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$\mathbb{F}_q R$-linear skew constacyclic codes and their application of constructing quantum codes

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Abstract
Let q be a prime power with $\gcd(q, 6) = 1$. Let $R = \mathbb{F}_{q^2} + u\mathbb{F}_{q^2} + v\mathbb{F}_{q^2} + uv\mathbb{F}_{q^2}$, where $u^2 = u$, $v^2 = v$ and $uv = vu$. In this paper, we give the definition of linear skew constacyclic codes over $\mathbb{F}_{q^2} R$. By the decomposition method, we study the structural properties and determine the generator polynomials and the minimal generating sets of linear skew constacyclic codes. We define a Gray map from $\mathbb{F}_{q^2}^{\alpha} \times R^{\beta}$ to $\mathbb{F}_{q^2}^{\alpha+4\beta}$ preserving the Hermitian orthogonality, where α and β are positive integers. As an application, by Hermitian construction, we obtain some good quantum error-correcting codes.

Keywords Linear skew constacyclic codes · Gray map · Hermitian orthogonality · Hermitian construction · Quantum codes

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1 Introduction

In 2007, Boucher et al. [12] studied skew cyclic codes over finite fields. These codes were constructed by non-commutative polynomial rings. The authors showed that

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some skew cyclic codes have larger minimum Hamming distances than previously best-known linear codes of the same lengths and dimensions. Inspired by this work, there are many papers on skew codes over finite fields. Abualrub et al. [1] studied skew quasi-cyclic codes over finite fields. Siap et al. [34] studied the structure of skew cyclic codes of arbitrary length and constructed some good linear codes over finite fields. Recently, the topic on skew codes has been generalized to finite rings. Boucher et al. [13] studied some structural properties of skew constacyclic codes over Galois rings. In [26], Jitman et al. generalized this issue to finite chain rings. Afterwards, many scholars studied skew cyclic codes and constacyclic codes over finite ring such as [2,11,14,22,24,36,38,42].

In past years, there are several papers on mixed alphabet codes. In 1973, Delsarte [17] introduced additive codes which can be viewed as subgroups of the underlying abelian group of the form $\mathbb{Z}_2^{\alpha} \times \mathbb{Z}_4^{\beta}$. Later, many scholars paid more attention to additive codes. Abualrub et al. [3] and Borges et al. [10] introduced $\mathbb{Z}_2\mathbb{Z}_4$-additive cyclic codes. They investigated the generator matrix and the duality of the family of codes. Aydogdu et al. [6,7] generalized $\mathbb{Z}_2\mathbb{Z}_4$-additive codes to $\mathbb{Z}_2\mathbb{Z}_{2^s}$-additive codes and $\mathbb{Z}_{p^r}\mathbb{Z}_{p^s}$-additive codes. Afterwards, some papers focused on additive codes appeared, such as [5,18,33,35].

Quantum error-correcting codes (QECCs) are based on the classical information theory and quantum mechanics. They play an important role in quantum computation and quantum secret communications, such as quantum signature schemes [41], quantum identities authentication schemes [16] and quantum key distribution protocol [40]. Recently, it has become a hot topic of constructing quantum error-correcting codes [8,14,20,30] and quantum error-avoiding codes [39]. The first quantum code was discovered by Shor [32]. Later, a construction method called CSS construction of quantum codes from classical error-correcting codes was given by Claderbank et al. [15]. Afterwards, many good quantum codes have been constructed from classical error-correcting codes.

The error-correcting codes over finite rings have richer algebraic structures than those over finite fields. Therefore, the quantum coding theory over the finite rings has received a lot of attention, recently. Many coding scholars have constructed new quantum codes with Euclidean and Hermitian orthogonality from cyclic and constacyclic codes over finite rings such as [21,23,29,31]. Recently, the structural properties of cyclic, constacyclic, skew constacyclic codes over the ring $\mathbb{F}_q + u\mathbb{F}_q + v\mathbb{F}_q + uv\mathbb{F}_q$ have been studied. Ashraf et al. [4] constructed quantum codes over $\mathbb{F}_5$ from cyclic codes over $\mathbb{F}_5 + u\mathbb{F}_5 + v\mathbb{F}_5 + uv\mathbb{F}_5$. Yao et al. [42] considered the structural properties of Euclidean dual codes of skew cyclic codes over $\mathbb{F}_q + u\mathbb{F}_q + v\mathbb{F}_q + uv\mathbb{F}_q$. Zheng et al. [43] studied some properties of Euclidean dual codes of constacyclic codes over $\mathbb{F}_p + u\mathbb{F}_p + v\mathbb{F}_p + uv\mathbb{F}_p$. Ma et al. [28] constructed some non-binary quantum codes from constacyclic codes over $\mathbb{F}_p[u, v]/\langle u^2 - 1, v^2 - v, uv - vu\rangle$. Skew constacyclic codes generalize cyclic codes and constacyclic codes and provide more flexibility in constructing of good quantum codes. In [14], the authors considered the structure of Euclidean dual codes of skew constacyclic codes over the ring $\mathbb{F}_q[u, v]/\langle u^2 - 1, v^2 - 1, uv - vu\rangle$, and some quantum codes were constructed from this family of codes. In [8], Aydin et al. introduced and studied additive skew

cyclic codes over the quaternary field $\mathbb{F}_4$. They showed that some optimal quantum codes can be obtained from additive skew cyclic codes. Motivated by the above work, in this paper, we consider structural properties of skew constacyclic codes with respect to the Hermitian inner product over the mixed alphabet $\mathbb{F}_{q^2}R$, where $R = \mathbb{F}_{q^2} + u\mathbb{F}_{q^2} + v\mathbb{F}_{q^2} + uv\mathbb{F}_{q^2}$ with $u^2 = u$, $v^2 = v$ and $uv = vu$. The contributions of our paper are listed as follows.

1. We discuss structural properties of skew λ-constacyclic codes over R. Moreover, we consider the dual codes of skew λ-constacyclic codes with respect to the Hermitian inner product. A sufficient and necessary condition for the existence of Hermitian dual-containing skew λ-constacyclic codes over R is given.
2. We study the algebraic structure of $\mathbb{F}_{q^2}R$-linear skew constacyclic codes and determine the generators and the minimal spanning sets of this family of codes.
3. We define an $\mathbb{F}_{q^2}$-linear Gray map from $\mathbb{F}_{q^2}^{\alpha} \times R^{\beta}$ to $\mathbb{F}_{q^2}^{\alpha+4\beta}$. The Gray image of any $\mathbb{F}_{q^2}R$-skew constacyclic code is the product of a cyclic code over $\mathbb{F}_{q^2}$ of length α and four skew constacyclic codes of length β.
4. As an application, by the Hermitian construction, we obtain some new quantum error-correcting codes. Moreover, our new quantum codes have better parameters than the ones appeared in previous studies.

This paper is organized as follows. In Sect. 2, we give some basic definitions and introduce the skew polynomial ring $R[x, \theta]$, where $R = \mathbb{F}_{q^2} + u\mathbb{F}_{q^2} + v\mathbb{F}_{q^2} + uv\mathbb{F}_{q^2}$ with $u^2 = u$, $v^2 = v$ and $uv = vu$. In Sect. 3, we study some structural properties of skew constacyclic codes over R. In Sect. 4, we introduce the definition and algebraic structure of $\mathbb{F}_{q^2}R$-linear skew constacyclic codes and determine their generating sets. In Sect. 5, we define an $\mathbb{F}_{q^2}$-linear Gray map from $\mathbb{F}_{q^2}^{\alpha} \times R^{\beta}$ to $\mathbb{F}_{q^2}^{\alpha+4\beta}$. Finally, we construct some good quantum codes from $\mathbb{F}_{q^2}R$-linear skew constacyclic codes by Hermitian construction.

2 Preliminaries

Let $\mathbb{F}_{q^2}$ be a finite field, where q is a prime power such that $\gcd(q, 6) = 1$. Let $R = \mathbb{F}_{q^2} + u\mathbb{F}_{q^2} + v\mathbb{F}_{q^2} + uv\mathbb{F}_{q^2}$, where $u^2 = u$, $v^2 = v$ and $uv = vu$. Clearly, R is not a finite chain ring. Let $e_1 = uv, e_2 = u - uv, e_3 = v - uv, e_4 = 1 - u - v + uv$. It is easy to show that $e_i^2 = e_i, e_i e_j = 0$ and $\sum_{i=1}^{4} e_i = 1$, where $i = 1, 2, 3, 4$ and $i \neq j$. From the Chinese remainder theorem, we have that $R = e_1\mathbb{F}_{q^2} \oplus e_2\mathbb{F}_{q^2} \oplus e_3\mathbb{F}_{q^2} \oplus e_4\mathbb{F}_{q^2}$. Thus, for any element $r \in R$, r can be expressed uniquely as $r = e_1 s + e_2 t + e_3 w + e_4 z$, where $s, t, w, z \in \mathbb{F}_{q^2}$.

We define the set

$$\mathbb{F}_{q^2}R = \{(x, r) | x \in \mathbb{F}_{q^2}, r \in R\}.$$

It is a ring but not an R-module under the operation of standard multiplication. For any $r = e_1 s + e_2 t + e_3 w + e_4 z = z + u(t - z) + v(w - z) + uv(s - t - w + z)$, define the following map

$$\delta : R \longrightarrow \mathbb{F}_{q^2}$$
$$r = e_1 s + e_2 t + e_3 w + e_4 z \mapsto z.$$

Clearly, δ is a well-defined ring homomorphism. For any $l \in R$, we define a multiplication $\star$ by $l\star(x, r) = (\delta(l)x, lr)$. It can be naturally generalized to $\mathbb{F}_{q^2}^{\alpha} \times R^{\beta}$ given by

$$l\star\mu = (\delta(l)x_0, \ldots, \delta(l)x_{\alpha-1}|lr'_0, \ldots, lr'_{\beta-1}),$$

where $l \in R$, $\mu = (x_0, \ldots, x_{\alpha-1}|r'_0, \ldots, r'_{\beta-1}) \in \mathbb{F}_{q^2}^{\alpha} \times R^{\beta}$.

From the above discussion, we have the following result directly.

Lemma 1 *The ring $\mathbb{F}_{q^2}^{\alpha} \times R^{\beta}$ is an R-module under the addition in the usual way and the above multiplication.*

In the following, we introduce an automorphism of R. Define

$$\theta : R \to R$$
$$a + ub + vc + uvd \mapsto a^q + ub^q + vc^q + uvd^q.$$

In this case, $\mathrm{ord}\langle\theta\rangle = 2$. Clearly, the invariant subring under the automorphism θ is $\mathbb{F}_q + u\mathbb{F}_q + v\mathbb{F}_q + uv\mathbb{F}_q$.

Definition 1 Let θ be an automorphism of R defined above. The skew polynomial ring $R[x, \theta]$ is a set of polynomials

$$R[x, \theta] = \{a(x) = a_0 + a_1 x + \cdots + a_t x^t | a_i \in R, \text{ for all } i = 0, 1, \ldots, t\},$$

where the addition of these polynomials is defined in the usual way, while multiplication is defined using the distributive law and the rule

$$(ax^i) * (bx^j) = a\theta^i(b)x^{i+j}.$$

The skew polynomial ring $R[x, \theta]$ is a non-commutative ring.

An element $g(x) \in R[x, \theta]$ is said to be a right divisor of $f(x)$ if there exists a polynomial $q(x) \in R[x, \theta]$ such that

$$f(x) = q(x) * g(x).$$

In this case, $f(x)$ is called a left multiple. Similarly, the left divisor can be given. In the following paper, we denote $g(x)$ a right divisor of $f(x)$ by $g(x)|_r f(x)$. Similar to the reference [26], we give the right division algorithm in $R[x, \theta]$.

Lemma 2 [26] *Let $f(x)$, $g(x) \in R[x, \theta]$, where the leading coefficient of $g(x)$ is a unit. Then, there exist unique polynomials $q(x)$, $r(x) \in R[x, \theta]$ such that*

$$f(x) = q(x) * g(x) + r(x),$$

where $r(x) = 0$ *or* $deg(r(x)) < deg(g(x))$.

The definition of left divisor algorithm is similar to the right divisor algorithm.

A non-empty subset C of $\mathbb{F}_{q^2}^{\alpha} \times R^{\beta}$ is called an $\mathbb{F}_{q^2} R$-linear code if it is a left R-submodule of $\mathbb{F}_{q^2}^{\alpha} \times R^{\beta}$. Let C be an $\mathbb{F}_{q^2} R$-linear code and C_{α} (respectively, C_{β}) be the canonical projection of C on the first α (respectively, on the last β) coordinates. The code C is called separable if C is the direct product of C_{α} and C_{β}, i.e. $C = C_{\alpha} \times C_{\beta}$.

Let $x = (x_0, \ldots, x_{\alpha-1} | x'_0, \ldots, x'_{\beta-1})$, $y = (y_0, \ldots, y_{\alpha-1} | y'_0, \ldots, y'_{\beta-1}) \in \mathbb{F}_{q^2}^{\alpha} \times R^{\beta}$, where $x'_j = e_1 s_{1,j} + e_2 t_{1,j} + e_3 w_{1,j} + e_4 z_{1,j}$ and $y'_j = e_1 s_{2,j} + e_2 t_{2,j} + e_3 w_{2,j} + e_4 z_{2,j}$, for $j = 0, 1, \ldots, \beta - 1$. The Hermitian inner product between x and y is defined by

$$\langle x, y \rangle_H = \frac{1}{12}(e_1 + e_2 + e_3) \sum_{i=0}^{\alpha-1} x_i \theta(y_i) + \sum_{j=0}^{\beta-1} x'_j \theta(y'_j).$$

For an $\mathbb{F}_{q^2} R$-linear code C of length $\alpha + \beta$, its Hermitian dual code is defined by

$$C = \{y \in \mathbb{F}_{q^2}^{\alpha} \times R^{\beta} | < x, y >_H = 0, \text{for any } x \in C\}.$$

A code is called Hermitian dual containing if $C^{\perp H} \subseteq C$.

3 Skew λ-constacyclic codes over R

In this section, we discuss the structural properties of skew λ-constacyclic codes over R.

For a linear code C of length n over R, define

$$\begin{aligned}
A_1 &= \{s \in \mathbb{F}_{q^2}^n | \exists\, t, w, z \in \mathbb{F}_{q^2}^n, \text{s.t. } e_1 s + e_2 t + e_3 w + e_4 z \in C\},\\
A_2 &= \{t \in \mathbb{F}_{q^2}^n | \exists\, s, w, z \in \mathbb{F}_{q^2}^n, \text{s.t. } e_1 s + e_2 t + e_3 w + e_4 z \in C\},\\
A_3 &= \{w \in \mathbb{F}_{q^2}^n | \exists\, s, t, z \in \mathbb{F}_{q^2}^n, \text{s.t. } e_1 s + e_2 t + e_3 w + e_4 z \in C\},\\
A_4 &= \{z \in \mathbb{F}_{q^2}^n | \exists\, s, t, w \in \mathbb{F}_{q^2}^n, \text{s.t. } e_1 s + e_2 t + e_3 w + e_4 z \in C\}.
\end{aligned} \tag{1}$$

Clearly, for any $i = 1, 2, 3, 4$, A_i is a linear code of length n over $\mathbb{F}_{q^2}$. Moreover, the linear code C can be uniquely expressed as $C = e_1 A_1 \oplus e_2 A_2 \oplus e_3 A_3 \oplus e_4 A_4$. The generator matrix of C is

$$G = \begin{pmatrix} e_1 G_1 \\ e_2 G_2 \\ e_3 G_3 \\ e_4 G_4 \end{pmatrix},$$

where G_i is the generator matrix of A_i, for $i = 1, 2, 3, 4$.

It is well known that the skew polynomial ring $R[x,\theta]$ is a non-commutative ring, then the ideal $\langle x^n-\lambda\rangle$ of $R[x,\theta]$ may not be two sided, where λ is a unit in R. It is easy to show that $\langle x^n-\lambda\rangle$ is a two sided ideal if and only if n is an even integer because of $\mathrm{ord}\langle\theta\rangle=2$. However, when n is odd, $R[x,\theta]/\langle x^n-\lambda\rangle$ is a left $R[x,\theta]$-module, where the left module multiplication is given by $f(x)*(g(x)+\langle x^n-\lambda\rangle)=f(x)*g(x)+\langle x^n-\lambda\rangle$, for $f(x)$ and $g(x)\in R[x,\theta]$.

Definition 2 Let λ be a unit in R. A linear code of length n over R is called a skew λ-constacyclic code if

(i) C is a left R-submodule of R^n;
(ii) C is closed under the ρ_λ-constacyclic shift, i.e.

$$\rho_\lambda(c)=(\lambda\theta(c_{n-1}),\theta(c_0),\ldots,\theta(c_{n-2}))\in C,$$

for any codeword $c=(c_0,c_1,\ldots,c_{n-1})\in C$.

When $\lambda=1$, C is called a skew cyclic code over R. When $\lambda=-1$, C is called a skew negacyclic code over R.

Let $R_n=R[x,\theta]/\langle x^n-\lambda\rangle$. To associate the vectors of R^n with the polynomials in R_n, we define an R-module isomorphism from R^n to R_n as

$$(c_0,c_1,\ldots,c_{n-1})\mapsto c_0+c_1x+\cdots+c_{n-1}x^{n-1}.$$

According to the above discussion, we can get the following result directly.

Lemma 3 *A linear code C of length n over R is a skew λ-constacyclic code if and only if C is a left $R[x,\theta]$-submodule of R_n.*

In the following, we will identity the skew λ-constacyclic code of length n over R with a left $R[x,\theta]$-submodule of R_n.

In [43], Zheng et al. gave the sufficient and necessary condition for the existence of units in the ring $\mathbb{F}_p+u\mathbb{F}_p+v\mathbb{F}_p+uv\mathbb{F}_p$. Similarly, we give the following lemma.

Lemma 4 *Let $\lambda=a+ub+vc+uvd$ be an element in R. Then, λ is a unit in R if and only if λ_1, λ_2, λ_3 and λ_4 are units in $\mathbb{F}_{q^2}$, where $\lambda_1=a+b+c+d$, $\lambda_2=a+b$, $\lambda_3=a+c$, $\lambda_4=a$.*

Now we give some results about skew λ-constacyclic codes over R. They are significant to study the generator polynomials of skew λ-constacyclic codes over $\mathbb{F}_{q^2}R$. For the sake of convenience in writing, we denote by λ and λ_i the following elements

$$\lambda=a+ub+vc+uvd,\quad \lambda_1=a+b+c+d,\quad \lambda_2=a+b,\quad \lambda_3=a+c,\quad \lambda_4=a.$$

Theorem 1 *Let $C=e_1A_1\oplus e_2A_2\oplus e_3A_3\oplus e_4A_4$ be a linear code of length n over R. Then, C is a skew λ-constacyclic code with respect to the automorphism θ if and only if A_i is the skew λ_i-constacyclic code over $\mathbb{F}_{q^2}$, for $i=1,2,3,4$.*

Proof Let $(s_0, s_1, \ldots, s_{n-1}) \in A_1$, $(t_0, t_1, \ldots, t_{n-1}) \in A_2$, $(w_0, w_1, \ldots, w_{n-1}) \in A_3$ and $(z_0, z_1, \ldots, z_{n-1}) \in A_4$. Suppose that $c_i = e_1 s_i + e_2 t_i + e_3 w_i + e_4 z_i$, for $i = 0, 1, \ldots, n-1$. Then, the vector $(c_0, c_1, \ldots, c_{n-1}) \in C$. Since C is a skew λ-constacyclic code with respect to the automorphism θ, then we have

$$(\lambda\theta(c_{n-1}), \theta(c_0), \ldots, \theta(c_{n-2})) \in C.$$

Note that $\lambda = e_1\lambda_1 + e_2\lambda_2 + e_3\lambda_3 + e_4\lambda_4$. Thus,

$$\begin{aligned}
&(\lambda\theta(c_{n-1}), \theta(c_0), \ldots, \theta(c_{n-2}))\\
&\quad = e_1(\lambda_1\theta(s_{n-1}), \theta(s_0), \ldots, \theta(s_{n-2})) + e_2(\lambda_2\theta(t_{n-1}), \theta(t_0), \ldots, \theta(t_{n-2}))\\
&\qquad + e_3(\lambda_3\theta(w_{n-1}), \theta(w_0), \ldots, \theta(w_{n-2})) + e_4(\lambda_4\theta(z_{n-1}), \theta(z_0), \ldots, \theta(z_{n-2})).
\end{aligned}$$

Therefore, $(\lambda_1\theta(s_{n-1}), \theta(s_0), \ldots, \theta(s_{n-2})) \in A_1$, $(\lambda_2\theta(t_{n-1}), \theta(t_0), \ldots, \theta(t_{n-2})) \in A_2$, $(\lambda_3\theta(w_{n-1}), \theta(w_0), \ldots, \theta(w_{n-2})) \in A_3$ and $(\lambda_4\theta(z_{n-1}), \theta(z_0), \ldots, \theta(z_{n-2})) \in A_4$, which implies that A_i is the skew λ_i-constacyclic code over $\mathbb{F}_{q^2}$, for $i = 1, 2, 3, 4$.

On the other hand, assume that $(c_0, c_1, \ldots, c_{n-1}) \in C$, where $c_i = e_1 s_i + e_2 t_i + e_3 w_i + e_4 z_i$, for $i = 0, 1, \ldots, n-1$. By Eq. (1), we have $(s_0, s_1, \ldots, s_{n-1}) \in A_1$, $(t_0, t_1, \ldots, t_{n-1}) \in A_2$, $(w_0, w_1, \ldots, w_{n-1}) \in A_3$ and $(z_0, z_1, \ldots, z_{n-1}) \in A_4$. For any $i = 1, 2, 3, 4$, if A_i is the skew λ_i-constacyclic code over $\mathbb{F}_{q^2}$, then

$$\begin{aligned}
&(\lambda\theta(c_{n-1}), \theta(c_0), \ldots, \theta(c_{n-2}))\\
&\quad = e_1(\lambda_1\theta(s_{n-1}), \theta(s_0), \ldots, \theta(s_{n-2})) + e_2(\lambda_2\theta(t_{n-1}), \theta(t_0), \ldots, \theta(t_{n-2}))\\
&\qquad + e_3(\lambda_3\theta(w_{n-1}), \theta(w_0), \ldots, \theta(w_{n-2})) + e_4(\lambda_4\theta(z_{n-1}), \theta(z_0), \ldots, \theta(z_{n-2}))\\
&\quad \in e_1A_1 \oplus e_2A_2 \oplus e_3A_3 \oplus e_4A_4.
\end{aligned}$$

Therefore, $(\lambda\theta(c_{n-1}), \theta(c_0), \ldots, \theta(c_{n-2})) \in C$, which implies that C is a skew λ-constacyclic code over R. □

In [22], we know that a skew λ-constacyclic code of length n over $\mathbb{F}_{q^2}$ is a left $\mathbb{F}_{q^2}[x, \theta]$-submodule of $\mathbb{F}_{q^2}[x, \theta]/\langle x^n - \lambda\rangle$ generated by a monic polynomial $f(x)$ with minimal degree in C and $f(x)|_r(x^n - \lambda)$. According to the result, we have the following theorem.

Theorem 2 *Let $C = e_1A_1 \oplus e_2A_2 \oplus e_3A_3 \oplus e_4A_4$ be a skew λ-constacyclic code of length n over R. Let $A_i = \langle g_i(x)\rangle$ be the left $\mathbb{F}_{q^2}[x, \theta]$-submodule of $\mathbb{F}_{q^2}[x, \theta]/\langle x^n - \lambda_i\rangle$, for $i = 1, 2, 3, 4$. Then, $C = \langle g(x)\rangle$, where $g(x) = e_1g_1(x) + e_2g_2(x) + e_3g_3(x) + e_4g_4(x)$ with $g(x)|_r(x^n - \lambda)$.*

Proof The proof process is similar to that of Theorem 6 in [22]. □

Theorem 3 *Let $C = e_1A_1 \oplus e_2A_2 \oplus e_3A_3 \oplus e_4A_4$ be a skew λ-constacyclic code of even length over R, where λ is fixed by θ of R. Then, $C^{\perp H} = e_1A_1^{\perp H} \oplus e_2A_2^{\perp H} \oplus e_3A_3^{\perp H} \oplus e_4A_4^{\perp H}$ is a skew λ^{-1}-constacyclic code over R, where $A_i^{\perp H}$ is the skew λ_i^{-1}-constacyclic code over $\mathbb{F}_{q^2}$, for $i = 1, 2, 3, 4$.*

Proof Let $x = (x_0, x_1, \ldots, x_{n-1}) \in C$ and $y = (y_0, y_1, \ldots, y_{n-1}) \in C^{\perp H}$. Then, $\rho_\lambda^{n-1}(x) = (\lambda\theta^{n-1}(x_1), \lambda\theta^{n-1}(x_2), \ldots, \lambda\theta^{n-1}(x_{n-1}), \theta^{n-1}(x_0)) \in C$. Note that λ is fixed by θ and n is even. Thus, we have

$$\begin{aligned} 0 &= \langle \rho_\lambda^{n-1}(x), y\rangle_H \\ &= \lambda\theta(x_1)\theta(y_0) + \lambda\theta(x_2)\theta(y_1) + \cdots + \lambda\theta(x_{n-1})\theta(y_{n-2}) + \theta(x_0)\theta(y_{n-1}). \end{aligned} \tag{2}$$

From Eq. (2), we obtain

$$\begin{aligned} 0 &= \theta(\langle \rho_\lambda^{n-1}(x), y\rangle_H) \\ &= \lambda(x_1y_0 + x_2y_1 + \cdots + x_{n-1}y_{n-2} + \lambda^{-1}x_0y_{n-1}), \end{aligned}$$

which implies that $x_1y_0 + x_2y_1 + \cdots + x_{n-1}y_{n-2} + \lambda^{-1}x_0y_{n-1} = 0$. Since $\rho_{\lambda^{-1}}(y) = (\lambda^{-1}\theta(y_{n-1}), \theta(y_0), \ldots, \theta(y_{n-2}))$, then

$$\begin{aligned} \langle x, \rho_{\lambda^{-1}}(y)\rangle_H &= x_0\theta(\lambda^{-1}\theta(y_{n-1})) + x_1\theta(\theta(y_0)) + \cdots + x_{n-1}\theta(\theta(y_{n-2})) \\ &= \lambda^{-1}x_0y_{n-1} + x_1y_0 + \cdots + x_{n-1}y_{n-2} \\ &= 0. \end{aligned}$$

Therefore, $\rho_{\lambda^{-1}}(y) \in C^{\perp H}$. Consequently, $C^{\perp H}$ is a skew λ^{-1}-constacyclic code over R. Similar to the proof of Theorem 1, we can get $C^{\perp H} = e_1A_1^{\perp H} \oplus e_2A_2^{\perp H} \oplus e_3A_3^{\perp H} \oplus e_4A_4^{\perp H}$ and $A_i^{\perp H}$ is a skew λ_i^{-1}-constacyclic code over $\mathbb{F}_{q^2}$, where $i = 1, 2, 3, 4$. □

Let $a(x) = a_0 + a_1x + \cdots + a_mx^m \in \mathbb{F}_{q^2}[x, \theta]$. Define $\varphi(\sum_{i=0}^m a_ix^i) = \sum_{i=0}^m x^{-i}a_i$ and $\phi(\sum_{i=0}^m a_ix^i) = \sum_{i=0}^m \theta(a_i)x^i$, which are introduced in [9]. According to [37], we have the following result.

Lemma 5 *Let $C = \langle g(x)\rangle$ be a skew λ-constacyclic code with respect to the automorphism θ of even length n over $\mathbb{F}_{q^2}$. Let $g(x) = \sum_{i=0}^{\deg(g(x))-1} g_ix^i + x^{\deg(g(x))}$ and $h(x) = \sum_{i=0}^{\deg(h(x))-1} h_ix^i + x^{\deg(h(x))}$ such that $x^n - \lambda = h(x) * g(x)$ in $\mathbb{F}_{q^2}[x, \theta]$. Then,*

$$C^{\perp H} = \langle \theta^{\deg(h(x))+1}(h_0^{-1})\phi(x^{\deg h(x)}\varphi(h(x)))\rangle.$$

From Theorem 3 and Lemma 5, we have the following theorem.

Theorem 4 *Let $C = \langle e_1g_1(x) + e_2g_2(x) + e_3g_3(x) + e_4g_4(x)\rangle$ be a skew λ-constacyclic code with respect to automorphism θ of even length n over R. Let $g_i(x) = \sum_{j=0}^{\deg(g_i(x))-1} g_{i,j}x^j + x^{\deg(g_i(x))}$ and $h_i(x) = \sum_{j=0}^{\deg(h_i(x))-1} h_{i,j}x^j + x^{\deg(h_i(x))}$ such that $x^n - \lambda_i = h_i(x) * g_i(x)$ in $\mathbb{F}_{q^2}[x, \theta]$, for $i = 1, 2, 3, 4$. Then,*

$$C^{\perp H} = \langle e_1h_1^\dagger(x) + e_2h_2^\dagger(x) + e_3h_3^\dagger(x) + e_4h_4^\dagger(x)\rangle,$$

where $h_i^\dagger(x) = \theta^{\deg(h_i(x))+1}(h_{i,0}^{-1})\phi(x^{\deg h_i(x)}\varphi(h_i(x)))$.

Lemma 6 *Let $C = \langle g(x)\rangle$ be a skew λ-constacyclic code of even length n with respect to the automorphism θ over $\mathbb{F}_{q^2}$. Let $\lambda = \pm 1$ and $g(x) = \sum_{i=0}^{\deg(g(x))-1} g_i x^i + x^{\deg(g(x))}$, $h(x) = \sum_{i=0}^{\deg(h(x))-1} h_i x^i + x^{\deg(h(x))}$ such that $x^n - \lambda = h(x) * g(x)$ in $\mathbb{F}_{q^2}[x,\theta]$. Then, $C^{\perp_H} \subseteq C$ if and only if $x^n - \lambda|_r h^\dagger(x) * h(x)$.*

Proof Let $\lambda = \pm 1$. Since n is even and $\text{ord}\langle\theta\rangle = 2$, then $x^n - \lambda$ commutes with any skew polynomial in $\mathbb{F}_{q^2}[x,\theta]$. Thus, we have $h(x) * (x^n - \lambda) = (x^n - \lambda) * h(x)$. Since $x^n - \lambda = h(x) * g(x)$ in $\mathbb{F}_{q^2}[x,\theta]$, then

$$h(x) * (h(x) * g(x)) = (h(x) * g(x)) * h(x) = h(x) * (g(x) * h(x)).$$

As the leading coefficient of $h(x)$ is a unit in $\mathbb{F}_{q^2}[x,\theta]$, divide both sides of the above equation by $h(x)$, then we get $h(x) * g(x) = g(x) * h(x)$. Assume that $C^{\perp_H} \subseteq C$, by Lemma 5, there exists a polynomial $q(x) \in \mathbb{F}_{q^2}[x,\theta]$ such that $h^\dagger(x) = q(x) * g(x)$. Multiplying both sides of it by $h(x)$ on the right, we have $h^\dagger(x) * h(x) = q(x) * g(x) * h(x)$. Thus, $h^\dagger(x) * h(x) = q(x) * (x^n - \lambda)$ implying that $x^n - \lambda|_r h^\dagger(x) * h(x)$.

On the other hand, if $x^n - \lambda|_r h^\dagger(x) * h(x)$, then there exists a polynomial $p(x) \in \mathbb{F}_{q^2}[x,\theta]$ such that $h^\dagger(x) * h(x) = p(x) * (x^n - \lambda) = p(x) * g(x) * h(x)$, which implies that $(h^\dagger(x) - p(x) * g(x)) * h(x) = 0$. Since $h(x)$ is not a zero divisor in $\mathbb{F}_{q^2}[x,\theta]$, then $h^\dagger(x) = p(x) * g(x)$, which implies that $C^{\perp_H} \subseteq C$. □

Theorem 5 *Let $C = e_1A_1 \oplus e_2A_2 \oplus e_3A_3 \oplus e_4A_4$ be a skew λ-constacyclic code of even length β over R, where $A_1 = \langle g_1(x)\rangle$, $A_2 = \langle g_2(x)\rangle$, $A_3 = \langle g_3(x)\rangle$ and $A_4 = \langle g_4(x)\rangle$ with $x^n - \lambda_1 = h_1(x) * g_1(x)$, $x^n - \lambda_2 = h_2(x) * g_2(x)$, $x^n - \lambda_3 = h_3(x) * g_3(x)$ and $x^n - \lambda_4 = h_4(x) * g_4(x)$. For any $i = 1,2,3,4$, if $\lambda_i = \pm 1$, then $C^{\perp_H} \subseteq C$ if and only if*

$$x^n - \lambda_1|_r h_1^\dagger(x) * h_1(x),\ x^n - \lambda_2|_r h_2^\dagger(x) * h_2(x),$$
$$x^n - \lambda_3|_r h_3^\dagger(x) * h_3(x),\ x^n - \lambda_4|_r h_4^\dagger(x) * h_4(x).$$

Proof Suppose that n is even and $\lambda_i = \pm 1$, for $i = 1,2,3,4$. If $x^n - \lambda_i|_r h_i^\dagger(x) h_i(x)$, by Lemma 6, we have $A_i^{\perp_H} \subseteq A_i$, which implies that $e_iA_i^{\perp_H} \subseteq e_iA_i$, for $i = 1,2,3,4$. Thus,

$$e_1A_1^{\perp_H} \oplus e_2A_2^{\perp_H} \oplus e_3A_3^{\perp_H} \oplus e_4A_4^{\perp_H} \subseteq e_1A_1 \oplus e_2A_2 \oplus e_3A_3 \oplus e_4A_4.$$

Hence, $C^{\perp_H} \subseteq C$.

Conversely, if $C^{\perp_H} \subseteq C$, then $e_1A_1^{\perp_H} \oplus e_2A_2^{\perp_H} \oplus e_3A_3^{\perp_H} \oplus e_4A_4^{\perp_H} \subseteq e_1A_1 \oplus e_2A_2 \oplus e_3A_3 \oplus e_4A_4$. Thus, $e_iA_i^{\perp_H} \subseteq e_iA_i$, for $i = 1,2,3,4$. Therefore, $A_i^{\perp_H} \subseteq A_i$, where $i = 1,2,3,4$. By Lemma 6, we have the result. □

4 Linear skew λ-constacyclic codes over $\mathbb{F}_{q^2}R$

In this section, we study linear skew λ-constacyclic codes over $\mathbb{F}_{q^2}R$. We give the definition of $\mathbb{F}_{q^2}R$-linear codes first.

Let $e_1 = uv, e_2 = u-uv, e_3 = v-uv, e_4 = 1-u-v+uv$. Since $e_1+e_2+e_3+e_4 = 1$, then for any $c = (x|y) \in \mathbb{F}_{q^2}^{\alpha} \times R^{\beta}$, we have $c = \left(\sum_{i=1}^{4} e_i x | y\right)$, where $x \in \mathbb{F}_{q^2}^{\alpha}$ and $y = e_1 s + e_2 t + e_3 w + e_4 z \in R^{\beta}$. For a linear code C of length $\alpha+\beta$ over $\mathbb{F}_{q^2}R$, define

$$
\begin{aligned}
C_1 &= \left\{(x|s) \in \mathbb{F}_{q^2}^{\alpha} \times \mathbb{F}_{q^2}^{\beta} \,|x \in \mathbb{F}_{q^2}^{\alpha}, s \in A_1\right\},\\
C_2 &= \left\{(x|t) \in \mathbb{F}_{q^2}^{\alpha} \times \mathbb{F}_{q^2}^{\beta} \,|x \in \mathbb{F}_{q^2}^{\alpha}, t \in A_2\right\},\\
C_3 &= \left\{(x|w) \in \mathbb{F}_{q^2}^{\alpha} \times \mathbb{F}_{q^2}^{\beta} \,|x \in \mathbb{F}_{q^2}^{\alpha}, w \in A_3\right\},\\
C_4 &= \left\{(x|z) \in \mathbb{F}_{q^2}^{\alpha} \times \mathbb{F}_{q^2}^{\beta} \,|x \in \mathbb{F}_{q^2}^{\alpha}, z \in A_4\right\},
\end{aligned}
$$

where A_i is defined as (1), for $i = 1, 2, 3, 4$. A linear code C of length $\alpha+\beta$ over $\mathbb{F}_{q^2}R$ can be expressed as $C = e_1C_1 \oplus e_2C_2 \oplus e_3C_3 \oplus e_4C_4$.

Definition 3 Let θ be an automorphisms of R and λ be a unit in R. A code C is called an $\mathbb{F}_{q^2}R$-linear skew λ-constacyclic codes of length $\alpha+\beta$ if

(i) C is a left R-submodule of $\mathbb{F}_{q^2}^{\alpha} \times R^{\beta}$;
(ii) C is closed under the $T_{\theta,\lambda}$-constacyclic shift, i.e.

$$T_{\theta,\lambda}(c) = (c_{\alpha-1}, c_0, \ldots, c_{\alpha-2}|\lambda\theta(c'_{\beta-1}), \theta(c'_0), \ldots, \theta(c'_{\beta-2})) \in C,$$

where $c = (c_0, c_1, \ldots, c_{\alpha-1}|c'_0, c'_1, \ldots, c'_{\beta-1}) \in C$ with $(c_0, c_1, \ldots, c_{\alpha-1}) \in \mathbb{F}_{q^2}^{\alpha}$ and $(c'_0, c'_1, \ldots, c'_{\beta-1}) \in R^{\beta}$.

Let $R_{\alpha,\beta} = \mathbb{F}_{q^2}[x]/\langle x^{\alpha}-1\rangle \times R[x,\theta]/\langle x^{\beta}-\lambda\rangle$. To associate the vectors of $\mathbb{F}_{q^2}^{\alpha} \times R^{\beta}$ with the polynomials in $R_{\alpha,\beta}$, we define an R-module isomorphism from $\mathbb{F}_{q^2}^{\alpha} \times R^{\beta}$ to $R_{\alpha,\beta}$ as

$$
\begin{aligned}
&(c_0, c_1, \ldots, c_{\alpha-1}|c'_0, c'_1, \ldots, c'_{\beta-1})\\
&\quad\mapsto (c_0 + c_1x + \cdots + c_{\alpha-1}x^{\alpha-1}|c'_0 + c'_1x + \cdots + c'_{\beta-1}x^{\beta-1}).
\end{aligned}
$$

Let $f(x) = f_0 + f_1x + \cdots + f_tx^t \in R[x,\theta]$ and $(c(x)|c'(x)) \in R_{\alpha,\beta}$. Define the multiplication operation

$$f(x)\star(c(x)|c'(x)) = (\delta(f(x))c(x)|f(x) * c'(x)),$$

where $\delta(f(x)) = \delta(f_0)+\delta(f_1)x+\cdots+\delta(f_t)x^t$ and $\delta(f(x))c(x) \in \mathbb{F}_{q^2}[x]/\langle x^{\alpha}-1\rangle$, $f(x) * c'(x) \in R[x,\theta]/\langle x^{\beta}-\lambda\rangle$. From the above discussion, we give the polynomial definition of $\mathbb{F}_{q^2}R$-linear skew λ-constacyclic codes as follows.

Definition 4 A code C is called an $\mathbb{F}_{q^2}R$-linear skew λ-constacyclic code of length $\alpha+\beta$ if

(i) C is a left R-submodule of $R_{\alpha,\beta}$;
(ii) If $(c(x)|c'(x)) \in C$, then

$$\begin{aligned} x\star(c(x)|c'(x)) &= (xc(x)|x * c'(x)) \\ &= (c_{\alpha-1} + c_0 x + \cdots + c_{\alpha-2}x^{\alpha-1}|\lambda\theta(c'_{\beta-1}) + \theta(c'_0)x \\ &\quad + \cdots + \theta(c'_{\beta-2})x^{\beta-1}) \in C, \end{aligned}$$

where $c(x) = c_0 + c_1 x + \cdots + c_{\alpha-1}x^{\alpha-1} \in \mathbb{F}_{q^2}[x]/\langle x^\alpha - 1\rangle$ and $c'(x) = c'_0 + c'_1 x + \cdots + c'_{\beta-1}x^{\beta-1} \in R[x,\theta]/\langle x^\beta - \lambda\rangle$.

By the above multiplication operation, we have the following result.

Lemma 7 *A code C is a linear skew λ-constacyclic code of length $\alpha+\beta$ over $\mathbb{F}_{q^2}R$ if and only if C is a left $R[x,\theta]$-submodule of $R_{\alpha,\beta}$.*

Theorem 6 *Let $C = e_1C_1 \oplus e_2C_2 \oplus e_3C_3 \oplus e_4C_4$ be a linear code of length $\alpha+\beta$ over $\mathbb{F}_{q^2}R$. Then, C is a skew λ-constacyclic code over $\mathbb{F}_{q^2}R$ if and only if C_i is the skew λ_i-constacyclic code of length $\alpha+\beta$ over $\mathbb{F}_{q^2}$, where $i = 1,2,3,4$.*

Proof Let $c = (c_0, c_1, \ldots, c_{\alpha-1}|c'_0, c'_1, \ldots, c'_{\beta-1}) \in C$, where $c'_j = e_1 s_j + e_2 t_j + e_3 w_j + e_4 z_j$, for $j = 0, 1, \ldots, \beta-1$. Then, the codeword c can be expressed as $c = e_1x_1 + e_2x_2 + e_3x_3 + e_4x_4$, where

$$\begin{aligned} x_1 &= (c_0, c_1, \ldots, c_{\alpha-1}|s_0, s_1, \ldots, s_{\beta-1}) \in C_1, \\ x_2 &= (c_0, c_1, \ldots, c_{\alpha-1}|t_0, t_1, \ldots, t_{\beta-1}) \in C_2, \\ x_3 &= (c_0, c_1, \ldots, c_{\alpha-1}|w_0, w_1, \ldots, w_{\beta-1}) \in C_3, \\ x_4 &= (c_0, c_1, \ldots, c_{\alpha-1}|z_0, z_1, \ldots, z_{\beta-1}) \in C_4. \end{aligned} \tag{3}$$

Assume that C is a skew λ-constacyclic code over $\mathbb{F}_{q^2}R$, then $T_{\theta,\lambda}(c) = (c_{\alpha-1}, c_0, \ldots, c_{\alpha-2}|\lambda\theta(c'_{\beta-1}), \theta(c'_0), \ldots, \theta(c'_{\beta-2})) \in C$. Note that

$$\begin{aligned} &\lambda\theta(c'_{\beta-1}) \\ &= \lambda(e_1\theta(s_{\beta-1}) + e_2\theta(t_{\beta-1}) + e_3\theta(w_{\beta-1}) + e_4\theta(z_{\beta-1})) \\ &= e_1\lambda_1\theta(s_{\beta-1}) + e_2\lambda_2\theta(t_{\beta-1}) + e_3\lambda_3\theta(w_{\beta-1}) + e_4\lambda_4\theta(z_{\beta-1}). \end{aligned}$$

Then, $T_{\theta,\lambda}(c) = e_1y_1 + e_2y_2 + e_3y_3 + e_4y_4$, where

$$\begin{aligned} y_1 &= (c_{\alpha-1}, c_0, \ldots, c_{\alpha-2}|\lambda_1\theta(s_{\beta-1}), \theta(s_0), \ldots, \theta(s_{\beta-2})) \in C_1, \\ y_2 &= (c_{\alpha-1}, c_0, \ldots, c_{\alpha-2}|\lambda_2\theta(t_{\beta-1}), \theta(t_0), \ldots, \theta(t_{\beta-2})) \in C_2, \\ y_3 &= (c_{\alpha-1}, c_0, \ldots, c_{\alpha-2}|\lambda_3\theta(w_{\beta-1}), \theta(w_0), \ldots, \theta(w_{\beta-2})) \in C_3, \\ y_4 &= (c_{\alpha-1}, c_0, \ldots, c_{\alpha-2}|\lambda_4\theta(z_{\beta-1}), \theta(z_0), \ldots, \theta(z_{\beta-2})) \in C_4. \end{aligned} \tag{4}$$

Combining (3) and (4), we can get C_i is the skew λ_i-constacyclic code in $\mathbb{F}_{q^2}^{\alpha} \times \mathbb{F}_{q^2}^{\beta}$, for $i = 1,2,3,4$.

Conversely, it has the similar proof, so we omit it. □

In the following, we consider the generators and the minimal spanning sets of linear skew λ-constacyclic codes over $\mathbb{F}_{q^2}R$. The proof process is similar to that of Theorem 4 in [5].

Theorem 7 *Let C be a linear skew λ-constacyclic code of length $\alpha+\beta$ over $\mathbb{F}_{q^2}R$. Then,*

$$C = \langle (f(x)|0), (l(x)|g(x)) \rangle,$$

where $f(x), l(x) \in \mathbb{F}_{q^2}[x]/\langle x^\alpha - 1\rangle$, $\deg(l(x)) < \deg(f(x))$, $f(x)|(x^\alpha - 1)$, $g(x) = e_1g_1(x) + e_2g_2(x) + e_3g_3(x) + e_4g_4(x)$, $g(x)|_r(x^\beta - \lambda)$ and $x^\beta - \lambda_i = h_i(x) * g_i(x)$, $i = 1, 2, 3, 4$.

Proof Let C be an $\mathbb{F}_{q^2}R$-linear skew constacyclic code of length $\alpha+\beta$. Define

$$\begin{aligned} \psi : \quad C &\to R[x,\theta]/\langle x^\beta - \lambda\rangle \\ (v(x)|v'(x)) &\mapsto v'(x), \end{aligned}$$

where $v(x) \in \mathbb{F}_{q^2}[x]/\langle x^\alpha - 1\rangle$ and $v'(x) \in R[x,\theta]/\langle x^\beta - \lambda\rangle$. For any $p(x) \in R[x,\theta]$, we have $\psi(p(x)\star(v(x)|v'(x))) = p(x) * \psi(v(x)|v'(x))$. Thus, ψ is a left $R[x,\theta]$-module homomorphism whose image is a left $R[x,\theta]$-submodule of $R[x,\theta]/\langle x^\beta - \lambda\rangle$. By Lemma 3 and Theorem 2, we obtain that $\psi(C) = \langle g(x)\rangle$, where $g(x) = e_1g_1(x) + e_2g_2(x) + e_3g_3(x) + e_4g_4(x)$ with $g(x)|_r(x^\beta - \lambda)$.

Define the set I to be

$$I = \left\{ f(x) \in \mathbb{F}_{q^2}[x]/\langle x^\alpha - 1\rangle | (f(x), 0) \in \ker(\psi) \right\}.$$

Clearly, I is an ideal of $\mathbb{F}_{q^2}[x]/\langle x^\alpha - 1\rangle$. Hence, I is a cyclic code in $\mathbb{F}_{q^2}[x]/\langle x^\alpha - 1\rangle$, which implies that $I = \langle f(x)\rangle$, where $f(x)$ is a divisor of $x^\alpha - 1$. For any $(v(x), 0) \in \ker(\psi)$, we have that $v(x) \in I = \langle f(x)\rangle$. Therefore, there exists a polynomial $p(x) \in R[x,\theta]$ such that $v(x) = \delta(p(x))f(x)$. Thus, $(v(x)|0) = p(x)\star(f(x)|0)$ which implies that $\ker(\psi)$ is a submodule of C generated by one element of the form $(f(x)|0)$, i.e. $\ker(\psi) = \langle (f(x)|0)\rangle$, where $f(x) \in \mathbb{F}_{q^2}[x]$ and $f(x)|(x^\alpha - 1)$. By the theorem of isomorphism, we have

$$C/\ker(\psi) \cong \langle g(x)\rangle.$$

Let $(l(x)|g(x)) \in C$ such that $\psi(l(x)|g(x)) = g(x)$. Then, C can be generated as a left $R[x,\theta]$-submodule of $R_{\alpha,\beta}$ by two elements of the form $(f(x)|0)$ and $(l(x)|g(x))$. Thus, any element in C can be written as

$$c(x)\star(f(x)|0) + d(x)\star(l(x)|g(x)),$$

where $c(x), d(x) \in R[x,\theta]$. Consequently,

$$C = \langle (f(x)|0), (l(x)|g(x)) \rangle.$$

Finally, we show that $\deg(l(x)) < \deg(f(x))$. Let $C = \langle (f(x)|0), (l(x)|g(x))\rangle$. Suppose that $\deg(l(x)) \geq \deg(f(x))$ and $\deg(l(x)) - \deg(f(x)) = t$. Let $D = \langle (f(x)|0), (l(x)|g(x)) + x^t \star (f(x), 0))\rangle$. Then, it can be regarded as

$$D = \langle (f(x)|0), (l(x) + x^t f(x)|g(x))\rangle.$$

Clearly, $D \subseteq C$. Moreover, $(l(x)|g(x)) = (l(x)+x^t f(x)|g(x)) - x^t \star (f(x)|0)$, which implies that $C \subseteq D$. Therefore, $C = D$, which implies a contradiction. □

Proposition 1 *Let the notations be the ones defined in Theorem 7. Then, we have* $f(x)|h_4(x)l(x)$ *in* $\mathbb{F}_{q^2}[x]/\langle x^\alpha - 1\rangle$.

Proof Let $x^\beta - \lambda_i = h_i(x) * g_i(x)$, for $i = 1, 2, 3, 4$. Then,

$$\begin{aligned}
&(e_1h_1(x) + e_2h_2(x) + e_3h_3(x) + e_4h_4(x)) * g(x)\\
&\quad = e_1h_1(x) * g_1(x) + e_2h_2(x) * g_2(x) + e_3h_3(x) * g_3(x) + e_4h_4(x) * g_4(x)\\
&\quad = e_1(x^\beta - \lambda_1) + e_2(x^\beta - \lambda_2) + e_3(x^\beta - \lambda_3) + e_4(x^\beta - \lambda_4) \qquad (5)\\
&\quad = (e_1 + e_2 + e_3 + e_4)x^\beta - (e_1\lambda_1 + e_2\lambda_2 + e_3\lambda_3 + e_4\lambda_4)\\
&\quad = x^\beta - \lambda.
\end{aligned}$$

By Eq. (5), we obtain

$$\begin{aligned}
&(e_1h_1(x) + e_2h_2(x) + e_3h_3(x) + e_4h_4(x)) \star (l(x)|g(x))\\
&\quad = (\delta(e_1h_1(x) + e_2h_2(x) + e_3h_3(x) + e_4h_4(x))l(x)|0)\\
&\quad = (h_4(x)l(x)|0) \in \ker(\psi).
\end{aligned}$$

From Theorem 7, we have that $f(x)|h_4(x)l(x)$ in $\mathbb{F}_{q^2}[x]/\langle x^\alpha - 1\rangle$. □

Theorem 8 *Let* $C = \langle (f(x)|0), (l(x)|g(x))\rangle$ *be a linear skew* λ*-constacyclic code of length* $\alpha + \beta$ *over* $\mathbb{F}_{q^2}R$*, where* β *is an even integer and* $g(x) = e_1g_1(x) + e_2g_2(x) + e_3g_3(x) + e_4g_4(x)$, $g(x)|_r(x^\beta - \lambda)$, $x^\beta - \lambda_i = h_i(x) * g_i(x)$, $i = 1, 2, 3, 4$. *Suppose that*

$$\begin{aligned}
S_1 &= \bigcup_{i=0}^{\alpha - \deg(f(x)) - 1} \{x^i \star (f(x)|0)\},\\
S_2 &= \bigcup_{i=0}^{\beta - \deg(h(x)) - 1} \{x^i \star (l(x)|g(x))\}.
\end{aligned}$$

Then,

$$S = S_1 \cup S_2$$

forms a minimal spanning set for C *with* $|C| = q^{2(\alpha - \deg(f(x)))}q^{4(\beta - \deg(h(x)))}$, *where* $h(x) = e_1h_1(x) + e_2h_2(x) + e_3h_3(x) + e_4h_4(x)$.

Proof Let $c(x) \in C = \langle (f(x)|0), (l(x)|g(x))\rangle$. Then, there exist polynomials $a(x)$, $b(x) \in R[x,\theta]$ such that $c(x) = a(x)\star(f(x)|0) + b(x)\star(l(x)|g(x))$. If $\deg(a(x)) \leq \alpha - \deg(f) - 1$, then $c(x) = a(x)\star(f(x)|0) \in \mathrm{Span}(S_1)$. Otherwise, by the right divisor algorithm, there exist polynomials $q(x)$ and $r(x) \in R[x,\theta]$ such that

$$\delta(a(x)) = \delta(q(x))\frac{x^\alpha - 1}{f(x)} + \delta(r(x)),$$

where $\delta(r(x)) = 0$ or $\deg(\delta(r(x))) < \deg\left(\frac{x^\alpha-1}{f(x)}\right)$. Hence,

$$\begin{aligned} a(x)\star(f(x)|0) &= (\delta(a(x))f(x)|0) \\ &= \left((\delta(q(x))\frac{x^\alpha - 1}{f(x)} + \delta(r(x)))f(x)|0\right) \\ &= (\delta(r(x))f(x)|0). \end{aligned}$$

Since $\deg(\delta(r(x))) < \deg\left(\frac{x^\alpha-1}{f(x)}\right)$, then $a(x)\star(f(x)|0) \in \mathrm{Span}(S_1)$.

Let $b(x) \in R[x,\theta]$. If $\deg(b(x)) \leq \beta - \deg(h(x)) - 1$, then $b(x)\star(l(x)|g(x)) \in \mathrm{Span}(S_2)$. Otherwise, by the right division algorithm, there exist polynomials $q_1(x)$, $r_1(x) \in R[x,\theta]$ such that

$$b(x) = q_1(x) * h(x) + r_1(x),$$

where $r_1(x) = 0$ or $\deg(r_1(x)) < \deg(h(x))$. Note that $h(x) * g(x) = x^\beta - \lambda$ in $R[x,\theta]/\langle x^\beta - \lambda\rangle$. Thus, $b(x)\star(l(x)|g(x)) = q_1(x)\star(\delta(h(x))l(x)|0) + r_1(x)\star(l(x)|g(x))$. Since $r_1(x)\star(l(x)|g(x)) \in \mathrm{Span}(S_2)$, by Proposition 1, we get $q_1(x)\star(\delta(h(x))l(x)|0) = q_1(x)\star(h_4(x)l(x)|0) \in \mathrm{Span}(S_1)$. Consequently, $c(x) = a(x)\star(f(x)|0) + b(x)\star(l(x)|g(x)) \in \mathrm{Span}(S_1 \cup S_2)$ and it is easy to show $|C| = q^{2(\alpha-\deg(f(x)))}q^{4(\beta-\deg(h(x)))}$. □

5 The Gray images of linear skew λ-constacyclic codes over $\mathbb{F}_{q^2}R$

For any $r = e_1s + e_2t + e_3w + e_4z \in R$, r can be expressed as $r = (s,t,w,z) \in \mathbb{F}_{q^2}^4$. Let $M = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix}$, $M^\top$ denotes the transpose matrix of M. Firstly, we need a Gray map from R to $\mathbb{F}_{q^2}^4$ given by $\Phi(r) = (s,t,w,z)M = rM$, where $r = e_1s + e_2t + e_3w + e_4z$. Here, for the sake of convenience in writing, we use r in place of vector (s,t,w,z). It can be extended to

$$\begin{aligned} &\Phi : \mathbb{F}_{q^2}^\alpha \times R^\beta \longrightarrow \mathbb{F}_{q^2}^{\alpha+4\beta} \\ &c = (c_0,\ldots,c_{\alpha-1}|c_0',\ldots,c_{\beta-1}') \mapsto (c_0,\ldots,c_{\alpha-1}|c_0'M,\ldots,c_{\beta-1}'M), \end{aligned}$$

where $c'_j = e_1 s_j + e_2 t_j + e_3 w_j + e_4 z_j$ and $c'_j M = (s_j, t_j, w_j, z_j)M$, for $j = 0, 1, \ldots, \beta - 1$. The Lee weight of an element $c = (c_0, \ldots, c_{\alpha-1}|c'_0, \ldots, c'_{\beta-1}) \in \mathbb{F}_{q^2}^{\alpha} \times R^{\beta}$ is defined as the Hamming weight of the extended Gray image, i.e.

$$w_L(c) = \sum_{i=0}^{\alpha-1} w_H(c_i) + \sum_{j=0}^{\beta-1} w_H(\Phi(c'_j)).$$

The Lee distance between two vectors x and y in $\mathbb{F}_{q^2}^{\alpha} \times R^{\beta}$ is defined as $d_L(x, y) = w_L(x - y)$. Based on the above definitions, we have the following result.

Proposition 2 *Let Φ be the Gray map defined above.*

(i) *Φ is an $\mathbb{F}_{q^2}$-linear distance preserving map from $\mathbb{F}_{q^2}^{\alpha} \times R^{\beta}$ (Lee distance) to $\mathbb{F}_{q^2}^{\alpha+4\beta}$ (Hamming distance).*

(ii) *If C is an $(\alpha+\beta, M, d_L)$ linear code over $\mathbb{F}_{q^2} R$, then $\Phi(C)$ is an $[\alpha+4\beta, \log_{q^2}^{M}, d_L]$ linear code over $\mathbb{F}_{q^2}$, where M denotes the number of codewords in C.*

Proof Let $x = (x_0, \ldots, x_{\alpha-1}|x'_0, \ldots, x'_{\beta-1})$ and $y = (y_0, \ldots, y_{\alpha-1}|y'_0, \ldots, y'_{\beta-1}) \in \mathbb{F}_{q^2}^{\alpha} \times R^{\beta}$, where $x'_j = e_1 s_{1,j} + e_2 t_{1,j} + e_3 w_{1,j} + e_4 z_{1,j}$ and $y'_j = e_1 s_{2,j} + e_2 t_{2,j} + e_3 w_{2,j} + e_4 z_{2,j}$, for $j = 0, 1, \ldots, \beta - 1$. Then, from the definition of the Gray map Φ, we have

$$\begin{aligned}
&\Phi(x + y)\\
&\quad = (x_0 + y_0, \ldots, x_{\alpha-1} + y_{\alpha-1}|(x'_0 + y'_0)M, \ldots, (x'_{\beta-1} + y'_{\beta-1})M)\\
&\quad = (x_0, \ldots, x_{\alpha-1}|x'_0 M, \ldots, x'_{\beta-1}M) + (y_0, \ldots, y_{\alpha-1}|y'_0 M, \ldots, y'_{\beta-1}M)\\
&\quad = \Phi(x) + \Phi(y).
\end{aligned}$$

Moreover, for any $a \in \mathbb{F}_{q^2}$, we have

$$\begin{aligned}
\Phi(ax) &= \Phi(ax_0, \ldots, ax_{\alpha-1}|ax'_0, \ldots, ax'_{\beta-1})\\
&= (ax_0, \ldots, ax_{\alpha-1}|ax'_0 M, \ldots, ax'_{\beta-1}M)\\
&= a\Phi(x).
\end{aligned}$$

Therefore, Φ is an $\mathbb{F}_{q^2}$-linear map. It is easy to show that Φ is an $\mathbb{F}_{q^2}$-linear distance preserving map. □

Proposition 3 *Let C be a linear Hermitian self-orthogonal code of length $\alpha + \beta$ over $\mathbb{F}_{q^2} R$. Then, $\Phi(C)$ is a linear Hermitian self-orthogonal code of length $\alpha + 4\beta$ over $\mathbb{F}_{q^2}$.*

Proof Let $x = (x_0, \ldots, x_{\alpha-1}|x'_0, \ldots, x'_{\beta-1})$, $y = (y_0, \ldots, y_{\alpha-1}|y'_0, \ldots, y'_{\beta-1}) \in \mathbb{F}_{q^2}^{\alpha} \times R^{\beta}$, where $x'_j = e_1 s_{1,j} + e_2 t_{1,j} + e_3 w_{1,j} + e_4 z_{1,j}$ and $y'_j = e_1 s_{2,j} + e_2 t_{2,j} +$

$e_3w_{2,j} + e_4z_{2,j}$, for $j = 0, 1, \ldots, \beta - 1$. If C is a linear Hermitian self-orthogonal code over $\mathbb{F}_{q^2}R$, then

$$\langle x, y\rangle_H = \frac{1}{12}(e_1 + e_2 + e_3)\sum_{i=0}^{\alpha-1} x_i\theta(y_i) + \sum_{j=0}^{\beta-1} x'_j\theta(y'_j) = 0,$$

which implies that

$$\begin{aligned}
\frac{1}{12}\sum_{i=0}^{\alpha-1} x_i\theta(y_i) + \sum_{j=0}^{\beta-1} s_{1,j}\theta(s_{2,j}) &= 0,\\
\frac{1}{12}\sum_{i=0}^{\alpha-1} x_i\theta(y_i) + \sum_{j=0}^{\beta-1} t_{1,j}\theta(t_{2,j}) &= 0,\\
\frac{1}{12}\sum_{i=0}^{\alpha-1} x_i\theta(y_i) + \sum_{j=0}^{\beta-1} w_{1,j}\theta(w_{2,j}) &= 0,\\
\sum_{j=0}^{\beta-1} z_{1,j}\theta(z_{2,j}) &= 0.
\end{aligned} \tag{6}$$

By Eq. (6), we obtain

$$\begin{aligned}
\sum_{j=0}^{\beta-1}(s_{1,j}\theta(s_{2,j}) + t_{1,j}\theta(t_{2,j}) + w_{1,j}\theta(w_{2,j})) &= -\frac{1}{4}\sum_{i=0}^{\alpha-1} x_i\theta(y_i),\\
\sum_{j=0}^{\beta-1} z_{1,j}\theta(z_{2,j}) &= 0.
\end{aligned}$$

Let $\theta(M) = (\theta(m_{i,j}))_{0\le i,j\le 3}$ for $M = (m_{i,j})_{0\le i,j\le 3}$. Note that

$$\begin{aligned}
&\langle \Phi(x), \Phi(y)\rangle_H\\
&= \sum_{i=0}^{\alpha-1} x_i\theta(y_i) + \sum_{j=0}^{\beta-1} x'_j M\theta(M)^\top\theta(y'_j)^\top\\
&= \sum_{i=0}^{\alpha-1} x_i\theta(y_i) + 4\sum_{j=0}^{\beta-1}(s_{1,j}\theta(s_{2,j}) + t_{1,j}\theta(t_{2,j}) + w_{1,j}\theta(w_{2,j}) + z_{1,j}\theta(z_{2,j}))\\
&= 0.
\end{aligned}$$

Therefore, $\Phi(C)$ is a linear Hermitian self-orthogonal code of length $\alpha + 4\beta$ over $\mathbb{F}_{q^2}$. □

6 Quantum codes from linear skew constacyclic codes over $\mathbb{F}_{q^2}R$

Let $\gcd(n, q) = 1$ and $m = \mathrm{ord}_n(q^2)$. Then, $\mathbb{F}_{q^{2m}}$ contains a primitive n*th* root of unity η and $x^n - 1 = \prod_{i=0}^{n-1}(x - \eta^i)$. Let s be an integer with $0 \leq s < n$. The q^2-cyclotomic coset mod n containing s is defined by $C_s = \{s, sq^2, s(q^2)^2, \ldots, s(q^2)^{r-1}\}$, where r is the smallest positive integer such that $s(q^2)^r \equiv s (\mathrm{mod}\ n)$.

Let $C = \langle g(x)\rangle$ be a cyclic code of length n over $\mathbb{F}_{q^2}$, where $g(x) = \prod_s \prod_{i\in C_s}(x - \eta^i)$ and s run through some subsets of q^2-cyclotomic cosets mod n. Let

$$Z = \left\{ i | g(\eta^i) = 0,\ for\ 0 \leq i \leq n-1 \right\}.$$

The set Z is called the defining set of C. Since $C^{\perp H} = (C^q)^{\perp}$, then the defining set of $C^{\perp H}$ is given by $Z^{-q} = \{-qZ(\mathrm{mod}\ n)|z \in Z\}$.

In [30], Mi et al. gave a sufficient and necessary condition for the existence of Hermitian dual-containing cyclic codes over $\mathbb{F}_{q^2}$ as follows.

Lemma 8 [30] *Let* $\gcd(q, n) = 1$. *A cyclic code of length n over* $\mathbb{F}_{q^2}$ *with defining set Z contains its Hermitian dual code if and only if* $Z \bigcap Z^{-q} = \emptyset$, *where* $Z^{-q} = \{-qZ(mod\ n)|z \in Z\}$.

Let C_α be a cyclic code over $\mathbb{F}_{q^2}$ and C_β be a skew λ-constacyclic code over R, respectively. If C is separable, then $C = C_\alpha \times C_\beta$, i.e. $C = \langle (f(x)|0), (0|g(x))\rangle$, where $C_\alpha = \langle f(x)\rangle$ with $f(x)|(x^\alpha - 1)$ and $C_\beta = \langle g(x)\rangle$ with $g(x)|_r(x^\beta - \lambda)$.

Lemma 9 *Let* $C = C_\alpha \times C_\beta$ *be a separable linear code of length* $\alpha + \beta$ *over* $\mathbb{F}_{q^2}R$. *Then,* $C^{\perp H} \subseteq C$ *if and only if* $C_\alpha^{\perp H} \subseteq C_\alpha$ *and* $C_\beta^{\perp H} \subseteq C_\beta$.

Theorem 9 [27] *Let* C_1 *and* C_2 *be* $[n, k_1, d_1]_{q^2}$ *and* $[n, k_2, d_2]_{q^2}$ *linear codes, respectively, where* $C_2^{\perp H} \subseteq C_1$. *Then, there exists a quantum error-correcting code C with parameters* $[[n, k_1 + k_2 - n, \geq \min\{d_1, d_2\}]]_q$. *In particular, if* $C_1^{\perp H} \subseteq C_1$, *then there exists a quantum error-correcting code with parameters* $[[n, 2k_1 - n, \geq d_1]]_q$.

Assume that $\gcd(\alpha, n) = 1$. Let C_α be a cyclic code of length α over $\mathbb{F}_{q^2}$ with $C_\alpha = \langle g_\alpha(x)\rangle$. Let C_β be a skew λ-constacyclic code with respect to θ of even length β over R and $C_\beta = \langle g_\beta(x)\rangle$, where $g_\beta(x) = e_1g_1 + e_2g_2 + e_3g_3 + e_4g_4$ with $x^\beta - \lambda_i = h_i(x) * g_i(x)$, for $i = 1, 2, 3, 4$ and $\lambda_i = \pm 1$. By Theorems 5 and 9, Proposition 3, Lemmas 8 and 9, we can get the following theorem.

Theorem 10 *Let* $C = C_\alpha \times C_\beta$ *be a* $(\alpha + \beta, M, d_L)$ *separable linear skew* λ-*constacyclic code over* $\mathbb{F}_{q^2}R$, *where* d_L *is the minimum Lee distance of C. If*

(i) $Z \bigcap Z^{-q} = \emptyset$;
(ii)

$$x^\beta - \lambda_1|_r h_1^\dagger(x) * h_1(x), \quad x^\beta - \lambda_2|_r h_2^\dagger(x) * h_2(x),$$
$$x^\beta - \lambda_3|_r h_3^\dagger(x) * h_3(x), \quad x^\beta - \lambda_4|_r h_4^\dagger(x) * h_4(x),$$

where $Z^{-q} = \{-qZ(mod\ n)|z \in Z\}$. *Then, there exists a quantum error-correcting code* C *with parameters* $[[\alpha + 4\beta, 2k - (\alpha + 4\beta), \geq d_L]]$, *where* k *is the dimension of the code* $\Phi(C)$ *and* d_L *is the minimum Hamming distance of* $\Phi(C)$.

Example 1 Let $C_\alpha = \langle g_\alpha(x)\rangle$ be a cyclic code of length 8 over $\mathbb{F}_{25}$, where $\mathbb{F}_{25} = \mathbb{F}_5[w]$ with $w^2 = w + 3$. Assume that $Z = \{1, 2\}$ is the defining set of C_α. Then, $g_\alpha(x) = x^2 + wx + w^9$. Since $Z^{-5} = \{3, 6\}$, then $Z \cap Z^{-5} = \emptyset$. By Lemma 8, C_α is a Hermitian dual-containing code with parameters $[8, 6, 3]_{25}$.

Let $R = \mathbb{F}_{25} + u\mathbb{F}_{25} + v\mathbb{F}_{25} + v\mathbb{F}_{25} + uv\mathbb{F}_{25}$, where $u^2 = u$, $v^2 = v$, $uv = vu$ and $\mathbb{F}_{25} = \mathbb{F}_5[w]$ with $w^2 = w+3$. Let θ be an automorphism of R denoted by $\theta(a) = a^5$ for every element $a \in \mathbb{F}_{25}$. Let $\beta = 8$. Then, we have

$$
\begin{aligned}
x^8 - 1 &= (x + 1) * (x + 4) * (x + 2) * (x + 3) * (x + w^7) * (x + w^{23}) * (x + w^9)^2,\\
x^8 - 1 &= (x + 2) * (x + 3) * (x + w^4) * (x + w^8) * (x + w^9)^2 * (x + w^3)^2,\\
x^8 - 1 &= (x + w^8) * (x + w^4) * (x + w) * (x + w^{17}) * (x + w^{23}) * (x + w^7)\\
&\quad * (x + w^{22}) * (x + w^2),\\
x^8 - 1 &= (x + w^4) * (x + w^8) * (x + w^9)^2 * (x + w^{14}) * (x + w^{10}) * (x + w^{15})^2.
\end{aligned}
$$

Let $C_\beta = e_1C_1 \oplus e_2C_2 \oplus e_3C_3 \oplus e_4C_4$ be a skew cyclic code of length 8 over R, where $C_1 = \langle g_1(x)\rangle$, $C_2 = \langle g_2(x)\rangle$, $C_3 = \langle g_3(x)\rangle$, $C_4 = \langle g_4(x)\rangle$ with $g_1(x) = x + w^9$, $g_2(x) = x + w^3$, $g_3(x) = x + w^2$ and $g_4(x) = x + w^{15}$. Since

$$
\begin{aligned}
h_1(x) &= x^7 + w^9x^6 + w^6x^5 + w^{15}x^4 + w^{12}x^3 + w^{21}x^2 + w^{18}x + w^3,\\
h_2(x) &= x^7 + w^3x^6 + w^{18}x^5 + w^{21}x^4 + w^{12}x^3 + w^{15}x^2 + w^6x + w^9,\\
h_3(x) &= x^7 + w^{22}x^6 + w^{12}x^5 + w^{10}x^4 + x^3 + w^{22}x^2 + w^{12}x + w^{10},\\
h_4(x) &= x^7 + w^{15}x^6 + w^{18}x^5 + w^9x^4 + w^{12}x^3 + w^3x^2 + w^6x + w^{21}.
\end{aligned}
$$

and

$$
\begin{aligned}
h_1^\dagger(x) &= x^7 + w^{15}x^6 + w^{18}x^5 + w^9x^4 + w^{12}x^3 + w^3x^2 + w^6x + w^{21},\\
h_2^\dagger(x) &= x^7 + w^{21}x^6 + w^6x^5 + w^3x^4 + w^{12}x^3 + w^9x^2 + w^{18}x + w^{15},\\
h_3^\dagger(x) &= x^7 + w^2x^6 + w^{12}x^5 + w^{14}x^4 + x^3 + w^2x^2 + w^{12}x + w^{14},\\
h_4^\dagger(x) &= x^7 + w^9x^6 + w^6x^5 + w^{15}x^4 + w^{12}x^3 + w^{21}x^2 + w^{18}x + w^3,
\end{aligned}
$$

then we have $x^8 - 1|_r h_i^\dagger(x)h_i(x)$ for $i = 1, 2, 3, 4$. By Theorem 5, C_β is a Hermitian dual-containing code with parameters $[32, 28, 3]_{25}$. Let $C = C_\alpha \times C_\beta$ be a separable skew cyclic code of length 16 over $\mathbb{F}_{25}R$. According to Lemma 9, we get $C^{\perp H} \subseteq C$. By Proposition 2, $\Phi(C)$ is a linear code over $\mathbb{F}_{25}$ with parameters $[40, 34, 3]$. By Theorem 10, we obtain a quantum code with parameters $[[40, 28, 3]]_5$. This quantum code has the larger dimension comparing with the known quantum code with parameters $[[40, 24, 3]]_5$ appeared in [28].

Table 1 New quantum codes $[[n, k, d]]_q$ from skew $\lambda = a + ub + vc + uvd$ constacyclic codes over R

Label	n	g_1	g_2	g_3	g_4	$\Phi(C)$	$[[n, k, d]]_q$
1	2	$w^{18}1$	$w^{6}1$	$w^{6}1$	$w^{6}1$	$[8, 4, 4]_{25}$	$[[8, 0, 4]]_5$
2	4	$w^{15}1$	$w^{9}1$	21	21	$[16, 12, 4]_{25}$	$[[16, 8, 4]]_5$
3	6	$w^{10}1$	$w^{14}1$	431	$w^{14}1$	$[24, 19, 4]_{25}$	$[[24, 14, 4]]_5$
4	6	$4w^{5}1$	$w^{10}1$	431	$4w^{17}1$	$[24, 17, 5]_{25}$	$[[24, 10, 5]]_5$
5	8	$w^{17}w^{8}1$	$w^{3}1$	$1w^{5}w^{2}w^{17}1$	$w^{19}w^{3}1$	$[32, 23, 6]_{25}$	$[[32, 14, 6]]_5$
6	8	$w^{3}1$	$w^{9}1$	$w^{15}1$	$w^{2}1$	$[32, 28, 3]_{25}$	$[[32, 24, 3]]_5$
7	10	$w^{4}1$	$w^{4}4w^{16}1$	$4w^{8}1$	$w^{20}w^{3}1$	$[40, 32, 5]_{25}$	$[[40, 24, 5]]_5$
8	10	$w^{4}1$	$w^{8}1$	$w^{10}1$	$w^{20}w^{3}1$	$[40, 35, 3]_{25}$	$[[40, 30, 3]]_5$
9	12	$w^{7}1$	$w1$	$3w^{10}1$	$w^{10}1$	$[48, 43, 4]_{25}$	$[[48, 38, 4]]_5$
10	6	41	$w^{34}1$	$w^{15}1$	$w^{33}1$	$[24, 20, 4]_{49}$	$[[24, 16, 4]]_7$
11	6	41	$w^{34}1$	651	$w^{20}w^{28}1$	$[24, 18, 5]_{49}$	$[[24, 12, 5]]_7$
12	6	651	$w^{34}1$	$w^{20}w^{28}1$	$w^{38}w^{20}1$	$[24, 17, 6]_{49}$	$[[24, 10, 6]]_7$
13	8	$w^{3}1$	$w^{9}1$	$w^{21}1$	$6w^{2}1$	$[32, 27, 4]_{49}$	$[[32, 22, 4]]_7$
14	4	$w^{39}1$	$w^{9}1$	$w^{27}1$	$w^{30}1$	$[16, 12, 4]_{169}$	$[[16, 8, 4]]_{13}$
15	6	31	$w^{10}1$	$w^{50}1$	$w^{90}1$	$[24, 20, 4]_{169}$	$[[24, 16, 4]]_{13}$
16	6	$w^{38}3w^{150}1$	$w^{10}1$	$w^{50}1$	$w^{90}1$	$[24, 18, 5]_{169}$	$[[24, 12, 5]]_{13}$
17	8	$w^{21}1$	$w^{63}1$	51	$w^{3}w^{3}1$	$[32, 27, 4]_{169}$	$[[32, 22, 4]]_{13}$

At the last of this section, we obtain some new quantum error-correcting codes. Table 1 contains some new non-binary quantum codes from skew λ-constacyclic codes. The second column of the table denotes the code length of C over R. The $g_i(x)$ are the generator polynomials of A_1, A_2, A_3 and A_4, respectively. The following column denotes the parameters of the Gray image of C. The last column denotes the associated quantum codes. In Table 2, λ is an element of R and λ_i are units of $\mathbb{F}_{q^2}$, respectively. The column five denotes the associated quantum codes, and the last column denotes the known quantum codes in comparison.

Remark 1 In Table 1, some quantum codes are constructed from skew λ- constacyclic codes $C_\beta = \langle e_1 g_1(x) + e_2 g_2(x) + e_3 g_3(x) + e_4 g_4(x) \rangle$ over R, where $\lambda = a + ub + vc + uvd$. Let $g_i(x) = g_0 + g_1 x + \cdots + g_t x^t$. For simplicity, we denote $g_i(x)$ by $g_0 g_1 \cdots g_t$.

In Table 2, our quantum codes $[[24, 10, 5]]_5$, $[[40, 24, 5]]_5$, $[[40, 30, 3]]_5$ have better parameters than the quantum codes $[[23, 6, 5]]_5$, $[[40, 24, 3]]_5$, $[[40, 24, 3]]_5$ in [28]. Moreover, our obtained quantum codes $[[8, 0, 4]]_5$, $[[16, 8, 4]]_5$, $[[24, 16, 4]]_7$, $[[16, 8, 4]]_{13}$ and $[[24, 16, 4]]_{13}$ are almost quantum MDS codes such that $n-k-2d = 2$. The rest of quantum codes $[[24, 14, 4]]_5$, $[[32, 24, 3]]_5$, $[[48, 38, 4]]_5$, $[[24, 12, 5]]_7$, $[[24, 10, 6]]_7$, $[[32, 22, 4]]_7$ and $[[32, 22, 4]]_{13}$ have the parameters satisfying $n - k - 2d + 2 = 4$.

Table 2 List of units and parameters used in Table 1

Label	n	λ	$(\lambda_1, \lambda_2, \lambda_3, \lambda_4)$	$[[n, k, d]]_q$	$[[n', k', d']]_q$
1	2	-1	$(-1, -1, -1, -1)$	$[[8, 0, 4]]_5$	$2d = n - k$
2	4	$1 - 2u$	$(-1, -1, 1, 1)$	$[[16, 8, 4]]_5$	$[[18, 4, 4]]_5$[23]
3	6	-1	$(-1, -1, -1, -1)$	$[[24, 14, 4]]_5$	$[[22, 10, 4]]_5$([23])
4	6	$1 - 2u - 2v + 4uv$	$(1, -1, -1, 1)$	$[[24, 10, 5]]_5$	$[[23, 6, 5]]_5$([19])
5	8	$-1 + 2u$	$(1, 1, -1, -1)$	$[[32, 14, 6]]_5$	–
6	8	1	$(1, 1, 1, 1)$	$[[32, 24, 3]]_5$	$[[32, 20, 3]]_5$([29])
7	10	$1 - 2v + 2uv$	$(1, 1, -1, 1)$	$[[40, 24, 5]]_5$	$[[40, 24, 3]]_5$([28])
8	10	$1 - 2v + 2uv$	$(1, 1, -1, 1)$	$[[40, 30, 3]]_5$	$[[40, 24, 3]]_5$([28])
9	12	$1 - 2u - 2v + 2uv$	$(-1, -1, -1, 1)$	$[[48, 38, 4]]_5$	$2d = n - k - 2$
10	6	$-1 + 2u$	$(1, 1, -1, -1)$	$[[24, 16, 4]]_7$	$[[27, 15, 4]]_7$([29])
11	6	$-1 + 2u + 2v - 2uv$	$(1, 1, 1, -1)$	$[[24, 12, 5]]_7$	$[[26, 12, 5]]_7$([19])
12	6	$-1 + 2u$	$(1, 1, -1, -1)$	$[[24, 10, 6]]_7$	$[[30, 12, 6]]_7$([19])
13	8	$-1 + 2u + 2v - 2uv$	$(1, 1, 1, -1)$	$[[32, 22, 4]]_7$	$2d = n - k - 2$
14	4	$1 - 2u - 2v + 2uv$	$(-1, -1, -1, 1)$	$[[16, 8, 4]]_{13}$	$[[16, 8, 2]]_{13}$([21])
15	6	$-1 + 2uv$	$(1, -1, -1, -1)$	$[[24, 16, 4]]_{13}$	$[[24, 12, 4]]_{13}$([20])
16	6	-1	$(-1, -1, -1, -1)$	$[[24, 12, 5]]_{13}$	$[[24, 8, 5]]_{13}$([20])
17	8	$-1 + 2u + 2v - 2uv$	$(1, 1, 1, -1)$	$[[32, 22, 4]]_{13}$	$[[36, 24, 4]]_{13}$([20])

Example 2 Let $R = \mathbb{F}_{25} + u\mathbb{F}_{25} + v\mathbb{F}_{25} + v\mathbb{F}_{25} + uv\mathbb{F}_{25}$, where $u^2 = u$, $v^2 = v$, $uv = vu$ and $\mathbb{F}_{25} = \mathbb{F}_5[w]$ with $w^2 = w + 3$. Let θ be an automorphism of R denoted by $\theta(a) = a^5$ for any element $a \in \mathbb{F}_{25}$. Let $f_1 = x + w^4$, $f_2 = x + w^8$, $f_3 = x^2 + w^3x + w^{20}$, $f_4 = x^3 + w^{16}x^2 + 4x + w^4$, $f_5 = x^2 + w^8x + 4$ and $f_6 = x + w^{10}$. It is easy to see that $f_1|(x^{10} - 1)$, $f_2|(x^{10} - 1)$, $f_3|(x^{10} - 1)$, $f_4|(x^{10} - 1)$, $f_5|(x^{10} + 1)$ and $f_6|(x^{10} + 1)$ in $\mathbb{F}_{25}[x, \theta]$. In Table 3, we list some examples of quantum codes over $\mathbb{F}_5$ obtained by Hermitian dual-containing skew constacyclic codes over $\mathbb{F}_{25} + u\mathbb{F}_{25} + v\mathbb{F}_{25} + uv\mathbb{F}_{25}$ of length 10.

Remark 2 In Table 3, we construct quantum codes of length 40 with parameters $[[40, 24, 5]]_5$, $[[40, 28, 4]]_5$, $[[40, 30, 3]]_5$, $[[40, 32, 2]]_5$. Comparing with the well-known quantum codes $[[40, 24, 3]]_5$ appeared in [28], our quantum codes $[[40, 24, 5]]_5$ and $[[40, 30, 3]]_5$ have the larger minimum distance and the larger dimension. Moreover, our quantum code $[[40, 32, 2]]_5$ has the larger dimension than the well-known quantum code $[[40, 24, 2]]_5$ in [25]. Therefore, our quantum codes will have better performances in the quantum channel.

Remark 3 In the last, we introduce our new contribution of this paper over the existing results in references [26,27,30,43].

In [26], Jitman et al. gave the properties of skew constacyclic codes over finite chain rings. The generators of Hermitian dual codes of skew constacyclic codes were determined. However, the authors did not introduce the application of this family of codes. In our paper, we discussed the properties of skew constacyclic codes over the finite non-chain ring $R = \mathbb{F}_{q^2} + u\mathbb{F}_{q^2} + v\mathbb{F}_{q^2} + uv\mathbb{F}_{q^2}$, where $u^2 = u, v^2 = v, uv = vu$.

Table 3 Quantum codes from Hermitian dual-containing skew constacyclic codes

n	g_1	g_2	g_3	g_4	$[[n, k, d]]_q$
10	f_1	f_4	f_5	f_3	$[[40, 24, 5]]_5$
10	f_1	f_2	f_5	f_3	$[[40, 28, 4]]_5$
10	f_1	f_2	f_6	f_3	$[[40, 30, 3]]_5$
10	f_1	f_2	f_6	f_1	$[[40, 32, 2]]_5$

Moreover, we extended this concept to mixed alphabet codes. Similarly, we gave the algebraic structure of $\mathbb{F}_{q^2} R$-linear skew constacyclic codes and determined their generating sets. More importantly, we constructed some good non-binary quantum codes.

In [27], the authors introduced non-binary stabilizer codes over finite fields. They established the self-orthogonality with respect to a trace-symplectic form. In our paper, by the theory of Hermitian construction in [27], we used Hermitian dual-containing skew constacyclic codes over R to construct quantum codes.

In [30], Mi et al. obtained some Hermitian dual-containing cyclic codes based on a characterization of q-cyclotomic cosets modulo n. But they only obtained quantum codes with odd length. In our paper, by considering the Hermitian dual-containing skew constacyclic codes over R, we got quantum codes of length $4n$, where n is a positive integer.

In [43], Zheng et al. only considered some structural properties of constacyclic codes under Euclidean inner product over $\mathbb{F}_p + u\mathbb{F}_p + v\mathbb{F}_p + uv\mathbb{F}_p$, where $u^2 = u$, $v^2 = v$, $uv = vu$. However, in our paper, we considered the structure of skew constacyclic codes with respect to Hermitian inner product. Moreover, we introduced linear skew constacyclic codes over $\mathbb{F}_{q^2} R$ and gave their structural properties. As an application, we constructed some good quantum codes in Tables 1 and 3.

7 Conclusion

In this paper, $\mathbb{F}_{q^2} R$-linear skew constacyclic codes of length $\alpha + \beta$ can be viewed as a left $R[x, \theta]$-submodules of $\mathbb{F}_{q^2}[x]/\langle x^\alpha - 1\rangle \times R[x, \theta]/\langle x^\beta - 1\rangle$, where $R = \mathbb{F}_{q^2} + u\mathbb{F}_{q^2} + v\mathbb{F}_{q^2} + uv\mathbb{F}_{q^2}$ with $u^2 = u$, $v^2 = v$ and $uv = vu$. Firstly, we discuss the structural properties of skew λ-constacyclic codes over R. Further, we study the Hermitian dual codes of skew λ-constacyclic codes over R. Secondly, we determine the generators and the minimal spanning sets of $\mathbb{F}_{q^2} R$-linear skew constacyclic codes of length $\alpha + \beta$. Finally, we define a Gray map from $\mathbb{F}_{q^2}^{\alpha} \times R^{\beta}$ to $\mathbb{F}_{q^2}^{\alpha+4\beta}$ preserving the Hermitian orthogonality. As an application, we obtain some quantum codes, which have better parameters than the known quantum codes. Quantum codes with good parameters have practical applications in the construction of secret sharing schemes in cryptography, improving the reliability of quantum computing and quantum communication. Moreover, they play an important role in quantum confidential communication. It is an interesting open problem to study how to apply quantum codes

from codes over rings into amplitude-damping qubit channel, phase-damping channel, depolarized-damping qubit channel and actual physical background.

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