathrm{Tr}(\rho[A, B]) + \frac{1}{2}\mathrm{Tr}(\rho\{A, B\}) - \mathrm{Tr}(\rho A)\mathrm{Tr}(\rho B)\right|}{1 - \frac{1}{2}\left|\mathrm{Tr}\left(\rho^{\frac{1}{2}}\left(\frac{A}{\sqrt{V_\rho(A)}} + e^{\mathrm{i}\alpha}\frac{B}{\sqrt{V_\rho(B)}}\right)\sigma\right)\right|^2}.$$

Take $c_A = c_B = 1$ in (20); using the same procedure described above, we can obtain

$$\begin{aligned} V_\rho(A) + V_\rho(B) &\geq 2\left|\mathrm{Cov}_\rho(A, B)\right| + \left|\mathrm{Tr}\left(\rho^{\frac{1}{2}}\left(A + e^{\mathrm{i}\alpha}B\right)\sigma\right)\right|^2 \\ &= \left|\mathrm{Tr}\left(\rho[A, B]\right) + \mathrm{Tr}\left(\rho\{A, B\}\right) - 2\mathrm{Tr}(\rho A)\mathrm{Tr}(\rho B)\right| \\ &\quad + \left|\mathrm{Tr}\left(\rho^{\frac{1}{2}}(A + e^{\mathrm{i}\alpha}B)\sigma\right)\right|^2. \end{aligned}$$

Removing the last term in the above inequality, we find the inequality

$$V_\rho(A) + V_\rho(B) \geq \left|\mathrm{Tr}\left(\rho[A, B]\right) + \mathrm{Tr}\left(\rho\{A, B\}\right) - 2\mathrm{Tr}(\rho A)\mathrm{Tr}(\rho B)\right|. \quad (22)$$

3 Uncertainty relations for three observables

One may generalize the Schrödinger uncertainty relation to three observables trivially, since

$$\begin{aligned} V_\rho(A) + V_\rho(B) &\geq 2\left|\mathrm{Cov_{æ}(A, B)}\right|, \\ V_\rho(B) + V_\rho(C) &\geq 2\left|\mathrm{Cov_{æ}(B, C)}\right|, \\ V_\rho(A) + V_\rho(C) &\geq 2\left|\mathrm{Cov_{æ}(A, C)}\right|, \end{aligned}$$

we have

$$V_\rho(A) + V_\rho(B) + V_\rho(C) \geq \left|\mathrm{Cov_{æ}(A, B)}\right| + \left|\mathrm{Cov_{æ}(B, C)}\right| + \left|\mathrm{Cov_{æ}(A, C)}\right|. \quad (23)$$

The lower bound of (23) can be null. So we will prove the following more stringent inequality exists.

Theorem 6 *Let $\rho \in D(H)$, $A, B \in S(H)$ and α, β be any real numbers. If $\sigma \in L^2(H)$ satisfies* $\mathrm{Tr}\left(\rho^{\frac{1}{2}}\sigma\right) = 0$ *and* $\|\sigma\|_2 = 1$, *then*

$$\begin{aligned} V_\rho(A) + V_\rho(B) + V_\rho(C) &\geq \left|\mathrm{Tr}\left(\rho^{\frac{1}{2}}\left(A + e^{-\mathrm{i}\alpha}B + e^{-\mathrm{i}\beta}C\right)\sigma\right)\right|^2 \\ &-2\mathrm{Re}\left\{e^{\mathrm{i}\alpha}\mathrm{Cov}_\rho(A, B) + e^{\mathrm{i}\beta}\mathrm{Cov}_\rho(A, C) + e^{\mathrm{i}(\beta-\alpha)}\mathrm{Cov}_\rho(B, C)\right\}. \end{aligned} \quad (24)$$

Proof Let Eq. (11) be valid. To the inequality (24), we introduce a general inequality

$$\left\| c_1(A-a)\rho^{\frac{1}{2}} + c_2 e^{\mathrm{i}\alpha}(B-b)\rho^{\frac{1}{2}} + c_3 e^{\mathrm{i}\beta}(C-c)\rho^{\frac{1}{2}} + t\left(\rho^{\frac{1}{2}} - A_\varphi\right)\right\|_2^2 \geq 0,$$

where $a = \mathrm{Tr}(\rho A)$, $b = \mathrm{Tr}(\rho B)$, $c = \mathrm{Tr}(\rho C)$ and c_1, c_2, c_3, t real constants. Calculating the square modulus, we find

$$c_1^2 V_\rho(A) + c_2^2 V_\rho(B) + c_3^2 V_\rho(C) \geq -t^2 r + c_1 c_2 c_3 \delta t + c_1 c_2 c_3 k, \tag{25}$$

where

$$\begin{aligned}
r &= 2\left(1-\cos\varphi\right), \\
k &= -2\mathrm{Re}\left\{\frac{1}{c_3} e^{\mathrm{i}\alpha}\mathrm{Cov}_\rho(A,B) + \frac{1}{c_2} e^{\mathrm{i}\beta}\mathrm{Cov}_\rho(A,C) + \frac{1}{c_1} e^{\mathrm{i}(\beta-\alpha)}\mathrm{Cov}_\rho(B,C)\right\}, \\
\delta &= 2\mathrm{Re}\left\{\mathrm{Tr}\left(\rho^{\frac{1}{2}}\left(\frac{A-a}{c_2c_3} + e^{-\mathrm{i}\alpha}\frac{B-b}{c_1c_3} + e^{-\mathrm{i}\beta}\frac{C-c}{c_1c_2}\right)A_\varphi\right)\right\}.
\end{aligned}$$

Now choose the value of t that maximizes the right-hand side of (25), namely $t = \frac{c_1c_2c_3\delta}{2r}$, then (25) becomes

$$c_1^2 V_\rho(A) + c_2^2 V_\rho(B) + c_3^2 V_\rho(C) \geq \frac{(c_1c_2c_3\delta)^2}{4r} + c_1c_2c_3k. \tag{26}$$

Put $c_1 = c_2 = c_3 = 1$, then (26) can be represented as

$$V_\rho(A) + V_\rho(B) + V_\rho(C) \geq \frac{\delta^2}{4r} + k,$$

with

$$\begin{aligned}
r &= 2\left(1-\cos\varphi\right), \\
k &= -2\mathrm{Re}\left\{e^{\mathrm{i}\alpha}\mathrm{Cov}_\rho(A,B) + e^{\mathrm{i}\beta}\mathrm{Cov}_\rho(A,C) + e^{\mathrm{i}(\beta-\alpha)}\mathrm{Cov}_\rho(B,C)\right\}, \\
\delta &= 2\mathrm{Re}\left\{\mathrm{Tr}\left(\rho^{\frac{1}{2}}\left(A-a+e^{-\mathrm{i}\alpha}(B-b)+e^{-\mathrm{i}\beta}(C-c)\right)A_\varphi\right)\right\}.
\end{aligned}$$

Similar to the proof of Theorem 4, if we take $D_0 = A-a+e^{-\mathrm{i}\alpha}(B-b)+e^{-\mathrm{i}\beta}(C-c)$, and then

$$\begin{aligned}
\lim_{\varphi\to 0}\frac{\delta^2}{4r} &= \lim_{\varphi\to 0}\frac{\left(\mathrm{Re}\left\{\mathrm{Tr}\left(\rho^{\frac{1}{2}}D_0A_\varphi\right)\right\}\right)^2}{2\left(1-\cos\varphi\right)} \\
&= \left|\mathrm{Tr}\left(\rho^{\frac{1}{2}}D_0\sigma\right)\right|^2 \\
&= \left|\mathrm{Tr}\left(\rho^{\frac{1}{2}}\left(A-a+e^{-\mathrm{i}\alpha}(B-b)+e^{-\mathrm{i}\beta}(C-c)\right)\sigma\right)\right|^2
\end{aligned}$$

$$= \left|\mathrm{Tr}\left(\rho^{\frac{1}{2}}\left(A + e^{-\mathrm{i}\alpha}B + e^{-\mathrm{i}\beta}C\right)\sigma\right)\right|^2.$$

Therefore, we can obtain the inequality (24). □

Remark 4 The inequality (24) can reduce to the inequality (6) when we choose an appropriate state ρ, a bound operator σ, and real constants α, β.

Indeed, if we put $\rho = |\psi\rangle\langle\psi|, \sigma = |\psi^{\perp}\rangle\langle\psi|, \alpha = \pm\frac{2\pi}{3}, \beta = \pm\frac{4\pi}{3}$, then $\mathrm{Tr}\left(\rho^{\frac{1}{2}}\sigma\right) = 0$ and $\|\sigma\|_2 = 1$, so the inequality (24) holds. Clearly,

$$V_\rho(A) = \Delta A^2,\ V_\rho(B) = \Delta B^2,\ V_\rho(C) = \Delta C^2,\ \mathrm{Cov}_\rho(A, B) = \langle AB\rangle - \langle A\rangle\langle B\rangle,$$
$$\mathrm{Cov}_\rho(A, C) = \langle AC\rangle - \langle A\rangle\langle C\rangle,\ \mathrm{Cov}_\rho(B, C) = \langle BC\rangle - \langle B\rangle\langle C\rangle.$$

It is easy to see that

$$\left|\mathrm{Tr}\left(\rho^{\frac{1}{2}}\left(A + e^{-\mathrm{i}\alpha}B + e^{-\mathrm{i}\beta}C\right)\sigma\right)\right|^2 = \left|\left\langle\psi\left|\left(A + e^{\mp\mathrm{i}\frac{2\pi}{3}}B + e^{\mp\mathrm{i}\frac{4\pi}{3}}C\right)\right|\psi^{\perp}\right\rangle\right|^2.$$

Moreover,

$$\begin{aligned} k &= -2\mathrm{Re}\left\{e^{\mathrm{i}\alpha}\mathrm{Cov}_\rho(A, B) + e^{\mathrm{i}\beta}\mathrm{Cov}_\rho(A, C) + e^{\mathrm{i}(\beta-\alpha)}\mathrm{Cov}_\rho(B, C)\right\} \\ &= -2\mathrm{Re}\left\{e^{\pm\mathrm{i}\frac{2\pi}{3}}(\langle AB\rangle - \langle A\rangle\langle B\rangle) + e^{\pm\mathrm{i}\frac{4\pi}{3}}(\langle AC\rangle - \langle A\rangle\langle C\rangle) + e^{\pm\mathrm{i}\frac{2\pi}{3}}(\langle BC\rangle - \langle B\rangle\langle C\rangle)\right\} \\ &= \frac{1}{2}\langle\{A, B, C\}\rangle \mp \mathrm{i}\frac{\sqrt{3}}{2}\langle[A, B, C]\rangle - \langle A\rangle\langle B\rangle - \langle A\rangle\langle C\rangle - \langle B\rangle\langle C\rangle, \end{aligned}$$

where $\langle\{A, B, C\}\rangle = \langle\{A, B\}\rangle + \langle\{A, C\}\rangle + \langle\{B, C\}\rangle$. Note that

$$\Delta(A + B + C)^2 = \Delta A^2 + \Delta B^2 + \Delta C^2 + \langle\{A, B, C\}\rangle - 2(\langle A\rangle\langle B\rangle + \langle A\rangle\langle C\rangle + \langle B\rangle\langle C\rangle).$$

Thus, $k = \frac{1}{2}\left(\Delta(A + B + C)^2 - \left(\Delta A^2 + \Delta B^2 + \Delta C^2\right)\right) \mp \mathrm{i}\frac{\sqrt{3}}{2}\langle[A, B, C]\rangle$. Therefore, we can obtain the following inequality:

$$\begin{aligned} &\Delta A^2 + \Delta B^2 + \Delta C^2 \geq \frac{1}{3}\Delta(A + B + C)^2 \mp \mathrm{i}\frac{\sqrt{3}}{3}\langle[A, B, C]\rangle + \frac{2}{3} \\ &\left|\left\langle\psi\left|\left(A + e^{\mp\mathrm{i}\frac{2\pi}{3}}B + e^{\mp\mathrm{i}\frac{4\pi}{3}}C\right)\right|\psi^{\perp}\right\rangle\right|^2, \end{aligned}$$

which is accordant to (6).

Remark 5 If we put $c_1 = \sqrt{V_\rho(B)V_\rho(C)}, c_2 = \sqrt{V_\rho(A)V_\rho(C)}, c_3 = \sqrt{V_\rho(A)V_\rho(B)}$ in the inequality (25), then

$$V_\rho(A)V_\rho(B)V_\rho(C) \leq \frac{4r(3 - k)}{\delta^2};$$

using the same procedure described above, we can obtain the following uncertainty relation:

$$\begin{aligned}&V_\rho(A)V_\rho(B)V_\rho(C)\\ \leq\ &\frac{3\sqrt{V_\rho(A)V_\rho(B)V_\rho(C)}+2\mathrm{Re}\{\sqrt{V_\rho(C)}e^{\mathrm{i}\alpha}\mathrm{Cov}_\rho(A,B)+\sqrt{V_\rho(B)}e^{\mathrm{i}\beta}\mathrm{Cov}_\rho(A,C)+\sqrt{V_\rho(A)}e^{\mathrm{i}(\beta-\alpha)}\mathrm{Cov}_\rho(B,C)\}}{\left|\mathrm{Tr}\left(\rho^{\frac{1}{2}}\left(\frac{A}{\sqrt{V_\rho(A)}}+e^{-\mathrm{i}\alpha}\frac{B}{\sqrt{V_\rho(B)}}+e^{-\mathrm{i}\beta}\frac{C}{\sqrt{V_\rho(C)}}\right)\sigma\right)\right|^2}.\end{aligned}$$

Remark 6 If we put $c_1=\sqrt{V_\rho(A)}$, $c_2=\sqrt{V_\rho(B)}$, $c_3=\sqrt{V_\rho(C)}$ in the inequality (25), then using the same procedure described above, the following uncertainty relation

$$\begin{aligned}&V_\rho^2(A)+V_\rho^2(B)+V_\rho^2(C)\\ \geq\ &\left|\mathrm{Tr}\left(\rho^{\frac{1}{2}}\left(\sqrt{V_\rho(A)}A+e^{-\mathrm{i}\alpha}\sqrt{V_\rho(B)}B+e^{-\mathrm{i}\beta}\sqrt{V_\rho(C)}C\right)\sigma\right)\right|^2\\ &-2\mathrm{Re}\left\{\sqrt{V_\rho(A)V_\rho(B)}e^{\mathrm{i}\alpha}\mathrm{Cov}_\rho(A,B)+\sqrt{V_\rho(A)V_\rho(C)}e^{\mathrm{i}\beta}\mathrm{Cov}_\rho(A,C)+\sqrt{V_\rho(B)V_\rho(C)}e^{\mathrm{i}(\beta-\alpha)}\mathrm{Cov}_\rho(B,C)\right\}\end{aligned}$$

is valid.

4 Conclusions

In this paper, we gave two stronger uncertainty relations, relating to the sum of variances with respect to density matrix:

$$V_\rho(A)+V_\rho(B)\geq\pm\mathrm{i}\mathrm{Tr}(\rho[A,B])+\left\|\rho^{\frac{1}{2}}(A\pm\mathrm{i}B)\rho_\perp^{\frac{1}{2}}\right\|_2^2,$$

and

$$V_\rho(A)+V_\rho(B)\geq\frac{1}{2}\left\|\rho^{\frac{1}{2}}(A+B)\rho_\perp^{\frac{1}{2}}\right\|_2^2,$$

where $\rho_\perp\in D(H)$, s.t. $\rho\rho_\perp=0$. These lower bounds are guaranteed to be nontrivial whenever the two observables are incompatible on the state of the system. If we choose appropriate ρ and $\rho_\perp$, the above results can reduce to (3) and (4), respectively, which were given by Maccone and Pati in [22]. In addition, a new bound for the sum of two variances of pure states is derived by theorem 3, that is corollary 3 including Maccone–Pati's inequality (4) as a special case if we choose $|\psi^\perp\rangle=\left|\psi_{A+B}^\perp\right\rangle$.

Moreover, two stronger uncertainty relations in terms of the product of the variances of two observables were established:

$$\sqrt{V_\rho(A)V_\rho(B)}\geq\frac{\pm\frac{\mathrm{i}}{2}\mathrm{Tr}(\rho[A,B])}{1-\frac{1}{2}\left|\mathrm{Tr}\left(\rho^{\frac{1}{2}}\left(\frac{A}{\sqrt{V_\rho(A)}}\pm\mathrm{i}\frac{B}{\sqrt{V_\rho(B)}}\right)\sigma\right)\right|^2},$$

which can recover to the inequality (5) when we choose appropriate ρ and σ, and

$$V_\rho(A)V_\rho(B) \geq \frac{\left|\mathrm{Cov}_\rho(A,B)\right|^2}{\left[1-\frac{1}{2}\left|\mathrm{Tr}\left(\rho^{\frac{1}{2}}\left(\frac{A}{\sqrt{V_\rho(A)}}+e^{\mathrm{i}\alpha}\frac{B}{\sqrt{V_\rho(B)}}\right)\sigma\right)\right|^2\right]^2},$$

where σ satisfies $\mathrm{Tr}\left(\rho^{\frac{1}{2}}\sigma\right) = 0$ and $\|\sigma\|_2 = 1$, α satisfies $e^{\mathrm{i}\alpha}\mathrm{Cov}_\rho(A,B) = |\mathrm{Cov}_\rho(A,B)|$. We use the square-modulus inequality and follow a procedure analogous to the one employed by [22]. However, the proof of theorem 4 is more complex, and lemma 2 is the key point to the proof. The limit given here is taken in the sense of $\|\cdot\|_2$.

Several stronger uncertainty relations for three observables were established, relating to the sum and product of variances with respect to density matrix, respectively. If we choose an appropriate state ρ, a bound operator σ, and real constants α, β, the inequality (24) can reduce to the inequality (6) which was given by Song and Qiao in [25].

Acknowledgements This subject was supported by the NNSF of China (Nos. 11701011, 11761001, 11761003), the NSF of Ningxia (No. 2020AAC03229), and the First-Class Disciplines Foundation of Ningxia (No.NXYLXK2017B09).

References

1. Heisenberg, W.: Über den anschaulichen Inhalt der quantentheoretischen Kinematik und Mechanik. Z. Phys. **43**, 172–198 (1927)
2. Pati, A.K., Sahu, P.K.: Sum uncertainty in quantum theorem. Phys. lett. A **367**, 177–181 (2007)
3. Huang, Y.: Variance-based uncertainty relations. Phys. Rev. A **86**, 024101 (2012)
4. Rivas, A., Luis, A.: Characterization of quantum angular-momentum fluctuations via principal components. Phys. Rev. A **77**, 022105 (2008)
5. Chen, B., Fei, S.M.: Sum uncertainty relations for arbitrary N incompatible observables. Sci. Rep. **5**, 14238 (2015)
6. Yao, Y., Xiao, X., Wang, X., Sun, C.P.: Implications and applications of the variance-based uncertainty equalities. Phys. Rev. A **91**, 062113 (2015)
7. Zheng, X., Ma, S.Q., Zhang, G.F., Fan, H., Liu, W.M.: Unified and exact framework for variance-based uncertainty relations. Sci. Rep. **10**, 150 (2020)
8. Furuichi, S., Yanagi, K.: Schrödinger uncertainty relation, Wigner-Yanase-Dyson skew information and metric adjusted correlation measure. J. Math. Anal. Appl. **388**, 1147–1156 (2012)
9. Furuichi, S.: Schrödinger uncertainty relation with Wigner-Yanase skew information. Phys. Rev. A **82**, 034101 (2010)
10. Luo, S., Zhang, Q.: On skew information. IEEE Trans. Inf. Theory **50**, 1778–1782 (2004)
11. Yanagi, K.: Uncertainty relation on Wigner-Yanase-Dyson skew information. J. Math. Anal. Appl. **365**, 12–18 (2010)
12. Yanagi, K.: Wigner-Yanase-Dyson skew information and uncertainty relation. J. Phys. Conf. Ser. **201**, 012015 (2010)
13. Horodecki, R., Horodecki, P., Horodecki, M., Horodecki, K.: Quantum entanglement. Rev. Mod. Phys. **81**, 865–942 (2009)
14. Gühne, O., Tóth, G.: Entanglement detection. Phys. Rep. **474**(1–6), 1–75 (2009)
15. Wehner, S., Winter, A.: Entropic uncertainty relations-a survey. New J. Phys. **12**, 025009 (2010)

16. Candès, E., Romberg, J., Tao, T.: Robust uncertainty principles: exact signal reconstruction from highly incomplete frequency information. IEEE Trans. Inf. Theory **52**(2), 489–509 (2006)
17. Zidan, M., Abdel-Aty, A.-H., Younes, A., Zanaty, E.A., El-khayat, I., Abdel-Aty, M.: A novel algorithm based on entanglement measurement for improving speed of quantum algorithms. Appl. Math. Inf. Sci. **12**(1), 265–269 (2018)
18. Zidan, M., Abdel-Aty, A.-H., Nguyene, D.M., Mohamed, A.S.A., Al-Sbou, Y., Eleuch, H., Abdel-Aty, M.: A quantum algorithm based on entanglement measure for classifying multivariate function into novel hidden classes. Results Phys. **15**, 102549 (2019)
19. Zidan, M., Abdel-Aty, A.-H., El-shafei, M., Feraig, M., Al-Sbou, Y., Eleuch, H., Abdel-Aty, M.: Quantum classification algorithm based on competitive learning neural network and entanglement measure. Appl. Sci. **9**(7), 1277 (2019)
20. Abdel-Aty, A.-H., Kadry, H., Zidan, M., Al-Sbou, Y., Zanaty, E.A., Abdel-Aty, M.: A quantum classification algorithm for classification incomplete patterns based on entanglement measure. J. Intell. Fuzzy Syst. **38**(3), 2809–2816 (2020)
21. Schrödinger, E.: About Heisenberg uncertainty relation. In: Proceedings of The Prussian Academy of Sciences, Physics-Mathematical Section XIX, vol. 293 (1930)
22. Maccone, L., Pati, A.K.: Strong uncertaionty relations for all incompatible observables. Phys. Rev. Lett. **113**, 260401 (2014)
23. Hofmann, H.F., Takeuchi, S.: Violation of local uncertainty relation as a signature of entanglement. Phys. Rev. A **68**, 032103 (2003)
24. Holevo, A.S.: A generalization of the Rao-Cramér inequality. Probab. Appl. **18**, 359–362 (1974)
25. Song, Q.C., Qiao, C.F.: Stronger Schrödinger-like uncertainty relations. Phys. Lett. A **380**(37), 2925–2930 (2016)