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Some new entanglement-assisted quantum error-correcting MDS codes from generalized Reed–Solomon codes

Fuyin Tian[1] · Shixin Zhu[1]

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Abstract
Entanglement-assisted quantum maximum distance separable (MDS) codes form a significant class of quantum codes. By using generalized Reed–Solomon (GRS) codes and extended GRS codes, we construct some new classes of q-ary entanglement-assisted quantum error-correcting MDS codes. Most of these codes are new in the sense that their parameters are not covered by the codes available in the literature.

Keywords Entanglement-assisted quantum error-correcting MDS codes · Generalized Reed–Solomon codes · Hermitian hull

Mathematics Subject Classification 94B05 · 81p70

1 Introduction

Quantum error-correcting codes (QECCs) are one of the necessary guarantees for the realization of quantum communication and quantum computer. The connections between quantum codes and classical codes were established by Calderbank et al. [1]. The establishment showed that QECCs can be constructed from self-orthogonal (or dual-containing) classical codes [2]. Since then, many classes of quantum codes have been constructed by using classical error-correcting codes. There are two main ways to construct quantum MDS codes, namely using constacyclic codes (see [3,4]) and generalized Reed–Solomon codes (see [5–8]). However, the self-orthogonal condition forms a barrier in the development of quantum coding theory. To break through the barrier, Brun et al. proposed the entanglement-assisted (EA) stabilizer formalism in [9].

✉ Shixin Zhu
zhushixinmath@hfut.edu.cn

Fuyin Tian
tianfuyin0825@163.com

1 School of Mathematics, HeFei University of Technology, Hefei 230009, China

By using preshared entanglement between the sender and the receiver, they proved that arbitrary classical linear error-correcting codes can be used to construct entanglement-assisted quantum error-correcting codes (EAQECCs). Since then, many scholars have been interested in EAQECCs and have made good progress.

Let q be a prime power. A q-ary EAQECC can be denoted as $[[n, k, d; c]]_q$, which encodes k logical qubits into n physical qubits with help of c pairs of maximally entangled states, where d is the minimum distance of the code. In particular, if $c = 0$, the code is a QECC. It is similar to the classical error-correcting codes, a quantum code with minimum distance d can detect up to $d - 1$ quantum errors and correct up to $\lfloor \frac{d-1}{2} \rfloor$ quantum errors. The singleton bound for an EAQECC is given in the following proposition:

Proposition 1 [10] *An* $[[n, k, d; c]]_q$ *EAQECC satisfies* $n + c - k \geq 2(d - 1)$, *where* $0 \leq c \leq n - 1$ *and* $d \leq \frac{n+2}{2}$.

An EAQECC attaining the singleton bound is called an entanglement-assisted quantum MDS (EAQMDS for short) code. By using Reed–Solomon codes and constacyclic codes, Fan et al. [11] constructed five classes of EAQMDS codes with the help of a few shared entanglement states. Chen et al. obtained four classes of EAQMDS codes from negacyclic codes with the help of 4 or 5 shared entanglement states [12]. In [13], Chen et al. obtained four classes of EAQMDS codes from constacyclic codes of length $n = \frac{q^2+1}{5}$. Subsequently, many researchers constructed many classes of EAQMDS codes with constacyclic codes [14–19]. In [20], Guenda et al. have shown that the number of shared pairs required to construct an EAQECC is related to the dimension of the hull of classical linear codes. Then, Luo et al. presented several infinite families of MDS codes with hulls of arbitrary dimensions by GRS codes and constructed several new infinite families of EAQMDS codes with flexible parameters [21]. Since then, many people have constructed many quantum codes by using GRS codes [22–24].

In this paper, we construct some classes of q-ary EAQMDS codes with parameters $[[n, n-2k+c, k+1; c]]_q$ and EAQECCs with parameters $[[n, c, n-k+1; n-2k+c]]_q$ from GRS codes. The specific values of n, k and c can be found in Theorems 1, 2, 3, 4 and 5. Moreover, we also construct some classes of q-ary EAQMDS codes with parameters $[[n + 1, n - 2k + c + 2, k + 1; c + 1]]_q$ and EAQECCs with parameters $[[n + 1, c + 1, n - k + 2; n - 2k + c + 2]]_q$ from extended GRS codes. The specific values of n, k and c can be found in Theorems 6, 7, 8, 9 and 10.

The paper is organized as follows. In Sect. 2, we recall the basic knowledge of linear codes, EAQECCs, GRS codes. In Sects. 3 and 4, we construct some classes of EAQMDS codes and EAQECCs from GRS codes. In Sect. 5, some classes of EAQMDS codes are constructed from extended GRS codes. Section 6 concludes this paper.

2 Preliminaries

Let $\mathrm{GF}(q^2)$ be a finite field with q^2 elements. Let $\mathrm{GF}(q^2)^n$ be the n-dimensional vector space over $\mathrm{GF}(q)$, where n is a positive integer. The Hamming weight of $\boldsymbol{x} \in \mathrm{GF}(q^2)^n$ is the number of nonzero coordinates of $\boldsymbol{x}$ and is denoted by $\mathrm{wt}(\boldsymbol{x})$. The Hamming distance of two vectors $\boldsymbol{x}$ and $\boldsymbol{y}$ is the Hamming weight of the $\boldsymbol{x}-\boldsymbol{y}$, denoted by $\mathrm{dist}(\boldsymbol{x},\boldsymbol{y})$.

A q^2-ary code C of length n is a subset of $\mathrm{GF}(q^2)^n$. The minimum distance of C, denoted by $d(C)$, is defined by $d(C)=\min\{\mathrm{dist}(\boldsymbol{x},\boldsymbol{y})|\boldsymbol{x}\neq\boldsymbol{y}\in C\}$. The code C is called a q^2-ary linear code of length n, if C is a subspace of $\mathrm{GF}(q^2)^n$. Clearly, the minimum Hamming distance of linear code C is equal to the minimum nonzero Hamming weight of all codewords in C. A q^2-ary linear code $[n,k,d]$ is a k-dimensional subspace of $\mathrm{GF}(q^2)^n$ and minimum distance d.

Let $\boldsymbol{x}=(x_1,x_2,\ldots,x_n)$ and $\boldsymbol{y}=(y_1,y_2,\ldots,y_n)$ be two vectors in $\mathrm{GF}(q^2)^n$, the Euclidean inner product is defined as $(\boldsymbol{x},\boldsymbol{y})_E=\sum_{i=1}^n x_i y_i$, and the Hermitian inner product is defined as $(\boldsymbol{x},\boldsymbol{y})_H=\sum_{i=1}^n x_i y_i^q$. For a q^2-ary linear code C of length n, the Euclidean dual of C, denoted by $C^{\perp_E}$, is defined by

$$C^{\perp_E}=\{\boldsymbol{x}\in\mathrm{GF}(q^2)^n|(\boldsymbol{x},\boldsymbol{y})_E=0,\ \text{for all}\ \boldsymbol{y}\in C\}.$$

If $C\subseteq C^{\perp_E}$, C is referred to as a Euclidean self-orthogonal code. Similarly, the Hermitian dual of C, denoted by $C^{\perp_H}$, is defined by

$$C^{\perp_H}=\{\boldsymbol{x}\in\mathrm{GF}(q^2)^n|(\boldsymbol{x},\boldsymbol{y})_H=0,\ \text{for all}\ \boldsymbol{y}\in C\}.$$

If $C\subseteq C^{\perp_H}$, C is referred to as a Hermitian self-orthogonal code.

For a vector $\boldsymbol{x}=(x_1,x_2,\ldots,x_n)$ in $\mathrm{GF}(q^2)^n$, define $\boldsymbol{x}^q=(x_1^q,x_2^q,\ldots,x_n^q)$. For a subset T of $\mathrm{GF}(q^2)^n$, define the set $T^q=\{\boldsymbol{x}^q|\boldsymbol{x}\in T\}$. It is easy to check that $C^{\perp_H}=(C^q)^{\perp_E}$ for a q^2-ary linear code C of length n.

For a linear code C over F_{q^2}, denoted by $Hull_h(C)$ the Hermitian hull $C\bigcap C^{\perp h}$ of C. Here are two propositions about Hermitian hull:

Proposition 2 [20] *Let C be a classical $[n,k,d]_{q^2}$ code with parity check matrix H and generator matrix G. Then, $rank(HH^\dagger)$ and $rank(GG^\dagger)$ are independent of H and G so that $rank(HH^\dagger)=n-k-dim(Hullh(C))=n-k-dim(Hullh(C^{\perp_h}))$, and $rank(GG^\dagger)=k-dim(Hullh(C))=k-dim(Hullh(C^{\perp_h}))$.*

Proposition 3 [20] *Let C be a classical $[n,k,d]_{q^2}$ code and let $C^{\perp h}$ be its Hermitian dual with parameters $[n,n-k,d^{\perp h}]_q$. Then, there exist $[[n,k-dim(Hullh(C)),d;n-k-dim(Hullh(C))]]_{q^2}$ and $[[n,n-k-dim(Hullh(C)),d^{\perp};k-dim(Hullh(C))]]_q$ EAQECCs. If C is MDS, then one of the two EAQECCs must be MDS.*

Let k,n be positive integers, and $\mathrm{GF}(q^2)[x]_k$ be the set of polynomials whose degree is less than k over $\mathrm{GF}(q^2)$. Select n distinct elements $a_1,a_2,\cdots,a_n$ of $\mathrm{GF}(q^2)$ and n nonzero elements $v_1,v_2,\cdots,v_n$ of $\mathrm{GF}(q^2)$. Let $\boldsymbol{a}=(a_1,a_2,\ldots,a_n)$ and $\boldsymbol{v}=(v_1,v_2,\ldots,v_n)$, then

$$\mathcal{G}RS_k(\boldsymbol{a},\boldsymbol{v}):=\{(v_1f(a_1),\ldots,v_nf(a_n))|f(x)\in\mathrm{GF}(q^2)[x]_k\}$$

is called a GRS code. Clearly,

$$G_k(\boldsymbol{a},\boldsymbol{v})=\begin{pmatrix} v_1a_1^0 & v_2a_2^0 & \cdots & v_na_n^0\\ v_1a_1^1 & v_2a_2^1 & \cdots & v_na_n^1\\ \vdots & \vdots & \cdots & \vdots\\ v_1a_1^{k-1} & v_2a_2^{k-1} & \cdots & v_na_n^{k-1}\end{pmatrix}$$

is a generator matrix of $\mathcal{G}RS_k(\boldsymbol{a},\boldsymbol{v})$. It is well known that $\mathcal{G}RS_k(\boldsymbol{a},\boldsymbol{v})$ is a q^2-ary $[n,k,n-k+1]$ MDS code.

Moreover, any GRS code of length n can be extended to a code of length $n+1$ and such code is called an extended GRS code. The definition of extended GRS code of length n can be given by

$$\mathcal{G}RS_k(\boldsymbol{a},\boldsymbol{v},\infty):=\{(v_1f(a_1),\ldots,v_nf(a_n),f_{k-1})|f(x)\in \mathrm{GF}(q^2)[x]_k\}$$

where f_{k-1} is the coefficient of x^{k-1} in $f(x)$ and $k\le n+1\le q^2+1$. It is known that $\mathcal{G}RS_k(\boldsymbol{a},\boldsymbol{v},\infty)$ has parameters $[n+1,k,n+2-k]_{q^2}$ and a generator matrix $[G_k(\boldsymbol{a},\boldsymbol{v})|\boldsymbol{u}^T]$, where $\boldsymbol{u}=(0,0,\ldots,0,1)$.

When all of the a_i, $i=1,2,\cdots,n$, are nonzero, let C_{k,k_1} be a q^2-ary linear code of length n with generator matrix

$$G_{k,k_1}(\boldsymbol{a},\boldsymbol{v})=\begin{pmatrix} v_1a_1^{k_1} & v_2a_2^{k_1} & \cdots & v_na_n^{k_1}\\ v_1a_1^{k_1+1} & v_2a_2^{k_1+1} & \cdots & v_na_n^{k_1+1}\\ \vdots & \vdots & \cdots & \vdots\\ v_1a_1^{k_1+k-1} & v_2a_2^{k_1+k-1} & \cdots & v_na_n^{k_1+k-1}\end{pmatrix}.$$

As $a_1,a_2,\ldots,a_n$ are n distinct nonzero elements of $\mathrm{GF}(q^2)$, put $v_i'=v_ia_i^{k_1}$, $i=1,2,\ldots,n$, then $C_{k,k_1}(\boldsymbol{a},\boldsymbol{v})=\mathcal{G}RS_k(\boldsymbol{a},\boldsymbol{v}')$. Hence, C_{k,k_1} is an MDS code with parameters $[n,k,n-k+1]$.

Similarly, when all of the a_i, $i=1,2,\ldots,n$, are nonzero, let $C_{k,k_1,\infty}$ be a q^2-ary linear code of length n with generator matrix $[G_{k,k_1}(\boldsymbol{a},\boldsymbol{v})|\boldsymbol{u}^T]$.

In the end of this section, we will give a useful lemma for the following construction.

Lemma 1 *Let C be a GRS code $\mathcal{G}RS_k(\boldsymbol{a},\boldsymbol{v})$ with generator matrix $G_k(\boldsymbol{a},\boldsymbol{v})=(\boldsymbol{g_0},\boldsymbol{g_1},\ldots,\boldsymbol{g_{k-1}})^T$ and $\hat{C}$ be an extended GRS code $\mathcal{G}RS_k(\boldsymbol{a},\boldsymbol{v},\infty)$ with generator matrix $\hat{G}_k(\boldsymbol{a},\boldsymbol{v})=[G_k(\boldsymbol{a},\boldsymbol{v})|\boldsymbol{u}^T]$, where $G_k(\boldsymbol{a},\boldsymbol{v})$ and $\boldsymbol{u}$ are defined as above. Then* $\dim(\mathrm{Hull}_h\hat{C})=\dim(\mathrm{Hull}_hC)-1$ *if $(\boldsymbol{g_i},\boldsymbol{g_{k-1}})_H=0$, where $i=0,\ldots,k-1$.*

Proof From Proposition 2, we know $\dim(\mathrm{Hull}h(C) = k - \mathrm{rank}(GG^{\dagger}))$.

$$\text{Since } G_k(\boldsymbol{a}, \boldsymbol{v})G_k(\boldsymbol{a}, \boldsymbol{v})^{\dagger} = \begin{pmatrix} \boldsymbol{g}_0\boldsymbol{g}_0^{\dagger} & \boldsymbol{g}_0\boldsymbol{g}_1^{\dagger} & \cdots & \boldsymbol{g}_0\boldsymbol{g}_{k-1}^{\dagger} \\ \boldsymbol{g}_1\boldsymbol{g}_0^{\dagger} & \boldsymbol{g}_1\boldsymbol{g}_1^{\dagger} & \cdots & \boldsymbol{g}_1\boldsymbol{g}_{k-1}^{\dagger} \\ \vdots & \vdots & \cdots & \vdots \\ \boldsymbol{g}_{k-1}\boldsymbol{g}_0^{\dagger} & \boldsymbol{g}_{k-1}\boldsymbol{g}_1^{\dagger} & \cdots & \boldsymbol{g}_{k-1}\boldsymbol{g}_{k-1}^{\dagger} \end{pmatrix},$$

$$\hat{G}_k(\boldsymbol{a}, \boldsymbol{v})\hat{G}_k(\boldsymbol{a}, \boldsymbol{v})^{\dagger} = \begin{pmatrix} \boldsymbol{g}_0\boldsymbol{g}_0^{\dagger} & \boldsymbol{g}_0\boldsymbol{g}_1^{\dagger} & \cdots & \boldsymbol{g}_0\boldsymbol{g}_{k-1}^{\dagger} \\ \boldsymbol{g}_1\boldsymbol{g}_0^{\dagger} & \boldsymbol{g}_1\boldsymbol{g}_1^{\dagger} & \cdots & \boldsymbol{g}_1\boldsymbol{g}_{k-1}^{\dagger} \\ \vdots & \vdots & \cdots & \vdots \\ \boldsymbol{g}_{k-1}\boldsymbol{g}_0^{\dagger} & \boldsymbol{g}_{k-1}\boldsymbol{g}_1^{\dagger} & \cdots & \boldsymbol{g}_{k-1}\boldsymbol{g}_{k-1}^{\dagger}+1 \end{pmatrix}.$$

It is easy to prove that $\mathrm{rank}(\hat{G}_k(\boldsymbol{a}, \boldsymbol{v})\hat{G}_k(\boldsymbol{a}, \boldsymbol{v})^{\dagger}) = G_k(\boldsymbol{a}, \boldsymbol{v})G_k(\boldsymbol{a}, \boldsymbol{v})^{\dagger} + 1$. Then $\dim(\mathrm{Hull}_h\hat{C}) = \dim(\mathrm{Hull}_h C) - 1$. □

3 The first construction

Throughout this section, let ω be a primitive element of $\mathrm{GF}(q^2)$ and $\gamma = \omega^h$, where h is an even integer such that $\frac{2(q-1)}{h} = 2\tau + 1$ for some $\tau \geq 1$. Let $n' = \frac{q^2-1}{h}$, then γ is a primitive n'-th root of unity.

For our construction, we need the following lemmas.

Lemma 2 *Let q, h, τ and n' be defined as above. Suppose that $0 \leq j, l \leq q-2$, then $jq + l + q + 1 \equiv 0 \pmod{n'}$ if and only if j, l satisfy one of the following three conditions: $j = l = \frac{s(q-1)}{h} - 1$, where s is even and $2 \leq s \leq h$; $j = \frac{(s-1)(q-1)}{h} + \tau - 1, l = \frac{(s-1)(q-1)}{h} + \tau + \frac{q-1}{2}$ where s is odd and $1 \leq s \leq \frac{h}{2} - 1$; or $j = \frac{(s-1)(q-1)}{h} + \tau, l = \frac{(s-1)(q-1)}{h} + \tau - \frac{q+1}{2}$ where s is odd and $\frac{h}{2} + 1 \leq s \leq h - 1$.*

Proof Suppose that $jq+l+q+1 \equiv 0 \pmod{n'}$. Note that $0 \leq j, l < q-2$, we have $q + 1 \leq jq + l + q + 1 < (q-1)(q+1)$, then there exists an integer s such that

$$jq + l = sn' - q - 1,$$

where $1 \leq s \leq h$. If s is odd, then

$$\begin{aligned} jq + l &= \left[\frac{(s-1)(q-1)}{h} - 1\right]q + \frac{(s-1)(q-1)}{h} + \frac{q^2-1}{h} - 1 \\ &= \left[\frac{(s-1)(q-1)}{h} - 1\right]q + \frac{(s-1)(q-1)}{h} + \frac{2(q-1)}{h} \cdot \frac{q+1}{2} - 1 \\ &= \left[\frac{(s-1)(q-1)}{h} + \tau - 1\right]q + \frac{(s-1)(q-1)}{h} + \frac{q-1}{2} + \tau. \end{aligned}$$

There are two cases.

Case 1 If $1 \leq s \leq \frac{h}{2}-1$, then

$$\begin{aligned}\frac{q+1}{2}+\tau-1 &\leq \frac{(s-1)(q-1)}{h}+\frac{q-1}{2}+\tau\\ &\leq q-2-\tau.\end{aligned}$$

It follows that $j=\frac{(s-1)(q-1)}{h}+\tau-1$ and $l=\frac{(s-1)(q-1)}{h}+\tau+\frac{q-1}{2}$.
Case 2 If $\frac{h}{2}+1 \leq s \leq h-1$, then

$$\begin{aligned}q+\tau-1 &\leq \frac{(s-1)(q-1)}{h}+\frac{q-1}{2}+\tau\\ &\leq q+\frac{q-1}{2}-\tau-2<2q.\end{aligned}$$

It follows that $j=\frac{(s-1)(q-1)}{h}+\tau$ and $l=\frac{(s-1)(q-1)}{h}+\tau-\frac{q+1}{2}$.
If s is even, then

$$jq+l=\left[\frac{s(q-1)}{h}-1\right]q+\left[\frac{s(q-1)}{h}-1\right].$$

Notice that $2 \leq s \leq h$, then we have

$$j=l=\frac{s(q-1)}{h}-1.$$

□

By using Lemma 2, we can obtain the GRS codes with the following parameters.

Lemma 3 *Let q, h, τ, n' be defined as above, and $n=n'(\frac{h}{4}+t)$, $(1 \leq t \leq \frac{h}{4})$. Then, there exist some GRS codes with following parameters:*

(1) *Let t_0, k be integers, where $0 \leq t_0 \leq t-2 (t \geq 2)$ and $\frac{q-1}{2}+\tau+\frac{2t_0(q-1)}{h} \leq k \leq \frac{q-1}{2}+\tau+\frac{2(t_0+1)(q-1)}{h}-1$, the code C has parameters $[n, k, n-k+1]$ and $Hull_h(C)$ has dimension $k-2t_0-2$.*
(2) *Let t_0, k be integers, where $0 \leq t_0 \leq \frac{h}{4}-t$ and $\frac{q-1}{2}+\tau+\frac{2(t+t_0-1)(q-1)}{h} \leq k \leq \frac{q-1}{2}+2\tau+\frac{2(t_0+t-1)(q-1)}{h}-1$, the code C has parameters $[n, k, n-k+1]$ and $Hull_h(C)$ has dimension $k-2t-3t_0$.*
(3) *Let t_0, k be integers, where $0 \leq t_0 \leq \frac{h}{4}-t-1$ and $\frac{q-1}{2}+2\tau+\frac{2(t+t_0-1)(q-1)}{h} \leq k \leq \frac{q-1}{2}+\tau+\frac{2(t_0+t)(q-1)}{h}-1$, the code C has parameters $[n, k, n-k+1]$ and $Hull_h(C)$ has dimension $k-2t-3t_0-1$.*

Proof Let γ, ω be defined as above. Set

$$\begin{aligned}&\boldsymbol{a}=(1, \gamma, \ldots, \gamma^{n'-1}, \omega, \omega\gamma, \ldots, \omega\gamma^{n'-1}, \ldots, \omega^{\frac{h}{4}+t}, \omega^{\frac{h}{4}+t}\gamma, \ldots, \omega^{\frac{h}{4}+t}\gamma^{n'-1}),\\ &\boldsymbol{v}=(u_0, u_0\gamma, \ldots, u_0\gamma^{n'-1}, u_1, u_1\gamma, \ldots, u_1\gamma^{n'-1}, \ldots, u_{\frac{h}{4}+t}, u_{\frac{h}{4}+t}\gamma,\\ &\quad \ldots, u_{\frac{h}{4}+t}\gamma^{n'-1}),\end{aligned}$$

where $u_0, u_1, \ldots, u_{\frac{h}{4}+t-1}$ are $\frac{h}{4}+t$ nonzero elements in $\mathrm{GF}(q^2)$.

We will prove that there exist $\frac{h}{4}+t$ nonzero elements $u_0, u_1, \cdots, u_{\frac{h}{4}+t}$ in $\mathrm{GF}(q^2)$ such that $\mathcal{GRS}_k(\boldsymbol{a}, \boldsymbol{v})$ has parameters above. In fact,

$$(\boldsymbol{a}^{qj+l}, \boldsymbol{v}^{q+1})_E = \sum_{i=0}^{\frac{h}{4}+t-1} \omega^{i(qj+l)} u_i^{q+1} \sum_{r=0}^{n'-1} \gamma^{r(qj+l+q+1)}.$$

Noticing that the order of γ is n', then

$$\sum_{r=0}^{n'-1} \gamma^{r\ell} = \begin{cases} 0 & \text{if } n' \nmid \ell, \\ n' & \text{if } n' \mid \ell. \end{cases}$$

It follows that $(\boldsymbol{a}^{qj+l}, \boldsymbol{v}^{q+1})_E = 0$ except for $n' \mid (qj+l+q+1)$. Now we assume that $0 \le j, l \le k-1$ such that $n' \mid (qj+l+q+1)$. From Lemma 2, if s is even and $2 \le s \le h$, then $j = l = \frac{s(q-1)}{h} - 1$; if s is odd and $1 \le s \le \frac{h}{2} - 1$, then $j = \frac{(s-1)(q-1)}{h} + \tau - 1, l = \frac{(s-1)(q-1)}{h} + \tau + \frac{q-1}{2}$; if s is odd and $\frac{h}{2}+1 \le s \le h-1$, then $j = \frac{(s-1)(q-1)}{h} + \tau, l = \frac{(s-1)(q-1)}{h} + \tau - \frac{q+1}{2}$.

Then, we will prove that $(\boldsymbol{a}^{qj+l}, \boldsymbol{v}^{q+1})_E = 0$ for $j = l = \frac{s(q-1)}{h} - 1$, where s is even and $1 \le \frac{s}{2} \le \frac{h}{4} + t - 1$. Then

$$(\boldsymbol{a}^{qj+l}, \boldsymbol{v}^{q+1})_E = n' \sum_{i=0}^{\frac{h}{4}+t-1} \omega^{i(q+1)[\frac{s}{2}\cdot\frac{2(q-1)}{h}-1]} u_i^{q+1}.$$

It suffices to prove that the system of $\frac{h}{4}+t-1$ equations $\sum_{i=0}^{\frac{h}{4}+t-1} \omega^{2imn-i(q+1)} u_i^{q+1} = 0$ for $1 \le m \le \frac{h}{4} + t - 1$ has a solution in $(\mathrm{GF}(q^2)^*)^{\frac{h}{4}+1}$. Take $y_i = (\omega^{-i} u_i)^{(q+1)}$ for $0 \le i \le \frac{h}{4} + t - 1$, then $y_i \in \mathrm{GF}(q)^*$. It suffices to prove that the system of the equations

$$\begin{pmatrix} 1 & \beta & \cdots & \beta^{\frac{h}{4}+t-1} \\ 1 & \beta^2 & \cdots & \beta^{2\cdot\frac{h}{4}+t-1} \\ \vdots & \vdots & \cdots & \vdots \\ 1 & \beta^{\frac{h}{4}+t-1} & \cdots & \beta^{(\frac{h}{4}+t-1)\cdot(\frac{h}{4}+t-1)} \end{pmatrix} \begin{pmatrix} y_0 \\ y_1 \\ \vdots \\ y_{\frac{h}{4}+t-1} \end{pmatrix} = 0$$

has a solution in $(\mathrm{GF}(q)^*)^{\frac{h}{4}+t}$, where $\beta = \omega^{2n'}$.

Because of $\mathrm{ord}(\beta) = \frac{h}{2}$, $\mathrm{ord}(\beta)$ divides $q-1$. Hence, $\beta \in \mathrm{GF}(q)^*$. Put $f(x) = \prod_{s=1}^{\frac{h}{4}+t-1}(x - \beta^s)$, then $f(x) \in \mathrm{GF}(q)[x]$ and $f(x) \mid (x^{\frac{h}{2}} - 1)$. Considering a q-ary cyclic code $\mathcal{C}$ of length $\frac{h}{2}$ with generator polynomial $f(x)$, it is easy for us to check that $\mathcal{C}$ is a $[\frac{h}{2}, \frac{h}{4} - t + 1, \frac{h}{4} + t]$ MDS code. Hence, all coefficients of $f(x) =$

$x^{\frac{h}{4}+t-1}+a_{\frac{h}{4}+t-2}x^{\frac{h}{4}+t-2}+\cdots+a_0$ are all nonzero. That means that the last system has a solution

$$(y_0,\ldots,y_{\frac{h}{4}+t-2},y_{\frac{h}{4}+t-1})=(a_0,\ldots,a_{\frac{h}{4}+t-2},1)\in(\mathrm{GF}(q)^*)^n.$$

For each $0\le i\le\frac{h}{4}+t-2$, since $a_i\in\mathrm{GF}(q)^*$, there exists $b_i\in\mathrm{GF}(q^2)^*$ such that $a_i=b_i^{q+1}$. Take

$$\boldsymbol{u}=(u_0,u_1,\ldots,u_{\frac{h}{4}+t-1})=(b_0,\omega b_1,\ldots,\omega^{\frac{h}{4}+t-2}b_{\frac{h}{4}+t-2},\omega^{\frac{h}{4}+t-1}),$$

then $(\boldsymbol{a}^{qj+l},\boldsymbol{v}^{q+1})_E=0$ for $j=l=\frac{s(q-1)}{h}-1$, where s is even and $0<\frac{s}{2}<\frac{h}{4}+t$.

Let $G=G_k(\boldsymbol{a},\boldsymbol{v})=\begin{pmatrix}\boldsymbol{g_0}\\ \boldsymbol{g_1}\\ \vdots\\ \boldsymbol{g_{k-1}}\end{pmatrix}$. Then $GG^\dagger=\begin{pmatrix}\boldsymbol{g_0g_0^\dagger} & \boldsymbol{g_0g_1^\dagger} & \cdots & \boldsymbol{g_0g_{k-1}^\dagger}\\ \boldsymbol{g_1g_0^\dagger} & \boldsymbol{g_1g_1^\dagger} & \cdots & \boldsymbol{g_1g_{k-1}^\dagger}\\ \vdots & \vdots & \cdots & \vdots\\ \boldsymbol{g_{k-1}g_0^\dagger} & \boldsymbol{g_{k-1}g_1^\dagger} & \cdots & \boldsymbol{g_{k-1}g_{k-1}^\dagger}\end{pmatrix}$.

In the first case: $0\le t_0\le t-2(t\ge 2)$ and $\frac{q-1}{2}+\tau+\frac{2t_0(q-1)}{h}\le k\le\frac{q-1}{2}+\tau+\frac{2(t_0+1)(q-1)}{h}-1$. It is easy to prove that $(\boldsymbol{a}^{qj+l},\boldsymbol{v}^{q+1})_E\ne 0$ if and only if $j=\frac{(s-1)(q-1)}{h}+\tau-1$, $l=\frac{(s-1)(q-1)}{h}+\tau+\frac{q-1}{2}$ where s is odd and $1\le s\le 2t_0+1$; or $j=\frac{(s-1)(q-1)}{h}+\tau$, $l=\frac{(s-1)(q-1)}{h}+\tau-\frac{q+1}{2}$ where s is odd and $\frac{h}{2}+1\le s\le\frac{h}{2}+2t_0+1$. There are $2t_0+2$ pairs (j,l) such that $(\boldsymbol{a}^{qj+l},\boldsymbol{v}^{q+1})_E\ne 0$.

By $\boldsymbol{g_j g_l^\dagger}=(\boldsymbol{a}^{qj+l},\boldsymbol{v}^{q+1})_E$, $\dim(\mathrm{Hull}_h(C))=k-\mathrm{rank}(GG^\dagger)=k-2t_0-2$.

In the second case: $0\le t_0\le\frac{h}{4}-t$ and $\frac{q-1}{2}+\tau+\frac{2(t+t_0-1)(q-1)}{h}\le k\le\frac{q-1}{2}+2\tau+\frac{2(t_0+t-1)(q-1)}{h}-1$. It is easy to prove that $(\boldsymbol{a}^{qj+l},\boldsymbol{v}^{q+1})_E\ne 0$ if and only if $j=l=\frac{s(q-1)}{h}-1$, where s is even and $\frac{h}{2}+2t<s\le\frac{h}{2}+2t+2t_0$; $j=\frac{(s-1)(q-1)}{h}+\tau-1$, $l=\frac{(s-1)(q-1)}{h}+\tau+\frac{q-1}{2}$ where s is odd and $1\le s\le 2t_0+2t-1$; or $j=\frac{(s-1)(q-1)}{h}+\tau$, $l=\frac{(s-1)(q-1)}{h}+\tau-\frac{q+1}{2}$ where s is odd and $\frac{h}{2}+1\le s\le\frac{h}{2}+2t_0+2t-1$. There are $3t_0+2t$ pairs (j,l) such that $(\boldsymbol{a}^{qj+l},\boldsymbol{v}^{q+1})_E\ne 0$.

By $\boldsymbol{g_j g_l^\dagger}=(\boldsymbol{a}^{qj+l},\boldsymbol{v}^{q+1})_E$, $\dim(\mathrm{Hull}_h(C))=k-\mathrm{rank}(GG^\dagger)=k-3t_0-2t$.

In the last case: $0\le t_0\le\frac{h}{4}-t-1$ and $\frac{q-1}{2}+2\tau+\frac{2(t+t_0-1)(q-1)}{h}\le k\le\frac{q-1}{2}+\tau+\frac{2(t_0+t)(q-1)}{h}-1$. It is easy to prove that $(\boldsymbol{a}^{qj+l},\boldsymbol{v}^{q+1})_E\ne 0$ if and only if $j=l=\frac{s(q-1)}{h}-1$, where s is even and $\frac{h}{2}+2t\le s\le\frac{h}{2}+2t+2t_0$; $j=\frac{(s-1)(q-1)}{h}+\tau-1$, $l=\frac{(s-1)(q-1)}{h}+\tau+\frac{q-1}{2}$ where s is odd and $1\le s\le 2t_0+2t-1$; or $j=\frac{(s-1)(q-1)}{h}+\tau$, $l=\frac{(s-1)(q-1)}{h}+\tau-\frac{q+1}{2}$ where s is odd and $\frac{h}{2}+1\le s\le\frac{h}{2}+2t_0+2t-1$. There are $3t_0+2t+1$ pairs (j,l) such that $(\boldsymbol{a}^{qj+l},\boldsymbol{v}^{q+1})_E\ne 0$.

By $\boldsymbol{g_j g_l^\dagger}=(\boldsymbol{a}^{qj+l},\boldsymbol{v}^{q+1})_E$, $\dim(\mathrm{Hull}_h(C))=k-\mathrm{rank}(GG^\dagger)=k-3t_0-2t-1$. □

Similar to (2) and (3) of Lemma 3, when $n=n'(\frac{h}{4}-t)$, $(0\le t\le\frac{h}{4}-1)$, we can get the following lemma.

Lemma 4 *Let q, h, τ, n' be defined as above, and $n = n'(\frac{h}{4} - t)$, $(0 \leq t \leq \frac{h}{4} - 1)$. Then, there exist some GRS codes with following parameters:*

(1) *Let t_0, k be integers, where $0 \leq t_0 \leq \frac{h}{4} - 1$ and $\frac{q-1}{2} + \tau + \frac{2t_0(q-1)}{h} \leq k \leq \frac{q-1}{2} + 2\tau + \frac{2(t_0+1)(q-1)}{h} - 1$, the code C has parameters $[n, k, n-k+1]$, and $Hullh(C)$ has dimension $k - t - 3t_0 - 3$.*

(2) *Let t_0, k be integers, where $0 \leq t_0 \leq \frac{h}{4} - 2$ and $\frac{q-1}{2} + 2\tau + \frac{2t_0(q-1)}{h} \leq k \leq \frac{q-1}{2} + \tau + \frac{2(t_0+1)(q-1)}{h} - 1$, the code C has parameters $[n, k, n-k+1]$, and $Hullh(C)$ has dimension $k - t - 3t_0 - 4$.*

From Lemmas 3, 4 and Proposition 3, we can get the following theorem.

Theorem 1 *Let q, n', h, τ be defined as above and k be an integer. Then, there exist $[[n, n-2k+c, k+1; c]]_q$ EAQMDS codes and $[[n, c, n-k+1; n-2k+c]]_q$ EAQECCs, if one of the following conditions holds:*

(1) *$n = n'(\frac{h}{4}+t)$, $c = 2t_0+2$, and $\frac{q-1}{2}+\tau+\frac{2t_0(q-1)}{h} \leq k \leq \frac{q-1}{2}+\tau+\frac{2(t_0+1)(q-1)}{h}-1$, where $2 \leq t \leq \frac{h}{4}$ and $0 \leq t_0 \leq t-2$.*

(2) *$n = n'(\frac{h}{4} + t)$, $c = 2t + 3t_0$, and $\frac{q-1}{2} + \tau + \frac{2(t+t_0-1)(q-1)}{h} \leq k \leq \frac{q-1}{2} + 2\tau + \frac{2(t_0+t-1)(q-1)}{h} - 1$, where $1 \leq t \leq \frac{h}{4}$ and $0 \leq t_0 \leq \frac{h}{4} - t$.*

(3) *$n = n'(\frac{h}{4} + t)$, $c = 2t + 3t_0 + 1$, $\frac{q-1}{2} + 2\tau + \frac{2(t+t_0-1)(q-1)}{h} \leq k \leq \frac{q-1}{2} + \tau + \frac{2(t_0+t)(q-1)}{h} - 1$, where $1 \leq t \leq \frac{h}{4}$ and $0 \leq t_0 \leq \frac{h}{4} - t - 1$.*

(4) *$n = n'(\frac{h}{4} - t)$, $c = t + 3t_0 + 3$, and $\frac{q-1}{2} + \tau + \frac{2t_0(q-1)}{h} \leq k \leq \frac{q-1}{2} + 2\tau + \frac{2(t_0+1)(q-1)}{h} - 1$, where $0 \leq t \leq \frac{h}{4} - 1$ and $0 \leq t_0 \leq \frac{h}{4} - 1$.*

(5) *$n = n'(\frac{h}{4} - t)$, $c = t + 3t_0 + 4$ and $\frac{q-1}{2} + 2\tau + \frac{2t_0(q-1)}{h} \leq k \leq \frac{q-1}{2} + \tau + \frac{2(t_0+1)(q-1)}{h} - 1$, where $0 \leq t \leq \frac{h}{4} - 1$ and $0 \leq t_0 \leq \frac{h}{4} - 2$.*

Remark 1 In Theorem 11 of [24], Fang et al. constructed a class of EAQMDS codes with parameters $[[n, k-l, n-k+1; n-k-l]]_q$ and $[[n, n-k-l, k+1; k-l]]_q$, where $n'|(q^2-1)$, $n = tn'$, $n_1 = \frac{n'}{gcd(n', q+1)}$, $1 \leq t \leq \frac{q-1}{n_1}$, $1 \leq k \leq \lfloor \frac{n+q}{q+1} \rfloor$ and $0 \leq l \leq k-1$. Compared with Theorem 1, we find that the range of k is disjoint for the same length. It means that the EAQMDS codes constructed by Theorem 1 are new.

Example 1 Let $q = 29$, $h = 8$, and then $\tau = 3$. Then, there exist $[[n, n-2k+c, k+1; c]]_q$ EAQMDS codes and $[[n, c, n-k+1; n-2k+c]]_q$ EAQECCs, where the values of n, k, c can be found in Table 1.

4 The second construction

Throughout this section, let ω be a primitive element of GF(q^2) and $\gamma = \omega^h$, where $h > 4$ is an even integer such that $\frac{2(q+1)}{h} = 2\tau + 1$ for some $\tau \geq 1$. Let $n' = \frac{q^2-1}{h}$, then γ is a primitive n'-th root of unity.

For our construction, we need the following lemmas.

Table 1 The values of n, k, c in Example 1

n	k	c	n	k	c
420	$17 \leq k \leq 23$	2	210	$17 \leq k \leq 19$	3
420	$24 \leq k \leq 26$	4	210	$20 \leq k \leq 23$	4
315	$17 \leq k \leq 19$	2	210	$17 \leq k \leq 19$	3
315	$20 \leq k \leq 23$	3	210	$20 \leq k \leq 23$	4
315	$24 \leq k \leq 26$	5	210	$24 \leq k \leq 26$	6

Lemma 5 *Let q, h, τ and n' be defined as above. Suppose that $0 \leq j, l \leq q-2$, then $jq + l + q + 1 \equiv 0 \pmod{n'}$ if and only if j, l satisfy one of the following three conditions: $j = \frac{s(q+1)}{h} - 2$ and $l = q - \frac{s(q+1)}{h} - 1$ where s is even and $2 \leq s \leq h$; $j = \frac{(s-1)(q+1)}{h} + \tau - 1$ and $l = \frac{q-3}{2} - \frac{(s-1)(q+1)}{h} - \tau$ where s is odd and $1 \leq s \leq \frac{h}{2} - 1$; or $j = \frac{(s-1)(q+1)}{h} + \tau - 2$ and $l = \frac{3q-3}{2} - \frac{(s-1)(q+1)}{h} - \tau$ where s is odd and $\frac{h}{2} + 1 \leq s \leq h - 1$.*

Proof Suppose that $jq + l + q + 1 \equiv 0 \pmod{n'}$. Note that $0 \leq j, l < q - 2$, we have $q + 1 \leq jq + l + q + 1 < (q-1)(q+1)$, then there exists an integer s such that

$$jq + l = sn' - q - 1,$$

where $1 \leq s \leq h$. If s is odd, then

$$\begin{aligned} jq + l &= \left[\frac{(s-1)(q+1)}{h} - 1\right] q - \frac{(s-1)(q+1)}{h} + \frac{q^2 - 1}{h} - 1 \\ &= \left[\frac{(s-1)(q+1)}{h} - 1\right] q - \frac{(s-1)(q+1)}{h} + \frac{2(q+1)}{h} \cdot \frac{q-1}{2} - 1 \\ &= \left[\frac{(s-1)(q+1)}{h} + \tau - 1\right] q + \left[\frac{q-3}{2} - \frac{(s-1)(q+1)}{h} - \tau\right]. \end{aligned}$$

There are two cases.

Case 1 If $1 \leq s \leq \frac{h}{2} - 1$, then

$$0 < \tau - 1 \leq \frac{q-3}{2} - \frac{(s-1)(q+1)}{h} - \tau \leq q - 3 - \tau < q.$$

It follows that $j = \frac{(s-1)(q+1)}{h} + \tau - 1$ and $l = \frac{q-3}{2} - \frac{(s-1)(q+1)}{h} - \tau$.
Case 2 If $\frac{h}{2} + 1 \leq s \leq h - 1$, then

$$-q < \tau - \frac{q-3}{2} \leq q - 3 - \tau \leq -\tau - 2 < 0.$$

It follows that $j = \frac{(s-1)(q+1)}{h} + \tau - 2$ and $l = \frac{3q-3}{2} - \frac{(s-1)(q+1)}{h} - \tau$.

If s is even, then

$$jq+l=\left[\frac{s(q+1)}{h}-2\right]q+\left[q-\frac{s(q+1)}{h}-1\right].$$

Notice that $2\le s\le h$, then we have

$$j=\frac{s(q+1)}{h}-2,\ l=q-\frac{s(q+1)}{h}-1.$$

□

4.1 The case $q\equiv 3\ (mod\ 4)$

By using Lemma 5, we can obtain the GRS codes with the following parameters. Firstly, we consider the case that $q\equiv 3\ (mod\ 4)$.

Lemma 6 *Let $q\equiv 3\,(mod\,4)$ and h,τ,n' be defined as above, and $n=n'(\frac{h}{4}+2t+2)$ for $0\le t\le\frac{h}{8}-1$. Then, there exist some GRS codes with following parameters:*

(1) *Let t_1,t_2,k be integers, where $0\le t_1\le t_2\le t$ and $\frac{q+3}{2}+\tau+\frac{2(t_1+t_2)(q+1)}{h}\le k\le\frac{q-3}{2}+\tau+\frac{2(t_1+t_2+1)(q+1)}{h}$, the code C has parameters $[n,k,n-k+1]$, and $Hull_h(C)$ has dimension $k-2t_1-2t_2-2$.*

(2) *Let t_1,t_2,k be integers, where $0\le t_1\le t_2\le\frac{h}{8}-1-t$ and $\frac{q+3}{2}+\tau+\frac{2(2t+t_1+t_2)(q+1)}{h}\le k\le\frac{q-3}{2}+\tau+\frac{2(2t+t_1+t_2+1)(q+1)}{h}$, the code C has parameters $[n,k,n-k+1]$, and $Hull_h(C)$ has dimension $k-4t-4t_1-2t_2-2$.*

Proof If $q\equiv 3\ (mod\ 4)$, then $8|h$. Let γ,ω be defined as above. Set

$$\begin{aligned}\boldsymbol{a}&=(1,\gamma,\ldots,\gamma^{n'-1},\omega,\ldots,\omega\gamma^{n'-1},\ldots,\omega^{\frac{h}{4}+2t+1},\ldots,\omega^{\frac{h}{4}+2t+1}\gamma^{n'-1}),\\ \boldsymbol{v}&=(u_0,u_0\gamma,\ldots,u_0\gamma^{n'-1},u_1,\ldots,u_1\gamma^{n'-1},\ldots,u_{\frac{h}{4}+2t+1},\ldots,u_{\frac{h}{4}+2t+1}\gamma^{n'-1}),\end{aligned}$$

where $u_0,u_1,\ldots,u_{\frac{h}{4}+2t+1}$ are $\frac{h}{4}+2t+2$ nonzero elements in $\mathrm{GF}(q^2)$.

We will prove that there exist $\frac{h}{4}+2t+2$ nonzero elements $u_0,u_1,\ldots,u_{\frac{h}{4}+2t+1}$ in $\mathrm{GF}(q^2)$ such that $G=G_{k,k_1}(\boldsymbol{a},\boldsymbol{v})$ has parameters above. In fact,

$$(\boldsymbol{a}^{qj+l},\boldsymbol{v}^{q+1})_E=\sum_{i=0}^{\frac{h}{4}+2t+1}\omega^{i(qj+l)}u_i^{q+1}\sum_{r=0}^{n'-1}\gamma^{r(qj+l+q+1)}.$$

Notice that the order of γ is n', then

$$\sum_{r=0}^{n'-1}\gamma^{r\ell}=\begin{cases}0 & \text{if } n'\nmid\ell,\\ n' & \text{if } n'\mid\ell.\end{cases}$$

It follows that $(\boldsymbol{a}^{qj+l}, \boldsymbol{v}^{q+1})_E = 0$ except for $n' \mid (qj+l+q+1)$. Now assume that $0 \le j, l \le k-1$ such that $n' \mid (qj+l+q+1)$. From Lemma 5, if s is even and $2 \le s \le h$, then $j = \frac{s(q+1)}{h} - 2$, $l = q - \frac{s(q+1)}{h} - 1$; if s is odd and $1 \le s \le \frac{h}{2} - 1$, then $j = \frac{(s-1)(q+1)}{h} + \tau - 1$, $l = \frac{q-3}{2} - \frac{(s-1)(q+1)}{h} - \tau$; if s is odd and $\frac{h}{2} + 1 \le s \le h-1$, then $j = \frac{(s-1)(q+1)}{h} + \tau - 2$, $l = \frac{3q-3}{2} - \frac{(s-1)(q+1)}{h} - \tau$.

Hence, we will prove that $(\boldsymbol{a}^{qj+l}, \boldsymbol{v}^{q+1})_E = 0$ for $j = \frac{s(q+1)}{h} - 2$ and $l = q - \frac{s(q+1)}{h} - 1$, where s is even and $\frac{h}{4} - 2t \le s \le \frac{3h}{4} + 2t$, which means $\frac{h}{8} - t \le \frac{s}{2} \le \frac{3h}{8} + t$ and

$$(\boldsymbol{a}^{qj+l}, \boldsymbol{v}^{q+1})_E = n' \sum_{i=0}^{\frac{h}{4}+2t+1} \omega^{i[\frac{s(q^2-1)}{h} - q - 1)]} u_i^{q+1}.$$

Let $m = \frac{s}{2} - \frac{h}{2}$, It suffices to prove that the system of $\frac{h}{4} + 2t + 1$ equations $\sum_{i=0}^{\frac{h}{4}+2t+1} \omega^{2i(m+\frac{h}{2})n' - i(q+1)} u_i^{q+1} = \sum_{i=0}^{\frac{h}{4}+2t+1} (\omega^{2n'})^{im} (\omega^{i(q-2)} u_i)^{q+1} = 0$ for $-\frac{h}{8} - t \le m \le \frac{h}{8} + t$ has a solution in $(\mathrm{GF}(q^2)^*)^{\frac{h}{4}+2t+2}$. Take $y_i = (\omega^{i(q-2)} u_i)^{(q+1)}$ for $0 \le i \le \frac{h}{4} + 2t + 1$, then $y_i \in \mathrm{GF}(q)^*$. That is to say, it suffices to prove that the system of the equations

$$\begin{pmatrix} 1 & \beta^{-\frac{h}{8}-t} & \cdots & \beta^{(-\frac{h}{8}-t)\cdot(\frac{h}{4}+2t+1)} \\ \vdots & \vdots & \cdots & \vdots \\ 1 & 1 & \cdots & 1 \\ \vdots & \vdots & \cdots & \vdots \\ 1 & \beta^{\frac{h}{8}+t} & \cdots & \beta^{(\frac{h}{8}+t)\cdot(\frac{h}{4}+2t+1)} \end{pmatrix} \begin{pmatrix} y_0 \\ y_1 \\ \vdots \\ y_{\frac{h}{4}+2t+1} \end{pmatrix} = 0$$

has a solution in $(\mathrm{GF}(q)^*)^{\frac{h}{4}+2t+2}$, where $\beta = \omega^{2n'}$.

Notice that $\mathrm{ord}(\beta) = \frac{h}{2}$, then $\mathrm{ord}(\beta)$ divides $q+1$. Hence, $\beta^{q+1} = \beta^q \beta = 1$, then $\beta^{-1} = \beta^q$ and $\beta^{-1} + \beta \in \mathrm{GF}(q)^*$. Put $f(x) = \prod_{m=-\frac{h}{8}-t}^{\frac{h}{8}+t} (x - \beta^m)$, then $f(x) \in \mathrm{GF}(q)[x]$ and $f(x) \mid (x^{\frac{h}{2}} - 1)$. Consider a q-ary cyclic code $\mathcal{C}$ of length $\frac{h}{2}$ with generator polynomial $f(x)$. It is easy to check that $\mathcal{C}$ is a $[\frac{h}{2}, \frac{h}{4} - 2t - 1, \frac{h}{4} + 2t + 2]$ MDS code. Hence, all coefficients of $f(x) = x^{\frac{h}{4}+2t+1} + a_{\frac{h}{4}+2t} x^{\frac{h}{4}+2t} + \cdots + a_0$ are all nonzero. That is, the last system has a solution

$$(y_0, \ldots, y_{\frac{h}{4}+2t}, y_{\frac{h}{4}+2t+1}) = (a_0, \ldots, a_{\frac{h}{4}+2t}, 1) \in (\mathrm{GF}(q)^*)^{\frac{h}{4}+2t+2}.$$

For each $0 \le i \le \frac{h}{4} + 2t + 1$, since $a_i \in \mathrm{GF}(q)^*$, there exists $b_i \in \mathrm{GF}(q^2)^*$ such that $a_i = b_i^{(q+1)}$. Therefore, taking

$$\boldsymbol{u} = (u_0, u_1, \ldots, u_{\frac{h}{4}+2t+1}) = (b_0, \omega b_1, \ldots, \omega^{\frac{h}{4}+2t+1} b_{\frac{h}{4}+2t+1}, \omega^{\frac{h}{4}+2t+1}),$$

we have $(\boldsymbol{a}^{qj+l}, \boldsymbol{v}^{q+1})_E = 0$ for $j = \frac{s(q+1)}{h} - 2$ and $l = q - \frac{s(q+1)}{h} - 1$, where s is even and $\frac{h}{4} - 2t \leq s \leq \frac{3h}{4} + 2t$.

Let $G = G_{k,k_1}(\boldsymbol{a}, \boldsymbol{v}) = \begin{pmatrix} \boldsymbol{g}_{k_1} \\ \boldsymbol{g}_{k_1+1} \\ \vdots \\ \boldsymbol{g}_{k_1+k-1} \end{pmatrix}$.

Then, $GG^\dagger = \begin{pmatrix} \boldsymbol{g}_{k_1}\boldsymbol{g}_{k_1}^\dagger & \boldsymbol{g}_{k_1}\boldsymbol{g}_{k_1+1}^\dagger & \cdots & \boldsymbol{g}_{k_1}\boldsymbol{g}_{k_1+k-1}^\dagger \\ \boldsymbol{g}_{k_1+1}\boldsymbol{g}_{k_1}^\dagger & \boldsymbol{g}_{k_1+1}\boldsymbol{g}_{k_1+1}^\dagger & \cdots & \boldsymbol{g}_{k_1+1}\boldsymbol{g}_{k_1+k-1}^\dagger \\ \vdots & \vdots & \cdots & \vdots \\ \boldsymbol{g}_{k_1+k-1}\boldsymbol{g}_{k_1}^\dagger & \boldsymbol{g}_{k_1+k-1}\boldsymbol{g}_{k_1+1}^\dagger & \cdots & \boldsymbol{g}_{k_1+k-1}\boldsymbol{g}_{k_1+k-1}^\dagger \end{pmatrix}$.

In the first case: $0 \leq t_1 \leq t_2 \leq t$, let $\frac{q-3}{4} - \tau - \frac{2t_1(q+1)}{h} \leq k_1 \leq \frac{q+1}{4} - 2 - \frac{2t_1(q+1)}{h}$ and $\frac{3q-1}{4} + \tau - 1 + \frac{2t_2(q+1)}{h} \leq k_1 + k - 1 \leq \frac{3q-1}{4} - 3 + \frac{2(t_2+1)(q+1)}{h}$. Then the range of k satisfies $\frac{q+3}{2} + \tau + \frac{2(t_1+t_2)(q+1)}{h} \leq k \leq \frac{q-3}{2} + \tau + \frac{2(t_1+t_2+1)(q+1)}{h}$. It is easy to prove that $(\boldsymbol{a}^{qj+l}, \boldsymbol{v}^{q+1})_E \neq 0$ if and only if $j = \frac{(s-1)(q+1)}{h} + \tau - 1$ and $l = \frac{q-3}{2} - \frac{(s-1)(q+1)}{h} - \tau$, where s is odd and $\frac{h}{4} + 1 - 2t_1 \leq s \leq \frac{h}{4} - 1 + 2t_1$; or $j = \frac{(s-1)(q+1)}{h} + \tau - 2$, $l = \frac{3q-3}{2} - \frac{(s-1)(q+1)}{h} - \tau$, where s is odd and $\frac{3h}{4} - 2t_2 - 1 \leq s \leq \frac{3h}{4} + 2t_2 + 1$. There are $2t_1 + 2t_2 + 2$ pairs (j, l) such that $(\boldsymbol{a}^{qj+l}, \boldsymbol{v}^{q+1})_E \neq 0$.

Since $\boldsymbol{g}_j\boldsymbol{g}_l^\dagger = (\boldsymbol{a}^{qj+l}, \boldsymbol{v}^{q+1})_E$, then $\dim(\mathrm{Hull}_h(C)) = k - \mathrm{rank}(GG^\dagger) = k - 2t_1 - 2t_2 - 2$.

In the second case: $0 \leq t_1 \leq t_2 \leq \frac{h}{8} - 1 - t$, let $\frac{q-3}{4} - \tau - \frac{2(t+t_1)(q+1)}{h} \leq k_1 \leq \frac{q+1}{4} - 2 - \frac{2(t+t_1)(q+1)}{h}$ and $\frac{3q-1}{4} + \tau - 1 + \frac{2(t+t_2)(q+1)}{h} \leq k_1 + k - 1 \leq \frac{3q-1}{4} - 3 + \frac{2(t+t_2+1)(q+1)}{h}$. Then the range of k satisfies $\frac{q+3}{2} + \tau + \frac{2(2t+t_1+t_2)(q+1)}{h} \leq k \leq \frac{q-3}{2} + \tau + \frac{2(2t+t_1+t_2+1)(q+1)}{h}$. It is easy to prove that $(\boldsymbol{a}^{qj+l}, \boldsymbol{v}^{q+1})_E \neq 0$ if and only if $j = \frac{(s-1)(q+1)}{h} + \tau - 1$ and $l = \frac{q-3}{2} - \frac{(s-1)(q+1)}{h} - \tau$, where s is odd and $\frac{h}{4} + 1 - 2t - 2t_1 \leq s \leq \frac{h}{4} - 1 + 2t + 2t_1$; $j = \frac{(s-1)(q+1)}{h} + \tau - 2, l = \frac{3q-3}{2} - \frac{(s-1)(q+1)}{h} - \tau$, where s is odd and $\frac{3h}{4} - 2t - 2t_2 - 1 \leq s \leq \frac{3h}{4} + 2t + 2t_2 + 1$. or $j = \frac{s(q+1)}{h} - 2, l = q - \frac{s(q+1)}{h} - 1$, where s is even and $0 < |s - \frac{h}{2}| - \frac{h}{4} - 2t \leq 2t_1$; There are $4t + 4t_1 + 2t_2 + 2$ pairs (j, l) such that $(\boldsymbol{a}^{qj+l}, \boldsymbol{v}^{q+1})_E \neq 0$.

By $\boldsymbol{g}_j\boldsymbol{g}_l^\dagger = (\boldsymbol{a}^{qj+l}, \boldsymbol{v}^{q+1})_E$, then $\dim(\mathrm{Hull}_h(C)) = k - \mathrm{rank}(GG^\dagger) = k - 4t - 4t_1 - 2t_2 - 2$. □

Similar to (2) of Lemma 6, when $n = n'(\frac{h}{4} - 2t)$, $(0 \leq t \leq \frac{h}{4} - 1)$, we can get the following lemma.

Lemma 7 *Let $q \equiv 3$ (mod 4) and h, τ, n' be defined as above, and $n = n'(\frac{h}{4} - 2t)$, for $0 \leq t \leq \frac{h}{8} - 1$. Then there exist some GRS codes with following parameters:*

Let t_1, t_2, k be integers, where $0 \leq t_1 \leq t_2 \leq \frac{h}{8} - 1$ and $\frac{q+3}{2} + \tau + \frac{2(t_1+t_2)(q+1)}{h} \leq k \leq \frac{q-3}{2} + \tau + \frac{2(t_1+t_2+1)(q+1)}{h}$, the code C has parameters $[n, k, n-k+1]$, and $Hull_h(C)$ has dimension $k - 2t - 2t_1 - 2t_2 - 4$.

Table 2 The values of n, k, c in Example 2

n	k	c	n	k	c	n	k	c
290	$33 \le k \le 35$	8	290	$38 \le k \le 40$	10	290	$43 \le k \le 45$	12
290	$43 \le k \le 45$	14	290	$48 \le k \le 50$	16	290	$53 \le k \le 55$	20
580	$33 \le k \le 35$	6	580	$38 \le k \le 40$	8	580	$43 \le k \le 45$	10
580	$43 \le k \le 45$	12	580	$48 \le k \le 50$	14	580	$53 \le k \le 55$	18
870	$33 \le k \le 35$	4	870	$38 \le k \le 40$	6	870	$43 \le k \le 45$	8
870	$43 \le k \le 45$	10	870	$48 \le k \le 50$	12	870	$53 \le k \le 55$	16
1160	$33 \le k \le 35$	2	1160	$38 \le k \le 40$	4	1160	$43 \le k \le 45$	8
1160	$48 \le k \le 50$	10	1160	$53 \le k \le 55$	14	1450	$33 \le k \le 35$	2
1450	$38 \le k \le 40$	4	1450	$43 \le k \le 45$	6	1450	$48 \le k \le 50$	8
1450	$53 \le k \le 55$	12	1740	$33 \le k \le 35$	2	1740	$38 \le k \le 40$	4
1740	$43 \le k \le 45$	6	1740	$48 \le k \le 50$	8	1740	$53 \le k \le 55$	10

From Lemmas 6, 7 and Proposition 3, we can get the following theorem.

Theorem 2 *Let q, n', h, τ be defined as above and t, t_1, t_2, k be integers. Then, there exist $[[n, n-2k+c, k+1; c]]_q$ EAQMDS codes and $[[n, c, n-k+1; n-2k+c]]_q$ EAQECCs, if one of the following holds:*

(1) $n = n'(\frac{h}{4}+2t+2)$, $c = 2t_1+2t_2+2$, $\frac{q+3}{2}+\tau+\frac{2(t_1+t_2)(q+1)}{h} \le k \le \frac{q-3}{2}+\tau+\frac{2(t_1+t_2+1)(q+1)}{h}$, *where* $0 \le t_1 \le t_2 \le t \le \frac{h}{8}-1$.

(2) $n = n'(\frac{h}{4}+2t+2)$, $c = 4t+4t_1+2t_2+2$, $\frac{q+3}{2}+\tau+\frac{2(2t+t_1+t_2)(q+1)}{h} \le k \le \frac{q-3}{2}+\tau+\frac{2(2t+t_1+t_2+1)(q+1)}{h}$, *where* $0 \le t_1 \le t_2 \le \frac{h}{8}-1-t$.

(3) $n = n'(\frac{h}{4}-2t)$, $c = 2t+4t_1+2t_2+4$, *and* $\frac{q+3}{2}+\tau+\frac{2(t_1+t_2)(q+1)}{h} \le k \le \frac{q-3}{2}+\tau+\frac{2(t_1+t_2+1)(q+1)}{h}$, *where* $0 \le t \le \frac{h}{8}-1$ *and* $0 \le t_1 \le t_2 \le \frac{h}{8}-1$.

Remark 2 Comparing Theorem 2 with Theorem 11 in [24], we find that most of the length is different and the range of k is disjoint for the same length. Comparing Theorem 2 with Corollary 3.4 of [22], we can conclude although some codes have the same parameters, there are still many codes with different parameters. Here is an example.

Example 2 Let $q = 59$ and $h = 24$, then $\tau = 2$. By Theorem 2, there exist $[[n, n-2k+c, k+1; c]]_q$ EAQMDS codes and $[[n, c, n-k+1; n-2k+c]]_q$ EAQECCs, where the values of n, k, c can be found in Table 2.

After removing the same parameters as in Reference [22], we can get the codes in Table 3.

In Theorem 2, the length of those codes is even times of n'. And then we move on to the case where length is positive integer multiples of n'.

Lemma 8 *Let $q \equiv 3 \pmod 4$ and h, τ, n' be defined as above, and $n = tn'$, for $1 \le t \le h$. Then, there exist some GRS codes with following parameters:*

Table 3 The values of n, k, c in Example 2 after deleting the same parameters

n	k	c	n	k	c	n	k	c
290	$34 \leq k \leq 35$	8	290	$38 \leq k \leq 40$	10	290	$43 \leq k \leq 45$	12
290	$43 \leq k \leq 45$	14	290	$48 \leq k \leq 50$	16	290	$53 \leq k \leq 55$	20
580	$38 \leq k \leq 40$	8	580	$43 \leq k \leq 45$	10	580	$43 \leq k \leq 45$	12
580	$48 \leq k \leq 50$	14	580	$53 \leq k \leq 55$	18	870	$43 \leq k \leq 45$	10
870	$48 \leq k \leq 50$	12	870	$53 \leq k \leq 55$	16	1160	40	4
1160	$48 \leq k \leq 50$	10	1160	$53 \leq k \leq 55$	14	1450	35	2
1450	40	4	1450	45	6	1450	50	8
1450	$53 \leq k \leq 55$	12	1740	40	4	1740	45	6
1740	50	8	1740	$54 \leq k \leq 55$	10			

Let t_1, t_2, k be integers, where $0 \leq t_1 \leq t_2 \leq \frac{h}{8} - 1$ and $\frac{q+3}{2} + \tau + \frac{2(t_1+t_2)(q+1)}{h} \leq k \leq \frac{q-3}{2} + \tau + \frac{2(t_1+t_2+1)(q+1)}{h}$, the code C has parameters $[n, k, n-k+1]$, and $Hull_h(C)$ has dimension $k - \frac{h}{4} - 4t_1 - 2t_2 - 3$.

Proof If $q \equiv 3 \ (mod\ 4)$, then $8|h$. Let γ, ω be defined as above. Set

$$\boldsymbol{a} = (1, \gamma, \ldots, \gamma^{n'-1}, \omega, \ldots, \omega\gamma^{n'-1}, \ldots, \omega^{t-1}, \ldots, \omega^{t-1}\gamma^{n'-1}),$$
$$\boldsymbol{v} = (u_0, u_0\gamma, \ldots, u_0\gamma^{n'-1}, u_1, \ldots, u_1\gamma^{n'-1}, \ldots, u_{t-1}, \ldots, u_{t-1}\gamma^{n'-1}).$$

And then, we have

$$(\boldsymbol{a}^{qj+l}, \boldsymbol{v}^{q+1})_E = \sum_{i=0}^{t-1} \omega^{i(qj+l)} u_i^{q+1} \sum_{r=0}^{n'-1} \gamma^{r(qj+l+q+1)}.$$

where $u_0, u_1, \ldots, u_t$ are t nonzero elements in GF(q^2) such that

$$\sum_{i=0}^{t-1} \omega^{i(qj+l)} u_i^{q+1} \neq 0.$$

Notice that the order of γ is n', then

$$\sum_{r=0}^{n'-1} \gamma^{r\ell} = \begin{cases} 0 & \text{if} n' \nmid \ell, \\ n' & \text{if} n' \mid \ell. \end{cases}$$

It follows that $(\boldsymbol{a}^{qj+l}, \boldsymbol{v}^{q+1})_E = 0$ except for $n' \mid (qj + l + q + 1)$. Now assume that $0 \leq j, l \leq k-1$ such that $n' \mid (qj + l + q + 1)$. From Lemma 5, if s is even and $2 \leq s \leq h$, then $j = \frac{s(q+1)}{h} - 2, l = q - \frac{s(q+1)}{h} - 1$; if s is odd and $1 \leq s \leq \frac{h}{2} - 1$, then $j = \frac{(s-1)(q+1)}{h} + \tau - 1, l = \frac{q-3}{2} - \frac{(s-1)(q+1)}{h} - \tau$; if s is odd and $\frac{h}{2} + 1 \leq s \leq h - 1$, then $j = \frac{(s-1)(q+1)}{h} + \tau - 2, l = \frac{3q-3}{2} - \frac{(s-1)(q+1)}{h} - \tau$.

Table 4 The values of n, k, c in Example 3

n	k	c	n	k	c
$145t$	$33 \le k \le 35$	9	$145t$	$38 \le k \le 40$	11
$145t$	$43 \le k \le 45$	13	$145t$	$43 \le k \le 45$	15
$145t$	$48 \le k \le 50$	17	$145t$	$53 \le k \le 55$	21

Let $0 \le t_1 \le t_2 \le \frac{h}{8} - 1$, $\frac{q-3}{4} - \tau - \frac{2t_1(q+1)}{h} \le k_1 \le \frac{q+1}{4} - 2 - \frac{2t_1(q+1)}{h}$ and $\frac{3q-1}{4} + \tau - 1 + \frac{2t_2(q+1)}{h} \le k_1 + k - 1 \le \frac{3q-1}{4} - 3 + \frac{2(t_2+1)(q+1)}{h}$. Then the range of k satisfies $\frac{q+3}{2} + \tau + \frac{2(t_1+t_2)(q+1)}{h} \le k \le \frac{q-3}{2} + \tau + \frac{2(t_1+t_2+1)(q+1)}{h}$. It is easy to prove that $(\boldsymbol{a}^{qj+l}, \boldsymbol{v}^{q+1})_E \ne 0$ if and only if $j = \frac{(s-1)(q+1)}{h} + \tau - 1$ and $l = \frac{q-3}{2} - \frac{(s-1)(q+1)}{h} - \tau$, where s is odd and $\frac{h}{4} + 1 - 2t_1 \le s \le \frac{h}{4} - 1 + 2t_1$; $j = \frac{(s-1)(q+1)}{h} + \tau - 2$, $l = \frac{3q-3}{2} - \frac{(s-1)(q+1)}{h} - \tau$, where s is odd and $\frac{3h}{4} - 2t_2 - 1 \le s \le \frac{3h}{4} + 2t_2 + 1$; or $j = \frac{s(q+1)}{h} - 2$, $l = q - \frac{s(q+1)}{h} - 1$, where s is even and $\frac{h}{4} - 2t_1 \le s \le \frac{3h}{4} + 2t_1$; There are $\frac{h}{4} + 4t_1 + 2t_2 + 3$ pairs (j, l) such that $(\boldsymbol{a}^{qj+l}, \boldsymbol{v}^{q+1})_E \ne 0$.

$$\text{Let } G = G_{k,k_1}(\boldsymbol{a}, \boldsymbol{v}) = \begin{pmatrix} \boldsymbol{g}_{k_1} \\ \boldsymbol{g}_{k_1+1} \\ \vdots \\ \boldsymbol{g}_{k_1+k-1} \end{pmatrix}.$$

By $\boldsymbol{g}_j \boldsymbol{g}_l^\dagger = (\boldsymbol{a}^{qj+l}, \boldsymbol{v}^{q+1})_E$, then $\dim(\mathrm{Hull}_h(C)) = k - \mathrm{rank}(GG^\dagger) = k - \frac{h}{4} - 4t_1 - 2t_2 - 3$. □

From Lemmas 8 and Proposition 3, we can get the following theorem.

Theorem 3 *Let q, n', h, τ be defined as above and t_1, t_2, k be integers. Then, there exist $[[n, n-2k+c, k+1; c]]_q$ EAQMDS codes and $[[n, c, n-k+1; n-2k+c]]_q$ EAQECCs, where $n = tn'$, $c = \frac{h}{4} + 4t_1 + 2t_2 + 3$, and $\frac{q+3}{2} + \tau + \frac{2(t_1+t_2)(q+1)}{h} \le k \le \frac{q-3}{2} + \tau + \frac{2(t_1+t_2+1)(q+1)}{h}$, $1 \le t \le h$, and $0 \le t_1 \le t_2 \le \frac{h}{8} - 1$.*

Remark 3 Comparing Theorem 3 with Theorem 11 in [24], we find that most of the length is different and the range of k is disjoint for the same length. Comparing Theorem 3 with Corollary 3.4 of [22], we can conclude if t is odd, the length are different. This means that the EAQMDS codes constructed in Theorem 3 are new when t is odd.

Example 3 Let $q = 59$ and $h = 24$, then $n' = 145$, $1 \le t \le 24$, and $\tau = 2$. By Theorem 3, there exist $[[n, n-2k+c, k+1; c]]_q$ EAQMDS codes and $[[n, c, n-k+1; n-2k+c]]_q$ EAQECCs, where the values of n, k, c can be found in Table 4.

4.2 The case $q \equiv 1$ (*mod* 4)

In this subsection, we consider the case that $q \equiv 1$ (*mod* 4). Similarly, using Lemma 5, we can obtain the GRS codes with the following parameters.

Lemma 9 *Let $q \equiv 1\ (mod\ 4)$ and h, τ, n' be defined as above, and $n = n'(\frac{h}{4} + 2t + 1)$, $(0 \le t \le \frac{h-4}{8})$. Then, there exist some GRS codes with following parameters:*

(1) *Let t_1, t_2, k be integers, where $0 \le t_1 \le t_2 \le t-1$ and $\frac{q+1}{2} + \frac{2(t_1+t_2)(q+1)}{h} \le k \le \frac{q-3}{2} + \frac{2(t_1+t_2+2)(q+1)}{h}$, the code C has parameters $[n, k, n-k+1]$, and $Hull_h(C)$ has dimension $k - 2t_1 - 2t_2 - 2$.*

(2) *Let t_1, t_2, k be integers, where $0 \le t_1 \le t_2 \le \frac{h-4}{8} - t$ and $\frac{q+1}{2} + \frac{2(2t+t_1+t_2)(q+1)}{h} \le k \le \frac{q+1}{2} - 3 + \frac{2(2t+t_1+t_2+1)(q+1)}{h}$, the code C has parameters $[n, k, n-k+1]$, and $Hull_h(C)$ has dimension $k - 4t - 4t_1 - 2t_2 - 2$.*

Proof If $q \equiv 1\ (mod\ 4)$, then $8|(h-4)$. Let γ, ω be defined as above. Set

$$\boldsymbol{a} = (1, \gamma, \ldots, \gamma^{n'-1}, \omega, \ldots, \omega\gamma^{n'-1}, \ldots, \omega^{\frac{h}{4}+2t+1}, \ldots, \omega^{\frac{h}{4}+2t+1}\gamma^{n'-1}),$$
$$\boldsymbol{v} = (u_0, u_0\gamma, \ldots, u_0\gamma^{n'-1}, u_1, \ldots, u_1\gamma^{n'-1}, \ldots, u_{\frac{h}{4}+2t+1}, \ldots, u_{\frac{h}{4}+2t+1}\gamma^{n'-1}),$$

where $u_0, u_1, \ldots, u_{\frac{h}{4}+2t+1}$ are $\frac{h}{4} + 2t + 2$ nonzero elements in $\mathrm{GF}(q^2)$.

From Lemma 5, we have $(\boldsymbol{a}^{qj+l}, \boldsymbol{v}^{q+1})_E = 0$ if and only if (j, l) takes the following values:

If s is even and $2 \le s \le h$, then $j = \frac{s(q+1)}{h} - 2$, $l = q - \frac{s(q+1)}{h} - 1$; if s is odd and $1 \le s \le \frac{h}{2} - 1$, then $j = \frac{(s-1)(q+1)}{h} + \tau - 1$, $l = \frac{q-3}{2} - \frac{(s-1)(q+1)}{h} - \tau$; if s is odd and $\frac{h}{2} + 1 \le s \le h - 1$, then $j = \frac{(s-1)(q+1)}{h} + \tau - 2$, $l = \frac{3q-3}{2} - \frac{(s-1)(q+1)}{h} - \tau$.

Similar to Lemma 6, we can find $\frac{h}{4} + 2t + 1$ nonzero elements $u_0, u_1, \ldots, u_{\frac{h}{4}+2t+1}$ in $\mathrm{GF}(q^2)$ such that $(\boldsymbol{a}^{qj+l}, \boldsymbol{v}^{q+1})_E = 0$ for $j = \frac{s(q+1)}{h} - 2$ and $l = q - \frac{s(q+1)}{h} - 1$, where s is even and $\frac{h}{4} + 1 - 2t \le s \le \frac{3h}{4} - 1 + 2t$.

Let $G = G_{k,k_1}(\boldsymbol{a}, \boldsymbol{v}) = \begin{pmatrix} \boldsymbol{g_{k_1}} \\ \boldsymbol{g_{k_1+1}} \\ \vdots \\ \boldsymbol{g_{k_1+k-1}} \end{pmatrix}$.

In the first case: $0 \le t_1 \le t_2 \le t-1$, let $\frac{(h-4)(q+1)}{4h} - \tau - 1 - \frac{2t_1(q+1)}{h} \le k_1 \le \frac{(h-4)(q+1)}{4h} + \tau - 1 - \frac{2t_1(q+1)}{h}$ and $\frac{(3h-4)(q+1)}{4h} + \tau - 2 + \frac{2t_2(q+1)}{h} \le k_1 + k - 1 \le \frac{(3h-4)(q+1)}{4h} + \tau - 3 + \frac{2(t_2+1)(q+1)}{h}$. Then the range of k satisfies $\frac{q+1}{2} + \frac{2(t_1+t_2)(q+1)}{h} \le k \le \frac{q-3}{2} + \frac{2(t_1+t_2+2)(q+1)}{h}$. It is easy to prove that $(\boldsymbol{a}^{qj+l}, \boldsymbol{v}^{q+1})_E \ne 0$ if and only if $j = \frac{(s-1)(q+1)}{h} + \tau - 1$ and $l = \frac{q-3}{2} - \frac{(s-1)(q+1)}{h} - \tau$, where s is odd and $\frac{h}{4} - 2t_1 \le s \le \frac{h}{4} + 2t_1$; or $j = \frac{(s-1)(q+1)}{h} + \tau - 2$, $l = \frac{3q-3}{2} - \frac{(s-1)(q+1)}{h} - \tau$, where s is odd and $\frac{3h}{4} - 2t_2 \le s \le \frac{3h}{4} + 2t_1$. There are $2t_1 + 2t_2 + 2$ pairs (j, l) such that $(\boldsymbol{a}^{qj+l}, \boldsymbol{v}^{q+1})_E \ne 0$.

By $\boldsymbol{g_j}\boldsymbol{g_l}^{\dagger} = (\boldsymbol{a}^{qj+l}, \boldsymbol{v}^{q+1})_E$, then $\dim(\mathrm{Hull}_h(C)) = k - \mathrm{rank}(GG^{\dagger}) = k - 2t_1 - 2t_2 - 2$.

In the second case: $0 \le t_1 \le t_2 \le \frac{h-4}{8} - t$, let $\frac{(h-4)(q+1)}{4h} - \frac{2(t+t_1)(q+1)}{h} \le k_1 \le \frac{(h-4)(q+1)}{4h} + \tau - 1 - \frac{2(t+t_1)(q+1)}{h}$ and $\frac{(3h-4)(q+1)}{4h} + \tau - 2 + \frac{2(t+t_2)(q+1)}{h} \le k_1 + k - 1 \le \frac{(3h-4)(q+1)}{4h} - 4 + \frac{2(t+t_2+1)(q+1)}{h}$. Then the range of k satisfies $\frac{q+1}{2} + \frac{2(2t+t_1+t_2)(q+1)}{h} \le$

$k \le \frac{q+1}{2} - 3 + \frac{2(2t+t_1+t_2+1)(q+1)}{h}$. It is easy to prove that $(\boldsymbol{a}^{qj+l}, \boldsymbol{v}^{q+1})_E \ne 0$ if and only if $j = \frac{(s-1)(q+1)}{h} + \tau - 1$ and $l = \frac{q-3}{2} - \frac{(s-1)(q+1)}{h} - \tau$, where s is odd and $\frac{h}{4} - 2t - 2t_1 \le s \le \frac{h}{4} + 2t + 2t_1$; $j = \frac{(s-1)(q+1)}{h} + \tau - 2, l = \frac{3q-3}{2} - \frac{(s-1)(q+1)}{h} - \tau$, where s is odd and $\frac{3h}{4} - 2t - 2t_2 \le s \le \frac{3h}{4} + 2t + 2t_2$; or $j = \frac{s(q+1)}{h} - 2, l = q - \frac{s(q+1)}{h} - 1$, where s is even and $0 < |s - \frac{h}{2}| + 1 - \frac{h}{4} - 2t \le 2t_1$; There are $4t + 4t_1 + 2t_2 + 2$ pairs (j, l) such that $(\boldsymbol{a}^{qj+l}, \boldsymbol{v}^{q+1})_E \ne 0$.

By $\boldsymbol{g}_j \boldsymbol{g}_l^\dagger = (\boldsymbol{a}^{qj+l}, \boldsymbol{v}^{q+1})_E$, then $\dim(\mathrm{Hull}_h(C)) = k - \mathrm{rank}(GG^\dagger) = k - 4t - 4t_1 - 2t_2 - 2$. □

Similar to (2) of Lemma 9, when $n = n'(\frac{h}{4} - 2t)$, $(0 \le t \le \frac{h}{4} - 1)$, we can get the following lemma.

Lemma 10 *Let $q \equiv 1 \ (mod\ 4)$ and h, τ, n' be defined as above, and $n = n'(\frac{h}{4} - 2t - 1)$, $(0 \le t \le \frac{h-4}{8} - 1)$. Then there exist some GRS codes with following parameters:*

Let t_1, t_2, k be integers, where $0 \le t_1 \le t_2 \le \frac{h-4}{8}$ and $\frac{q+1}{2} + \frac{2(t_1+t_2)(q+1)}{h} \le k \le \frac{q+1}{2} - 3 + \frac{2(t_1+t_2+1)(q+1)}{h}$, the code C has parameters $[n, k, n-k+1]$, and $Hull_h(C)$ has dimension $k - 4t - 4t_1 - 2t_2 - 4$.

From Lemmas 9, 10 and Proposition 3, we can get the following theorem.

Theorem 4 *Let q, n', h, τ be defined as above and t_1, t_2, k be an integer. Then, there exist $[[n, n-2k+c, k+1; c]]_q$ EAQMDS codes and $[[n, c, n-k+1; n-2k+c]]_q$ EAQECCs, if one of the following holds:*

(1) $n = n'(\frac{h}{4} + 2t + 1)$, $c = 2t_1 + 2t_2 + 2$, $\frac{q+1}{2} + \frac{2(t_1+t_2)(q+1)}{h} \le k \le \frac{q-3}{2} + \frac{2(t_1+t_2+2)(q+1)}{h}$, *where* $0 \le t_1 \le t_2 \le t - 1 \le \frac{h-4}{8} - 1$.
(2) $n = n'(\frac{h}{4} + 2t + 1)$, $c = 4t + 4t_1 + 2t_2 + 2$, $\frac{q+1}{2} + \frac{2(2t+t_1+t_2)(q+1)}{h} \le k \le \frac{q+1}{2} - 3 + \frac{2(2t+t_1+t_2+1)(q+1)}{h}$, *where* $0 \le t \le \frac{h-4}{8}$ *and* $0 \le t_1 \le t_2 \le \frac{h-4}{8} - t$.
(3) $n = n'(\frac{h}{4} - 2t - 1)$, $c = 2t + 4t_1 + 2t_2 + 4$, *and* $\frac{q+1}{2} + \frac{2(t_1+t_2)(q+1)}{h} \le k \le \frac{q+1}{2} - 3 + \frac{2(t_1+t_2+1)(q+1)}{h}$, *where* $0 \le t \le \frac{h-4}{8} - 1$ *and* $0 \le t_1 \le t_2 \le \frac{h-4}{8}$.

Remark 4 Comparing Theorem 4 with Theorem 11 of [24], we find that most of the length is different and the range of k is disjoint for the same length. Comparing Theorem 4 with Corollary 3.4 of [22], we can conclude although some codes have the same parameters, there are still many codes with different parameters. Here is an example.

Example 4 Let $q = 49$ and $h = 20$, then $\tau = 2$. By Theorem 4, there exist $[[n, n-2k+c, k+1; c]]_q$ EAQMDS codes and $[[n, c, n-k+1; n-2k+c]]_q$ EAQECCs, where the values of n, k, c can be found in Table 5.

After removing the same parameters as in Reference [22], we can get the codes in Table 6.

Lemma 11 *Let $q \equiv 1 \ (mod\ 4)$ and h, τ, n' be defined as above, and $n = tn'$, $(1 \le t \le h)$. Then, there exist some GRS codes with following parameters:*

Table 5 The values of n, k, c in Example 4

n	k	c	n	k	c	n	k	c
240	$25 \leq k \leq 27$	6	240	$30 \leq k \leq 32$	8	240	$35 \leq k \leq 37$	10
240	$35 \leq k \leq 37$	12	240	$40 \leq k \leq 42$	14	240	$45 \leq k \leq 47$	18
480	$25 \leq k \leq 27$	4	480	$30 \leq k \leq 32$	6	480	$35 \leq k \leq 37$	8
480	$35 \leq k \leq 37$	10	480	$40 \leq k \leq 42$	12	480	$45 \leq k \leq 47$	16
720	$25 \leq k \leq 27$	2	720	$30 \leq k \leq 32$	4	720	$35 \leq k \leq 37$	6
720	$35 \leq k \leq 37$	8	720	$40 \leq k \leq 42$	10	720	$45 \leq k \leq 47$	14
960	$25 \leq k \leq 33$	2	960	$35 \leq k \leq 37$	6	960	$40 \leq k \leq 42$	8
960	$45 \leq k \leq 47$	12	1200	$25 \leq k \leq 33$	2	1200	$30 \leq k \leq 38$	4
1200	$35 \leq k \leq 43$	8	1200	$45 \leq k \leq 47$	10			

Table 6 The values of n, k, c in Example 4 after deleting the same parameters

n	k	c	n	k	c	n	k	c
240	$30 \leq k \leq 32$	8	240	$35 \leq k \leq 37$	10	240	$35 \leq k \leq 37$	12
240	$40 \leq k \leq 42$	14	240	$45 \leq k \leq 47$	18	480	$35 \leq k \leq 37$	8
480	$35 \leq k \leq 37$	10	480	$40 \leq k \leq 42$	12	480	$45 \leq k \leq 47$	16
720	$35 \leq k \leq 37$	8	720	$40 \leq k \leq 42$	10	720	$45 \leq k \leq 47$	14
960	$30 \leq k \leq 33$	2	960	40	8	960	$45 \leq k \leq 47$	12
1200	$30 \leq k \leq 33$	2	1200	$35 \leq k \leq 38$	4	1200	$35 \leq k \leq 40$	8
1200	$45 \leq k \leq 47$	10						

Let t_1, t_2, k be integers, where $0 \leq t_1 \leq t_2 \leq \frac{h-4}{8}$ and $\frac{q+1}{2} + \frac{2(t_1+t_2)(q+1)}{h} \leq k \leq \frac{q+1}{2} - 3 + \frac{2(t_1+t_2+1)(q+1)}{h}$, the code C has parameters $[n, k, n-k+1]$, and $Hull_h(C)$ has dimension $k - \frac{h}{4} - 4t_1 - 2t_2 - 2$.

Proof If $q \equiv 1 \ (mod\ 4)$, then $8|(h-4)$. Let γ, ω be defined as above. Set

$$\boldsymbol{a} = (1, \gamma, \ldots, \gamma^{n'-1}, \omega, \ldots, \omega\gamma^{n'-1}, \ldots, \omega^{t-1}, \ldots, \omega^{t-1}\gamma^{n'-1}),$$
$$\boldsymbol{v} = (u_0, u_0\gamma, \ldots, u_0\gamma^{n'-1}, u_1, \ldots, u_1\gamma^{n'-1}, \ldots, u_{t-1}, \ldots, u_{t-1}\gamma^{n'-1}),$$

And then, we have

$$(\boldsymbol{a}^{qj+l}, \boldsymbol{v}^{q+1})_E = \sum_{i=0}^{t-1} \omega^{i(qj+l)} u_i^{q+1} \sum_{r=0}^{n'-1} \gamma^{r(qj+l+q+1)}.$$

where $u_0, u_1, \ldots, u_{t-1}$ are t nonzero elements in $\mathrm{GF}(q^2)$ such that

$$\sum_{i=0}^{t-1} \omega^{i(qj+l)} u_i^{q+1} \neq 0.$$

From Lemma 5, we have $(\boldsymbol{a}^{qj+l}, \boldsymbol{v}^{q+1})_E = 0$ if and only if (j, l) takes the following values:

If s is even and $2 \le s \le h$, then $j = \frac{s(q+1)}{h} - 2, l = q - \frac{s(q+1)}{h} - 1$; If s is odd and $1 \le s \le \frac{h}{2} - 1$, then $j = \frac{(s-1)(q+1)}{h} + \tau - 1, l = \frac{q-3}{2} - \frac{(s-1)(q+1)}{h} - \tau$; If s is odd and $\frac{h}{2} + 1 \le s \le h - 1$, then $j = \frac{(s-1)(q+1)}{h} + \tau - 2, l = \frac{3q-3}{2} - \frac{(s-1)(q+1)}{h} - \tau$.

Let $G = G_{k,k_1}(\boldsymbol{a}, \boldsymbol{v}) = \begin{pmatrix} \boldsymbol{g_{k_1}} \\ \boldsymbol{g_{k_1+1}} \\ \vdots \\ \boldsymbol{g_{k_1+k-1}} \end{pmatrix}$.

Let $0 \le t_1 \le t_2 \le \frac{h-4}{8}$, $\frac{(h-4)(q+1)}{4h} - \frac{2t_1(q+1)}{h} \le k_1 \le \frac{(h-4)(q+1)}{4h} + \tau - 1 - \frac{2t_1(q+1)}{h}$ and $\frac{(3h-4)(q+1)}{4h} + \tau - 2 + \frac{2t_2(q+1)}{h} \le k_1 + k - 1 \le \frac{(3h-4)(q+1)}{4h} - 4 + \frac{2(t_2+1)(q+1)}{h}$. Then the range of k satisfies $\frac{q+1}{2} + \frac{2(t_1+t_2)(q+1)}{h} \le k \le \frac{q+1}{2} - 3 + \frac{2(t_1+t_2+1)(q+1)}{h}$. It is easy to prove that $(\boldsymbol{a}^{qj+l}, \boldsymbol{v}^{q+1})_E \ne 0$ if and only if $j = \frac{(s-1)(q+1)}{h} + \tau - 1$ and $l = \frac{q-3}{2} - \frac{(s-1)(q+1)}{h} - \tau$, where s is odd and $\frac{h}{4} - 2t_1 \le s \le \frac{h}{4} + 2t_1$; $j = \frac{(s-1)(q+1)}{h} + \tau - 2, l = \frac{3q-3}{2} - \frac{(s-1)(q+1)}{h} - \tau$, where s is odd and $\frac{3h}{4} - 2t_2 \le s \le \frac{3h}{4} + 2t_2$; or $j = \frac{s(q+1)}{h} - 2, l = q - \frac{s(q+1)}{h} - 1$, where s is even and $\frac{h}{4} - 2t_1 \le s \le \frac{3h}{4} + 2t_1$; There are $\frac{h}{4} + 4t_1 + 2t_2 + 2$ pairs (j, l) such that $(\boldsymbol{a}^{qj+l}, \boldsymbol{v}^{q+1})_E \ne 0$.

By $\boldsymbol{g_j g_l^{\dagger}} = (\boldsymbol{a}^{qj+l}, \boldsymbol{v}^{q+1})_E$, then $\dim(\mathrm{Hull}_h(C)) = k - \mathrm{rank}(GG^{\dagger}) = k - \frac{h}{4} - 4t_1 - 2t_2 - 2$.

□

From Lemmas 11 and Proposition 3, we can get the following theorem.

Theorem 5 *Let q, n', h, τ be defined as above and t_1, t_2, k be integers. Then, there exist $[[n, n-2k+c, k+1; c]]_q$ EAQMDS codes and $[[n, c, n-k+1; n-2k+c]]_q$ EAQECCs, where $n = tn'$, $c = \frac{h}{4} + 4t_1 + 2t_2 + 2$, and $\frac{q+1}{2} + \frac{2(t_1+t_2)(q+1)}{h} \le k \le \frac{q+1}{2} - 3 + \frac{2(t_1+t_2+1)(q+1)}{h}$, $1 \le t \le h$, and $0 \le t_1 \le t_2 \le \frac{h-4}{8}$.*

Remark 5 Comparing Theorem 5 with Theorem 11 in [24], we find that most of the length is different and the range of k is disjoint for the same length. Comparing Theorem 5 with Corollary 3.4 of [22], we can conclude if t is odd, the length is different. That means that the EAQMDS codes constructed in Theorem 5 are new when t is odd.

Example 5 Let $q = 29$ and $h = 12$, then $n' = 70$, $1 \le t \le 12$, and $\tau = 2$. By Theorem 5, there exist $[[n, n-2k+c, k+1; c]]_q$ EAQMDS codes and $[[n, c, n-k+1; n-2k+c]]_q$ EAQECCs, where the values of n, k, c can be found in Table 7.

5 The third construction

In this section, similar to the previous two sections, we will construct some new EAQECCs by extended GRS codes.

Table 7 The values of n, k, c in Example 5

n	k	c
$70t$	$15 \le k \le 17$	5
$70t$	$20 \le k \le 22$	7
$70t$	$25 \le k \le 27$	11

In the proofs of Lemmas 3, 4, 6, 7, 8, 9, 10 and 11, if the generator matrices of GRS code satisfy the condition in Lemma 1, we can construct the following extended GRS codes:

Lemma 12 *Let q, h, τ, n' be defined as in Sect. 3, and $n = n'(\frac{h}{4} + t)$, $(1 \le t \le \frac{h}{4})$. Then, there exist some extended GRS codes with following parameters:*

(1) *Let t_0, k be integers, where $0 \le t_0 \le t - 2 (t \ge 2)$ and $\frac{q-1}{2} + \tau + \frac{2t_0(q-1)}{h} < k \le \frac{q-1}{2} + \tau + \frac{2(t_0+1)(q-1)}{h} - 1$, the code C has parameters $[n+1, k, n-k+2]$, and $Hull_h(C)$ has dimension $k - 2t_0 - 3$.*
(2) *Let t_0, k be integers, where $0 \le t_0 \le \frac{h}{4} - t$ and $\frac{q-1}{2} + \tau + \frac{2(t+t_0-1)(q-1)}{h} < k \le \frac{q-1}{2} + 2\tau + \frac{2(t_0+t-1)(q-1)}{h} - 1$, the code C has parameters $[n+1, k, n-k+2]$, and $Hull_h(C)$ has dimension $k - 2t - 3t_0 - 1$.*
(3) *Let t_0, k be integers, where $0 \le t_0 \le \frac{h}{4} - t - 1$ and $\frac{q-1}{2} + 2\tau + \frac{2(t+t_0-1)(q-1)}{h} < k \le \frac{q-1}{2} + \tau + \frac{2(t_0+t)(q-1)}{h} - 1$, the code C has parameters $[n+1, k, n-k+2]$, and $Hull_h(C)$ has dimension $k - 2t - 3t_0 - 2$.*

Lemma 13 *Let q, h, τ, n' be defined as in Sect. 3, and $n = n'(\frac{h}{4} - t)$, $(0 \le t \le \frac{h}{4} - 1)$. Then, there exist some extended GRS codes with following parameters:*

(1) *Let t_0, k be integers, where $0 \le t_0 \le \frac{h}{4} - 1$ and $\frac{q-1}{2} + \tau + \frac{2t_0(q-1)}{h} < k \le \frac{q-1}{2} + 2\tau + \frac{2(t_0+1)(q-1)}{h} - 1$, the code C has parameters $[n+1, k, n-k+2]$, and $Hull_h(C)$ has dimension $k - t - 3t_0 - 4$.*
(2) *Let t_0, k be integers, where $0 \le t_0 \le \frac{h}{4} - 2$ and $\frac{q-1}{2} + 2\tau + \frac{2t_0(q-1)}{h} < k \le \frac{q-1}{2} + \tau + \frac{2(t_0+1)(q-1)}{h} - 1$, the code C has parameters $[n+1, k, n-k+2]$, and $Hull_h(C)$ has dimension $k - t - 3t_0 - 5$.*

Lemma 14 *Let $q \equiv 3 \pmod 4$ and h, τ, n' be defined as in Sect. 4.1, and $n = n'(\frac{h}{4} + 2t + 2)$, $(0 \le t \le \frac{h}{8} - 1)$. Then, there exist some extended GRS codes with following parameters:*

(1) *Let t_1, t_2, k be integers, where $0 \le t_1 \le t_2 \le t$ and $\frac{q+3}{2} + \tau + \frac{2(t_1+t_2)(q+1)}{h} < k \le \frac{q-3}{2} + \tau + \frac{2(t_1+t_2+1)(q+1)}{h}$, the code C has parameters $[n+1, k, n-k+2]$, and $Hull_h(C)$ has dimension $k - 2t_1 - 2t_2 - 3$.*
(2) *Let t_1, t_2, k be integers, where $0 \le t_1 \le t_2 \le \frac{h}{8} - 1 - t$ and $\frac{q+3}{2} + \tau + \frac{2(2t+t_1+t_2)(q+1)}{h} < k \le \frac{q-3}{2} + \tau + \frac{2(2t+t_1+t_2+1)(q+1)}{h}$, the code C has parameters $[n+1, k, n-k+2]$, and $Hull_h(C)$ has dimension $k - 4t - 4t_1 - 2t_2 - 3$.*

Lemma 15 *Let $q \equiv 3 \pmod 4$ and h, τ, n' be defined as in Sect. 4.1, and $n = n'(\frac{h}{4} - 2t)$, $(0 \le t \le \frac{h}{8} - 1)$. Then, there exist some extended GRS codes with following parameters:*

Let t_1, t_2, k *be integers, where* $0 \le t_1 \le t_2 \le \frac{h}{8} - 1$ *and* $\frac{q+3}{2} + \tau + \frac{2(t_1+t_2)(q+1)}{h} < k \le \frac{q-3}{2} + \tau + \frac{2(t_1+t_2+1)(q+1)}{h}$, *the code* C *has parameters* $[n+1, k, n-k+2]$, *and* $Hull_h(C)$ *has dimension* $k - 2t - 2t_1 - 2t_2 - 5$.

Lemma 16 *Let* $q \equiv 3 \pmod 4$ *and* h, τ, n' *be defined as in Sect. 4.1, and* $n = tn'$, *for* $1 \le t \le h$. *Then, there exist some extended GRS codes with following parameters:*

Let t_1, t_2, k *be integers, where* $0 \le t_1 \le t_2 \le \frac{h}{8} - 1$ *and* $\frac{q+3}{2} + \tau + \frac{2(t_1+t_2)(q+1)}{h} < k \le \frac{q-3}{2} + \tau + \frac{2(t_1+t_2+1)(q+1)}{h}$, *the code* C *has parameters* $[n, k, n-k+1]$, *and* $Hull_h(C)$ *has dimension* $k - \frac{h}{4} - 4t_1 - 2t_2 - 4$.

Lemma 17 *Let* $q \equiv 1 \pmod 4$ *and* h, τ, n' *be defined as in Sect. 4.2, and* $n = n'(\frac{h}{4} + 2t + 1)$, $(0 \le t \le \frac{h-4}{8})$. *Then, there exist some extended GRS codes with following parameters:*

(1) *Let* t_1, t_2, k *be integers, where* $0 \le t_1 \le t_2 \le t - 1$ *and* $\frac{q+1}{2} + \frac{2(t_1+t_2)(q+1)}{h} < k \le \frac{q-3}{2} + \frac{2(t_1+t_2+2)(q+1)}{h}$, *the code* C *has parameters* $[n+1, k, n-k+2]$, *and* $Hull_h(C)$ *has dimension* $k - 2t_1 - 2t_2 - 3$.
(2) *Let* t_1, t_2, k *be integers, where* $0 \le t_1 \le t_2 \le \frac{h-4}{8} - t$ *and* $\frac{q+1}{2} + \frac{2(2t+t_1+t_2)(q+1)}{h} < k \le \frac{q+1}{2} - 3 + \frac{2(2t+t_1+t_2+1)(q+1)}{h}$, *the code* C *has parameters* $[n+1, k, n-k+2]$, *and* $Hull_h(C)$ *has dimension* $k - 4t - 4t_1 - 2t_2 - 3$.

Lemma 18 *Let* $q \equiv 1 \pmod 4$ *and* h, τ, n' *be defined as in Sect. 4.2, and* $n = n'(\frac{h}{4} - 2t - 1)$, $(0 \le t \le \frac{h-4}{8})$. *Then, there exist some extended GRS codes with following parameters:*

Let t_1, t_2, k *be integers, where* $0 \le t_1 \le t_2 \le \frac{h-4}{8}$ *and* $\frac{q+1}{2} + \frac{2(t_1+t_2)(q+1)}{h} < k \le \frac{q+1}{2} + \frac{2(t_1+t_2+1)(q+1)}{h}$, *the code* C *has parameters* $[n+1, k, n-k+2]$, *and* $Hull_h(C)$ *has dimension* $k - 4t - 4t_1 - 2t_2 - 5$.

Lemma 19 *Let* $q \equiv 1 \pmod 4$ *and* h, τ, n' *be defined as in Sect. 4.2, and* $n = tn'$, $(1 \le t \le h)$. *Then, there exist some extended GRS codes with following parameters:*

Let t_1, t_2, k *be integers, where* $0 \le t_1 \le t_2 \le \frac{h-4}{8}$ *and* $\frac{q+1}{2} + \frac{2(t_1+t_2)(q+1)}{h} < k \le \frac{q+1}{2} - 3 + \frac{2(t_1+t_2+1)(q+1)}{h}$, *the code* C *has parameters* $[n, k, n-k+1]$, *and* $Hull_h(C)$ *has dimension* $k - \frac{h}{4} - 4t_1 - 2t_2 - 3$.

From Lemmas 12, 13, 14, 15, 16, 17, 18, 19 and Proposition 3, we can get the following theorems.

Theorem 6 *Let* q, n', h, τ *be defined as in Sect. 3 and* k *be an integer. Then, there exist* $[[n+1, n-2k+c+2, k+1; c+1]]_q$ *EAQMDS codes and* $[[n+1, c+1, n-k+2; n-2k+c+2]]_q$ *EAQECCs, if one of the following holds:*

(1) $n = n'(\frac{h}{4}+t)$, $c = 2t_0+2$, *and* $\frac{q-1}{2} + \tau + \frac{2t_0(q-1)}{h} < k \le \frac{q-1}{2} + \tau + \frac{2(t_0+1)(q-1)}{h} - 1$, *where* $2 \le t \le \frac{h}{4}$ *and* $0 \le t_0 \le t - 2$.
(2) $n = n'(\frac{h}{4}+t)$, $c = 2t + 3t_0$, *and* $\frac{q-1}{2} + \tau + \frac{2(t+t_0-1)(q-1)}{h} < k \le \frac{q-1}{2} + 2\tau + \frac{2(t_0+t-1)(q-1)}{h} - 1$, *where* $1 \le t \le \frac{h}{4}$ *and* $0 \le t_0 \le \frac{h}{4} - t$.
(3) $n = n'(\frac{h}{4}+t)$, $c = 2t + 3t_0 + 1$, $\frac{q-1}{2} + 2\tau + \frac{2(t+t_0-1)(q-1)}{h} < k \le \frac{q-1}{2} + \tau + \frac{2(t_0+t)(q-1)}{h} - 1$, *where* $1 \le t \le \frac{h}{4}$ *and* $0 \le t_0 \le \frac{h}{4} - t - 1$.

(4) $n = n'(\frac{h}{4} - t)$, $c = t + 3t_0 + 3$, *and* $\frac{q-1}{2} + \tau + \frac{2t_0(q-1)}{h} < k \le \frac{q-1}{2} + 2\tau + \frac{2(t_0+1)(q-1)}{h} - 1$, *where* $0 \le t \le \frac{h}{4} - 1$ *and* $0 \le t_0 \le \frac{h}{4} - 1$.
(5) $n = n'(\frac{h}{4} - t)$, $c = t + 3t_0 + 4$ *and* $\frac{q-1}{2} + 2\tau + \frac{2t_0(q-1)}{h} < k \le \frac{q-1}{2} + \tau + \frac{2(t_0+1)(q-1)}{h} - 1$, *where* $0 \le t \le \frac{h}{4} - 1$ *and* $0 \le t_0 \le \frac{h}{4} - 2$.

Theorem 7 *Let* q, n', h, τ *be defined as in Sect.* 4.1 *and* t_1, t_2, k *be an integer. Then, there exist* $[[n+1, n-2k+c+2, k+1; c+1]]_q$ *EAQMDS codes and* $[[n+1, c+1, n-k+2; n-2k+c+2]]_q$ *EAQECCs, if one of the following holds:*

(1) $n = n'(\frac{h}{4} + 2t + 2)$, $c = 2t_1 + 2t_2 + 2$, $\frac{q+3}{2} + \tau + \frac{2(t_1+t_2)(q+1)}{h} < k \le \frac{q-3}{2} + \tau + \frac{2(t_1+t_2+1)(q+1)}{h}$, *where* $0 \le t_1 \le t_2 \le t \le \frac{h}{8} - 1$.
(2) $n = n'(\frac{h}{4} + 2t + 2)$, $c = 4t + 4t_1 + 2t_2 + 2$, $\frac{q+3}{2} + \tau + \frac{2(2t+t_1+t_2)(q+1)}{h} < k \le \frac{q-3}{2} + \tau + \frac{2(2t+t_1+t_2+1)(q+1)}{h}$, *where* $0 \le t_1 \le t_2 \le \frac{h}{8} - 1 - t$.
(3) $n = n'(\frac{h}{4} - 2t)$, $c = 2t + 4t_1 + 2t_2 + 4$, *and* $\frac{q+3}{2} + \tau + \frac{2(t_1+t_2)(q+1)}{h} < k \le \frac{q-3}{2} + \tau + \frac{2(t_1+t_2+1)(q+1)}{h}$, *where* $0 \le t \le \frac{h}{8} - 1$ *and* $0 \le t_1 \le t_2 \le \frac{h}{8} - 1$.

Theorem 8 *Let* q, n', h, τ *be defined as in Sect.* 4.1 *and* t_1, t_2, k *be integers. Then, there exist* $[[n+1, n-2k+c+2, k+1; c+1]]_q$ *EAQMDS codes and* $[[n+1, c+1, n-k+2; n-2k+c+2]]_q$ *EAQECCs, where* $n = tn'$, $c = \frac{h}{4} + 4t_1 + 2t_2 + 3$, *and* $\frac{q+3}{2} + \tau + \frac{2(t_1+t_2)(q+1)}{h} < k \le \frac{q-3}{2} + \tau + \frac{2(t_1+t_2+1)(q+1)}{h}$, $1 \le t \le h$, *and* $0 \le t_1 \le t_2 \le \frac{h}{8} - 1$.

Theorem 9 *Let* q, n', h, τ *be defined as in Sect.* 4.2 *and* t_1, t_2, k *be an integer. Then, there exist* $[[n+1, n-2k+c+2, k+1; c+1]]_q$ *EAQMDS codes and* $[[n+1, c+1, n-k+2; n-2k+c+2]]_q$ *EAQECCs, if one of the following holds:*

(1) $n = n'(\frac{h}{4} + 2t + 1)$, $c = 2t_1 + 2t_2 + 2$, $\frac{q+1}{2} + \frac{2(t_1+t_2)(q+1)}{h} < k \le \frac{q-3}{2} + \frac{2(t_1+t_2+1)(q+1)}{h}$, *where* $0 \le t_1 \le t_2 \le t \le \frac{h-4}{8}$.
(2) $n = n'(\frac{h}{4} + 2t + 1)$, $c = 4t + 4t_1 + 2t_2 + 2$, $\frac{q+1}{2} + \frac{2(2t+t_1+t_2)(q+1)}{h} < k \le \frac{q+1}{2} - 3 + \frac{2(2t+t_1+t_2+1)(q+1)}{h}$, *where* $0 \le t_1 \le t_2 \le \frac{h-4}{8} - t$.
(3) $n = n'(\frac{h}{4} - 2t - 1)$, $c = 2t + 4t_1 + 2t_2 + 4$, $\frac{q+1}{2} + \frac{2(t_1+t_2)(q+1)}{h} < k \le \frac{q+1}{2} - 3 + \frac{2(t_1+t_2+1)(q+1)}{h}$, *where* $0 \le t \le \frac{h-4}{8}$ *and* $0 \le t_1 \le t_2 \le \frac{h-4}{8}$.

Theorem 10 *Let* q, n', h, τ *be defined as in Sect.* 4.2 *and* t_1, t_2, k *be integers. Then, there exist* $[[n+1, n-2k+c+2, k+1; c+1]]_q$ *EAQMDS codes and* $[[n+1, c+1, n-k+2; n-2k+c+2]]_q$ *EAQECCs, where* $n = tn'$, $c = \frac{h}{4} + 4t_1 + 2t_2 + 2$, *and* $\frac{q+1}{2} + \frac{2(t_1+t_2)(q+1)}{h} < k \le \frac{q+1}{2} - 3 + \frac{2(t_1+t_2+1)(q+1)}{h}$, $1 \le t \le h$, *and* $0 \le t_1 \le t_2 \le \frac{h-4}{8}$.

Remark 6 Similar to Remark 1, 2, 3, 4 and 5, we can see that most of EAQMDS codes constructed in Theorem 6, 7, 8, 9 and 10 are new.

Table 8 Quantum MDS codes with length $\frac{a(q^2-1)}{b}$ and $\frac{a(q^2-1)}{b}+1$

Class	Length	Distance	Preshared maximally entangled states	References
1	$n=b\frac{q^2-1}{2a}$ $q=2am-1$ $1\le b\le 2a$	$cm+2\le d\le (a+\lceil\frac{c}{2}\rceil)m$	$1\le c\le 2a-1$	[22]
2	$n=b\frac{q^2-1}{2a+1}$ $q=(2a+1)m-1$ $1\le b\le 2a$	$cm+2\le d\le (a+1+\lceil\frac{c}{2}\rceil)m$	$1\le c\le 2a$	[22]
3	$n=tn^{'}, 1\le t\le \frac{q-1}{n_1}$ $q>2, n^{'}\|(q^2-1)$ and $n_1=\frac{n^{'}}{gcd(n^{'},q+1)}$	$2\le d\le \lceil\frac{n+q}{q+1}\rceil+1$	$1\le c\le d$	[24]
4	$n=tn^{'}+1, 1\le t\le \frac{q-1}{n_1}$ $q>2, n^{'}\|(q^2-1)$ and $n_1=\frac{n^{'}}{gcd(n^{'},q+1)}$	$2\le d\le \lceil\frac{n+q}{q+1}\rceil+1$	$0\le c\le d$	[24]
5	$n=\lambda(q+1), q\ge 7$ q,λ is odd, $\lambda\|q-1$ and $\lambda\ge 3$	$q+\lambda+2\le d\le q+2\lambda$	$c=4$	[12]
6	$n=2\lambda(q+1), q\ge 13$ $q\equiv 1\ (mod\ 4)$, λ is odd, $\lambda\|q-1$ and $\lambda\ge 3$	$q+2\lambda+2\le d\le q+4\lambda$	$c=4$	[12]
7	$n=\frac{q^2-1}{3}$	$\frac{2(q+1)}{3}\le d\le q$ $q+1\le d\le \frac{4(q+1)}{3}-2$	$c=1$ $c=3$	[15]
8	$n=\frac{q^2-1}{5}$	$\frac{3(q+1)}{5}\le d\le \frac{4(q+1)}{5}-1$ $\frac{4(q+1)}{5}\le d\le q$ $q+1\le d\le \frac{6(q+1)}{5}-1$	$c=1$ $c=3$ $c=5$	[15]

Table 8 continued

Class	Length	Distance	Preshared maximally entangled states	References
9	$n=\frac{q^2-1}{7}$	$\frac{4(q+1)}{7}\leq d\leq\frac{5(q+1)}{7}-1$ $\frac{5(q+1)}{7}\leq d\leq\frac{6(q+1)}{7}-1$ $\frac{6(q+1)}{7}\leq d\leq q$ $q+1\leq d+\leq\frac{8(q+1)}{7}-1$	$c=1$ $c=3$ $c=5$ $c=7$	[15]
10	$n=\frac{q^2-1}{4}$	$\frac{3(q+1)}{4}\leq d\leq q$ $q+1\leq d\leq\frac{5(q+1)}{4}-1$	$c=2$ $c=4$	[15]
11	$n=\frac{q^2-1}{6}$	$\frac{4(q+1)}{6}\leq d\leq\frac{5(q+1)}{6}-1$ $\frac{5(q+1)}{6}\leq d\leq q$ $q+1\leq d\leq\frac{7(q+1)}{6}-1$	$c=2$ $c=4$ $c=6$	[15]
12	$n=\frac{q^2-1}{at}$, q is odd $q=atm+1$, a is even or a is odd and t is even	$2\leq d\leq(\frac{at}{2}+1)m+1$ $(\frac{at}{2}+1)m+2\leq d\leq(\frac{at}{2}+2)m+1$ $(\frac{at}{2}+2)m+2\leq d\leq(\frac{at}{2}+3)m+1$	$c=0$ $c=2$ $c=4$	[16]
13	$n=\frac{q^2-1}{30}$ q is odd, $q=30m+11$	$8m+4\leq d\leq 11m+5$ $11m+6\leq d\leq 14m+7$	$c=2$ $c=4$	[16]
14	$n=\frac{q^2-1}{30}$ q is odd, $q=30m+19$	$8m+6\leq d\leq 11m+7$ $11m+8\leq d\leq 13m+8$ $13m+9\leq d\leq 16m+10$	$c=2$ $c=4$ $c=6$	[16]
15	$n=\frac{q^2-1}{12}$ q is odd, $q=12m+5$	$5m+3\leq d\leq 7m+3$ $7m+4\leq d\leq 8m+3$	$c=2$ $c=4$	[16]
16	$n=q^2-1$	$2\leq d\leq 2q-2$	$c=1$	[11]

Table 8 continued

Class	Length	Distance	Preshared maximally entangled states	References
17	$n=\frac{q^2-1}{2}$, q is odd	$\frac{q+1}{2}+2\le d\le \frac{3}{2}q-\frac{1}{2}$	$c=2$	[11]
18	$n=\frac{q^2-1}{t}$, q is odd $t\|q+1$, $t\ge 3$ is odd	$\frac{(t-1)(q+1)}{t}+2\le d\le \frac{(t+1)(q+1)}{t}-2$	$c=t$	[11]
19	$n=\frac{q^2-1}{2h}$, q is odd $2h\|q+1$, $h\in\{3,5,7\}$	$\frac{q+1}{h}+1\le d\le \frac{(q+1)(h+3)}{2h}-1$	$c=1$	[25]
20	$n=2\lambda(q-1)$, q,λ is odd $8\|q+1$, $\lambda\|q+1$, $1\le i\le 2$	$\frac{q-1}{2}(i-1)+4\lambda+1\le d\le \frac{q-1}{2}+2(i+1)\lambda$	$c=2i$	[25]
21	$n=\frac{q^2-1}{a}$, $aa'=q-1$ $q=2a+1$	$2\le d\le a'$ $\beta a'+1\le d\le(\beta+1)a'$, $\beta\in[1,a]$	$c=0$ $c=\beta$	[26]
22	$n=\frac{q^2-1}{a}$, $aa'=q-1$ $q\ge 3a+1$	$2\le d\le a'$ $\beta a'+1\le d\le(\beta+1)a'$, $\beta\in[1,a]$ $\beta a'+\lceil\frac{\beta}{a}\rceil\le d\le(\beta+1)a'$ and $\beta\in[a+1,\frac{q-1}{2}]$	$c=0$ $c=\beta$ $c=\beta+$ $\sum_{k-a+1}^{\beta}2(\lceil\frac{\beta}{a}\rceil-1)$	[26]
23	$n=\frac{q^2-1}{a}$, $aa'=q-1$ $q\ge 2a+1$, a is even	$2\le d\le \frac{(a+2)a'}{2}+1$ $\beta a'+\lceil\frac{\beta-\frac{a}{2}}{a}\rceil+1\le d\le(\beta+1)a'+1$ and $\beta\in[\frac{a+2}{2},\frac{q-1}{2}]$	$c=0$ $c=$ $\sum_{\frac{a+2}{2}}^{\beta}2\lceil\frac{k-\frac{a}{2}}{a}\rceil$	[26]

6 Conclusions

In this paper, we constructed some classes of EAQMDS codes and EAQECCs and evaluated the dimensions of their Hermitian hulls. According to the entanglement-assisted quantum singleton bound, the resulting EAQMDS codes are optimal. In Table 8 we summarize the parameters of all precious quantum MDS codes with length $\frac{a(q^2-1)}{b}$ and $\frac{a(q^2-1)}{b}+1$ (where $b|(q^2-1)$ and a is a positive integer). From the tables, we can easily see most of these q-ary EAQMDS codes are new in the sense that their parameters are not covered by the codes available in the literature. GRS code is a powerful tool for constructing EAQMDS codes. In the future work, we look forward to getting more EAQMDS codes with large minimum distance from GRS codes.

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