



Monogamous nature of Dicke-class of states with two distinct Majorana spinors

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Abstract

Monogamy relations place restrictions on the shareability of quantum correlations in multipartite states. Being an intrinsic quantum feature, monogamy property throws light on *residual* entanglement, an entanglement which is not accounted for by the pairwise entanglement in the state. Expressed in terms of suitable pairwise entanglement measures such as concurrence, the monogamy inequality leads to the evaluation of *tangle*, a measure of residual entanglement. In this work, we explore monogamy relations in pure symmetric multiqubit states constituted by two distinct spinors, the so-called *Dicke-class* of states. Pure symmetric N -qubit states constituted by permutation of two orthogonal qubits form the well-known Dicke states. Those N -qubit pure symmetric states constructed by permutations of two non-orthogonal qubits are a one-parameter class of generalized Dicke states. With the help of Majorana geometric representation and angular momentum algebra, we analyze the bounds on monogamy inequality, expressed in terms of squared concurrence/squared negativity of partial transpose. We show that the states with equal distribution of the two spinors are more monogamous and hence possess larger residual entanglement when compared to other inequivalent classes with different degeneracy configurations.

Keywords Monogamy of quantum entanglement · Majorana representation of symmetric multiqubit states · Dicke-class of states · Residual entanglement · Concurrence tangle · Negativity tangle · Bound on monogamy inequality

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1 Introduction

Quantum entanglement is an important resource for several quantum information processing tasks which are impossible in information processing using classically correlated states. One among the distinct properties of quantum entanglement that separates it from classical correlations is its restricted shareability. While classical correlations are infinitely shareable, there is a limitation on the manner in which quantum entanglement is distributed among its subsystems. For instance, in a tripartite system, entanglement of one party with another limits its entanglement with the third party. This unique feature of quantum entanglement is termed ‘*monogamy of entanglement*’ [1] and has evoked a lot of interest [1–53] in the quantum information community. The importance of multipartite quantum states exhibiting monogamous nature is due to their applicability in quantum communication tasks such as secure quantum key distribution [4,24] and reliable quantum teleportation [53].

Quantifying ‘*tripartite entanglement*’ or the so-called *residual entanglement*, which is not accounted for by the pairwise entanglement in the state, is another important issue which can be addressed using monogamous nature of composite quantum states. The restricted shareability in a multiqubit state is captured in the monogamy inequality

$$D_{A_1:A_2A_3\cdots A_N} \geq (D_{A_1A_2} + D_{A_1A_3} + D_{A_1A_4} + \cdots + D_{A_1A_N}). \quad (1)$$

Here $D_{A_iA_j}$, $i \neq j = 1, 2, 3 \dots N$ is a suitable measure of pairwise entanglement, and $D_{A_1:A_2A_3\cdots A_N}$ quantifies entanglement between one party (say A_1) and all other parties $A_2, A_3, \dots A_N$ taken together. Equation (1) indicates that the sum of pairwise entanglements in a composite state can never exceed the entanglement between one party and the remaining parties. Quantification of three-party entanglement can be done through the nonzero quantity

$$D_{A_1:A_2A_3\cdots A_N} - (D_{A_1A_2} + D_{A_1A_3} + D_{A_1A_4} + \cdots + D_{A_1A_N}) \quad (2)$$

called *tangle* with respect to the chosen measure $D_{A_iA_j}$ of pairwise entanglement.

Choosing squared concurrence C [54,55] as a measure of pairwise entanglement, Coffman et al. [1] proposed $\tau_c = C_{A:BC}^2 - (C_{AB}^2 + C_{AC}^2)$, the so-called *three-tangle* or *concurrence tangle* as a measure of *residual entanglement* in *three-qubit pure states*. Its generalization to N -qubit pure states has been carried out by Osborne and Verstraete [2], and the measure of residual entanglement given by

$$\tau_N = C_{A_1:A_2A_3A_4\cdots A_N}^2 - (C_{A_1A_2}^2 + C_{A_1A_3}^2 + C_{A_1A_4}^2 + \cdots + C_{A_1A_N}^2) \quad (3)$$

is termed the N -concurrence tangle [2].

Monogamy inequality in terms of different measures of entanglement such as squared negativity of partial transpose [56–58] and square of entanglement of formation [54,55] is proposed in Refs. [9] and [34]. It is shown that [40] W-class of states have vanishing concurrence tangle [1] but nonzero negativity tangle [9,40]. Choosing a particular measure of residual entanglement, among the several available choices, is a non-trivial task. Despite this difficulty, the choice of a *single* convenient measure

for an entire class of states serves to quantify the residual entanglement, *with respect to the chosen measure*.

Symmetric multiqubit states form an important class of states due to their theoretical significance and experimental relevance. There has been a considerable experimental progress in controlled generation of multiqubit Dicke states in physical systems like photons, cold atoms and trapped ions [59–63]. Innovative experimental schemes have also been proposed to produce a large variety of symmetric multiqubit photonic states [64,65]. Establishing an entirely non-classical feature such as restricted shareability of quantum entanglement/monogamous nature in symmetric multiqubit states is bound to have immense impact in quantum information technology in general, and secure quantum communications in particular.

In this paper, we focus our attention on monogamy property of N -qubit pure symmetric states constituted by two distinct qubits. This set of states, i.e., N -qubit pure symmetric states characterized by only two distinct qubits, is defined as the *Dicke-class of states* [66,67]. Dicke states, the common eigenstates of collective angular momentum operators \hat{J}^2, \hat{J}_z , consist of two orthogonal spinors [66,67]. Pure symmetric states of N -qubits, constructed by permutations of any two arbitrary non-orthogonal spinors, form a generalized class of Dicke states. Both Dicke and generalized Dicke states are represented by two distinct points on the Bloch sphere based on Majorana’s geometric description [68–73].

The paper is organized as follows: In Sect. 2, we employ Majorana geometric representation [68–73] to obtain a simplified, one-parameter form of states in the Dicke-class. Using this simplified form and with the help of well-established angular momentum techniques, we obtain the structure of two-qubit and single-qubit marginals of the Dicke-class of states in Sect. 3. With the help of these reduced density matrices, we explore monogamous nature of Dicke-class of states in Sect. 4. A summary of results is given in Sect. 5.

2 Majorana geometric representation of pure symmetric N -qubit states with two distinct spinors

Ettore Majorana, in his novel 1932 paper [68], proposed that a pure spin $j = \frac{N}{2}$ quantum state can be represented as a *symmetrized* combination of N constituent spinors as follows:

$$|\Psi_{\text{sym}}\rangle = \mathcal{N} \sum_P \hat{P} \{|\epsilon_1, \epsilon_2, \dots, \epsilon_N\rangle\}, \tag{4}$$

where

$$|\epsilon_l\rangle = \left(\cos(\alpha_l/2) |0\rangle + \sin(\alpha_l/2) |1\rangle e^{i\beta_l/2} \right), \quad l = 1, 2, \dots, N \tag{5}$$

denote arbitrary states of spinors (qubits) $|\epsilon_l\rangle$. In Eq. (4), the symbol \hat{P} corresponds to the set of all $N!$ permutations of the qubits and \mathcal{N} corresponds to an overall normalization factor. The name Majorana *geometric* representation is owing to the fact that

it leads to an intrinsic picture of the state in terms of N points on the unit sphere. The spinors $|\epsilon_l\rangle, l = 1, 2, \dots, N$ of Eq. (5) correspond geometrically to N points on the Bloch sphere S^2 , with the pair of angles (α_l, β_l) determining the orientation of each point on the sphere. Pure symmetric N -qubit states consisting of two distinct qubits (orthogonal as well as non-orthogonal) are given by [69–71],

$$|D_{N-k,k}\rangle = \mathcal{N} \sum_P \hat{P} \{|\underbrace{\epsilon_1, \epsilon_1, \dots, \epsilon_1}_{N-k}; \underbrace{\epsilon_2, \epsilon_2, \dots, \epsilon_2}_k\rangle\}, \quad k = 1, 2, 3, \dots \left\lfloor \frac{N}{2} \right\rfloor \tag{6}$$

It may be noted that in Eq. (6), one of the spinors say $|\epsilon_1\rangle$ occurs $N - k$ times, whereas the other spinor $|\epsilon_2\rangle$ occurs k times in each term of the symmetrized combination ($\lfloor \frac{N}{2} \rfloor = \frac{N}{2}$ when N is even and $\lfloor \frac{N}{2} \rfloor = \frac{N-1}{2}$ when N is odd). It has been shown in Refs. [28,71] that the states $|D_{N-k,k}\rangle$ are equivalent, under identical¹ local unitary transformations, to a canonical form, characterized by only one real parameter. More specifically, pure symmetric multiqubit states $|D_{N-k,k}\rangle$ in Eq. (6) can be brought [28] to the form

$$|D_{N-k,k}\rangle \equiv \sum_{r=0}^k \beta_r^{(k)} \left| \frac{N}{2}, \frac{N}{2} - r \right\rangle, \tag{7}$$

$$\beta_r^{(k)} = \mathcal{N} \sqrt{\frac{N!(N-r)!}{r!}} \frac{a^{k-r} b^r}{(N-k)!(k-r)!}, \quad 0 \leq a < 1, \quad b = \sqrt{1-a^2}.$$

Here $\left| \frac{N}{2}, \frac{N}{2} - r \right\rangle, r = 0, 1, 2 \dots, N$ denote the Dicke states, which are common eigenstates of the collective angular momentum operators \hat{J}^2, \hat{J}_z and are the basis states of the symmetric subspace of collective angular momentum space, with dimension $N + 1$. The generalized Dicke states $|D_{N-k,k}\rangle$ are characterized by only one real parameter ‘ a ,’ and they belong to the one-parameter family, the Dicke-class of states. The Dicke-class consists of both Dicke states (when $a = 0$) and generalized Dicke states (when $0 < a < 1$). While the Dicke states having parameter $a = 0$ are characterized by two orthogonal spinors $|0\rangle, |1\rangle$, the states $|D_{N-k,k}\rangle$ with $0 < a < 1$ are characterized by two non-orthogonal spinors $|\epsilon_1\rangle, |\epsilon_2\rangle$. The parameter ‘ a ’ can thus be termed as *non-orthogonality parameter*.

It is important to notice that in the Dicke-class of states $|D_{N-k,k}\rangle$, different values of k ($k = 1, 2, 3, \dots \lfloor \frac{N}{2} \rfloor$), correspond to SLOCC inequivalent classes [69–71,74]. In fact, different values of k lead to different *degeneracy configurations* [69–71] of the two

¹ For any symmetric state to be transformed into another symmetric state through local unitary transformations, the unitary transformations are to be identical in order to retain the symmetry of the state (see Ref. [71]).

spinors. For instance, when $N = 4$, there are two possible degeneracy configurations² corresponding to $k = 3$ and $k = 2$. The states $|D_{2,2}\rangle$ belong to the class $\{\mathcal{D}_{2,2}\}$, and the states $|D_{3,1}\rangle$ belong to $\{\mathcal{D}_{3,1}\}$. Both $\{\mathcal{D}_{2,2}\}$ and $\{\mathcal{D}_{3,1}\}$ are the only possible classes when $N = 4$. In general, for any N , the Dicke-class of states is a collection of all inequivalent classes $\{\mathcal{D}_{N,k}\}$, $k = 1, 2, 3, \dots, \lfloor \frac{N}{2} \rfloor$. In the next section, we evaluate the two-qubit and single-qubit marginal density matrices of the state $|D_{N-k,k}\rangle$ for different values of $k = 1, 2, 3, \dots, \lfloor \frac{N}{2} \rfloor$. We then proceed to analyze the monogamy property of the states belonging to different inequivalent classes $\{\mathcal{D}_{N,k}\}$ with respect to two well-known measures of two-qubit entanglement.

3 Single- and two-qubit reduced density matrices of generalized Dicke states

The monogamy relation in Eq. (1) requires evaluation of measures of entanglement between any two qubits and also between a given qubit and all other qubits in the state. For quantification of pairwise entanglement through any suitable measure of entanglement, one needs to evaluate the two-qubit reduced density matrix of the multiqubit state under consideration. Toward this end, we proceed to evaluate single- and two-qubit reduced density matrices of pure symmetric states belonging to Dicke-class. For the pure symmetric multiqubit state $|D_{N-k,k}\rangle$, owing to exchange symmetry, all reduced density matrices are identical. Thus, the two-qubit marginal density matrix $\rho_2^{(k)}$ corresponding to any random pair of qubits in the N -qubit symmetric state $|D_{N-k,k}\rangle$ is obtained by tracing over the remaining $N - 2$ qubits in it.

On using the form of the state $|D_{N-k,k}\rangle$ given in Eq. (7), we have

$$\begin{aligned} \rho_2^{(k)} &= \text{Tr}_{N-2} (|D_{N-k,k}\rangle\langle D_{N-k,k}|), \quad k = 1, 2, 3, \dots, \lfloor \frac{N}{2} \rfloor; \\ &= \text{Tr}_{N-2} \left(\sum_{r,r'=0}^k \beta_r^{(k)} \beta_{r'}^{(k)} \left| \frac{N}{2}, \frac{N}{2} - r \right\rangle \left\langle \frac{N}{2}, \frac{N}{2} - r' \right| \right). \end{aligned} \tag{8}$$

To facilitate the tracing operation over $N - 2$ qubits, we partition the state $|D_{N-k,k}\rangle$ into $N - 2$ qubits and two qubits. As a pure symmetric state with $N - 2$ qubits is equivalent to an angular momentum state $|j_1, m_1\rangle$ with $j_1 = \frac{N-2}{2}$ and a two-qubit pure symmetric state is equivalent to the state $|j_2, m_2\rangle$ with $j_2 = 1$, we use the addition

² In $|D_{2,2}\rangle$, the spinors $|\epsilon_1\rangle, |\epsilon_2\rangle$ appear two times in each term of the symmetrized combination shown in Eq. (6). That is,

$$|D_{2,2}\rangle = \mathcal{N} [|\epsilon_1\epsilon_1\epsilon_2\epsilon_2\rangle + |\epsilon_1\epsilon_2\epsilon_1\epsilon_2\rangle + |\epsilon_1\epsilon_2\epsilon_2\epsilon_1\rangle + |\epsilon_2\epsilon_1\epsilon_2\epsilon_1\rangle + |\epsilon_2\epsilon_2\epsilon_1\epsilon_1\rangle].$$

Similarly,

$$|D_{3,1}\rangle = \mathcal{N} [|\epsilon_1\epsilon_1\epsilon_1\epsilon_2\rangle + |\epsilon_1\epsilon_1\epsilon_2\epsilon_1\rangle + |\epsilon_1\epsilon_2\epsilon_1\epsilon_1\rangle + |\epsilon_2\epsilon_1\epsilon_1\epsilon_1\rangle].$$

Here, spinor $|\epsilon_1\rangle$ appears three times, whereas $|\epsilon_2\rangle$ appears only once in each term of the symmetrized combination.

of angular momenta relation [75]

$$|j, m\rangle = \sum_{m_2=-j_2}^{j_2} C(j_1, j_2, j; m - m_2, m_2, m) (|j_1, m - m_2\rangle \otimes |j_2, m_2\rangle) \quad (9)$$

for the required partition of the N -qubit state $|D_{N-k, k}\rangle$. Here $C(j_1, j_2, j; m - m_2, m_2, m)$ are the Clebsch–Gordan coefficients [75] in the addition of angular momenta, with quantum numbers j_1 and j_2 . With $j_1 = \frac{N}{2} - 1, j_2 = 1$, we obtain

$$\begin{aligned} & \left| \frac{N}{2}, \frac{N}{2} - r \right\rangle \\ &= \sum_{m_2=-1,0,1} c_{m_2}^{(r)} \left| \frac{N}{2} - 1, \frac{N}{2} - r - m_2 \right\rangle \otimes |1, m_2\rangle, \quad r = 0, 1, \dots, N, \end{aligned} \quad (10)$$

where we have denoted

$$\begin{aligned} c_{m_2}^{(r)} &= C\left(\frac{N}{2} - 1, 1, \frac{N}{2}; m - m_2, m_2, m\right), \\ m &= -\frac{N}{2}, -\frac{N}{2} + 1, \dots, \frac{N}{2}, \quad m_2 = -1, 0, 1, \quad r = 0, 1, \dots, N. \end{aligned} \quad (11)$$

We thus obtain [see Eqs. (8) and (10)]

$$\begin{aligned} \rho_2^{(k)} &= \text{Tr}_{N-2} \left\{ \sum_{r,r'=0}^k \beta_r^{(k)} \beta_{r'}^{(k)} \sum_{m_2,m'_2} \left[c_{m_2}^{(r)} c_{m'_2}^{(r')} \right. \right. \\ & \quad \left. \left. \left| \frac{N}{2} - 1, \frac{N}{2} - r - m_2 \right\rangle \left\langle \frac{N}{2} - 1, \frac{N}{2} - r' - m'_2 \right| \right] \right. \\ & \quad \left. \otimes |1, m_2\rangle \langle 1, m'_2| \right\}. \end{aligned}$$

More explicitly, tracing over $N - 2$ qubits leads us to

$$\begin{aligned} \rho_2^{(k)} &= \left\{ \sum_{r,r'=0}^k \beta_r^{(k)} \beta_{r'}^{(k)} \sum_{m_2,m'_2} c_{m_2}^{(r)} c_{m'_2}^{(r')} \left[\left| \frac{N}{2} - 1, \frac{N}{2} - r - m_2 \right\rangle \left\langle \frac{N}{2} - 1, \frac{N}{2} - r' - m'_2 \right| \right] |1, m_2\rangle \langle 1, m'_2| \right\} \\ &= \left\{ \sum_{r,r'=0}^k \sum_{m_2,m'_2} \beta_r^{(k)} \beta_{r'}^{(k)} c_{m_2}^{(r)} c_{m'_2}^{(r')} \sum_{m_1=(-N/2)+1}^{(N/2)-1} \left\langle \frac{N}{2} - 1, m_1 \left| \frac{N}{2} - 1, \frac{N}{2} - r' - m'_2 \right\rangle \right. \right. \\ & \quad \left. \left. \times \left\langle \frac{N}{2} - 1, \frac{N}{2} - r - m_2 \left| \frac{N}{2} - 1, m_1 \right\rangle \right| |1, m_2\rangle \langle 1, m'_2| \right\}. \end{aligned} \quad (12)$$

In the second line of Eq. (12), we have made use of the completeness relation

$$I = \sum_{m_1=(-N/2)+1}^{(N/2)-1} \left| \frac{N}{2} - 1, m_1 \right\rangle \left\langle \frac{N}{2} - 1, m_1 \right|,$$

with I being the identity matrix in the space of $N - 2$ qubits. We finally obtain the two-qubit reduced density matrix of any random pair of qubits of the N -qubit state $|D_{N-k, k}\rangle$. The two-qubit reduced density matrix $\rho_2^{(k)}$ now turns out to be

$$\rho_2^{(k)} = \sum_{m_2, m'_2=1,0,-1} \rho_{m_2, m'_2}^{(k)} |1, m_2\rangle \langle 1, m'_2|, \tag{13}$$

where

$$\begin{aligned} \rho_{m_2, m'_2}^{(k)} &= \sum_{r, r'=0}^k \beta_r^{(k)} \beta_{r'}^{(k)} c_{m_2}^{(r)} c_{m'_2}^{(r')} \left\{ \sum_{m_1=(-N/2)+1}^{(N/2)-1} \left\langle \frac{N}{2} - 1, m_1 \right| \frac{N}{2} - 1, \frac{N}{2} - r' - m'_2 \right\rangle \right. \\ &\quad \left. \times \left\langle \frac{N}{2} - 1, \frac{N}{2} - r - m_2 \right| \frac{N}{2} - 1, m_1 \right\rangle \right\} \end{aligned} \tag{14}$$

denote the matrix elements of $\rho_2^{(k)}$ in the basis $\{|1, m_2\rangle, m_2 = -1, 0, 1\}$. The associated Clebsch–Gordan coefficients $c_{m_2}^{(r)}$ are explicitly given by [75]

$$\begin{aligned} c_1^{(r)} &= \sqrt{\frac{(N-r)(N-r-1)}{N(N-1)}}, \quad c_{-1}^{(r)} = \sqrt{\frac{r(r-1)}{N(N-1)}}, \\ c_0^{(r)} &= \sqrt{\frac{2r(N-r)}{N(N-1)}}. \end{aligned} \tag{15}$$

Expressing the spin-1 states in terms of the constituent two-qubit states i.e.,

$$|1, 1\rangle = |0_A, 0_B\rangle, \quad |1, 0\rangle = (|0_A, 1_B\rangle + |1_A, 0_B\rangle)/\sqrt{2}, \quad |1, -1\rangle = |1_A, 1_B\rangle,$$

the following simplified form [76,77] is realized for the symmetric two-qubit reduced density matrix:

$$\rho_2^{(k)} = \begin{pmatrix} A^{(k)} & B^{(k)} & B^{(k)} & C^{(k)} \\ B^{(k)} & D^{(k)} & D^{(k)} & E^{(k)} \\ B^{(k)} & D^{(k)} & D^{(k)} & E^{(k)} \\ C^{(k)} & E^{(k)} & E^{(k)} & F^{(k)} \end{pmatrix}. \tag{16}$$

The elements $A^{(k)}$, $B^{(k)}$, $C^{(k)}$, $D^{(k)}$, $E^{(k)}$ and $F^{(k)}$ are real and are explicitly given by [66]

$$\begin{aligned}
 A^{(k)} &= \sum_{r=0}^k \left(\beta_r^{(k)}\right)^2 \left(c_1^{(r)}\right)^2, & B^{(k)} &= \frac{1}{\sqrt{2}} \sum_{r=0}^{k-1} \beta_r^{(k)} \beta_{r+1}^{(k)} c_1^{(r)} c_0^{(r+1)} \\
 C^{(k)} &= \sum_{r=0}^{k-2} \beta_r^{(k)} \beta_{r+2}^{(k)} c_1^{(r)} c_{-1}^{(r+2)}, & D^{(k)} &= \frac{1}{2} \sum_{r=1}^k \left(\beta_r^{(k)}\right)^2 \left(c_0^{(r)}\right)^2 \\
 E^{(k)} &= \frac{1}{\sqrt{2}} \sum_{r=0}^{k-1} \beta_r^{(k)} \beta_{r+1}^{(k)} c_0^{(r)} c_{-1}^{(r+1)}, & F^{(k)} &= \sum_{r=0}^k \left(\beta_r^{(k)}\right)^2 \left(c_{-1}^{(r)}\right)^2
 \end{aligned} \tag{17}$$

In order to evaluate the entanglement between a single qubit say A_1 and the remaining qubits $A_2 A_3 \cdots A_N$, we recall that when the N -qubit state is pure, the partition containing $N - 1$ qubits can be treated as a single qubit [1]. This observation allows any measure of pairwise entanglement suitable for quantifying entanglement between one qubit and the remaining qubits in a *pure* multiqubit state. In particular, when the measure of entanglement is either concurrence [54] or negativity of partial transpose [56–58], it is seen that [1,9]

$$C_{A_1:A_2 A_3 \cdots A_N} = N_{A_1:A_2 A_3 \cdots A_N} = 2 \sqrt{\det \rho_{A_1}}, \tag{18}$$

where ρ_{A_1} is the density matrix of the qubit A_1 .

The state $|D_{N-k}, k\rangle$ being symmetric, the marginal $\rho_1^{(k)}$ is independent of the choice of qubit. We obtain the single-qubit marginal $\rho_1^{(k)}$ either by tracing $N - 1$ qubits in the state $|D_{N-k}, k\rangle$ or by tracing a single qubit from the two-qubit reduced density matrix $\rho_2^{(k)}$ [see Eq. (16)]. The reduced density matrix $\rho_1^{(k)}$ characterizing any single qubit of the pure symmetric N -qubit state $|D_{N-k}, k\rangle$ is thus obtained as

$$\rho_1^{(k)} = \begin{pmatrix} A^{(k)} + D^{(k)} & B^{(k)} + E^{(k)} \\ B^{(k)} + E^{(k)} & D^{(k)} + F^{(k)} \end{pmatrix}. \tag{19}$$

The two-qubit and single-qubit marginals $\rho_2^{(k)}$, $\rho_1^{(k)}$ obtained explicitly through relations (16, 19), help in setting up the monogamy relations with respect to any suitable measure of pairwise entanglement. In the following section, we analyze the bounds on monogamy relations for the state $|D_{N-k}, k\rangle$, choosing concurrence and negativity of partial transpose as measures of pairwise entanglement.

4 Monogamy relation for Dicke-class of states

Having obtained the form of single- and two-qubit reduced density matrices $\rho_1^{(k)}$ and $\rho_2^{(k)}$ [see Eqs. (16, 19)] of the state $|D_{N-k}, k\rangle$, we will use them here to set up the monogamy relation for the Dicke-class of states. For this purpose, we choose squared concurrence and squared negativity of partial transpose as two different, yet suitable measures of entanglement. As the Dicke-class of states contains several inequivalent

classes $\{D_{N,k}\}$, with different degeneracy configuration [69–71] of the two spinors (corresponding to $k = 1, 2, \dots [N/2]$), we analyze the bound on monogamy relation in each class.

4.1 Monogamy relation for generalized Dicke states in terms of squared concurrence

We recall here that the monogamy relation in terms of squared concurrence was set up for three-qubit pure states by Coffman et al. [1]. It was generalized to N -qubit pure states in Ref. [2]. Concurrence is a convenient measure of pairwise entanglement in two-qubit states (pure as well as mixed). For the two-qubit state $\rho_2^{(k)}$, concurrence C is given by

$$C_{k_2} = \max \left(0, \sqrt{\lambda_1} - \sqrt{\lambda_2} - \sqrt{\lambda_3} - \sqrt{\lambda_4} \right).$$

Here $\lambda_i, i = 1, 2, 3, 4$ are the eigenvalues of the matrix $\rho_2^{(k)} \rho_2'^{(k)}$ with $\rho_2'^{(k)} = (\sigma_y \otimes \sigma_y) \rho_2^{*(k)} (\sigma_y \otimes \sigma_y)$ being the spin-flipped density matrix [54,55].

With the explicit form of $\rho_2^{(k)}$ obtained in Sect. 3 [see Eq. (16)], the concurrence C_{k_2} can readily be evaluated.

While the concurrence C_{k_2} quantifies the pairwise entanglement between any two qubits of the state $|D_{N-k,k}\rangle$, the entanglement between any qubit and the remaining $N - 1$ qubits is given by [see Eq. (18)]

$$C_{A_1:A_2A_3\dots A_N} \equiv C_{k_1} = 2 \sqrt{\det \rho_1^{(k)}} \tag{20}$$

Notice here that, due to the symmetry of the state $|D_{N-k,k}\rangle$ under permutation of qubits, the choice of qubit A_1 in $C_{A_1:A_2A_3\dots A_N} \equiv C_{k_1}$ is arbitrary and hence Eq. (20) gives the entanglement between any qubit and remaining $N - 1$ qubits. With the single-qubit marginal $\rho_1^{(k)}$ of the state $|D_{N-k,k}\rangle$ being explicitly given in Sect. 3 [see Eq. (19)] along with Eqs. (15, 17), one can readily evaluate the concurrence C_{k_1} [see Eq. (20)] which quantifies the entanglement between any qubit and the remaining qubits. With reference to an arbitrary qubit of the N -qubit pure symmetric state $|D_{N-k,k}\rangle$, there are $N - 1$ pairs of qubits [see Eqs. (2), (3)] and entanglement between them is quantified through squared concurrence $C_{k_2}^2$. The monogamy inequality for the Dicke-class of states [see Eq. (3)] is therefore given by

$$C_{k_1}^2 > (N - 1)C_{k_2}^2. \tag{21}$$

The bound on monogamy inequality can be ascertained by evaluating the *concurrence tangle* τ_c which quantifies residual entanglement (as entanglement of N -qubit states which is not captured by pairwise entanglement measures alone) in terms of squared concurrence [1]. For the states $|D_{N-k,k}\rangle$ belonging to the Dicke-class, the concurrence tangle is given by

$$\tau_N^{(k)} = C_{k_1}^2 - (N - 1)C_{k_2}^2. \tag{22}$$

It may be noted that larger value of residual entanglement $\tau_N^{(k)}$ indicates that the N -qubit generalized Dicke states $|D_{N-k, k}\rangle$ are more monogamous. In other words, Eq. (22) is useful to verify constrained shareability of entanglement among the N -qubits because of monogamy property of quantum entanglement. We have explicitly evaluated the concurrence tangle $\tau_N^{(k)}$ for the states belonging to inequivalent classes $\{\mathcal{D}_{N-k, k}\}, k = 1, 2, 3, 4, 5$ as a function of N and the parameter ‘ a .’ In Figs. 1, 2, 3 and 4, the variation of concurrence tangle $\tau_N^{(k)}$ (measure of residual entanglement in terms of squared concurrence) with the parameter ‘ a ’ is shown.

The states $|D_{N-1, 1}\rangle$ belong to the W-class of states [40] $\{\mathcal{D}_{N-1, 1}\}$ in which one of the spinors appears only once in each term of the symmetrized combination [see Eq. (7)]. The well-known W-states $|\frac{N}{2}, \frac{N}{2} - 1\rangle$ belong to this family and correspond to the parameter $a = 0$ of the state $|D_{N-1, 1}\rangle$ [see Eq. (7)]. It has been shown in Ref. [40] that the W-class of states saturate the *monogamy inequality in terms of squared concurrence*, i.e.,

$$C_{k_1}^2 = (N - 1)C_{k_2}^2 \implies \tau_N^{(k)} = C_{k_1}^2 - (N - 1)C_{k_2}^2 = 0$$

for W-class of states. This has the physical implication that the N -qubit pure state $|D_{N-1, 1}\rangle$ possesses only pairwise entanglement, when squared concurrence is chosen as measure of two-qubit entanglement. It also means that entanglement between one qubit and the remaining qubits is equally shared among the pairwise entanglement in the $N - 1$ pairs of qubits, when the entanglement is measured in terms of squared concurrence. We will illustrate in the next section that, when expressed in terms of squared negativity of partial transpose, the W-class of states are shown to have nonzero residual entanglement.

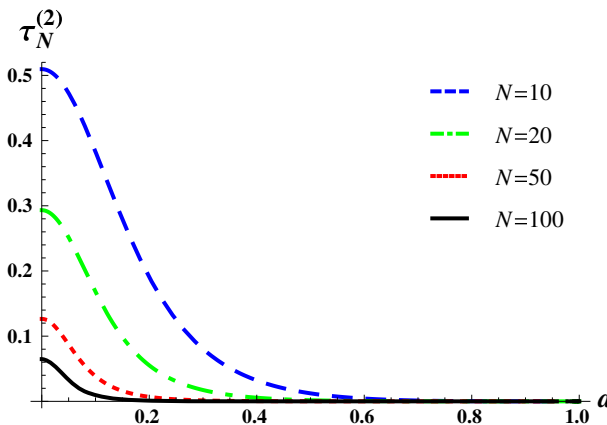


Fig. 1 The plot of concurrence tangle $\tau_N^{(2)}$ as a function of the real parameter ‘ a ’ ($0 \leq a \leq 1$), for the states $|D_{N-2, 2}\rangle$

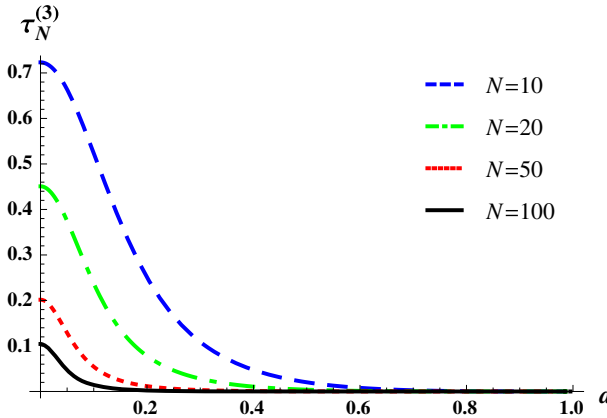


Fig. 2 The plot of concurrence tangle $\tau_N^{(3)}$ as a function of the real parameter ‘ a ’ ($0 \leq a \leq 1$), for the states $|D_{N-3,3}\rangle$

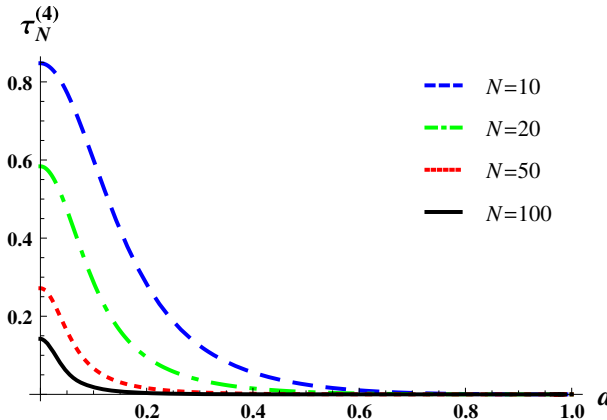


Fig. 3 The plot of concurrence tangle $\tau_N^{(4)}$ as a function of the real parameter ‘ a ’ for the states $|D_{N-4,4}\rangle$

From Figs. 1, 2, 3 and 4, one can draw the following conclusions about the bound on monogamy relation, for Dicke-class of states.

1. For the Dicke states $|\frac{N}{2}, \frac{N}{2} - r\rangle, r = 1, 2, \dots, N$, characterized by two orthogonal spinors the bound on monogamy is larger compared to their companion states with non-orthogonal spinors. Figures 1, 2, 3 and 4 readily illustrate that the concurrence tangle $\tau_N^{(k)}$ is maximum when the parameter $a = 0$ and monotonically decreases when a ($0 < a < 1$) increases. For separable states corresponding to $a = 1$, the concurrence tangle vanishes, as expected.
2. The concurrence tangle for the states $|D_{N-k,k}\rangle$ increases with the increase in the value of k ($k = 2, 3, 4, 5$). In particular, the state $|D_{N-k,k}\rangle$ with $k = \lfloor \frac{N}{2} \rfloor$ is more monogamous. This implies that the Dicke-class of states $|D_{N-k,k}\rangle$ with equal distribution of two spinors has larger bound on the monogamy relation and possesses larger residual entanglement.

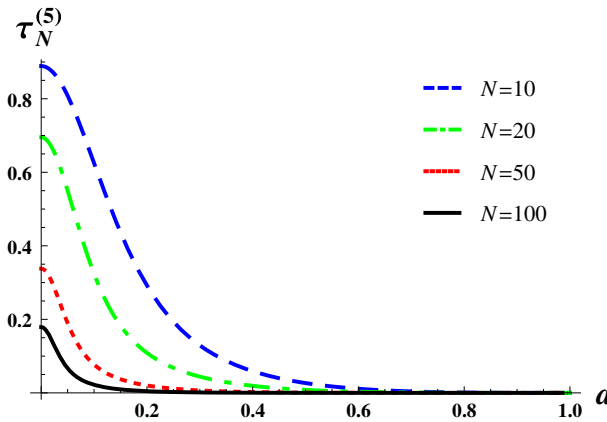


Fig. 4 The plot of concurrence tangle $\tau_N^{(5)}$ as a function of the real parameter ‘ a ’ ($0 \leq a \leq 1$), for the states $|D_{N-5}, 5\rangle$,

3. The residual entanglement (and hence the bound on monogamy relation with respect to squared concurrence) reduces with the increase in number of qubits N , as expected.

4.2 Monogamy relation for generalized Dicke states in terms of squared negativity of partial transpose

Monogamous nature of pure three-qubit states with respect to negativity of partial transpose was examined in Ref. [9]. This work gave an indication that any *single* measure of entanglement serves to provide a bound on monogamy inequality, which is specific to that measure and the state under consideration. For instance, it is shown in Ref. [9] that three-qubit W-state and its non-symmetric generalization have nonzero bound on monogamy inequality, set up in terms of negativity of partial transpose. While concurrence tangle of W- and non-symmetric generalization of W-states vanishes, their negativity tangle is nonzero. It is also shown in Ref. [40] that for the one-parameter family $\{\mathcal{D}_{N-1, 1}\}$ of W-class of states $|D_{N-1, 1}\rangle$ (generalized W-states), consisting of two orthogonal/non-orthogonal qubits, concurrence tangle is zero but negativity tangle has a maximum value for W-states $|\frac{N}{2}, \frac{N}{2} - 1\rangle$ and decreases monotonically with parameter a ($0 < a \leq 1$) of $|D_{N-1, 1}\rangle$. We thus focus on investigating monogamous nature of Dicke-class of states with respect to squared negativity of partial transpose.

With the knowledge of two-qubit reduced density matrix [see Eq. (16)] of the state $|D_{N-k, k}\rangle$, we can readily evaluate its negativity of partial transpose [56–58]. The partially transposed density matrix $(\rho_2^{(k)})^T$ of the two-qubit state $\rho_2^{(k)}$ [see Eq. (16)] is explicitly given by

$$\left(\rho_2^{(k)}\right)^T = \begin{pmatrix} A^{(k)} & B^{(k)} & B^{(k)} & D^{(k)} \\ B^{(k)} & D^{(k)} & C^{(k)} & E^{(k)} \\ B^{(k)} & C^{(k)} & D^{(k)} & E^{(k)} \\ D^{(k)} & E^{(k)} & E^{(k)} & F^{(k)} \end{pmatrix}, \tag{23}$$

as [56] $\left(\rho_2^{(k)}\right)_{ij;kl}^T = \left(\rho_2^{(k)}\right)_{il;kj} = \left(\rho_2^{(k)}\right)_{kj;il}$ are the elements of $\left(\rho_2^{(k)}\right)^T$.

The negativity of partial transpose [56–58] is defined as

$$N_{k_2} = \left(\left\| \left(\rho_2^{(k)}\right)^T \right\| - 1 \right) / 2,$$

where $\left\| \left(\rho_2^{(k)}\right)^T \right\|$ is the trace norm³ of the partially transposed density matrix $\left(\rho_2^{(k)}\right)^T$.

It may be noted that negativity of partial transpose of any two-qubit system varies from 0 to 0.5. Here we adopt the convention in Ref. [9] and redefine N_{k_2} as

$$N_{k_2} = \left\| \left(\rho_2^{(k)}\right)^T \right\| - 1 \tag{24}$$

so that it takes values in the range 0 to 1.

We now proceed to evaluate the entanglement N_{k_1} between any qubit of the state $|D_{N-k,k}\rangle$ and its remaining $N - 1$ -qubits, in terms of squared negativity of partial transpose. On recalling the result [9] that negativity of partial transpose between a single qubit and the remaining qubits of any pure N -qubit state matches with the corresponding concurrence, we have

$$N_{k_1} = C_{k_1} = 2\sqrt{\det \rho_1^{(k)}} \tag{25}$$

for the pure symmetric state $|D_{N-k,k}\rangle$.

As the pairwise entanglement between all the $N - 1$ pairs of qubits is equal, due to symmetry of the states belonging to Dicke-class, the monogamy inequality [see Eq. (1)] turns out to be [9],

$$N_{k_1}^2 \geq (N - 1)N_{k_2}^2 \tag{26}$$

in terms of squared negativity of partial transpose. The bound on monogamy inequality or, equivalently the measure of residual entanglement of the states $|D_{N-k,k}\rangle$, is given by

$$\xi_N^{(k)} = N_{k_1}^2 - (N - 1)N_{k_2}^2. \tag{27}$$

³ The trace norm [58] $\left\| \left(\rho_2^{(k)}\right)^T \right\|$ is the sum of the square roots of eigenvalues of the positive-definite matrix $\left(\rho_2^{(k)}\right)^T \left(\rho_2^{(k)}\right)^{\dagger}$.

where $\xi_N^{(k)}$ denotes the negativity tangle of the states $|D_{N-k,k}\rangle$ belonging to the Dicke-class.

In fact, for $k = 1$, i.e., for $\{\mathcal{D}_{N-1,1}\}$, the so-called W-class of states, concurrence tangle $\tau_N^{(1)} = 0$, whereas negativity tangle $\xi_N^{(1)}$ is nonzero for all values of $N \geq 3$ [40]. This feature is seen in Fig. 9.

It may be seen from Figs. 5, 6, 7 and 8 that the nature of variation of negativity tangle $\xi_N^{(k)}$ is quite similar to that of concurrence tangle $\tau_N^{(k)}$ (see Figs. 1, 2, 3 and 4) for each value of k ($k = 2, 3, 4, 5$) and N . We also notice that $\xi_N^{(k)} \geq \tau_N^{(k)}$ (see Figs. 5, 6, 7, 8 and 9). Thus, the bound on monogamy inequality in terms of squared negativity of partial transpose exceeds the bound in terms of squared concurrence. In

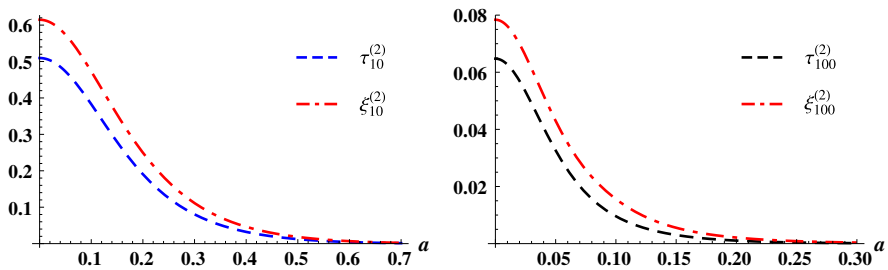


Fig. 5 Comparison of concurrence tangle $\tau_N^{(2)}$ with negativity tangle $\xi_N^{(2)}$ for $N = 10$ and $N = 100$

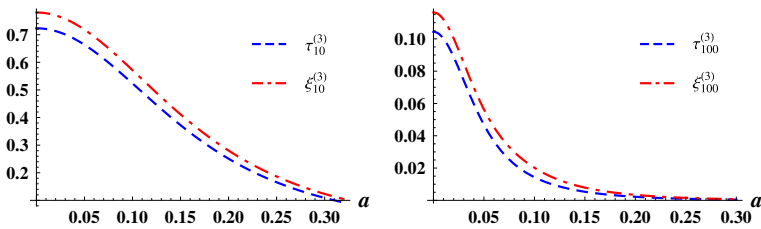


Fig. 6 Comparison of concurrence tangle $\tau_N^{(3)}$ with negativity tangle $\xi_N^{(3)}$ for $N = 10$ and $N = 100$

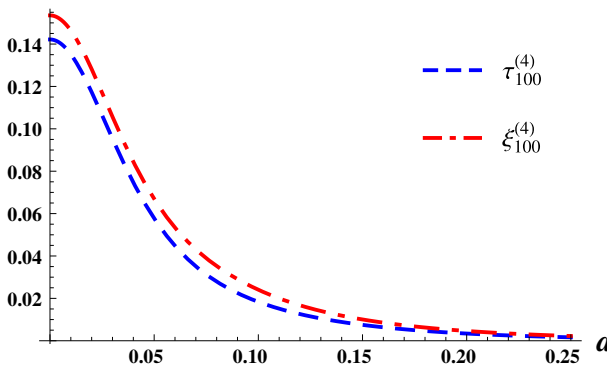


Fig. 7 Comparison of concurrence tangle $\tau_N^{(4)}$ with negativity tangle $\xi_N^{(4)}$ for $N = 100$

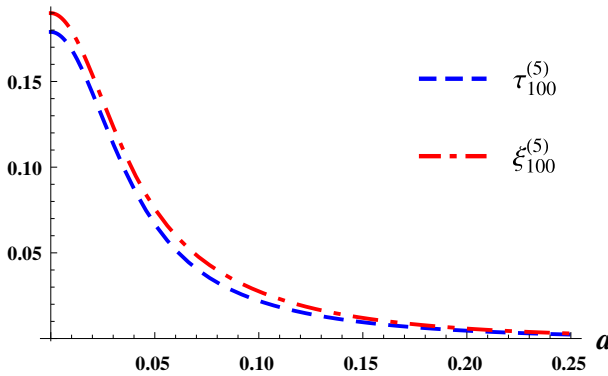


Fig. 8 Comparison of concurrence tangle $\tau_N^{(5)}$ with negativity tangle $\xi_N^{(5)}$ for $N = 100$

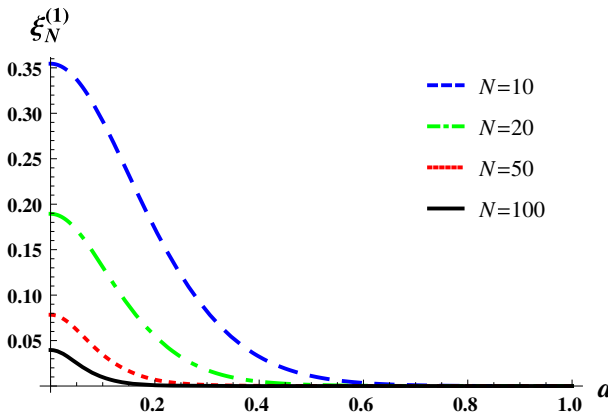


Fig. 9 The variation of negativity tangle $\xi_N^{(1)}$ as a function of the parameter a for different values of N

other words, negativity tangle $\xi_N^{(k)}$ is greater than or equal to concurrence tangle $\tau_N^{(k)}$, quite in accordance with the result obtained in Ref. [9] for N -qubit states of the W-class. We therefore conclude that for the entire Dicke-class $\{\mathcal{D}_{N-1,1}\}$, containing all possible SLOCC inequivalent classes of states $|D_{N-k,k}\rangle$, the nature of variation of the bound on monogamy inequality, expressed in squared concurrence, squared negativity of partial transpose, is in agreement with that observed for W-class of states [40].

5 Conclusion

In this article, we have analyzed the monogamous nature of N -qubit pure symmetric states with two distinct Majorana spinors—the so-called Dicke-class of states, using squared concurrence, and squared negativity of partial transpose as measures of two-qubit entanglement. Toward this end, we have made use of Majorana geometric representation to obtain a simplified, one-parameter structure of Dicke-class of states. The familiar angular momentum algebra relating to addition of angular momentum of

two spin systems enables us to partition this simplified form of the generalized Dicke states into two of its subsystems. Using this partitioning, we obtain the general form of two-qubit and single-qubit density matrices of Dicke-class of states. The reduced density matrices so obtained allow us to determine the pairwise entanglement and residual entanglement (which goes beyond the pairwise entanglement) in the system.

The bound on monogamy relation, using squared concurrence as a measure of entanglement, is analyzed for all SLOCC inequivalent families of states belonging to Dicke-class. Moreover, our results reveal that, among the several inequivalent families, corresponding to different degeneracy configuration of the two spinors, the states belonging to the family having equal distribution of its two spinors are found to have less shareability among its qubits (more monogamous), thereby possessing larger residual entanglement.

The monogamy inequality in terms of squared negativity of partial transpose is also analyzed, and it is seen that for the entire Dicke-class of states, the residual entanglement quantified through negativity tangle exceeds the one quantified through concurrence tangle. Our results confirm that both the measures of entanglement are analogous in establishing monogamous nature in Dicke-class of states and either of the two measures can be chosen for the evaluation of their residual entanglement.

Summarizing, our work accomplishes the task of analyzing the monogamous nature of the entire one-parameter family of pure symmetric multiqubit states characterized by two distinct spinors, for the first time. The results of this work open up avenues for potential applications of Dicke-class of states in the field of quantum information processing, as limited shareability of entanglement among subsystems/monogamous nature happens to be one of the highlighted *non-classical* features of multiparty quantum systems.

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