




Numerical detection of Gaussian entanglement and its application to the identification of bound entangled Gaussian states

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Abstract

We present a numerical method for solving the separability problem of Gaussian quantum states in continuous-variable quantum systems. We show that the separability problem can be cast as an equivalent problem of determining the feasibility of a set of linear matrix inequalities. Thus, it can be efficiently solved using existent numerical solvers. We apply this method to the identification of bound entangled Gaussian states. We show that the proposed method can be used to identify bound entangled Gaussian states that could be simple enough to be producible in quantum optics.

Keywords Entanglement · Separability · Gaussian states · Bound entanglement · Continuous variable

1 Introduction

Quantum entanglement plays a central role in quantum information technologies, e.g., in quantum computation, quantum communication, and quantum metrology [1–7]. In

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recent years, a great deal of research effort has been put into the analysis of the entanglement properties of multiparticle systems [8–17]. While most of the effort has been devoted to systems with finite-dimensional Hilbert spaces, in particular discrete-variable qubit states, recently there has been considerable interest in the continuous-variable (CV) case [18–26]. Gaussian states, as a particularly useful class of CV states, are commonly produced in quantum optics laboratories. Given a Gaussian state of a bipartite CV system, the most fundamental problem in CV quantum information theory is to determine whether the state is entangled or not with respect to the splitting. Consider two CV quantum systems A with m modes and B with n modes having infinite-dimensional Hilbert spaces \mathcal{H}_A and \mathcal{H}_B , respectively. The global bipartite system $A + B$ with $m + n$ modes has a Hilbert space $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$. By definition, a quantum state $\hat{\rho}$ of the global bipartite system $A + B$ is said to be separable if it can be written as a convex sum of pure product states, namely,

$$\hat{\rho} = \sum_j p_j \hat{\rho}_j^A \otimes \hat{\rho}_j^B, \quad (1)$$

where $p_j \geq 0$ and $\sum_j p_j = 1$ [4,27]. Note that in Eq. (1), the sum can also be an integral and the probabilities are then replaced by a continuous probability density function. Physically, Eq. (1) means that separable states can be produced from product states by means of local operations and classical communications (LOCCs). By definition, entangled states are states which are not separable. Then, the so-called separability problem is to determine whether a given quantum state is separable or not. Despite considerable progress made in recent years, the separability problem is still far from being completely solved [28–34].

Perhaps, the most commonly used tool in quantum information theory for checking if a given state is separable or not is based on the partial transpose [35,36]. For a separable quantum state $\hat{\rho}$ as in Eq. (1), the partial transpose with respect to one of the two subsystems yields again a legitimate density operator and, in particular, positive, i.e., $\hat{\rho}^{T_A} = \sum_j p_j \left(\hat{\rho}_j^A\right)^T \otimes \hat{\rho}_j^B \geq 0$. Hence, the positivity of the partial transpose (PPT) provides us a necessary condition for separability. However, it should be noted that the PPT criterion is, in general, not a sufficient condition for separability. In fact, a 2×2 -mode Gaussian state, which has positive partial transpose but nevertheless is entangled, has been constructed in Ref. [30]. This type of Gaussian state is known as a bound entangled Gaussian state. Bound entangled Gaussian states are entangled states but their entanglement cannot be distilled into maximally entangled pure states with LOCCs [37–42].

For Gaussian quantum states, all the entanglement information is contained in the covariance matrix of position and momentum observables [4,24]. Thus, the separability problem can be investigated at the level of covariance matrices. In fact, both the separability problem and the PPT criterion have been successfully reformulated in terms of the covariance matrix language in Ref. [30]. Built upon this work, Ref. [31] proposes a nonlinear iterative procedure to check the separability of a Gaussian state. Reference [33] developed a numerical method for finding an optimal entanglement witness that robustly detects the entangled state. Both methods are nice and effective for Gaussian states.

In this paper, we show that the separability problem can be cast as an equivalent problem of solving a set of linear matrix inequalities. If there exist solutions to the set of linear matrix inequalities, then the Gaussian state is separable; otherwise, it is entangled. Thus, the feasibility of the linear matrix inequalities serves as a necessary and sufficient condition for the separability of the corresponding Gaussian state. On the other hand, solving linear matrix inequalities is a mature technology [43]. There are many efficient numerical methods that can be used to solve linear matrix inequalities. For example, the linear matrix inequalities solver SeDuMi [44] or SDTP3 [45] with YALMIP interface [46] can perform very well. Thus, Gaussian entanglement can, in principle, be detected by using this method. Strictly speaking, the numerical method proposed in this paper may be considered as a special case of the numerical method proposed in Ref. [33]. It receives particular attention in this paper since the method of solving linear matrix inequalities is quite familiar to researchers from engineering [43]. By using the above LMI solvers, Gaussian entanglement can be detected very efficiently. As an application, we use this numerical method to identify bound entangled Gaussian states. It is well known that bound entanglement is a rare phenomenon [1] and the detection of bound entanglement is a challenging problem [47]. In this paper, we use the proposed numerical method to identify a 2×2 bound entangled Gaussian state that is easily prepared in quantum optics. We first parametrize the covariance matrices of 2×2 Gaussian states using some free parameters. Then, we choose the values of these parameters and see if the resulting covariance matrix passes the PPT and entanglement tests. If so, then we know the corresponding Gaussian state is a bound entangled Gaussian state. Based on the decomposition result, we can further provide a theoretical optical implementation for the generation of the bound entangled state. Finally, we highlight some important characteristics of this bound entangled state example, which may make it the best candidate for a realistic experimental verification of bound entanglement.

2 Preliminaries

We consider CV quantum systems with Gaussian states. Quantum continuous variables describe quantum mechanics applied to an infinite-dimensional Hilbert space equipped with a set of canonical quadrature operators \hat{q}_j and \hat{p}_j ($j = 1, 2, \dots, n$). Here, \hat{q}_j and \hat{p}_j are position and momentum operators, respectively. They obey canonical commutation relations $[\hat{q}_j, \hat{p}_k] = i\delta_{jk}$ (in natural units, $\hbar = 1$). The quadrature operators \hat{q}_j and \hat{p}_j are collected to form a vector of operators $\hat{\xi} = (\hat{q}_1, \hat{p}_1, \dots, \hat{q}_n, \hat{p}_n)^T$. Then, the commutation relations can be written as

$$[\hat{\xi}_j, \hat{\xi}_k] = i\Omega_{jk}, \tag{2}$$

where Ω_{jk} is the generic entry of the $2n \times 2n$ matrix $\Omega := \omega^{\oplus n} = \begin{pmatrix} \omega & 0 \\ & \ddots \\ 0 & \omega \end{pmatrix}$,

$$\omega := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

Gaussian states are CV states with Gaussian characteristic functions. Gaussian states are completely characterized by a real displacement vector $\langle \hat{\xi} \rangle = \text{tr}(\hat{\rho} \hat{\xi})$ and a real covariance matrix γ with elements

$$\gamma_{jk} = \text{tr}(\hat{\rho}\{\hat{\xi}_j - \langle \hat{\xi}_j \rangle, \hat{\xi}_k - \langle \hat{\xi}_k \rangle\}) \tag{3}$$

where we define the anticommutator product as $\{\hat{A}, \hat{B}\} = \hat{A}\hat{B} + \hat{B}\hat{A}$. The covariance matrix γ is real and symmetric, and due to the commutation rules (2), the covariance matrix γ satisfies the uncertainty relation

$$\gamma + i\Omega \geq 0. \tag{4}$$

Inequality (4) is a necessary and sufficient condition for a real symmetric matrix γ to correspond to a physical quantum state [48]. The displacement vector $\langle \hat{\xi} \rangle$ contains no entanglement information. All the entanglement information is contained in the covariance matrix of the state. Thus, we will restrict ourselves to the analysis of covariance matrices γ . For a Gaussian state $\hat{\rho}_{AB}$ pertaining to an $(m + n)$ -mode bipartite system $A + B$, a necessary and sufficient condition has been developed in Ref. [30] for the separability of the state. It states that an $(m + n)$ -mode Gaussian state $\hat{\rho}_{AB}$ with covariance matrix γ is separable if and only if there exist an m -mode covariance matrix $\gamma_A \geq i\Omega_A$ and an n -mode covariance matrix $\gamma_B \geq i\Omega_B$ such that

$$\gamma \geq \gamma_A \oplus \gamma_B. \tag{5}$$

Here, $\gamma_A \oplus \gamma_B$ denotes the matrix direct sum of γ_A and γ_B ; that is, $\gamma_A \oplus \gamma_B = \begin{pmatrix} \gamma_A & 0 \\ 0 & \gamma_B \end{pmatrix}$. The condition (5) is a necessary and sufficient condition for the separability of a Gaussian state. Given the covariance matrix γ , if we can find γ_A and γ_B that satisfy the inequality (5), then the state is separable; otherwise, it is entangled.

A very convenient method for checking if a given state is separable or not is the PPT criterion. The PPT criterion provides a necessary condition for a state to be separable. The partial transpose of a state corresponds physically to a local time reversal. For a bipartite Gaussian system $A + B$, the partial transpose with respect to system A transforms the covariance matrix γ into $\tilde{\gamma} = (\Lambda \oplus I_B)\gamma(\Lambda \oplus I_B)$, where $\Lambda = \bigoplus_{k=1}^m \text{diag}(1, -1)$ corresponds to a sign change of the momentum variables belonging to system A and I_B is the n -mode identity matrix. Clearly, if the partial transpose of $\hat{\rho}$ is a valid density operator, we must have $\tilde{\gamma} + i\Omega \geq 0$. But this is equivalent to $\gamma + i\tilde{\Omega} \geq 0$, where $\tilde{\Omega} = (-\Omega_A) \oplus \Omega_B$. Summarizing, suppose γ is the covariance matrix of a Gaussian state, with finite second moments, which has positive partial transpose. Then, we have

$$\gamma + i\tilde{\Omega} \geq 0, \text{ where } \tilde{\Omega} = \begin{pmatrix} -\Omega_A & 0 \\ 0 & \Omega_B \end{pmatrix}. \tag{6}$$

The PPT criterion (6) is a necessary condition for separability. This can be seen from the fact that the inequality (6) directly follows from the inequality (5). In general, the

PPT criterion is not sufficient to guarantee separability. There exist Gaussian states that satisfy the PPT criterion but nevertheless are entangled. Such states are known as bound entangled Gaussian states. Bound entangled states are entangled states, but they cannot be distilled into pure entangled states using LOCCs. Bound entangled states have practical applications in quantum cryptography [49], channel discrimination [50] and many quantum information protocols [40].

3 Detection of Gaussian entanglement via solving linear matrix inequalities

As mentioned in Eq. (5), an $(m + n)$ -mode Gaussian state $\hat{\rho}_{AB}$ with covariance matrix γ is separable if and only if there exist an m -mode covariance matrix $\gamma_A \geq i\Omega_A$ and an n -mode covariance matrix $\gamma_B \geq i\Omega_B$ such that $\gamma \geq \gamma_A \oplus \gamma_B$. Since the inequality $\gamma_A \geq i\Omega_A$ is equivalent to $\begin{pmatrix} \gamma_A & \Omega_A \\ \Omega_A^T & \gamma_A \end{pmatrix} \geq 0$ and the inequality $\gamma_B \geq i\Omega_B$ is equivalent to $\begin{pmatrix} \gamma_B & \Omega_B \\ \Omega_B^T & \gamma_B \end{pmatrix} \geq 0$ (see Lemma 2 in Ref. [31] for a proof), the separability problem of a Gaussian state $\hat{\rho}_{AB}$ can be recast as an equivalent feasibility problem involving linear matrix inequalities as follows:

Separability problem

$$\begin{aligned} &\textbf{find} \quad \gamma_A \text{ and } \gamma_B \\ &\textbf{subject to} \quad \gamma - \begin{pmatrix} \gamma_A & 0 \\ 0 & \gamma_B \end{pmatrix} \geq 0, \end{aligned} \tag{7}$$

$$\begin{pmatrix} \gamma_A & \Omega_A \\ \Omega_A^T & \gamma_A \end{pmatrix} \geq 0, \tag{8}$$

$$\begin{pmatrix} \gamma_B & \Omega_B \\ \Omega_B^T & \gamma_B \end{pmatrix} \geq 0. \tag{9}$$

The separability problem is thus to determine whether the constraints (7), (8) and (9) are consistent for a given bipartite Gaussian state γ , and if so, find a pair of positive definite matrices γ_A and γ_B satisfying them. If the above problem is feasible, that is, there exist solutions γ_A and γ_B to the constraints (7), (8), and (9), then we can conclude that the covariance matrix γ is separable; otherwise, it is entangled. The linear matrix inequalities (7), (8) and (9) can be solved efficiently by using existent numerical solvers such as SeDuMi [44] and the SDTP3 [45].

It should be mentioned that although the linear matrix inequalities (7), (8) and (9) are necessary and sufficient for checking separability of γ , one has to be very careful when the state γ lies very close to the boundary of the set of separable states or to the boundary of the set of physical states (i.e., the smallest eigenvalue of $\gamma + i\Omega$ is very close to zero). For such cases, since the constraints (7), (8) and (9) are non-strict linear matrix inequalities, the unavoidable round-off errors caused by floating point computations may have a significant impact on the solvability of the problem. We consider an example to illustrate this fact.

Example. Consider the Gaussian state constructed in Ref. [30]. The covariance matrix of this 2×2 Gaussian state is given by

$$\gamma = \begin{pmatrix} 2 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 2 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 & 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 4 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 2 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 4 \end{pmatrix}. \tag{10}$$

The eigenvalues of $\gamma + i\Omega$ are $0, 3 - \sqrt{3}, 3, 3 + \sqrt{3}$, each with multiplicity 2. Because 0 is an eigenvalue of $\gamma + i\Omega$, the state lies just on the boundary of the set of physical states. For this particular γ , even if numerical computation shows that the problem with constraints (7), (8) and (9) is infeasible, it is still too early to make a conclusion on whether the state is entangled or not, since we have not yet ruled out the possibility that the infeasibility is a result of the application of floating point computations to non-strict inequalities. Fortunately, for this particular γ , we can relax the problem and solve the following strict inequalities instead:

$$\begin{aligned} &\mathbf{find} \quad \gamma_A \text{ and } \gamma_B \\ &\mathbf{subject\ to} \quad \gamma - \begin{pmatrix} \gamma_A & 0 \\ 0 & \gamma_B \end{pmatrix} > -\epsilon I, \end{aligned} \tag{11}$$

$$\begin{pmatrix} \gamma_A & \Omega_A \\ \Omega_A^T & \gamma_A \end{pmatrix} > -\epsilon I, \tag{12}$$

$$\begin{pmatrix} \gamma_B & \Omega_B \\ \Omega_B^T & \gamma_B \end{pmatrix} > -\epsilon I. \tag{13}$$

Here, $\epsilon > 0$ is a small number such that the LMI problem (11), (12) and (13) slightly relaxes the LMI problem (7), (8) and (9). Typically, we may choose $10^{-6} < \epsilon < 10^{-9}$. For any $\epsilon > 0$, if there is no solution γ_A, γ_B to the inequalities (11), (12) and (13), then there is no solution γ_A, γ_B to the LMI problem (7), (8) and (9), and we can safely conclude that the state γ is an entangled state. For this particular example, we choose $\epsilon = 10^{-8}$. We find that even in this case, there is still no solution γ_A, γ_B to the relaxed inequalities (11), (12) and (13). So we can safely conclude that the state γ in Eq. (10) is an entangled state. On the other hand, a direct calculation shows that γ has a positive partial transpose, i.e., $\gamma + i\tilde{\Omega} \geq 0$. Thus, the state with covariance matrix (10) is a bound entangled Gaussian state with respect to the bipartite splitting $\{\{1, 2\}, \{3, 4\}\}$.

Remark 1 e make some remarks about how to solve linear matrix inequalities numerically. Linear matrix inequalities are essentially convex constraints and can be solved efficiently using many existing numerical methods [43,51]. A simple algorithm that is guaranteed to solve linear matrix inequality problems is the ellipsoid algorithm [52]. A more computationally efficient algorithm is the interior point method [53]. The

basic idea of these algorithms can be found in [43]. Based on these algorithms, several software packages for solving linear matrix inequalities have been produced such as the solvers SeDuMi [44] and the SDTP3 [45]. We refer the reader to the appendices of Ref. [54] for some code examples on how to use these solvers.

4 Identification of 2 × 2 bound entangled Gaussian states

Bound entangled Gaussian states are a class of Gaussian states that satisfy the PPT criterion, but nevertheless are entangled. In order to guarantee a bound entangled state, we need to make sure that the following two conditions hold: (1) the covariance matrix γ satisfies the PPT criterion (6); (2) the linear matrix inequalities (7)–(9) are infeasible; that is, there exist no solutions γ_A and γ_B to the constraints (7)–(9).

According to a theorem by Williamson [55], every positive-definite real symmetric matrix of even dimension can be diagonalized through a symplectic transformation. In particular, this theorem can be applied to covariance matrices of Gaussian states. Given an arbitrary n -mode Gaussian state with covariance matrix γ , there exists a symplectic matrix S such that

$$\gamma = S \left[\bigoplus_{k=1}^n \nu_k I_2 \right] S^T, \tag{14}$$

where S is a symplectic matrix, i.e., $S\Omega S^T = \Omega$. The n positive quantities ν_k are called the symplectic eigenvalues of γ and can also be computed by taking the modulus of the standard eigenspectrum of the matrix $i\Omega\gamma$. The symplectic spectrum expresses the fundamental properties of the corresponding Gaussian quantum state. For example, the uncertainty principle (4) is equivalent to $\nu_k \geq 1$. When $\nu_k = 1$ for all k , the resulting covariance matrix $\gamma = SS^T$ corresponds to a pure Gaussian state [48,56,57].

Furthermore, the symplectic matrix S in Eq. (14) can be decomposed using the Euler decomposition [58,59]. In fact, every n -mode symplectic matrix S can be written as

$$S = K \left[\bigoplus_{k=1}^n S(r_k) \right] L, \tag{15}$$

where K and L are symplectic and orthogonal matrices, and $S(r_k) = \begin{pmatrix} e^{-r_k} & 0 \\ 0 & e^{r_k} \end{pmatrix}$ is a set of single-mode squeezing matrices. It is worth mentioning that the symplectic and orthogonal matrices K and L correspond to passive interferometers which can be implemented using a network of beam splitters and phase shifters in quantum optics. Combining Eq. (14) and Eq. (15), we obtain that an arbitrary n -mode covariance matrix γ can be written as

$$\gamma = K \left[\bigoplus_{k=1}^n S(r_k) \right] L \left[\bigoplus_{k=1}^n \nu_k I_2 \right] L^T \left[\bigoplus_{k=1}^n S(r_k) \right] K^T. \tag{16}$$

Physically, Eq. (16) means that every n -mode zero-mean Gaussian quantum state can be prepared beginning with n thermal states described by the diagonal covariance matrix $\mathbf{v} = \bigoplus_{k=1}^n v_k I_2$, followed by applying an interferometer L , then single-mode squeezers $S(r_k)$ and finally an interferometer K .

In this section, our main objective is to construct an example of a 2×2 bound entangled Gaussian state that is simple to be prepared in quantum optics. This is done by using Eq. (16) with the help of the numerical detection method discussed in Sect. 3. Firstly, we consider the symplectic and orthogonal matrices K and L in Eq. (16). We rearrange the entries of the matrix Ω such that we have $J := P^T \Omega P = \begin{pmatrix} 0 & I_4 \\ -I_4 & 0 \end{pmatrix}$ where

$$P = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

Let us define $O := P^T K P$. Then, it can be shown that O is orthogonal (i.e., $O^T O = I$) and satisfies $O J O^T = J$, which means that the matrix O can be written as $O = \begin{pmatrix} X & Y \\ -Y & X \end{pmatrix}$, where $XX^T + YY^T = I$ and $XY^T - YX^T = 0$. This implies that $Q := X + iY$ is a unitary matrix. Therefore, if we choose a unitary matrix Q , and let $X := \text{Re}(Q)$ and $Y := \text{Im}(Q)$, the matrix $O = \begin{pmatrix} X & Y \\ -Y & X \end{pmatrix}$ is orthogonal and satisfies $O J O^T = J$. As a result, the corresponding symplectic and orthogonal matrix K can be obtained by $K = P O P^T$. We mention that the class of 4×4 unitary matrices has been parametrized in Ref. [60]. After determining K and L , we choose symplectic eigenvalues v_k and squeezing matrices $S(r_k)$ in Eq. (16) and we obtain a Gaussian state γ . This Gaussian state γ is then tested by the numerical detection method discussed in Sect. 3. If the obtained Gaussian state γ satisfies the PPT criterion (6), but nevertheless is entangled, then we obtain a bound entangled state. Otherwise, we try different parameters until we get a bound entangled state.

Using the idea above, we have successfully obtained a 2×2 bound entangled Gaussian state. The covariance matrix of this Gaussian state is calculated as

$$\gamma = \begin{pmatrix} 1.8605 & 0 & -0.7593 & 0 & 0.1030 & 0 & 0.6384 & 0 \\ 0 & 1.8687 & 0 & -0.3556 & 0 & 0.6854 & 0 & 1.0340 \\ -0.7593 & 0 & 2.3534 & 0 & 1.0738 & 0 & -0.7593 & 0 \\ 0 & -0.3556 & 0 & 1.9334 & 0 & 0.5029 & 0 & -0.3556 \\ 0.1030 & 0 & 1.0738 & 0 & 2.4990 & 0 & 0.1030 & 0 \\ 0 & 0.6854 & 0 & 0.5029 & 0 & 2.9027 & 0 & 0.6854 \\ 0.6384 & 0 & -0.7593 & 0 & 0.1030 & 0 & 1.8605 & 0 \\ 0 & 1.0340 & 0 & -0.3556 & 0 & 0.6854 & 0 & 1.8687 \end{pmatrix}. \tag{17}$$

We obtain the covariance matrix (17) by using the following procedure. First, we choose the symplectic eigenvalues $\nu_k \geq 1$ in Eq. (16). Here, we have chosen

$$\mathbf{v} = \bigoplus_{k=1}^n \nu_k I_2 = \begin{pmatrix} 1.01 & & & & & & & \mathbf{0} \\ & 1.01 & & & & & & \\ & & 1.01 & & & & & \\ & & & 1.01 & & & & \\ & & & & 3.2 & & & \\ & & & & & 3.2 & & \\ & & & & & & 3.2 & \\ \mathbf{0} & & & & & & & 3.2 \end{pmatrix}. \tag{18}$$

Second, we choose the symplectic and orthogonal matrix L and the squeezing operators $S(r_k)$. Here, for simple implementation of the resulting state, we have chosen $L = I$ and

$$S(r_1) = S(r_3) = \begin{pmatrix} 1.1 & 0 \\ 0 & \frac{1}{1.1} \end{pmatrix}, \tag{19}$$

$$S(r_2) = S(r_4) = \begin{pmatrix} \frac{1}{1.1} & 0 \\ 0 & 1.1 \end{pmatrix}. \tag{20}$$

Finally, we choose the symplectic matrix K . Using the method described before, we

choose the unitary matrix $Q = \begin{pmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{4} & -\frac{\sqrt{2}}{4} & \frac{1}{2} \\ 0 & \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0 \\ 0 & -\frac{1}{2} & \frac{1}{2} & \frac{\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{4} & -\frac{\sqrt{2}}{4} & \frac{1}{2} \end{pmatrix}$. The resulting symplectic and

orthogonal matrix K is then calculated as

$$K = \begin{pmatrix} \frac{\sqrt{2}}{2} & 0 & \frac{\sqrt{2}}{4} & 0 & -\frac{\sqrt{2}}{4} & 0 & \frac{1}{2} & 0 \\ 0 & \frac{\sqrt{2}}{2} & 0 & \frac{\sqrt{2}}{4} & 0 & -\frac{\sqrt{2}}{4} & 0 & \frac{1}{2} \\ 0 & 0 & \frac{\sqrt{2}}{2} & 0 & \frac{\sqrt{2}}{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{\sqrt{2}}{2} & 0 & \frac{\sqrt{2}}{2} & 0 & 0 \\ 0 & 0 & -\frac{1}{2} & 0 & \frac{1}{2} & 0 & \frac{\sqrt{2}}{2} & 0 \\ 0 & 0 & 0 & -\frac{1}{2} & 0 & \frac{1}{2} & 0 & \frac{\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} & 0 & \frac{\sqrt{2}}{4} & 0 & -\frac{\sqrt{2}}{4} & 0 & \frac{1}{2} & 0 \\ 0 & -\frac{\sqrt{2}}{2} & 0 & \frac{\sqrt{2}}{4} & 0 & -\frac{\sqrt{2}}{4} & 0 & \frac{1}{2} \end{pmatrix}.$$

Substituting the above values of \mathbf{v} , L , $S(r_k)$ and K into the decomposition (16), we will obtain the covariance matrix (17). The physicality of the state (17) is guaranteed by our choice of the symplectic eigenvalues $\nu_k \geq 1$. Next, we show that it is a bound entangled state. It is found that $\min \text{eig}(\gamma + i\Omega) = 0.0840 > 0$; thus, the covariance matrix (17) satisfies the PPT criterion (6) and is not distillable. On the other hand, we find by numerical computation that the set of strict inequalities (11)-(13) with $\epsilon = 10^{-8}$ is infeasible given the covariance matrix (17). Hence, it is an entangled state. In conclusion, the 2×2 Gaussian state (17) satisfies the PPT criterion (6), but is an entangled state. So it is a bound entangled state.

Now, we show how to construct an optical system to generate the bound entangled Gaussian state (17). We note that the symplectic map K corresponds to the following linear unitary transformation on the annihilation operators

$$\begin{pmatrix} \hat{c}_1 \\ \hat{c}_2 \\ \hat{c}_3 \\ \hat{c}_4 \end{pmatrix} = \begin{pmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{4} & -\frac{\sqrt{2}}{4} & \frac{1}{2} \\ 0 & \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0 \\ 0 & -\frac{1}{2} & \frac{1}{2} & \frac{\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{4} & -\frac{\sqrt{2}}{4} & \frac{1}{2} \end{pmatrix} \begin{pmatrix} \hat{b}_1 \\ \hat{b}_2 \\ \hat{b}_3 \\ \hat{b}_4 \end{pmatrix}.$$

Here, $\hat{b}_1 - \hat{b}_4$ and $\hat{c}_1 - \hat{c}_4$ are annihilation operators; see Fig. 1 for details. Furthermore, using the result developed in Ref. [61,62], this unitary transformation can be realized as a network of three beam splitters as follows:

$$\begin{pmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{4} & -\frac{\sqrt{2}}{4} & \frac{1}{2} \\ 0 & \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0 \\ 0 & -\frac{1}{2} & \frac{1}{2} & \frac{\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{4} & -\frac{\sqrt{2}}{4} & \frac{1}{2} \end{pmatrix} = B_3 B_2 B_1, \tag{21}$$

where $B_1 - B_3$, representing beam splitter transformations, are given by

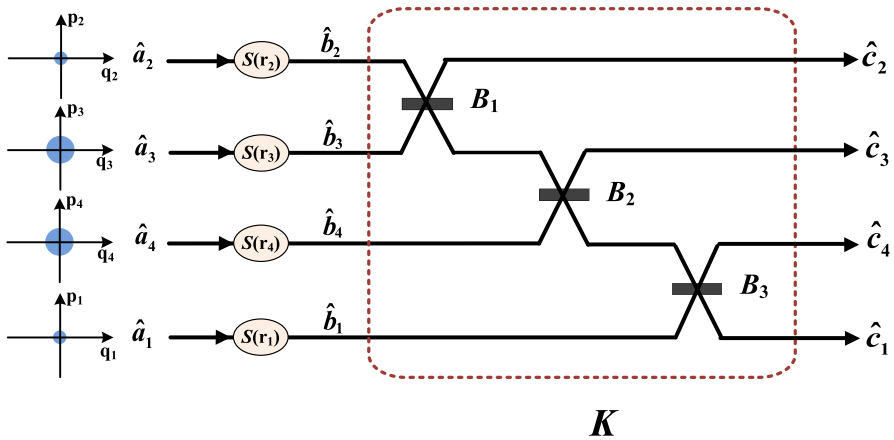


Fig. 1 Diagram for preparation of the bound entangled Gaussian state (17) in quantum optics. The initial inputs $\hat{a}_1, \hat{a}_2, \hat{a}_3$ and \hat{a}_4 are in thermal states described by the covariance matrix ν shown in Eq. (18). B_1, B_2 and B_3 are balanced beam splitters and realize the corresponding unitary transformations (22) and (23); $S(r_1), \dots, S(r_4)$ are a set of single-mode squeezers as described by Eqs. (19) and (20). The output Gaussian state $(\hat{c}_1, \dots, \hat{c}_4)$ has the covariance matrix (17) and is a 2×2 bound entangled state with respect to the bipartite splitting $\{\{\hat{c}_1, \hat{c}_2\}, \{\hat{c}_3, \hat{c}_4\}\}$

$$B_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0 \\ 0 & -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, B_2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ 0 & 0 & -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{pmatrix}, \tag{22}$$

$$B_3 = \begin{pmatrix} \frac{\sqrt{2}}{2} & 0 & 0 & \frac{\sqrt{2}}{2} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\frac{\sqrt{2}}{2} & 0 & 0 & \frac{\sqrt{2}}{2} \end{pmatrix}. \tag{23}$$

Hence, the symplectic map K is implemented by a network of beam splitters as described in the dotted box in Fig. 1. B_1, B_2 and B_3 are balanced beam splitters (50 : 50). According to Eq. (16), the output Gaussian state $(\hat{c}_1, \dots, \hat{c}_4)$ generated by the optical system described in Fig. 1 has the covariance matrix (17) and is a bound entangled state with respect to the bipartite splitting $\{\{\hat{c}_1, \hat{c}_2\}, \{\hat{c}_3, \hat{c}_4\}\}$.

The bound entangled Gaussian state (17) has some nice characteristics as follows that make it simple enough to be producible in quantum optics.

1. $L = I$ in Eq. (16). It means that we do not need to implement another beam splitter network before the single-mode squeezers $S(r_1), \dots, S(r_4)$.
2. The optical system only consists of three beam splitters B_1, B_2 and B_3 . It happens that this is the minimum number of beam splitters to prepare a 2×2 bound entangled Gaussian state. Besides, all the beam splitters B_1, B_2 and B_3 are balanced (50 : 50), and hence they can be easily implemented experimentally.
3. As shown in Eq. (18), all the symplectic eigenvalues ν_k are strictly larger than 1. In other words, the initial inputs $\hat{a}_1 - \hat{a}_4$ are all in thermal states. This is clearly realistic

since in actual implementations we cannot prepare precisely vacuum inputs due to the presence of noise and imperfections. For comparison, the preparation of the bound entangled state (10) requires two vacuum inputs as illustrated in Ref. [42].

4. The state (17) is fairly robust for experimental verification. Numerical simulations show that small imperfections in the implementation (such as input fields and squeezers) can still generate bound entanglement (though not the original state). An important reason is that the state (17) has strictly positive partial transpose, i.e., $\gamma + i\tilde{\Omega} > 0$. Hence, small imperfections in experimental implementations still lead to a PPT state.

Because of the above characteristics, the Gaussian state (17) may serve as a good candidate for a realistic experimental verification of bound entanglement.

5 Conclusion

In this paper, we have shown that the separability problem of Gaussian quantum states can be cast as an equivalent problem of determining feasibility of a set of linear matrix inequalities. Thus, Gaussian entanglement can be directly detected by checking the feasibility of the corresponding linear matrix inequalities. We have applied this method to the identification of bound entangled Gaussian states. By choosing some parameters in the decomposition of a covariance matrix, we can find bound entangled states that are simple enough to be producible in quantum optics. We have provided an optical scheme for generating a particular bound entangled Gaussian state in quantum optics. We have highlighted some characteristics of the bound entangled Gaussian state which may make it a good candidate for a realistic experimental verification of bound entanglement. In future work, it would be interesting to investigate potential applications of bound entangled Gaussian states to quantum engineering areas such as multiparty quantum communication [40,49] and quantum metrology [6,7].

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