



Non-Markovian dynamics and quantum interference in open three-level quantum systems

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Abstract

The exact analytical solutions for the dynamics of the dissipative three-level V-type and Λ -type atomic systems in the vacuum Lorentzian environments are presented. Quantum interference between the spontaneous emissions of different decaying channels for the V-type atomic system is observed. For the dissipative Λ -type atomic system, however, similar phenomenon of quantum interference does not exist. We demonstrate that quantum interference can be used to protect effectively the quantum entanglement and quantum coherence. The control of the transition from Markovian to non-Markovian processes is discussed.

Keywords Non-Markovian dynamics · Quantum interference · Quantum entanglement · Quantum coherence

1 Introduction

The evolution of open quantum systems can be divided into two basic types, i.e., Markovian and non-Markovian processes. The early researches usually involve only the Markovian processes, which make use of the Born–Markov approximation to neglect all the memory effects, leading to the Lindblad master equation [1,2]. Recently, researchers are attracted deeply by the non-Markovian processes. Much attention has been devoted to the study of non-Markovian process of open quantum systems, including the measure of non-Markovianity [3–16], the positivity of dynamics [17–19], and some other dynamical properties [20–29]. Many possible applications of the dynamical non-Markovianity also have been discovered, including the applications in quantum metrology [30–32], quantum communication [33–36], and quantum control [37].

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The dynamics of open quantum systems is very sophisticated, and only very rare of them can be solved exactly. The case of a two-level atom dissipating in a vacuum environment is one of the few examples that can be solved exactly. Due to the advantage of the exact analytical solution, the dissipative two-level system becomes the paradigm for the investigation of non-Markovian dynamics. Multilevel open quantum systems, especially multilevel dissipative systems, due to their complexity, are relatively seldom involved. Though several works have already involved the study of dissipative three-level systems [38,39], the exact analytical expression that relates the evolved state to its initial state (or Kraus representation) has not yet been established explicitly. Due to the potential applications such as the quantum cryptography [40,41], the fault-tolerant quantum computation, and quantum error correction [42], it is very worthwhile to pour more efforts into the study of dynamics of multilevel open quantum systems. In this paper, we will present the exact analytical solutions for the dissipative three-level V-type and Λ -type atoms in the vacuum Lorentzian environments, which would be very helpful in the study of the non-Markovian dynamics of open quantum systems.

Quantum interference is a kind of unique phenomenon of multilevel atomic systems different from two-level systems. The transitions from different channels may take place interference, leading to many interesting physical phenomena, such as the well-known electromagnetically induced transparency (EIT) [43], ultranarrow spectral lines [44], and spontaneous emission cancelation [45]. Quantum entanglement and quantum coherence are two important concepts of quantum theory, which are very useful in quantum information processing. As the application of the exact analytical solutions, we first use them to discuss the problem of quantum interference of the three-level atomic systems in the process of spontaneous emissions. Then we further study the applications of the quantum interference in the protection of quantum entanglement and quantum coherence.

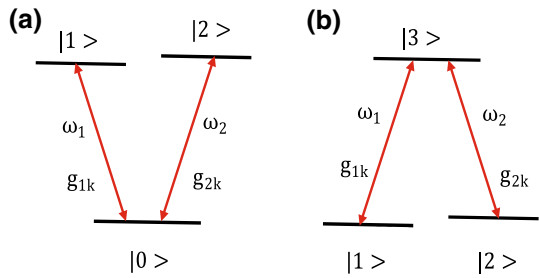
The paper is organized as follows. In Sect. 2, we introduce the two dynamical models for the V-type and Λ -type atomic systems interacting with vacuum environments and present the exact analytical solutions. In Sect. 3, we study the phenomenon of quantum interference between different decaying channels for the two considered models. In Sect. 4, we study the protective roles of quantum interference to the quantum entanglement and quantum coherence, for the dissipative V-type atomic system. In Sect. 5, we discuss the non-Markovianity of the two types of dynamical models as the last application of the exact solutions. Finally, we give the conclusions in Sect. 6.

2 Dynamical models and their solutions

2.1 V-type three-level atom

Consider a V-type three-level atom with transition frequencies ω_1 and ω_2 (see Fig. 1a), which is embedded in a zero-temperature bosonic reservoir modeled by an infinite chain of quantum harmonic oscillators. The Hamiltonian for the whole system may be written as

Fig. 1 Schematic diagram of energy level for **a** V-type atom and **b** Λ -type atom



$$H_V = \omega_1|1\rangle\langle 1| + \omega_2|2\rangle\langle 2| + \sum_k \omega_k b_k^\dagger b_k + \sum_k [g_{1k} b_k |1\rangle\langle 0| + g_{2k} b_k |2\rangle\langle 0| + \text{h.c.}], \tag{1}$$

where we set the energy of level $|0\rangle$ to be zero and thus levels $|1\rangle$ and $|2\rangle$ have energies ω_1 and ω_2 ($\hbar = 1$), respectively, b_k and b_k^\dagger are the annihilation and creation operators for the k th harmonic oscillator of the reservoir, and g_{1k} and g_{2k} are the coupling strengths between reservoir and the two transition channels, respectively.

Suppose that the initial state of the whole system is

$$|\Psi(0)\rangle = [c_0(0)|0\rangle + c_1(0)|1\rangle + c_2(0)|2\rangle] \otimes |0\rangle_R, \tag{2}$$

where $|0\rangle_R$ denotes the vacuum state of environment. Employing the conservativeness of excitation numbers of Jaynes–Cummings model, the dynamical state at any time t may be written as

$$|\Psi(t)\rangle = [c_0(t)|0\rangle + c_1(t)|1\rangle + c_2(t)|2\rangle] \otimes |0\rangle_R + \sum_k c_k(t)|0\rangle \otimes |1_k\rangle_R, \tag{3}$$

where $|1_k\rangle_R$ indicates that there is a photon in the k th mode of the environment. Tracing over the environmental degrees of freedom, then the reduced state of the atom in its natural bases is

$$\rho_s(t) = \begin{bmatrix} 1 - |c_1(t)|^2 - |c_2(t)|^2 & c_0(t)c_1^*(t) & c_0(t)c_2^*(t) \\ c_0^*(t)c_1(t) & |c_1(t)|^2 & c_1(t)c_2^*(t) \\ c_0^*(t)c_2(t) & c_1^*(t)c_2(t) & |c_2(t)|^2 \end{bmatrix} \tag{4}$$

The evolution of coefficients $c_i(t)$ is determined by the Schrödinger equation $i\partial|\Psi(t)\rangle/\partial t = H_V|\Psi(t)\rangle$, which satisfy the following set of equations

$$\begin{cases} c_0(t) = c_0(0), & (5a) \end{cases}$$

$$\begin{cases} i\dot{c}_1(t) = \omega_1 c_1(t) + \sum_k g_{1k} c_k(t), & (5b) \end{cases}$$

$$\begin{cases} i\dot{c}_2(t) = \omega_2 c_2(t) + \sum_k g_{2k} c_k(t), & (5c) \end{cases}$$

$$\begin{cases} i\dot{c}_k(t) = \omega_k c_k(t) + g_{1k}^* c_1(t) + g_{2k}^* c_2(t). & (5d) \end{cases}$$

Without loss of generality, we assume in the following g_{1k} and g_{2k} to be real. Formally integrating Eq. (5d) and plugging it into Eqs. (5b), (5c), by use of the continuum limitation of environmental modes $\sum_k g_{ik}g_{jk} \rightarrow \int d\omega J_{ij}(\omega)$ with $J_{ij}(\omega)$ ($i, j = 1, 2$) being the spectral density function, one obtains

$$\dot{c}_1(t) = -i\omega_1 c_1(t) - \int_0^t d\tau f_{11}(t - \tau)c_1(\tau) - \int_0^t d\tau f_{12}(t - \tau)c_2(\tau), \tag{6}$$

$$\dot{c}_2(t) = -i\omega_2 c_2(t) - \int_0^t d\tau f_{22}(t - \tau)c_2(\tau) - \int_0^t d\tau f_{21}(t - \tau)c_1(\tau), \tag{7}$$

where $f_{ij}(t - \tau) = \int d\omega J_{ij}(\omega)e^{-i\omega(t-\tau)}$ is the two-point correlation function of environment. In the above deduction, we have used the initial condition $c_k(0) = 0$. Denoting $c_i(p)$ and $f_{ij}(p)$ the Laplace transformation of $c_i(t)$ and $f_{ij}(t)$, respectively, then Eqs. (7) and (8) lead to

$$c_1(p) = \frac{[p + i\omega_2 + f_{22}(p)]c_1(0) - f_{12}(p)c_2(0)}{Q(p)}, \tag{8}$$

$$c_2(p) = \frac{[p + i\omega_1 + f_{11}(p)]c_2(0) - f_{21}(p)c_1(0)}{Q(p)}, \tag{9}$$

with $Q(p) = [p + i\omega_1 + f_{11}(p)][p + i\omega_2 + f_{22}(p)] - f_{12}(p)f_{21}(p)$. In principle, the inverse Laplace transformation of these equations gives the time evolution of $c_1(t)$ and $c_2(t)$.

To go further, we need to specify the spectral density of the environment. As an exemplification, we choose the Lorentzian spectrum

$$J_{ij}(\omega) = \frac{1}{2\pi} \cdot \frac{\gamma_{ij}\lambda^2}{(\omega_0 - \omega)^2 + \lambda^2}, \tag{10}$$

where ω_0 is the central frequency and λ defines the spectral width. The parameter $\gamma_{ii} \equiv \gamma_i$ with $i = 1, 2$ describes the spontaneous emission rates of level $|i\rangle$, and γ_{ij} with $i \neq j$ describes the correlation between the two transitions in Fig. 1a. When the dipole moments of the two transitions are parallel, the relation $\gamma_{12} = \gamma_{21} = \sqrt{\gamma_1\gamma_2}$ is met. In this paper, we consider only this case.

For the Lorentzian spectrum, the correlation function becomes as $f_{ij}(t - \tau) = \frac{\gamma_{ij}\lambda}{2}e^{-M(t-\tau)}$ and the corresponding Laplace transformation reads $f_{ij}(p) = B_{ij}/(p + M)$ with $B_{ij} = \gamma_{ij}\lambda/2$ and $M = \lambda + i\omega_0$. In the case that the dipole moments of the two transitions are parallel, one also has $B_{11}B_{22} - B_{12}B_{21} = 0$. Taking these results into consideration, Eqs. (8)–(9) become,

$$c_1(p) = \frac{[(p + i\omega_2)(p + M) + B_{22}]c_1(0) - B_{12}c_2(0)}{p^3 + h_1p^2 + h_2p + h_3}, \tag{11}$$

$$c_2(p) = \frac{[(p + i\omega_1)(p + M) + B_{11}]c_2(0) - B_{21}c_1(0)}{p^3 + h_1p^2 + h_2p + h_3}, \tag{12}$$

where $h_1 = M + i(\omega_1 + \omega_2)$, $h_2 = B_{11} + B_{22} - \omega_1\omega_2 + iM(\omega_1 + \omega_2)$, $h_3 = -\omega_1\omega_2M + i(\omega_1B_{22} + \omega_2B_{11})$.

Observing that the numerator and denominator of Eqs. (11)–(12) are the second- and third-order polynomials of p , if the roots b_i ($i = 1, 2, 3$) of the polynomial equation $p^3 + h_1p^2 + h_2p + h_3 = 0$ are non-degenerate, then we can make the decomposition

$$\begin{cases} c_1(p) = \sum_{i=1}^3 \frac{D_i}{p - b_i}, & (13a) \\ c_2(p) = \sum_{i=1}^3 \frac{D'_i}{p - b_i}. & (13b) \end{cases}$$

For the degenerate case, the decomposition has different forms which are discussed in “Appendix A.” Noting that b_i are determined by the atomic and environmental structure parameters, the degenerative probability is very small and may be avoided by adjusting these structure parameters. The parameters D_i (D'_i) are the residues of $c_1(p)$ ($c_2(p)$) at the points $p = b_i$, which in our problem may be written as

$$\begin{cases} D_i = E_i c_1(0) + F_i c_2(0), & (14a) \\ D'_i = G_i c_2(0) + H_i c_1(0), & (14b) \end{cases}$$

with

$$\begin{aligned} E_i &= \frac{(b_i + i\omega_2)(b_i + M) + B_{22}}{3b_i^2 + 2h_1b_i + h_2}, \\ F_i &= -\frac{B_{12}}{3b_i^2 + 2h_1b_i + h_2}, \\ G_i &= \frac{(b_i + i\omega_1)(b_i + M) + B_{11}}{3b_i^2 + 2h_1b_i + h_2}, \\ H_i &= -\frac{B_{21}}{3b_i^2 + 2h_1b_i + h_2}. \end{aligned}$$

Finally, the inverse Laplace transformation of Eqs. (13a)–(13b) gives

$$\begin{cases} c_1(t) = E(t)c_1(0) + F(t)c_2(0), & (15a) \\ c_2(t) = G(t)c_2(0) + H(t)c_1(0), & (15b) \end{cases}$$

with

$$\begin{aligned} E(t) &= \sum_{i=1}^3 E_i e^{b_i t}, \\ F(t) &= \sum_{i=1}^3 F_i e^{b_i t}, \end{aligned}$$

$$G(t) = \sum_{i=1}^3 G_i e^{b_i t},$$

$$H(t) = \sum_{i=1}^3 H_i e^{b_i t}.$$

So far, we have completed the solving process of the open V-type three-level atom and the exact analytical results are given by Eqs. (4), (5a), (15a), (15b). We can also express this dynamical process as the map ε that satisfies,

$$\begin{aligned} \varepsilon(|0\rangle\langle 0|) &= |0\rangle\langle 0|, \\ \varepsilon(|1\rangle\langle 1|) &= [1 - |E(t)|^2 - |H(t)|^2]|0\rangle\langle 0| + |E(t)|^2|1\rangle\langle 1| + |H(t)|^2|2\rangle\langle 2| \\ &\quad + E(t)H^*(t)|1\rangle\langle 2| + E^*(t)H(t)|2\rangle\langle 1|, \\ \varepsilon(|2\rangle\langle 2|) &= [1 - |F(t)|^2 - |G(t)|^2]|0\rangle\langle 0| + |F(t)|^2|1\rangle\langle 1| + |G(t)|^2|2\rangle\langle 2| \\ &\quad + F(t)G^*(t)|1\rangle\langle 2| + F^*(t)G(t)|2\rangle\langle 1|, \\ \varepsilon(|1\rangle\langle 0|) &= E(t)|1\rangle\langle 0| + H(t)|2\rangle\langle 0|, \\ \varepsilon(|2\rangle\langle 0|) &= F(t)|1\rangle\langle 0| + G(t)|2\rangle\langle 0|, \\ \varepsilon(|2\rangle\langle 1|) &= [-F(t)E^*(t) - G(t)H^*(t)]|0\rangle\langle 0| + F(t)E^*(t)|1\rangle\langle 1| \\ &\quad + G(t)H^*(t)|2\rangle\langle 2| + F(t)H^*(t)|1\rangle\langle 2| + E^*(t)G(t)|2\rangle\langle 1|. \end{aligned}$$

This map will be used in the following for studying the evolution of entanglement.

2.2 Λ -type three-level atom

Besides open V-type three-level atoms, the dynamics of an open Λ -type three-level atom dissipating in a zero-temperature bosonic reservoir can also be solved exactly. Figure 1b shows the energy-level structure of a Λ -type three-level atom with $|1\rangle$ and $|2\rangle$ the ground and meta-stable states and $|3\rangle$ the excited state. The Hamiltonian for the combined system of the atom and its environment reads

$$H_\Lambda = -\omega_1|1\rangle\langle 1| - \omega_2|2\rangle\langle 2| + \sum_k \omega_k b_k^\dagger b_k + \sum_k [g_{1k} b_k |3\rangle\langle 1| + g_{2k} b_k |3\rangle\langle 2| + \text{h.c.}]. \tag{16}$$

Here ω_1 and ω_2 are, respectively, the frequencies of the transitions $|1\rangle \leftrightarrow |3\rangle$ and $|2\rangle \leftrightarrow |3\rangle$, and we set the energy of level $|3\rangle$ to be zero. The meanings of other symbols are the same as before.

Suppose that the initial state of the combined system is

$$|\Upsilon(0)\rangle = [c_1(0)|1\rangle + c_2(0)|2\rangle + c_3(0)|3\rangle] \otimes |0\rangle_R, \tag{17}$$

with $|0\rangle_R$ the vacuum state of environment, then the dynamical state at any time t may be written as

$$|\mathcal{Y}(t)\rangle = [c_1(t)|1\rangle + c_2(t)|2\rangle + c_3(t)|3\rangle] \otimes |0\rangle_R + \sum_k c_k(t)|1\rangle \otimes |1_k\rangle_R + \sum_k d_k(t)|2\rangle \otimes |1_k\rangle_R. \tag{18}$$

As before $|1_k\rangle_R$ indicates the single-photon state of the k th mode of environment. The reduced state of the atom in its natural bases now becomes

$$\rho_s(t) = \begin{bmatrix} |c_1(t)|^2 + \sum_k |c_k(t)|^2 & c_1(t)c_2^*(t) + \sum_k c_k(t)d_k^*(t) & c_1(t)c_3^*(t) \\ c_2(t)c_1^*(t) + \sum_k d_k(t)c_k^*(t) & |c_2(t)|^2 + \sum_k |d_k(t)|^2 & c_2(t)c_3^*(t) \\ c_3(t)c_1^*(t) & c_3(t)c_2^*(t) & |c_3(t)|^2 \end{bmatrix}. \tag{19}$$

Here the evolution of coefficients is determined by following set of equations

$$\begin{cases} i\dot{c}_1(t) = -\omega_1 c_1(t), & (20a) \\ i\dot{c}_2(t) = -\omega_2 c_2(t), & (20b) \\ i\dot{c}_3(t) = \sum_k g_{1k}c_k(t) + \sum_k g_{2k}d_k(t), & (20c) \\ i\dot{c}_k(t) = -\omega_1 c_k(t) + \omega_k c_k(t) + g_{1k}^* c_3(t), & (20d) \\ i\dot{d}_k(t) = -\omega_2 d_k(t) + \omega_k d_k(t) + g_{2k}^* c_3(t). & (20e) \end{cases}$$

Obviously, the solving process of this problem is more complex than the case of V-type atom, because we need to know the evolution of $c_k(t)$ and $d_k(t)$, besides $c_1(t)$, $c_2(t)$ and $c_3(t)$. The evolution of $c_1(t)$ and $c_2(t)$ can be easily obtained

$$\begin{cases} c_1(t) = c_1(0)e^{i\omega_1 t}, & (21a) \\ c_2(t) = c_2(0)e^{i\omega_2 t}. & (21b) \end{cases}$$

To get the evolution of the other coefficients, we formally integrate Eqs. (20d), (20e) in the condition $c_k(0) = d_k(0) = 0$ and get

$$\begin{cases} c_k(t) = -i \int_0^t d\tau g_{1k}^* c_3(\tau) e^{i(\omega_1 - \omega_k)(t - \tau)}, & (22a) \\ d_k(t) = -i \int_0^t d\tau g_{2k}^* c_3(\tau) e^{i(\omega_2 - \omega_k)(t - \tau)}. & (22b) \end{cases}$$

By plugging them into Eq. (20c), in the continuum limitation $\sum_k g_{ik}g_{jk} \rightarrow \int d\omega J_{ij}(\omega)$ with $i, j = 1, 2$, we obtain

$$\dot{c}_3(t) = - \int_0^t d\tau [f_1(t - \tau) + f_2(t - \tau)]c_3(\tau); \tag{23}$$

here the correlation function is defined by $f_j(t - \tau) = \int d\omega J_{jj}(\omega)e^{i(\omega_j - \omega)(t - \tau)}$.

Now we also assume the Lorentzian spectrum Eq. (10) with $\gamma_{jj} \equiv \gamma_j$ ($j = 1, 2$) that describes the spontaneous emissions of levels $|3\rangle$ to $|j\rangle$. Assume that the transitions have parallel dipole moments so that $\gamma_{12} = \gamma_{21} = \sqrt{\gamma_1\gamma_2}$. Under these conditions, we have $f_j(t - \tau) = \frac{\gamma_j\lambda}{2}e^{-M_j(t-\tau)}$ with $M_j = \lambda + i(\omega_0 - \omega_j)$. The solution of Eq. (23) may be written as

$$c_3(t) = \xi(t)c_3(0), \tag{24}$$

with $\xi(t) = (\mathcal{D}_1e^{\beta_1t} + \mathcal{D}_2e^{\beta_2t} + \mathcal{D}_3e^{\beta_3t})$. Here β_i are the roots of equation $R(p) = p^3 + (M_1 + M_2)p^2 + (M_1M_2 + \frac{\gamma_1\lambda}{2} + \frac{\gamma_2\lambda}{2})p + \frac{\gamma_1\lambda}{2}M_2 + \frac{\gamma_2\lambda}{2}M_1 = 0$, which are assumed non-degenerate. As the degenerative probability is very small and can always be avoided by adjusting the structure parameters, we thus have no longer presented the expresses in detail. The coefficients \mathcal{D}_i are given by

$$\mathcal{D}_i = \frac{(\beta_i + M_1)(\beta_i + M_2)}{3\beta_i^2 + 2(M_1 + M_2)\beta_i + M_1M_2 + \frac{\gamma_1\lambda}{2} + \frac{\gamma_2\lambda}{2}}. \tag{25}$$

Having $c_3(t)$ in hand, we can then obtain the evolution of $c_k(t)$ and $d_k(t)$. Equations (22a), (22b), in transition to continuum limitation, lead to

$$\sum_k |c_k(t)|^2 = \int_0^t d\tau \int_0^{\tau} d\tau' f_1(\tau' - \tau)c_3(\tau)c_3^*(\tau'), \tag{26}$$

$$\sum_k |d_k(t)|^2 = \int_0^t d\tau \int_0^{\tau} d\tau' f_2(\tau' - \tau)c_3(\tau)c_3^*(\tau'), \tag{27}$$

and

$$\begin{aligned} \sum_k c_k(t)d_k^*(t) &= \int_0^t d\tau \int_0^{\tau} d\tau' \int d\omega J_{12}(\omega)e^{i\omega(\tau-\tau')}e^{i(\omega_1t-\omega_2t-\omega_1\tau+\omega_2\tau')}c_3(\tau)c_3^*(\tau') \\ &= \frac{\gamma_{12}\lambda}{2}e^{i(\omega_1-\omega_2)t} \int_0^t d\tau \int_0^{\tau} d\tau' c_3(\tau)c_3^*(\tau')e^{-\lambda|\tau'-\tau|+i(\delta_1\tau-\delta_2\tau')}. \end{aligned} \tag{28}$$

where

$$f_j(\tau' - \tau) = \int d\omega J_{jj}(\omega) \exp [i(\omega_j - \omega)(\tau' - \tau)] = \frac{\gamma_j\lambda}{2} \exp [-\lambda|\tau' - \tau| - i\delta_j(\tau' - \tau)],$$

$\delta_j = \omega_0 - \omega_j$, and we have used

$$\int d\omega J_{12}(\omega)e^{i\omega(\tau-\tau')} = \frac{\gamma_{12}\lambda}{2}e^{-\lambda|\tau'-\tau|-i\omega_0(\tau'-\tau)}$$

in the second equality of Eq. (28).

The double integrals in Eqs. (26)–(28) can be further solved. Using

$$|\tau' - \tau| = \begin{cases} \tau' - \tau & \text{if } \tau' \geq \tau, \\ \tau - \tau' & \text{if } \tau' < \tau, \end{cases}$$

to divide the integral with respect to τ' into two parts, after tedious but straightforward calculations, we finally obtain

$$\left\{ \begin{aligned} \sum_k |c_k(t)|^2 &= \alpha_1(t)|c_3(0)|^2, & (29a) \\ \sum_k |d_k(t)|^2 &= \alpha_2(t)|c_3(0)|^2, & (29b) \\ \sum_k c_k(t)d_k^*(t) &= \Theta(t)|c_3(0)|^2, & (29c) \end{aligned} \right.$$

where

$$\alpha_1(t) = \frac{\gamma_1 \lambda}{2} \sum_{j,l=1}^3 \mathcal{D}_j \mathcal{D}_l^* \left\{ \Omega_1^{jl} e^{(\beta_j + \beta_l^*)t} + \Omega_2^{jl} e^{(-\lambda + \beta_j + i\delta_1)t} + \Omega_3^{jl} e^{(-\lambda + \beta_l^* - i\delta_1)t} + \Omega_4^{jl} \right\}, \tag{30}$$

$$\Theta(t) = \frac{\gamma_{12} \lambda}{2} \sum_{j,l=1}^3 \mathcal{D}_j \mathcal{D}_l^* \left\{ W_1^{jl} e^{(\beta_j + \beta_l^*)t} + W_2^{jl} e^{(-\lambda + \beta_j + i\delta_2)t} + W_3^{jl} e^{(-\lambda + \beta_l^* - i\delta_1)t} + W_4^{jl} e^{i(\omega_1 - \omega_2)t} \right\}, \tag{31}$$

with

$$\begin{aligned} \Omega_1^{jl} &= \frac{1}{(-\lambda + \beta_l^* - i\delta_1)(\lambda + \beta_j + i\delta_1)} - \frac{2\lambda}{(\lambda + \beta_l^* - i\delta_1)(-\lambda + \beta_l^* - i\delta_1)(\beta_j + \beta_l^*)}, \\ \Omega_2^{jl} &= \frac{1}{(\lambda - \beta_j - i\delta_1)(\lambda + \beta_l^* - i\delta_1)}, \\ W_1^{jl} &= \frac{1}{(-\lambda + \beta_l^* - i\delta_2)(\lambda + \beta_j + i\delta_1)} - \frac{2\lambda}{(\lambda + \beta_l^* - i\delta_2)(-\lambda + \beta_l^* - i\delta_2)[\beta_j + \beta_l^* + i(\omega_2 - \omega_1)]}, \\ W_2^{jl} &= \frac{1}{(\lambda + \beta_l^* - i\delta_2)(\lambda - \beta_j - i\delta_1)}. \end{aligned}$$

The other four coefficients can be obtained through the following replacement

$$\begin{aligned} \Omega_1^{jl} \xrightarrow{\lambda \rightarrow -\lambda} \Omega_4^{jl}, \quad \Omega_2^{jl} \xrightarrow{\lambda \rightarrow -\lambda} \Omega_3^{jl}, \\ W_1^{jl} \xrightarrow{\lambda \rightarrow -\lambda} W_4^{jl}, \quad W_2^{jl} \xrightarrow{\lambda \rightarrow -\lambda} W_3^{jl}. \end{aligned}$$

In addition, the coefficient $\alpha_2(t)$ in Eq. (29b) can be obtained by making the replacement

$$\alpha_1(t) \xrightarrow{\gamma_1 \rightarrow \gamma_2, \omega_1 \rightarrow \omega_2} \alpha_2(t). \tag{32}$$

Finally, the reduced density matrix Eq. (19) can be written as

$$\rho_S(t) = \begin{bmatrix} |c_1|^2 + \alpha_1(t)|c_3|^2 & c_1c_2^*e^{i(\omega_1-\omega_2)t} + \Theta(t)|c_3|^2 & T_1(t)c_1c_3^* \\ c_2c_1^*e^{i(\omega_2-\omega_1)t} + \Theta^*(t)|c_3|^2 & |c_2|^2 + \alpha_2(t)|c_3|^2 & T_2(t)c_2c_3^* \\ T_1^*(t)c_3c_1^* & T_2^*(t)c_3c_2^* & \alpha_3(t)|c_3|^2 \end{bmatrix}, \tag{33}$$

where

$$T_1(t) = e^{i\omega_1t} \left(\sum_{j=1}^3 \mathcal{D}_j^* e^{\beta_j^*t} \right),$$

$$T_2(t) = e^{i\omega_2t} \left(\sum_{j=1}^3 \mathcal{D}_j^* e^{\beta_j^*t} \right),$$

$$\alpha_3(t) = \left| \sum_{j=1}^3 \mathcal{D}_j e^{\beta_j t} \right|^2,$$

and $\alpha_1(t)$, $\alpha_2(t)$, $\Theta(t)$ are given by Eqs. (30)–(32). For simplicity, we use the abbreviation $c_i(0) \equiv c_i$ with $i = 1, 2, 3$ in Eq. (33). This dynamics also can be expressed in terms of a map Λ ,

$$\begin{aligned} \Lambda(|1\rangle\langle 1|) &= |1\rangle\langle 1|, \\ \Lambda(|2\rangle\langle 2|) &= |2\rangle\langle 2|, \\ \Lambda(|3\rangle\langle 3|) &= \alpha_1(t)|1\rangle\langle 1| + \alpha_2(t)|2\rangle\langle 2| + \alpha_3(t)|3\rangle\langle 3| \\ &\quad + \Theta(t)|1\rangle\langle 2| + \Theta^*(t)|2\rangle\langle 1|, \\ \Lambda(|1\rangle\langle 2|) &= \exp[i(\omega_1 - \omega_2)t]|1\rangle\langle 2|, \\ \Lambda(|1\rangle\langle 3|) &= |1\rangle\langle 3|, \\ \Lambda(|2\rangle\langle 3|) &= |2\rangle\langle 3|. \end{aligned}$$

3 Quantum interference

Having the exact analytical solutions presented above, we may study various dynamical properties of the open three-level systems, whether Markovian or non-Markovian processes. In this section, we use the solutions to demonstrate the phenomenon of quantum interference of the spontaneous emissions.

For the open V-type three-level atom involved above, by observing the structure of $E(t)$, $F(t)$, $G(t)$, $H(t)$ in Eqs. (15a), (15b), we know that the boundedness of $|C_1(t)|^2$ and $|C_2(t)|^2$ in the limit $t \rightarrow \infty$ requires that each of the roots b_i ($i = 1, 2, 3$) of the equation $p^3 + h_1p^2 + h_2p + h_3 = 0$ should have non-positive real part. When one or more of the roots have zero real parts (i.e., pure imaginary roots), $|C_1(t)|^2$ and $|C_2(t)|^2$ may have nonzero asymptotic values. Otherwise, the asymptotic values are zero. The occurrence of the nonzero asymptotic values can be regarded as the result of

quantum interference between the transitions $|1\rangle \rightarrow |0\rangle$ and $|2\rangle \rightarrow |0\rangle$, as explained in the following.

By setting the ansatz $p = i\chi$ into the equation $p^3 + h_1p^2 + h_2p + h_3 = 0$, we find that the necessary condition of pure imaginary root is $\omega_1 = \omega_2$. This is just the necessary condition of optical interference. Under this condition, can one observe the quantum interference really?

In Fig. 2, we plot the time evolution of the populations $|c_1(t)|^2$ and $|c_2(t)|^2$ under the interference condition, where we set $\omega_1 = \omega_2 = \omega_0 = 20\gamma$, $\lambda = 2\gamma$. In Fig. 2a, b, we choose $\gamma_1 = \gamma_2 \equiv \gamma$ and $c_1(0) = -c_2(0)$ (Fig. 2a) or $c_1(0) = c_2(0)$ (Fig. 2b). We find that the populations $|c_1(t)|^2$ and $|c_2(t)|^2$ keep almost unchanged when $c_1(0)$ and $c_2(0)$ have opposite signs, and reduce quickly to nearly zero when they have same signs. This is actually the result of destructive interference and constructive interference. The opposite signs between $c_1(0)$ and $c_2(0)$ mean opposite phases between the transitions $|1\rangle \rightarrow |0\rangle$ and $|2\rangle \rightarrow |0\rangle$, leading to destructive interference of the transitions which prohibits the decaying of the populations $|c_1(t)|^2$ and $|c_2(t)|^2$. On the contrary, same signs between $c_1(0)$ and $c_2(0)$ lead to constructive interference which speeds up the decaying of the populations.

In Fig. 2c, d, we let $\gamma_1 \neq \gamma_2$ but satisfying $\gamma_1|c_1(0)|^2 = \gamma_2|c_2(0)|^2$ (where $\gamma_1 = \gamma$ and $\gamma_2 = 2\gamma, 3\gamma, 4\gamma$ for the blue, red, black lines, respectively). It is shown similar results: The populations $|c_1(t)|^2$ and $|c_2(t)|^2$ keep almost unchanged when $c_1(0)$ and $c_2(0)$ have opposite signs, and reduce quickly when they have same signs. This can be explained as follows: Though the decaying rates of the two excited states γ_1 and γ_2 are different, the condition $\gamma_1|c_1(0)|^2 = \gamma_2|c_2(0)|^2$ guarantees that the decaying strengths from the two transitions are same. Thus, the destructive interference in Fig. 2c is complete and the populations can maintain unchanged. On the contrary, for the case of $\gamma_1|c_1(0)|^2 \neq \gamma_2|c_2(0)|^2$ (Fig. 2e, f, where $\gamma_1 = \gamma$ and $\gamma_2 = 3\gamma, 4\gamma, 2\gamma$ for the blue, red, black lines, respectively), the difference of the decaying strengths from the two transitions gives rise to incompletely destructive interference, leading to the change of populations. The light emitted from the higher-strength decaying channel has excess part which can excite in turn the lower-strength channel, leading to the increase in asymptotic population that corresponds to lower-strength decaying channel (Fig. 2e). In addition, for the constructive interference, as the decaying rates γ_1 and γ_2 in Fig. 2b are small, the energy lost in the environment spreads out timely, leading to the monotonic decrease in the populations. However, in Fig. 2d, f, with the increase in γ_2 , the energy lost in the environment cannot spread out timely, leading to the re-excitation of the populations. This is actually the commonly so-called memory effect.

We can also discuss in a similar way the quantum interference of Λ -type atom in the process of spontaneous emissions. From Eq. (24), we see that the necessary condition for nonzero asymptotic population of the upper-level state is that the real parts of β_i must be zero. By setting ansatz $p = i\chi$ into the equation $R(p) = p^3 + (M_1 + M_2)p^2 + (M_1M_2 + \frac{\gamma_1\lambda}{2} + \frac{\gamma_2\lambda}{2})p + \frac{\gamma_1\lambda}{2}M_2 + \frac{\gamma_2\lambda}{2}M_1 = 0$, one obtains

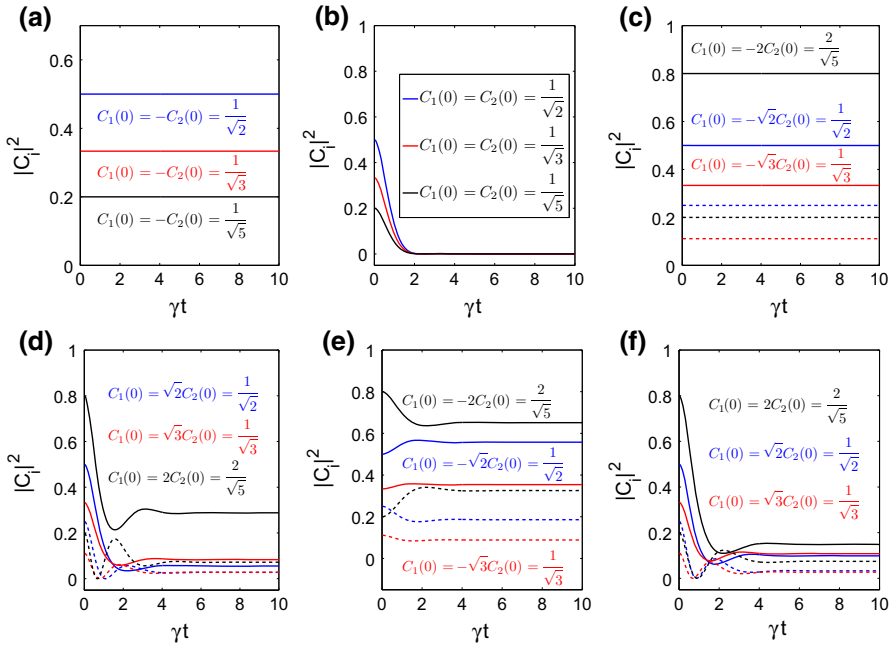
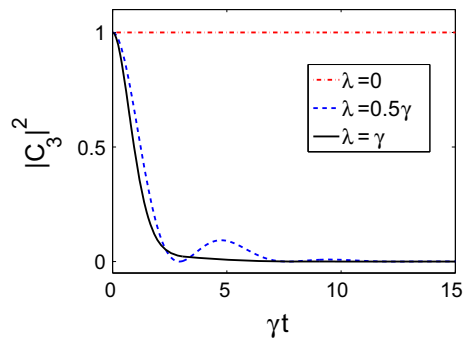


Fig. 2 Evolution of the populations versus dimensionless time γt for the open V-type atom with initial state given by Eq. (2), where $\omega_1 = \omega_2 = \omega_0 = 20\gamma$, $\lambda = 2\gamma$. The solid lines denote $|c_1(t)|^2$, and dash lines denote $|c_2(t)|^2$. The curves with different colors correspond to, respectively, the initial states with the same colors. **a** and **b** $\gamma_1 = \gamma_2 = \gamma$; **c** and **d** $\gamma_1 |c_1(0)|^2 = \gamma_2 |c_2(0)|^2$; **e** and **f** $\gamma_1 |c_1(0)|^2 \neq \gamma_2 |c_2(0)|^2$

Fig. 3 Time evolution of the population of the upper-level state for the A-type atom, where we set $\gamma_1 = \gamma_2 = \gamma$ and the other parameters are $\lambda = 0$, $\omega_0 = 91\gamma$, $\omega_1 = \omega_2 = 90\gamma$ for the red dot-dash line; $\lambda = 0.5\gamma$, $\omega_1 = \omega_2 = \omega_0 = 90\gamma$ for the blue dash line; $\lambda = \gamma$, $\omega_1 = 90\gamma$, $\omega_2 = 92\gamma$, $\omega_0 = 91\gamma$ for the black solid line (Color figure online)



$$\begin{cases} 2\chi^2 + (\delta_1 + \delta_2)\chi - \frac{\lambda}{2}(\gamma_1 + \gamma_2) = 0, & (34a) \\ \chi^3 + (\delta_1 + \delta_2)\chi^2 - \left[\lambda^2 - \delta_1\delta_2 + \frac{\lambda}{2}(\gamma_1 + \gamma_2) \right] \chi - \frac{\lambda}{2}(\gamma_1\delta_2 + \gamma_2\delta_1) = 0. & (34b) \end{cases}$$

Though there are several adjustable structure parameters γ_j , δ_j and λ , this set of equations have no real root for χ when $\lambda \neq 0$ (see the proof in ‘‘Appendix B’’). Thus, the asymptotic population of the upper-level state must be zero when $t \rightarrow \infty$. Note that when $\lambda = 0$ the set of equations have real roots $\chi = 0$ and $\chi = -\delta_1$, with the former valid for any structure parameters and the latter valid for $\delta_1 = \delta_2$. In fact, when $\lambda = 0$, the correlation functions $f_j(t-\tau)$ ($j = 1, 2$) in Eq. (23) are zero. Thus, $|c_3(t)|^2$ remains its initial value unchanged. We plot the time evolution of the population $|c_3(t)|^2$ as in Fig. 3 for several different sets of structure parameters. The oscillation of the blue dash line originates from the non-Markovian effect. This analysis suggests that there is no quantum interference between the two decaying channels when a Λ -type atom takes place spontaneous emission in the realistic Lorentzian environment ($\lambda \neq 0$). Due to the absence of the quantum interference for the spontaneous emission of the Λ -type atom, we will in the following section take the V-type atom as an exemplum for investigation so as to highlight the roles of quantum interference.

4 Protection of quantum entanglement and quantum coherence

Entanglement and coherence describe the two different aspects of a quantum state—quantum correlation and purity. Both of them are the important resource in quantum information processing. We will find that the quantum interference of the open V-type atom system has good protective roles to both the quantum entanglement and coherence.

We employ the notion of entanglement negativity as the description of quantum entanglement. For a bipartite system state ρ_{AB} , entanglement negativity is defined as [46,47]

$$N(\rho_{AB}) = \sum_k |\eta_k^{T_A(-)}| = \frac{\sum_k |\eta_k^{T_A}| - 1}{2}, \tag{35}$$

where $\eta_k^{T_A(-)}$ and $\eta_k^{T_A}$ are, respectively, the negative and all eigenvalues of the partial transpose of ρ_{AB} with respect to subsystem A.

Now assume that two V-type atoms are initially in a Werner-like state [48],

$$\rho_\varepsilon = \frac{(1 - \varepsilon)}{9} \mathbb{I} + \varepsilon |\Psi^{AB}\rangle \langle \Psi^{AB}|, \tag{36}$$

where \mathbb{I} denotes the three-dimensional identity matrix and $|\Psi^{AB}\rangle = \frac{1}{\sqrt{3}}(|00\rangle + |11\rangle + |22\rangle)$ is the maximally entangled state of the two atoms. The Werner-like state is separable for $0 \leq \varepsilon \leq 1/4$ and entangled for $1/4 < \varepsilon \leq 1$.

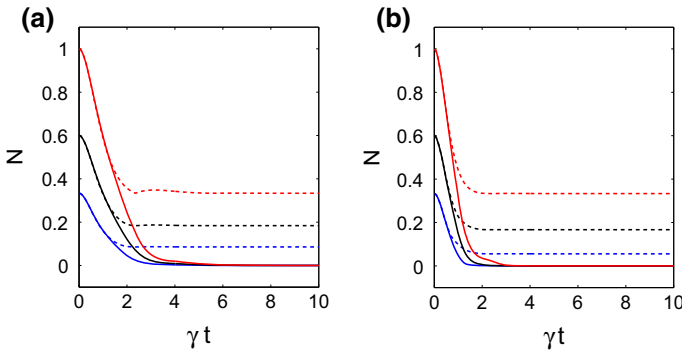


Fig. 4 Evolution of entanglement negativity versus dimensionless time γt for the open V-type atom with initial Werner-like state of Eq. (36), where the red, black and blue lines correspond to $\varepsilon = 1, 0.7, 0.5$, respectively. The dash lines correspond to the case of quantum interference with $\omega_1 = \omega_2 = \omega_0 = 90\gamma$, and the solid lines correspond to the case without quantum interference with $\omega_1 = 90\gamma$, $\omega_2 = 92\gamma$, $\omega_0 = 91\gamma$. Other parameters are chosen as $\gamma_1 = \gamma_2 = \gamma$, $\lambda = 2\gamma$. **a** Corresponds to unilateral environment and **b** corresponds to bilateral environment

Obviously, for studying the evolution of entanglement, the key step is to obtain the evolved state $\rho_\varepsilon(t)$. This can be reached by using the dynamical map presented in Sect. 2. We discuss the problem in two cases: unilateral environment and bilateral environment. The former means that only the atom A is influenced by the noisy environment but atom B keeps noise-free; the latter means that both of the two atoms are influenced by noises. In Fig. 4, we show the time evolution of the entanglement negativity of the Werner-like state for these two cases. It is shown that the entanglement negativity for both unilateral environment and bilateral environment has similar decaying behaviors. An interesting phenomenon is that the entanglement negativity reduces to zero in the case without quantum interference (solid lines), but to a nonzero asymptotic value in the case with quantum interference (dash lines). This result demonstrates that the quantum interference has good protective roles on quantum entanglement. The reason for this protection is readily comprehensible: Without quantum interference, all the population decays to the ground level of the V-type atom so that the entanglement disappears. When quantum interference exists, the two upper levels can sustain certain population and thus the entanglement is preserved effectively.

Quantum coherence is a very important notion in quantum physics, but a rigorous quantification of it has been lacked. Up to very recently, Baumgratz, Cramer and Plenio [49] established a rigorous framework for the quantification of coherence from the point of resource theory. Two typical measures, i.e., the l_1 norm of coherence and the relative entropy of coherence (REC) in the framework, were presented. In a reference basis $\{|i\rangle\}_{i=1,\dots,d}$ of a d -dimensional quantum system, the l_1 norm of coherence is simply defined as the sum of the absolute value of all the off-diagonal elements of the system density matrix,

$$C_{l_1}(\rho) = \sum_{i,j|i \neq j} |\rho_{i,j}|. \quad (37)$$

Obviously, this is a very simple and intuitive definition. The REC is defined as

$$C_{\text{rel}}(\rho) = S(\rho_{\text{diag}}) - S(\rho), \quad (38)$$

where S is the von Neumann entropy function and ρ_{diag} denotes the state obtained from ρ by deleting all off-diagonal elements. Note that quantum coherence is basis-dependent, and in the following we will use the atomic energy levels as the reference bases.

Figure 5 gives the time evolution of the two measures of coherence for the Werner-like state Eq. (36) of the V-type atomic system. It is shown that though the coherence disappears quickly in the case without quantum interference, it has large asymptotic value in the case of quantum interference. This asymptotic value is even much larger than its initial value. This is to say that quantum interference can protect or even enhance the coherence of quantum states. The physical explanation to this issue is similar to that of the protection of entanglement: Quantum interference sustains population in the upper levels of the V-type atom, and thus, coherence may be preserved. Otherwise, the population decays to the ground level and coherence disappears.

Another result revealed by Fig. 5 is that the two measures of coherence are not always compatible. For example, in the beginning stage of Fig. 5b, the blue dash line increases, but the red dash line decreases; Also, the blue solid line increases firstly and then decreases, but the red solid line drops directly. Figure 5d also reveals distinct difference of the two kinds of measure. This is to say the two measures of coherence may be incompatible in some situations, though both of them satisfy the requirements of the resource theory.

5 Non-Markovianity

The exact solutions for the open three-level atomic systems presented in Sect. 2 are very suitable for the study of non-Markovian dynamics, because any approximation may erase the non-Markovianity of quantum processes. We have already mentioned the effect of non-Markovianity in Sect. 3. Now let us further discuss the problem for both the V-type and Λ -type atomic systems.

One typical method for quantifying non-Markovianity of open quantum processes is the entanglement-based measure [5], which is actually equivalent to the evolution of entanglement for a maximal entangled state in the unilateral environment. In Fig. 6, we show, respectively, the time evolution of entanglement negativity for the open V-type and Λ -type atomic systems in the case of unilateral environment, where the initial states are $\frac{1}{\sqrt{3}}(|00\rangle + |11\rangle + |22\rangle)$ and $\frac{1}{\sqrt{3}}(|11\rangle + |22\rangle + |33\rangle)$, respectively. We see from the oscillation of the curves that the non-Markovianity decreases with the increase in λ , and the transition from Markovian to non-Markovian dynamics can be realized by controlling the spectral width λ . Note that the asymptotic entanglement for the V-type atom vanishes, because all the populations on the two upper levels decays ultimately into the single lower level; however, the asymptotic entanglement for the Λ -type atom will not vanish in general, because the populations on the two lower levels can produce entanglement.

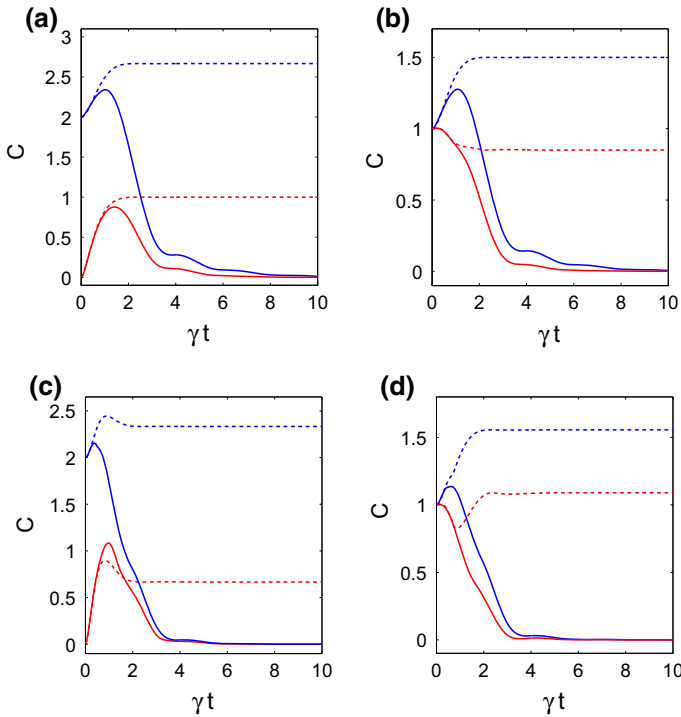


Fig. 5 Time evolution of quantum coherence for the open V-type atom with initial Werner-like state, where $\gamma_1 = \gamma_2 = \gamma, \lambda = 2\gamma$. **a** Unilateral environment and $\varepsilon = 1$; **b** unilateral environment and $\varepsilon = 0.5$; **c** bilateral environment and $\varepsilon = 1$; **d** bilateral environment and $\varepsilon = 0.5$; The blue lines correspond to C_{I_1} , and red lines correspond to REC. Dash lines correspond to the case of quantum interference with $\omega_1 = \omega_2 = \omega_0 = 90\gamma$, and solid lines to the case without quantum interference with $\omega_1 = 90\gamma, \omega_2 = 92\gamma, \omega_0 = 91\gamma$ (Color figure online)

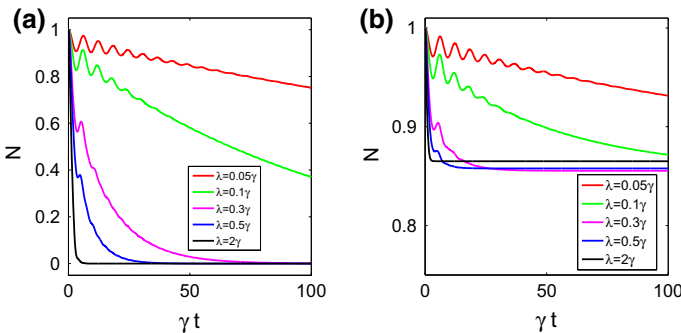


Fig. 6 Time evolution of entanglement negativity for the unilateral environment for different spectral widths. **a** V-type atom with initial state $\frac{1}{\sqrt{3}}(|00\rangle + |11\rangle + |22\rangle)$ and **b** A-type atom with initial state $\frac{1}{\sqrt{3}}(|11\rangle + |22\rangle + |33\rangle)$, where $\gamma_1 = \gamma_2 = \gamma, \omega_1 = 90\gamma, \omega_2 = 92\gamma, \omega_0 = 91\gamma$

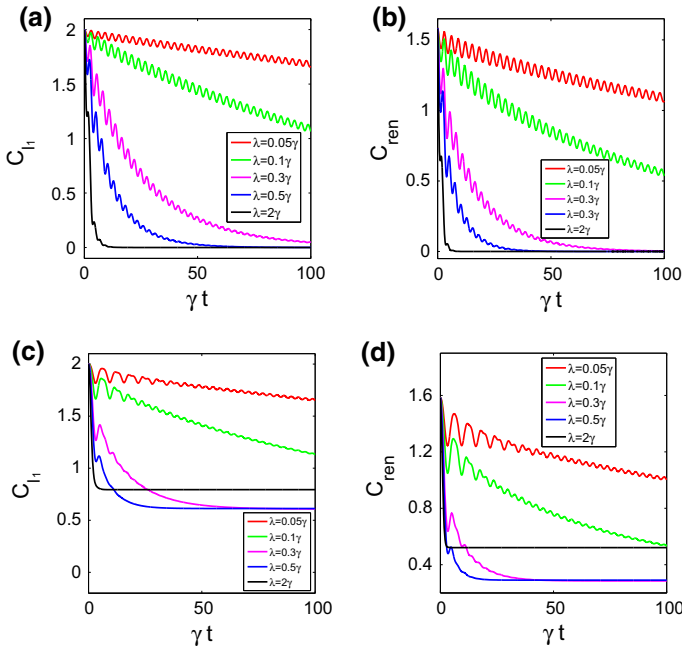


Fig. 7 Time evolution of quantum coherence: **a** and **b** for V-type atom with initial state $\frac{1}{\sqrt{3}}(|0\rangle + |1\rangle + |2\rangle)$, **c** and **d** for A-type atom with initial state $\frac{1}{\sqrt{3}}(|1\rangle + |2\rangle + |3\rangle)$, where $\gamma_1 = \gamma_2 = \gamma$, $\omega_1 = 90\gamma$, $\omega_2 = 92\gamma$, $\omega_0 = 91\gamma$

In Fig. 7, we show the time evolution of the l_1 -norm coherence and the REC for the single V-type and A-type atomic systems, respectively, where the initial states are set to be a uniform superposition of the three levels. For comparison, we choose the same parameters as in Fig. 6. The non-Markovianity and the transition from Markovian to non-Markovian dynamics are visibly seen from the oscillation of the curves. The evolutionary rules of the l_1 -norm coherence and the REC are similar; both of them have more visible oscillations compared with the evolution of entanglement, in the same parameter conditions (compare with Fig. 6). This implies that the measure of non-Markovianity-based coherence is more sensitive than that based on entanglement for the considered dynamics. In the same reason as in the evolution of entanglement, here the asymptotic coherence for the A-type atom will not vanish in general, due to the coherent population on the two lower levels.

One point needs to be noted: For the A-type atomic system, though the entanglement and coherence drop more rapidly as the spectral width λ increases, their asymptotic values may become larger (compare the black lines with the blue and purple lines in Figs. 6b, 7c, d). The physical mechanism for this phenomenon is worthwhile to be found.

6 Conclusions

In conclusion, we have presented the exact analytical solutions for the dynamics of the dissipative three-level V-type and Λ -type atomic systems in the vacuum Lorentzian environments. On this basis, we have discussed the phenomenon of quantum interference and demonstrated its protective role to the quantum entanglement and quantum coherence. The control of the non-Markovianity for the two types of dynamical processes has also been discussed.

Starting from the property of the asymptotic populations, we have verified that quantum interference only takes place in the dissipative process of the V-type atomic system, but does not exist in the dissipative Λ -type atomic system. We have derived the necessary condition of the quantum interference between the two decaying transitions of the V-type atom, which are completely consistent with the interference condition of classical light. Further analysis demonstrated that the quantum interference can be distinguished as the destructive and constructive one. We believe that these results will be valuable to the theory of quantum interference of the spontaneous emission processes.

Quantum entanglement and coherence are very important notions in quantum mechanics. Both of them are important physical resources in quantum information processing. We have demonstrated that the quantum interference for the dissipative V-type atomic system can be used to protect effectively the quantum entanglement and coherence. Based on the notations of entanglement and coherence, we have discussed the non-Markovianity for both the V-type and Λ -type atomic dynamics and revealed the transition from Markovian to non-Markovian processes by controlling the spectral width of the environment.

It is worthwhile to point out that the interference condition $\omega_1 = \omega_2$ of the spontaneous emission derived from the exact solution is different from the usual result derived from the approximation approaches [44,45]. The underlying explanation to this point needs to be explored further. In practice, due to the broadening of atomic level, the condition $\omega_1 = \omega_2$ only can be met approximately.

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Appendix A: The inverse Laplace transformation of Eqs. (11), (12) for the degenerate cases

If the polynomials $p^3 + h_1 p^2 + h_2 p + h_3 = 0$ have a twofold root b_1 and a single root b_3 , then the decomposition of Eqs. (11) and (12) becomes

$$\begin{cases} c_1(p) = \frac{\hat{D}_2}{(p - b_1)^2} + \frac{\hat{D}_1}{p - b_1} + \frac{\hat{D}_3}{p - b_3}, & \text{(A.1a)} \\ c_2(p) = \frac{\hat{D}'_2}{(p - b_1)^2} + \frac{\hat{D}'_1}{p - b_1} + \frac{\hat{D}'_3}{p - b_3}, & \text{(A.1b)} \end{cases}$$

where

$$\begin{cases} \hat{D}_i = \hat{E}_i c_1(0) + \hat{F}_i c_2(0), & \text{(A.2a)} \\ \hat{D}'_i = \hat{G}_i c_2(0) + \hat{H}_i c_1(0), & \text{(A.2b)} \end{cases}$$

with

$$\begin{aligned} \hat{E}_1 &= \frac{b_1^2 - (2b_1 + M + i\omega_2)b_3 - i\omega_2 M - B_{22}}{(b_1 - b_3)^2}, \\ \hat{F}_1 &= \frac{B_{12}}{(b_1 - b_3)^2}, \\ \hat{E}_2 &= \frac{(b_1 + i\omega_2)(b_1 + M) + B_{22}}{b_1 - b_3}, \\ \hat{F}_2 &= -\frac{B_{12}}{b_1 - b_3}, \\ \hat{E}_3 &= \frac{(b_3 + i\omega_2)(b_3 + M) + B_{22}}{(b_1 - b_3)^2}, \\ \hat{F}_3 &= -\frac{B_{12}}{(b_1 - b_3)^2}, \end{aligned}$$

and

$$\begin{aligned} \hat{G}_1 &= \frac{b_1^2 - (2b_1 + M + i\omega_1)b_3 - i\omega_1 M - B_{11}}{(b_1 - b_3)^2}, \\ \hat{H}_1 &= \frac{B_{21}}{(b_1 - b_3)^2}, \\ \hat{G}_2 &= \frac{(b_1 + i\omega_1)(b_1 + M) + B_{11}}{b_1 - b_3}, \\ \hat{H}_2 &= -\frac{B_{21}}{b_1 - b_3}, \\ \hat{G}_3 &= \frac{(b_3 + i\omega_1)(b_3 + M) + B_{11}}{(b_1 - b_3)^2}, \\ \hat{H}_3 &= -\frac{B_{21}}{(b_1 - b_3)^2}. \end{aligned}$$

The inverse Laplace transformation of Eqs. (A.1a) and (A.1b) gives the evolution

$$\begin{cases} c_1(t) = \hat{E}(t)c_1(0) + \hat{F}(t)c_2(0), & \text{(A.3a)} \\ c_2(t) = \hat{G}(t)c_2(0) + \hat{H}(t)c_1(0), & \text{(A.3b)} \end{cases}$$

with

$$\hat{E}(t) = (\hat{E}_1 + \hat{E}_2 t)e^{b_1 t} + \hat{E}_3 e^{b_3 t},$$

$$\begin{aligned} \hat{F}(t) &= (\hat{F}_1 + \hat{F}_2 t)e^{bt} + \hat{F}_3 e^{b_3 t}, \\ \hat{G}(t) &= (\hat{G}_1 + \hat{G}_2 t)e^{bt} + \hat{G}_3 e^{b_3 t}, \\ \hat{H}(t) &= (\hat{H}_1 + \hat{H}_2 t)e^{bt} + \hat{H}_3 e^{b_3 t}. \end{aligned}$$

If the polynomials $p^3 + h_1 p^2 + h_2 p + h_3 = 0$ have only one threefold root b , then Eqs. (11) and (12) become

$$\begin{cases} c_1(p) = \frac{\check{D}_3}{(p-b)^3} + \frac{\check{D}_2}{(p-b)^2} + \frac{\check{D}_1}{p-b}, & \text{(A.4a)} \\ c_2(p) = \frac{\check{D}'_3}{(p-b)^3} + \frac{\check{D}'_2}{(p-b)^2} + \frac{\check{D}'_1}{p-b}, & \text{(A.4b)} \end{cases}$$

where

$$\begin{aligned} \check{D}_3 &= [(b + i\omega_2)(b + M) + B_{22}]c_1(0) - B_{12}c_2(0), \\ \check{D}_2 &= (2b + M + i\omega_2)c_1(0), \\ \check{D}_1 &= 2c_1(0), \\ \check{D}'_3 &= [(b + i\omega_1)(b + M) + B_{11}]c_2(0) - B_{21}c_1(0), \\ \check{D}'_2 &= (2b + M + i\omega_1)c_2(0), \\ \check{D}'_1 &= 2c_2(0). \end{aligned}$$

The inverse Laplace transformation of Eqs. (A.4a) and (A.4b) gives

$$\begin{cases} c_1(t) = \check{E}(t)c_1(0) + \check{F}(t)c_2(0), & \text{(A.5a)} \\ c_2(t) = \check{G}(t)c_2(0) + \check{H}(t)c_1(0), & \text{(A.5b)} \end{cases}$$

with $\check{E}(t) = \{\frac{1}{2}[(b + i\omega_2)(b + M) + B_{22}]t^2 + (2b + M + i\omega_2)t + 2\}e^{bt}$, $\check{F}(t) = -\frac{1}{2}B_{12}t^2 e^{bt}$, $\check{G}(t) = \{\frac{1}{2}[(b + i\omega_1)(b + M) + B_{11}]t^2 + (2b + M + i\omega_1)t + 2\}e^{bt}$, $\check{H}(t) = -\frac{1}{2}B_{21}t^2 e^{bt}$.

Appendix B: Proof of no real root of Eqs. (34a), (34b)

If $\lambda \neq 0$, then $\chi \neq 0$. Multiplying Eq. (34a) by $\chi/2$ and then subtracting Eq. (34b), one has

$$(\delta_1 + \delta_2)\chi^2 - \left[2\lambda^2 - 2\delta_1\delta_2 + \frac{\lambda}{2}(\gamma_1 + \gamma_2) \right] \chi - \lambda(\gamma_1\delta_2 + \gamma_2\delta_1) = 0 \quad \text{(B.1)}$$

If $\delta_1 + \delta_2 \neq 0$, multiplying Eq. (34a) by $(\delta_1 + \delta_2)/2$ and then subtracting Eq. (B.1), one gets

$$\chi = \frac{\lambda\gamma_1(3\delta_2 - \delta_1) + \lambda\gamma_2(3\delta_1 - \delta_2)}{8\lambda^2 + 2(\delta_1 - \delta_2)^2 + 2\lambda(\gamma_1 + \gamma_2)}. \quad \text{(B.2)}$$

Plugging it into Eq. (34a), we can obtain the quadratic equation with respect to ω_0 ,

$$a\omega_0^2 + b\omega_0 + c = 0 \tag{B.3}$$

where $a = 4\lambda^2(u + v)^2 + 4\lambda n(u + v)$, $b = -8\lambda^2(u + v)(\omega_1 u + \omega_2 v) - 2\lambda n[(3\omega_1 + \omega_2)u + (3\omega_2 + \omega_1)v]$ and $c = 4\lambda^2(\omega_1 u + \omega_2 v)^2 + 2\lambda n(\omega_1 + \omega_2)(\omega_1 u + \omega_2 v) + \frac{1}{2}\lambda n^2(u + v)$, with $u = \gamma_1 - 3\gamma_2$, $v = \gamma_2 - 3\gamma_1$ and $n = 2(\omega_1 - \omega_2)^2 + 8\lambda^2 + 2\lambda(\gamma_1 + \gamma_2)$. The discriminant of Eq. (B.3) is

$$\begin{aligned} \Delta &= b^2 - 4ac \\ &= -256\lambda^2 n^2 [\gamma_1 \gamma_2 (\omega_1 - \omega_2)^2 + \lambda^2 (\gamma_1 + \gamma_2)^2] < 0, \end{aligned}$$

which implies that ω_0 is a complex number.

If $\delta_1 + \delta_2 = 0$, then Eqs. (34a) and (B.1) reduce, respectively, to

$$4\chi^2 - \lambda(\gamma_1 + \gamma_2) = 0, \tag{B.4}$$

$$[4\lambda^2 + 4\delta_1^2 + \lambda(\gamma_1 + \gamma_2)]\chi - \lambda\delta_1(\gamma_1 - \gamma_2) = 0. \tag{B.5}$$

Combining them to eliminate χ , we get the quadratic equation with respect to δ_1

$$\pm \sqrt{(\gamma_1 + \gamma_2)\lambda}\delta_1^2 - \lambda(\gamma_1 - \gamma_2)\delta_1 \pm \sqrt{(\gamma_1 + \gamma_2)\lambda^5} \pm \frac{1}{4}[(\gamma_1 + \gamma_2)\lambda]^{3/2} = 0. \tag{B.6}$$

This equation also leads to δ_1 only having complex roots.

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