Quantum Information Processing (2020) 19:2
https://doi.org/10.1007/s11128-019-2491-3

A weak limit theorem for a class of long-range-type quantum walks in 1d

Kazuyuki Wada[1]

Received: 13 March 2019 / Accepted: 26 October 2019 / Published online: 13 November 2019

Abstract
We derive the weak limit theorem for a class of long-range-type quantum walks. To do it, we analyze spectral properties of a time evolution operator and prove that modified wave operators exist and are complete.

Keywords Quantum walks · Weak limit theorem · Scattering theory

Mathematics Subject Classification 46N50 · 47A40 · 47B47 · 60F05

1 Introduction

Quantum walks have been introduced as a quantum counter part of classical random walks. Several prototypes of quantum walks have been introduced (for example, see e.g., [1,17,24]). In a view point of quantum information theory, Grover [16] and Shor [35] introduced algorithms based on quantum mechanics related to database searching and prime factorization, respectively. After that, Ambainis et al. [2] considered discrete time quantum walks based on quantum information theory. It is thought that their contribution is a trigger that quantum walks are paid attention by various field researchers.

Quantum walks are roughly divided into two types, discrete time and continuous time. In continuous-time cases, time evolution is governed by the Laplacian on graphs which is a self-adjoint operator. On the other hand, in discrete-time cases, time evolution is governed by a unitary operator. One of the typical unitary operators is an Ambainis type [2] which is a product of a shift operator and a coin operator. Especially, in discrete-time cases, a generator of time evolution operator is unknown in general. In these aspects, generators of a class of discrete time quantum walks are studied by Segawa and Suzuki [33]. However, in order to know some properties of

✉ Kazuyuki Wada
wada-g@hachinohe.kosen-ac.jp

1 National Institute of Technology, Hachinohe College, Hachinohe 039-1192, Japan

discrete time quantum walks, we need technical treatment for unitary operators themselves. A relation between continuous time and discrete time is studied in [6,34,36]. Unitary equivalent classes between two discrete time quantum walks are investigated in [26,27].

It is known that discrete-time quantum walks have remarkable properties which are not seen in classical random walks. One of these properties appears in a "weak limit theorem." In [19], Konno firstly derived the limit distribution of quantum walks. He also revealed that the shape of limit distribution is quite different from the normal distribution. Limit distribution of various types of quantum walks is investigated [7,8,10–12,15,21,22,30,40]. Second feature is a "localization." Localization is a phenomenon that the existence probability of a quantum walker is strictly positive after infinitely many time evolutions on some positions. An occurrence of localization is deeply connected to the existence of eigenvalues of time evolution operators. One-dimensional one defect models are precisely considered in [5] by using the CGMV method. Konno [20] also considered a one-dimensional one defect model. He showed the existence of new types of localization. Localization is also considered in other models [13,14,18,25]. Third feature is "quantum tunneling." In [23], it is shown that a quantum walker can tunnel through a double well under some conditions. It is believed that this phenomenon corresponds to "resonances." However, studies related to resonances on quantum walks are few. Other properties of quantum walks are summarized in [38] and references therein.

From now, we explain some results related to weak limit theorem in detail. First, we briefly introduce a mathematical framework of quantum walks. The Hilbert space is

$$\mathcal{H} := l^2(\mathbb{Z};\mathbb{C}^2) = \left\{ \Psi : \mathbb{Z} \to \mathbb{C}^2 | \sum_{x\in\mathbb{Z}} \|\Psi(x)\|^2_{\mathbb{C}^2} < \infty \right\},$$

and time evolution operator is $U := SC$ where

$$(S\Psi)(x) = \begin{bmatrix} \Psi^{(1)}(x+1) \\ \Psi^{(2)}(x-1) \end{bmatrix}, \quad (C\Psi)(x) = C(x)\Psi(x), \quad \Psi \in \mathcal{H}, \quad x \in \mathbb{Z},$$

and $\{C(x)\}_{x\in\mathbb{Z}} \subset U(2)$. Let $\Psi_0 \in \mathcal{H}$ ($\|\Psi\| = 1$) be an initial state of a quantum walker. Then, the quantum state after time $t \in \mathbb{Z}$ is given by $U^t\Psi_0$.

For $\Psi_0 \in \mathcal{H}$ with $\|\Psi_0\| = 1$ and $t \in \mathbb{Z}$, X_t be a $\mathbb{Z}$-valued random variable whose probability distribution is given by $\mathbb{P}(X_t = x) = \|(U^t\Psi_0)(x)\|^2_{\mathbb{C}^2}$. Our interest is to find the random variable V such that X_t with a suitable scaling converges to V as $t \to \infty$. Konno considered space-homogeneous quantum walks in one dimension. It means that $C(x) = C_0$ ($x \in \mathbb{Z}$) for some $C_0 \in U(2)$. He assumes that the initial state $\Psi_0 \in \mathcal{H}$ has a form of

$$\Psi(x) = \begin{cases} \begin{bmatrix} \alpha \\ \beta \end{bmatrix} & x = 0, \\ \begin{bmatrix} 0 \\ 0 \end{bmatrix} & \text{otherwise}, \end{cases} \qquad (|\alpha|^2 + |\beta|^2 = 1).$$

Then, he showed that the existence of $\mathbb{R}$-valued random variable V such that $X_t/t \to V$ as $t \to \infty$ in a weak sense through combinatorial arguments [19]. After that, Grimmett et al. gave a simple proof for Konno's result, extended to d-dimensional space-homogeneous quantum walks and removed the assumption related to initial states [15]. Their proof is based on an application of the discrete Fourier transform. A crucial contribution is to find the self-adjoint operator $\hat{V}_0$ which induces the random variable V. $\hat{V}_0$ is called an "asymptotic velocity operator." To find the limit distribution of X_t/t as $t \to \infty$, it suffices to find a suitable asymptotic velocity operator. Recently, a nonlinear quantum walk is considered in [22].

If we allow a coin operator C to be depend on $x \in \mathbb{Z}$, it becomes difficult to obtain the weak limit theorem since the discrete Fourier transform does not work. To overcome this difficulty, Suzuki [37] introduced the idea of spectral scattering theory for quantum walks. Here, we introduce the notion of short-range-type and long-range-type conditions:

Definition 1.1 A coin operator C satisfies a short (resp. long) range-type condition if there exists $C_0 \in U(2)$, $\kappa > 0$, and $\gamma > 1$ (resp. $1 \geq \gamma > 0$) such that

$$\|C(x) - C_0\|_{\mathcal{B}(\mathbb{C}^2)} \leq \kappa(1 + |x|)^{-\gamma}, \quad x \in \mathbb{Z},$$

where $\| \cdot \|_{\mathcal{B}(\mathbb{C}^2)}$ is the operator norm on $\mathbb{C}^2$.

We assume that C satisfies the short-range-type condition. We set $U_0 := SC_0$. Then, the following wave operator

$$W_{\pm} := \text{s-}\lim_{t \to \pm\infty} U^{-t} U_0^t \Pi_{\mathrm{ac}}(U_0)$$

exist and are complete (i.e., $\mathrm{Ran} W_{\pm} = \mathcal{H}_{\mathrm{ac}}(U)$). It means that the quantum state $U^t \Psi$ is approximated by $U_0^t \Psi_{\pm}$ for some $\Psi_{\pm} \in \mathcal{H}$ as $t \to \pm\infty$. Moreover, we can show the absence of singular continuous spectrum of U by Mourre theory [4,29]. We denote the asymptotic velocity operator of U_0 by $\hat{V}_0$. Suzuki showed that the limit distribution of X_t/t as $t \to \infty$ is derived from a sum of the orthogonal projection onto the set of eigenvectors of U and the spectral measure of $W_+^* \hat{V}_0 W_+$.

On the other hand, in the long-range-type condition, wave operators do not exist in general [39]. It means that the quantum state $U^t \Psi$ cannot be approximated by a vector whose form is $U_0^t \Psi_+$ for some $\Psi_+ \in \mathcal{H}$ as $t \to \infty$. In the short-range-type case, a coin operator $C(x)$ converges to C_0 sufficiently fast. Thus, it is assumed that a quantum walker far from origin is approximately driven by not $U = SC$ but $U_0 = SC_0$. However, in the long-range-type case, a coin operator $C(x)$ slowly converges to C_0. Thus, it is assumed that a quantum walker still gets a influence of $C(x)$ even though a quantum walker is in a position far from origin. In this sense, it is not trivial problem how to get the limit distribution of X_t/t.

In scattering theory for quantum mechanics, it is known that we have to introduce modified wave operators instead of wave operators. There are lots of results related to long-range-type scattering theory [9,28]. To introduce modified wave operators, it is important to introduce a suitable “modifier” induced by the Hamiltonian. However, it is difficult to introduce a modifier in a context of quantum walks straightforwardly since the Hamiltonian corresponds to $U = SC$ is unknown in general.

To overcome the difficulty mentioned in above, we consider a suitable unitary transform J such that the difference of U and $\tilde{U}_0 := JU_0J^{-1}$ behaves like the short-range-type condition. We assume that J is a multiplication operator and coin operator C has a form of

$$C(x) = \begin{bmatrix} e^{-i\xi(x)} & 0 \\ 0 & e^{i\xi(x)} \end{bmatrix} C_0 = \begin{bmatrix} ae^{-i\xi(x)}e^{i\alpha} & be^{-i\xi(x)}e^{i\beta} \\ -be^{i\xi(x)}e^{-i\beta+i\delta} & a^*e^{i\xi(x)}e^{-i\alpha+i\delta} \end{bmatrix},$$

for some $\xi : \mathbb{Z} \to \mathbb{R}$ and $C_0 \in U(2)$. For details, see Sect. 2. Since only phases of components of $C(x)$ are position dependent, transition probabilities that a quantum walker moves to right or left are essentially same as the case of C_0. In this sense, a time evolution operator $U = SC$ can be regarded as a simple perturbation of $U_0 = SC_0$. However, we can not apply straightforwardly results in [37] if $\xi(x)$ slowly converges to 0.

Under an additional assumption for $\xi(x)$, we can choose a suitable unitary operator J and derive the weak limit theorem. As far as we know, this is the first result related to long-range-type quantum walks. To derive the weak limit theorem, it is important to show the absence of singular continuous spectrum of U and existence of modified wave operators. We apply commutator theory for unitary operators under two Hilbert space settings established by Richard et al. [29] and Kato–Rosenblum-type theorem established by Suzuki [37].

Contents of this paper are as follows. In Sect. 2, we give a definition of a model in quantum walks and some fundamental properties are explained. In Sect. 3, some facts in the commutator theory are introduced. In Sect. 4, we show the absence of singular continuous spectrum of U by applying the commutator theory explained in Sect. 3. In Sect. 5, we derive the weak limit theorem which is a main result in this paper. In Sect. 6, we give comments for future problem as a conclusion of this paper.

2 Definition of a model

In this section, we review some notations and fundamental results for quantum walks. The Hilbert space is given by

$$\mathcal{H} := l^2(\mathbb{Z}; \mathbb{C}^2) = \left\{ \Psi : \mathbb{Z} \to \mathbb{C}^2 \,\middle|\, \sum_{x\in\mathbb{Z}} \|\Psi(x)\|^2_{\mathbb{C}^2} < \infty \right\}, \tag{2.1}$$

where $\|\cdot\|_{\mathbb{C}^2}$ is the norm on $\mathbb{C}^2$. We denote its inner product and norm by $\langle\cdot,\cdot\rangle_{\mathcal{H}}$ (linear in the right vector) and $\|\cdot\|_{\mathcal{H}}$, respectively. If there is no danger of confusion, then we omit the subscript $\mathcal{H}$ of them. We introduce the following dense subspace of $\mathcal{H}$:

$$\mathcal{H}_{\text{fin}} := \{\Psi \in \mathcal{H} | \exists N \in \mathbb{N} \text{ such that } \Psi(x) = 0 \text{ for all } |x| \geq N\}. \tag{2.2}$$

Next, we introduce two unitary operators U and U_0. The shift operator S is defined by

$$(S\Psi)(x) := \begin{bmatrix} \Psi^{(1)}(x+1) \\ \Psi^{(2)}(x-1) \end{bmatrix}, \quad \Psi \in \mathcal{H} \quad x \in \mathbb{Z}. \tag{2.3}$$

Let C_0 be a 2×2 unitary matrix. We introduce the coin operator C as follows:

$$(C\Psi)(x) := C(x)\Psi(x), \quad C(x) := \begin{bmatrix} e^{-i\xi(x)} & 0 \\ 0 & e^{i\xi(x)} \end{bmatrix} C_0, \quad x \in \mathbb{Z}, \tag{2.4}$$

where ξ is a real-valued function on $\mathbb{Z}$. Throughout in this paper, we identify C_0 as a unitary operator on $\mathcal{H}$ such that $(C_0\Psi)(x) = C_0\Psi(x)$, $x \in \mathbb{Z}$. We set $U := SC$ and $U_0 := SC_0$.

Next, we recall spectral properties of $U_0 = SC_0$. We denote the discrete Fourier transform which is unitary from $\mathcal{H}$ to $\mathcal{K} := L^2([0, 2\pi), \mathrm{d}k/2\pi; \mathbb{C}^2)$ and

$$(\mathcal{F}\phi)(k) := \hat{\phi}(k) = \sum_{x \in \mathbb{Z}} \phi(x) e^{-ikx}, \quad k \in [0, 2\pi), \quad \phi \in \mathcal{H}_{\text{fin}}.$$

We set $\hat{U}_0 := \mathcal{F} U_0 \mathcal{F}^{-1}$. It is seen that $\hat{U}_0$ is a $U(2)$-valued multiplication operator given by

$$\hat{U}_0(k) = \begin{bmatrix} e^{ik} & 0 \\ 0 & e^{-ik} \end{bmatrix} C_0, \quad k \in [0, 2\pi).$$

Note that C_0 has a form of

$$C_0 = \begin{bmatrix} ae^{i\alpha} & be^{i\beta} \\ -be^{-i\beta+i\delta} & ae^{-i\alpha+i\delta} \end{bmatrix},$$

where $a, b \in [0, 1]$ with $a^2 + b^2 = 1$, $\alpha, \beta \in [0, 2\pi)$ and $e^{i\delta}$ ($\delta \in [0, 2\pi)$) is the determinant of C_0. We denote an eigenvalue and a corresponding normalized eigenvector by $\lambda_j(k)$ and $u_j(k)$ ($j = 1, 2$), respectively.

Let B be a unitary or self-adjoint operator on $\mathcal{H}$. The sets $\sigma(B)$, $\sigma_{\text{p}}(B)$, $\sigma_{\text{c}}(B)$, $\sigma_{\text{ess}}(B)$ and $\sigma_{\text{ac}}(B)$ are called spectrum, pure point spectrum, continuous spectrum, essential spectrum and absolutely continuous spectrum of B, respectively.

Proposition 2.1 [29, Lemma 4.1]

(1) *If* $a = 0$, *then*

$$\lambda_1(k) = ie^{i\delta/2}, \quad \lambda_2(k) = -ie^{i\delta/2},$$

and

$$\sigma(U_0) = \sigma_p(U_0) = \{ie^{i\delta}, -ie^{i\delta}\}.$$

(2) *If* $0 < a < 1$, *then*

$$\lambda_j(k) = e^{i\delta/2}(\tau(k) + i(-1)^{j-1}\eta(k)), \quad j = 1, 2,$$

where $\tau(k) := a\cos(k + \alpha - \delta/2)$ *and* $\eta(k) := \sqrt{1 - \tau(k)^2}$. *Moreover, it follows that*

$$\sigma(U_0) = \sigma_c(U_0) = \{e^{it} | t \in [\delta/2 + \zeta, \pi + \delta/2 - \zeta] \cup [\pi + \delta/2 + \zeta, 2\pi + \delta/2 - \zeta]\},$$

where $\zeta := \arccos(a)$.

(3) *If* $a = 1$, *then*

$$\lambda_1(k) = e^{i(k+\alpha)}, \quad \lambda_2(k) = e^{-i(k+\alpha-\delta)},$$

and

$$\sigma(U_0) = \sigma_c(U_0) = \mathbb{T} := \{e^{it} | t \in [0, 2\pi)\}.$$

In what follows, we assume that $a \in (0, 1]$ (C_0 is not off-diagonal) to avoid a trivial case.

For a given coin operator C defined in (2.4), we introduce an important assumption:

Assumption 2.1 Let $\xi : \mathbb{Z} \to \mathbb{R}$ be a function such that $\lim_{t \to \pm\infty} \xi(x) = 0$. Then, there exists $\theta : \mathbb{Z} \to \mathbb{R}$ such that

$$\begin{cases} \left|\xi(x) - \left\{\theta(x+1) - \theta(x)\right\}\right| \le \kappa(1 + |x|)^{-1-\epsilon_0}, \\ \left|\xi(x) - \left\{\theta(x) - \theta(x-1)\right\}\right| \le \kappa(1 + |x|)^{-1-\epsilon_0}, \end{cases} \quad x \in \mathbb{Z},$$

with some constants $\kappa > 0$ and $\epsilon_0 > 0$.

Example 2.1 If $\xi(x) = (1 + |x|)^{-1}$, $x \in \mathbb{Z}$. Then, we choose θ as follows:

$$\theta(x) = \begin{cases} \log(1 + x), & \text{if } x \ge 0 \\ -\log(1 - x), & \text{if } x < 0 \end{cases}$$

Then, there exists $\kappa > 0$ such that

$$\begin{cases} \left|\xi(x) - \left\{\theta(x+1) - \theta(x)\right\}\right| \le \kappa(1 + |x|)^{-2}, \\ \left|\xi(x) - \left\{\theta(x) - \theta(x-1)\right\}\right| \le \kappa(1 + |x|)^{-2}, \end{cases} \quad x \in \mathbb{Z}.$$

Example 2.2 We can consider a generalization of Example 2.1. For $0 < p < 1$, we set $\xi(x) = (1+|x|)^{-p}$, $x \in \mathbb{Z}$. Then, we choose θ as

$$\theta(x) = \begin{cases} \dfrac{1}{1-p}(1+x)^{1-p}, & \text{if } x \geq 0, \\ -\dfrac{1}{1-p}(1-x)^{1-p}, & \text{if } x < 0 \end{cases}$$

Then, there exists $\kappa > 0$ such that

$$\begin{cases} \left|\xi(x) - \left\{\theta(x+1) - \theta(x)\right\}\right| \leq \kappa(1+|x|)^{-1-p}, \\ \left|\xi(x) - \left\{\theta(x) - \theta(x-1)\right\}\right| \leq \kappa(1+|x|)^{-1-p}, \end{cases} \quad x \in \mathbb{Z}.$$

In what follows, we assume the existence of θ which satisfies Assumption 2.1. We introduce the $U(2)$-valued multiplication operator J as follows:

$$(J\Psi)(x) := J(x)\Psi(x), \quad J(x) = \begin{bmatrix} e^{i\theta(x)} & 0 \\ 0 & e^{i\theta(x)} \end{bmatrix}, \quad x \in \mathbb{Z}, \quad \Psi \in \mathcal{H}. \tag{2.5}$$

It is obvious that J is unitary on $\mathcal{H}$. We set $\tilde{U_0} := JU_0J^{-1}$. Then, we can express $\tilde{U_0}$ as $\tilde{U_0} = S\tilde{C}_0$, where $\tilde{C}_0 := S^{-1}JSC_0J^{-1}$.

Proposition 2.2 *$\tilde{C}_0$ is a $U(2)$-valued multiplication operator on $\mathbb{Z}$ such that*

$$\tilde{C}_0(x) = \begin{bmatrix} e^{-i\{\theta(x)-\theta(x-1)\}} & 0 \\ 0 & e^{i\{\theta(x+1)-\theta(x)\}} \end{bmatrix} C_0, \quad x \in \mathbb{Z}.$$

Proof Since J^{-1} and C_0 commute, it suffices to consider the form of $(S^{-1}JSJ^{-1})(x)$. For any $\Psi \in \mathcal{H}$, it is seen that

$$(JSJ^{-1}\Psi)(x) = \begin{bmatrix} e^{i\{\theta(x)-\theta(x+1)\}}\Psi^{(1)}(x+1) \\ e^{i\{\theta(x)-\theta(x-1)\}}\Psi^{(2)}(x-1) \end{bmatrix}.$$

Moreover, it follows that

$$\begin{aligned} (S^{-1}JSJ^{-1}\Psi)(x) &= \begin{bmatrix} (JSJ^{-1}\Psi)^{(1)}(x-1) \\ (JSJ^{-1}\Psi)^{(2)}(x+1) \end{bmatrix} \\ &= \begin{bmatrix} e^{-i\{\theta(x)-\theta(x-1)\}}\Psi^{(1)}(x) \\ e^{i\{\theta(x+1)-\theta(x)\}}\Psi^{(2)}(x) \end{bmatrix} \\ &= \begin{bmatrix} e^{-i\{\theta(x)-\theta(x-1)\}} & 0 \\ 0 & e^{i\{\theta(x+1)-\theta(x)\}} \end{bmatrix} \begin{bmatrix} \Psi^{(1)}(x) \\ \Psi^{(2)}(x) \end{bmatrix} \end{aligned}$$

Thus, the desired result follows. □

By proposition 2.2 and $|e^{is} - 1| \leq |s|$ for $s \in \mathbb{R}$, we have the following proposition:

Proposition 2.3 *For any $x \in \mathbb{Z}$, it follows that*

$$\left\| C(x) - \tilde{C}_0(x) \right\|_{\mathcal{B}(\mathbb{C}^2)} \leq 2\kappa (1 + |x|)^{-1-\epsilon_0},$$

where $\| \cdot \|_{\mathcal{B}(\mathbb{C}^2)}$ is the operator norm on $\mathbb{C}^2$.

We introduce "modified wave operators" as follows:

$$W_\pm(U, U_0, J) := \text{s-}\lim_{t \to \pm\infty} U^{-t} J U_0^t \Pi_{\mathrm{ac}}(U_0),$$

where $\Pi_{\mathrm{ac}}(U_0)$ is the orthogonal projection onto the absolutely continuous subspace of U_0.

Theorem 2.1 *$W_\pm(U, U_0, J)$ exist and are complete.*

Proof From Proposition 2.3, we can show that $C - \tilde{C}_0$ is trace class [37, Lemma 2.1]. Thus, $U - \tilde{U}_0$ is trace class. Then, it is seen that

$$W_\pm(U, \tilde{U}_0) := \text{s-}\lim_{t \to \pm\infty} U^{-t} \tilde{U}_0^t \Pi_{\mathrm{ac}}(\tilde{U}_0)$$

exist and are complete ($\mathrm{Ran} W_\pm = \mathcal{H}_{\mathrm{ac}}(U)$) [37, Theorem 2.3]. Since $\tilde{U}_0^t = J U_0^t J^{-1}$ and $\Pi_{\mathrm{ac}}(\tilde{U}_0) = J \Pi_{\mathrm{ac}}(U_0) J^{-1}$, it is seen that

$$\begin{aligned} \text{s-}\lim_{t \to \pm\infty} U^{-t} J U_0^t \Pi_{\mathrm{ac}}(U_0) &= \text{s-}\lim_{t \to \pm\infty} U^{-t} J U_0^t J^{-1} J \Pi_{\mathrm{ac}}(U_0) J^{-1} J \\ &= \text{s-}\lim_{t \to \pm\infty} U^{-t} \tilde{U}_0^t \Pi_{\mathrm{ac}}(\tilde{U}_0) J \\ &= W_\pm(U, \tilde{U}_0) J. \end{aligned}$$

This implies the existence of $W_\pm(U, U_0, J)$. Since $W_\pm(U, \tilde{U}_0)$ are complete, we have $\mathrm{Ran}(W_\pm(U, \tilde{U}_0)) = \mathcal{H}_{\mathrm{ac}}(U)$. Since U_0 has purely absolutely continuous spectrum (see Proposition 4.1 below), J maps $\mathcal{H}_{\mathrm{ac}}(U_0)$ to $\mathcal{H}_{\mathrm{ac}}(\tilde{U}_0)$. Thus, the completeness of $W_\pm(U, U_0, J)$ follows. □

Proposition 2.4 *It follows that*

$$\begin{aligned} \sigma_{\mathrm{ess}}(U) &= \sigma_{\mathrm{ess}}(U_0) \\ &= \begin{cases} \{e^{it} | t \in [\delta/2 + \zeta, \pi + \delta/2 - \zeta] \cup [\pi + \delta/2 + \zeta, 2\pi + \delta/2 - \zeta]\}, & \text{if } 0 < a < 1, \\ \mathbb{T} & \text{if } a = 1. \end{cases} \end{aligned}$$

Proof From Proposition 2.3, $C - \tilde{C}_0$ is a compact operator. This implies that the compactness of $U - \tilde{U}_0 = S(C - \tilde{C}_0)$. By Lemma 2.2 of [25] and unitary invariance of essential spectrum, we have $\sigma_{\mathrm{ess}}(U) = \sigma_{\mathrm{ess}}(\tilde{U}_0) = \sigma_{\mathrm{ess}}(U_0)$. The last equality follows from Proposition 2.1. □

3 Commutator theory

In this section, we recall some definitions and notations related to commutator theory. We mainly refer [2,29]. We denote the set of bounded linear operators from a Hilbert space $\mathcal{H}_0$ to $\mathcal{H}$ by $\mathcal{B}(\mathcal{H}_0, \mathcal{H})$ and $\mathcal{B}(\mathcal{H}) := \mathcal{B}(\mathcal{H}, \mathcal{H})$. Moreover, we denote the set of compact operators from $\mathcal{H}_0$ to $\mathcal{H}$ by $\mathcal{K}(\mathcal{H}_0, \mathcal{H})$ and $\mathcal{K}(\mathcal{H}) := \mathcal{K}(\mathcal{H}, \mathcal{H})$.

Let $T \in \mathcal{B}(\mathcal{H})$ and let A be a self-adjoint operator on $\mathcal{H}$. We say that $T \in C^k(A)$ ($k \in \mathbb{N}$) if a $\mathcal{B}(\mathcal{H})$-valued map $\mathbb{R} \ni t \mapsto e^{-itA}Te^{itA}$ belongs to C^k class strongly. Especially in the case where $k = 1$, it is known that $T \in C^1(A)$ if and only if a following form

$$D(A) \ni \phi \mapsto \langle A\phi, T\phi\rangle - \langle \phi, TA\phi\rangle$$

can be continuously extended to the form on $\mathcal{H}$. We denote the operator corresponds to continuous extension of the above form by $[A, T]$.

Here, we introduce three regularity conditions which are stronger than $T \in C^1(A)$. $T \in C^{1,1}(A)$ means that $T \in C^1(A)$ and

$$\int_0^1 \|e^{-itA}Te^{itA} + e^{itA}Te^{-itA} - 2S\|_{\mathcal{B}(\mathcal{H})} \frac{\mathrm{d}t}{t^2} < \infty.$$

$T \in C^{1+0}(A)$ means that $T \in C^1(A)$ and

$$\int_0^1 \|e^{-itA}[A, S]e^{itA} - [A, S]\|_{\mathcal{B}(\mathcal{H})} \frac{\mathrm{d}t}{t} < \infty.$$

$T \in C^{1+\epsilon}$ for some $\epsilon > 0$ means that $T \in C^1(A)$ and

$$\|e^{-itA}[A, S]e^{itA} - [A, S]\|_{\mathcal{B}(\mathcal{H})}\| \leq \text{Const.}t^{\epsilon} \quad \text{for all } t \in (0, 1).$$

For above conditions, following inclusion relation holds [2, Sect. 5.2.4]:

$$C^2(A) \subset C^{1+\epsilon}(A) \subset C^{1+0}(A) \subset C^{1,1}(A) \subset C^1(A).$$

Next, we introduce two functions which are useful to consider the commutator theory for unitary operators which is introduced in [29]. For self-adjoint cases, see e.g., [2, Sect. 7.2]. We assume that $U \in C^1(A)$. For $T, S \in \mathcal{B}(\mathcal{H})$, we write $T \gtrsim S$ if there exists a compact operator $K \in \mathcal{K}(\mathcal{H})$ such that $T + K \geq S$. For $\theta \in \mathbb{T}$ and $\epsilon > 0$, we set

$$\Theta(\theta, \epsilon) := \{\theta' \in \mathbb{T} | |\arg(\theta - \theta')| < \epsilon\}, \quad E^U(\theta; \epsilon) := E^U(\Theta(\theta; \epsilon)).$$

where, $E^U(\cdot)$ is the spectral measure of U. Under above preparations, we introduce functions $\rho_U^A : \mathbb{T} \mapsto (-\infty, \infty]$ and $\tilde{\rho}_U^A : \mathbb{T} \mapsto (-\infty, \infty]$ by

$$\rho_U^A(\theta) := \sup\{a \in \mathbb{R} | \exists \epsilon \text{ such that } E^U(\theta; \epsilon)U^{-1}[A, U]E^U(\theta; \epsilon) \geq aE^U(\theta; \epsilon)\},$$

and

$$\tilde{\rho}_U^A(\theta) := \sup\{a \in \mathbb{R} | \exists \epsilon > 0 \text{ such that } E^U(\theta;\epsilon)U^{-1}[A,U]E^U(\theta;\epsilon) \gtrsim aE^U(\theta;\epsilon)\}.$$

General facts related to commutator theory for unitary operators in one Hilbert space are considered in [29, Sect. 3.3]. The following fact is important to show the absence of singular continuous spectrum:

Theorem 3.1 [29, Theorem 3.6] *Let U be a unitary operator and A be a self-adjoint operator on $\mathcal{H}$. We assume either that U has a spectral gap and $U \in C^{1,1}(A)$ or $U \in C^{1+0}(A)$. Moreover, we also assume that there exists an open set $\Theta \subset \mathbb{T}$, $a > 0$, and an operator $K \in \mathcal{K}(\mathcal{H})$ such that*

$$E^U(\Theta)U^{-1}[A,U]E^U(\Theta) \geq aE^U(\Theta) + K.$$

Then, U has at most finitely many eigenvalues in Θ, each one of finite multiplicity, and U has no singular continuous spectrum in Θ.

To show Theorem 3.1, in addition, we introduce the commutator theory in a two Hilbert space setting. We consider an another triple $(\mathcal{H}_0, U_0, A_0)$ in addition to $(\mathcal{H}, U, A)$, where $\mathcal{H}_0$ is a Hilbert space, U_0 is a unitary operator on $\mathcal{H}$, and A_0 is a self-adjoint operator on $\mathcal{H}_0$. We also introduce a identification operator $J \in \mathcal{B}(\mathcal{H}_0, \mathcal{H})$. Following general result is important:

Theorem 3.2 [29, Theorem 3.7] *We assume that*

1. *$U_0 \in C^1(A_0)$ and $U \in C^1(A)$,*
2. *$JU_0^{-1}[A_0, U_0]J^* - U^{-1}[A,U] \in \mathcal{K}(\mathcal{H})$,*
3. *$JU_0 - UJ \in \mathcal{K}(\mathcal{H}_0, \mathcal{H})$,*
4. *For each $f \in C(\mathbb{C}, \mathbb{R})$, $f(U)(JJ^* - 1)f(U) \in \mathcal{K}(\mathcal{H})$.*

Then, it follows that $\tilde{\rho}_U^A \geq \tilde{\rho}_{U_0}^{A_0}$

To apply the commutator theory for time evolution operator U introduced in Sect. 2, in what follows, we consider two triples $(\mathcal{H}, U, JA_0J^*)$ and $(\mathcal{H}, U_0, A_0)$. A following fact is useful to check the condition $U \in C^1(A)$ and the second condition in Theorem 4.2:

Theorem 3.3 [29, Corollary 3.11, Corollary 3.12] *Let $U_0 \in C^1(A_0)$. Suppose that JA_0J^* is essentially self-adjoint on a set $\mathcal{D}$, and assume that*

$$\overline{BA_0 \restriction D(A_0)} \in \mathcal{B}(\mathcal{H}), \quad \overline{B_*A_0 \restriction D(A_0)} \in \mathcal{K}(\mathcal{H}),$$

where $B := JU_0 - UJ_0$ and $B_* := JU_0^* - U^*J$. Then, $U \in C^1(JA_0J^*)$ and $JU_0^{-1}[A_0, U_0]J^* - U^{-1}[JA_0J^*, U] \in \mathcal{K}(\mathcal{H})$.

4 Spectral analysis for quantum walks

In this section, we show the absence of singular continuous spectrum of U. First, we introduce the asymptotic velocity operator of $U_0 = SC_0$ by

$$\widehat{V_0\psi}(k) = \sum_{j=1,2} \frac{i\lambda_j(k)}{\lambda_j(k)} \langle u_j(k), \hat{\psi}(k)\rangle_{\mathbb{C}^2} u_j(k), \quad x \in [0, 2\pi), \quad \psi \in \mathcal{H}.$$

Note that V_0 is bounded and self-adjoint on $\mathcal{H}$.

Let $C([0, 2\pi); \mathbb{C}^2)$ be the set of $\mathbb{C}^2$-valued continuous functions on $[0, 2\pi)$.

For any $\psi, \phi \in C([0, 2\pi); \mathbb{C}^2)$, we introduce the operator $|\psi\rangle\langle\phi| : C([0, 2\pi); \mathbb{C}^2) \to C([0, 2\pi); \mathbb{C}^2)$ by

$$\big(|\psi\rangle\langle\phi| f\big)(k) := \langle \psi(k), f(k)\rangle_{\mathbb{C}^2} \phi(k), \quad f \in C([0, 2\pi; \mathbb{C}^2), \quad k \in [0, 2\pi).$$

This operator can be continuously extended to a bounded operator on $\mathcal{H}$. Moreover, we introduce the self-adjoint operator P in $\mathcal{K}$ as follows:

$$D(P) := \{f \in \mathcal{K} | f \text{ is absolutely continuous }, \quad f' \in \mathcal{K}, \quad \text{and} \quad f(0) = f(2\pi)\},$$
$$(Pf) := -if', \quad f \in D(P).$$

Under above notations, we introduce the operator X by

$$\widehat{X} f(k) := - \sum_{j=1,2} (|u_j\rangle\langle u_j| P - i|u_j\rangle\langle u'_j|) f, \quad f \in \mathcal{F}\mathcal{H}_{\text{fin}}.$$

X is essentially self-adjoint [29, Lemma 4.3], and we denote the closure of X by the same symbol. Moreover, we introduce the following operator:

$$A_0 := \frac{1}{2}(XV_0 + V_0X).$$

A_0 is self-adjoint and essentially self-adjoint on $\mathcal{H}_{\text{fin}}$.

Proposition 4.1 [29, Proposition 4.5] *Following properties hold:*

1. $U_0 \in C^1(A_0)$ *and* $U_0^{-1}[A_0, U_0] = V_0^2$.
2. $\rho_{U_0}^{A_0} = \tilde{\rho}_{U_0}^{A_0}$ *and*

   (a) *if* $a \in (0, 1)$, *then* $\tilde{\rho}_{U_0}^{A_0}(\theta) > 0$ *for* $\theta \in Int(\sigma(U_0))$, $\tilde{\rho}_{U_0}^{A_0}(\theta) = 0$ *for* $\theta \in \partial\sigma(U_0)$, *and* $\tilde{\rho}_{U_0}^{A_0}(\theta) = \infty$ *otherwise,*

   (b) *if* $a = 1$, *then* $\tilde{\rho}_{U_0}^{A_0}(\theta) = 1$ *for all* $\theta \in \mathbb{T}$.

3. *If* $a \in (0, 1)$, *then* U_0 *has purely absolutely continuous spectrum and*

$$\sigma(U_0) = \sigma_{ac}(U_0) = \{e^{i\gamma} | \gamma \in [\delta/2 + \zeta, \pi + \delta/2 - \zeta] \cup [\pi + \delta/2 + \zeta, 2\pi + \delta/2 - \zeta]\}$$

4. *If* $a = 1$*, then* U_0 *has purely absolutely continuous spectrum and* $\sigma(U_0) = \sigma_{ac}(U_0) = \mathbb{T}$.

In what follows, we set $A := JA_0J^*$. We show conditions in Theorem 3.2 for two triples $(\mathcal{H}, U, A)$ and $(\mathcal{H}, U_0, A_0)$.

Lemma 4.1 *It follows that* $U \in C^1(A)$ *and* $JU_0^{-1}[A_0, U_0]J^* - U^{-1}[A, U] \in \mathcal{K}(\mathcal{H})$.

Proof From Proposition 4.1, we know $U_0 \in C^1(A_0)$. Moreover, JA_0J^{-1} is essentially self-adjoint on $\mathcal{D} = \mathcal{H}_{\text{fin}}$ since J is unitary and $J\mathcal{H}_{\text{fin}} = \mathcal{H}_{\text{fin}}$. Now, we check two conditions in Theorem 4.3. We note that A_0 has a following form on $\mathcal{H}_{\text{fin}}$:

$$A_0 = QK + \frac{i}{2}H_0$$

for some $K, H_0 \in \mathcal{B}(\mathcal{H})$, where Q is the position operator defined by

$$D(Q) := \left\{ \psi \in \mathcal{H} \middle| \sum_{x \in \mathbb{Z}} x^2 \|\psi(x)\|_{\mathbb{C}^2} < \infty \right\}, \quad (Q\psi)(x) := x\psi(x), \quad x \in \mathbb{Z}.$$

For more details, see the proof of [12, Lemma 4.10]. On $\mathcal{H}_{\text{fin}}$, it follows that

$$\begin{aligned} BA_0 = (JU_0J^* - U)J\left(QK + \frac{i}{2}H_0\right) &= (\tilde{U}_0 - U)QJK + \frac{i}{2}(\tilde{U}_0 - U)H_0 \\ &= S(\tilde{C}_0 - C)QJK + \frac{i}{2}(\tilde{U}_0 - U)H_0, \end{aligned}$$

where we used the commutativity of J and Q. From Proposition 2.3, we see that $\tilde{C}_0 - C$ is a compact operator and $(\tilde{C}_0 - C)Q$ can be extended to a compact operator on $\mathcal{H}$. Thus, we have $\overline{BA_0 \restriction D(A_0)} \in \mathcal{K}(\mathcal{H}) \subset \mathcal{B}(\mathcal{H})$. By the similar manner, it follows on $\mathcal{H}_{\text{fin}}$ that

$$\begin{aligned} B_*A_0 &= (JU_0^*J^* - U^*)J\left(QK + \frac{i}{2}H_0\right) \\ &= (\tilde{C}_0^* - C^*)SJQ\mathcal{F}^{-1}K\mathcal{F} + \frac{i}{2}(\tilde{U}_0 - U)^*JH_0 \\ &= (\tilde{C}_0 - C)^*QSJK + (\tilde{C}_0 - C)^*(SQ - QS)JK + \frac{i}{2}(\tilde{U}_0 - U)^*JH_0 \end{aligned}$$

Since $(\tilde{C}_0 - C)^*$ and $(\tilde{U}_0 - U)^*$ are compact, $(\tilde{C}_0 - C)^*Q$ can be extended to a compact operator on $\mathcal{H}$ and $SQ - QS$ can be extended to a bounded operator on $\mathcal{H}$, we have $\overline{B_*A_0 \restriction D(A_0)} \in \mathcal{K}(\mathcal{H})$. An application of Theorem 3.3 implies the desired result. □

Since J is unitary, $JJ^* = 1$ holds. Moreover, $JU_0 - UJ = (JU_0J - U)J = (\tilde{U}_0 - U)J \in \mathcal{K}(\mathcal{H})$ since $\tilde{U}_0 - U$ is compact. Therefore, we checked conditions in Theorem 4.2. We introduce the set of threshold of U by $\tau(U) := \partial\sigma(U_0)$, where $\partial\sigma(U_0)$ is the set of boundary of $\sigma(U_0)$ in $\mathbb{T}$. We note that $\tau(U)$ contains at most 4 values.

Proposition 4.2 *We have $\tilde{\rho}_U^A \geq \tilde{\rho}_{U_0}^{A_0}$. In particular, if $\theta \in \sigma(U_0)\setminus\tau(U)$, then $\tilde{\rho}_{U_0}^{A_0}(\theta) > 0$.*

Proof $\tilde{\rho}_U^A \geq \tilde{\rho}_{U_0}^{A_0}$ follows by an application of Theorem 3.2. The latter assertion follows from Proposition 4.1. □

To apply Theorem 3.1, we have to check a regularity of U more detail.

Lemma 4.2 *For any $\epsilon \in (0,1)$ with $\epsilon \leq \epsilon_0$, $U \in C^{1+\epsilon_0}(A)$. Here, $\epsilon_0 > 0$ is a constant introduced in Assumption 2.1.*

Proof This proof is a slight modification of [29, Lemma 4.13]. In the proof of Proposition 4.5 of [29], we see that $U_0 \in C^2(A_0)$. Since J is unitary, it follows that $\tilde{U}_0 \in C^2(A) \subset C^{1+\epsilon}(A)$. We decompose U as $U = \tilde{U}_0 + (U - \tilde{U}_0)$. Thus, it suffices to show that $U - \tilde{U}_0 \in C^{1+\epsilon}(A)$. We see that

$$D_0 := A(U - \tilde{U}_0) - (U - \tilde{U}_0)A$$

on $\mathcal{H}_{\text{fin}}$ can be extended to a bounded operator on $\mathcal{H}$. We denote it by the same symbol. According to [2, p. 325–328] or [29, Lemma 4.13], following estimate holds:

$$\begin{aligned}\|e^{-itA}D_0e^{itA} - D_0\|_{\mathcal{B}(\mathcal{H})} &\leq \text{Const.}(\|\sin(tA)D_0\|_{\mathcal{B}(\mathcal{H})} + \sin(tA)D_0^*\|_{\mathcal{B}(\mathcal{H})}) \\ &\leq \text{Const.}(\|tA(tA+i)^{-1}D_0\|_{\mathcal{B}(\mathcal{H})} + \|tA(tA+i)^{-1}D_0^*\|_{\mathcal{B}(\mathcal{H})}).\end{aligned}$$

We set $A_t := tA(tA+i)^{-1}$ and $\Lambda_t := t\langle Q\rangle(\langle Q\rangle + i)^{-1}$ with $\langle Q\rangle := \sqrt{Q^2+1}$. We note that $A\langle Q\rangle^{-1} \in \mathcal{B}(\mathcal{H})$. Then, it follows that

$$A_t = (A_t + i(tA+i)^{-1}A\langle Q\rangle^{-1})\Lambda_t.$$

Since $A_t + i(tA+i)^{-1}A\langle Q\rangle^{-1}$ is bounded, it suffices to show that

$$\|\Lambda_t D_0\|_{\mathcal{B}(\mathcal{H})} + \|\Lambda_t D_0^*\|_{\mathcal{B}(\mathcal{H})} \leq \text{Const.}\ t^\epsilon \quad t \in (0,1).$$

We have to show that operators $\langle Q\rangle^\epsilon D_0$ and $\langle Q\rangle^\epsilon D_0$ defined on the form sense on $\mathcal{H}_{\text{fin}}$ extended to a bounded operator on $\mathcal{H}$. We note that $\langle Q\rangle^{1+\epsilon}(C - \tilde{C}_0) \in \mathcal{B}(\mathcal{H})$ and $\langle Q\rangle^{-1}A_0$ defined in the form sense on $\mathcal{H}_{\text{fin}}$ extend to a bounded operator on $\mathcal{H}$. This implies that $\langle Q\rangle^\epsilon D_0$ and $\langle Q\rangle^\epsilon D_0^*$ defined in the form sense on $\mathcal{H}_{\text{fin}}$ extend to bounded operators on $\mathcal{H}$. Thus, the proof is completed. □

By Theorem 3.1, Proposition 4.2 and Lemma 4.2, we have the following result.

Theorem 4.1 *For any closed set $\Theta \subset \mathbb{T}\setminus\tau(U)$, the operator U has at most finitely many eigenvalues in Θ, each one of finite multiplicity, and U has no singular continuous spectrum in Θ.*

Recall that $\tau(U)$ is a finite set. From Theorem 4.1, U has no singular continuous spectrum.

5 Derivation of weak limit theorem

We set $Q_0(t) := U_0^{-t} Q U_0^t$.

Theorem 5.1 [37, Theorem 4.1] *It follows that*

$$s\text{-}\lim_{t\to\infty} e^{i\xi Q_0(t)/t} = e^{i\xi V_0}, \quad \xi \in \mathbb{R}.$$

Let X_t be a random variable which describes the position of a quantum walker with U and an initial state Ψ_0 at time $t \in \mathbb{Z}$. The probability distribution of X_t is given by

$$\mathbb{P}(\{X_t = x\}) = \|(U^t \Psi_0)(x)\|_{\mathbb{C}^2}^2, \quad x \in \mathbb{Z}.$$

Moreover, we also introduce the characteristic function of the average velocity X_t/t of a quantum walker by

$$\mathbb{E}[e^{i\xi X_t/t}] := \langle \Psi_0, e^{itQ(t)/t}\Psi_0\rangle, \quad \xi \in \mathbb{R},$$

where $Q(t) := U^{-t} Q U^t$. Our interest is the limit of X_t/t in a weak sense.

Theorem 5.2 *We set* $V_J^+ := W_+(U, U_0, J) V_0 W_+(U, U_0, J)^*$. *Then, for any* $\xi \in \mathbb{R}$, *it follows that*

$$s\text{-}\lim_{t\to\infty} e^{i\xi Q(t)/t} = \Pi_p(U) + e^{i\xi V_J^+}\Pi_{ac}(U),$$

where $\Pi_p(U)$ *is the orthogonal projection onto a subspace generated by eigenvectors of* U.

Proof Since U have no continuous spectrum, we can decompose that

$$\text{s-}\lim_{t\to\infty} e^{i\xi Q(t)/t} = \text{s-}\lim_{t\to\infty} \big(e^{i\xi Q(t)/t}\Pi_{\rm p}(U) + e^{i\xi Q(t)/t}\Pi_{\rm ac}(U)\big).$$

By [37, Theorem 4.2], we have $\text{s-}\lim_{t\to\infty} e^{i\xi Q(t)/t}\Pi_{\rm p}(U) = \Pi_{\rm p}(U)$. For the absolutely continuous part, we consider the following decomposition:

$$\begin{aligned}
&e^{i\xi Q(t)/t}\Pi_{\rm ac}(U) - e^{i\xi V_J^+}\Pi_{\rm ac}(U)\\
&\quad = U^{-t} e^{i\xi Q/t} U^t \Pi_{\rm ac}(U) - W_+(U, U_0, J) e^{i\xi V_0} W_+(U, U_0, J)^* \Pi_{\rm ac}(U)\\
&\quad = U^{-t} J U_0^t (U_0^{-t} e^{i\xi Q/t} U_0^t) U_0^{-t} J^{-1} U^t \Pi_{\rm ac}(U)\\
&\qquad - W_+(U, U_0, J) e^{i\xi V_0} W_+(U, U_0, J)^* \Pi_{\rm ac}(U)\\
&\quad = U^{-t} J U_0^t e^{i\xi Q_0(t)/t} \big(U_0^{-t} J^{-1} U^t \Pi_{\rm ac}(U) - W_+(U, U_0, J)^*\big)\Pi_{\rm ac}(U)\\
&\qquad + U^{-t} J U_0^t \big(e^{i\xi Q(t)_0/t} - e^{i\xi V_0}\big) W_+(U, U_0, J)^* \Pi_{\rm ac}(U)\\
&\qquad + \big(U^{-t} J U_0^t - W_+(U, U_0, J)\big) e^{i\xi V_0} W_+(U, U_0, J)^* \Pi_{\rm ac}(U),
\end{aligned}$$

where we used the strong commutativity of Q and J. We note that $W_+(U, U_0, J)^*$ maps $\mathcal{H}_{ac}(U)$ to $\mathcal{H}_{ac}(U_0)$ and V_0 leaves $\mathcal{H}_{ac}(U_0)$ invariant. By Theorem 5.1, it is seen that $\text{s-}\lim_{t\to\infty} e^{i\xi Q_0/t} = e^{i\xi V_0}$. By taking a limit $t \to \infty$, the desired result follows. □

Theorem 5.3 *Let $\Psi_0 \in \mathcal{H}$ be an initial state with $\|\Psi_0\| = 1$ and V be the random variable whose probability distribution is given by*

$$\mu_V(\mathrm{d}v) := \|\Pi_p(U)\Psi_0\|^2\delta_0 \mathrm{d}v + \|E^+_{V_J}(\cdot)\Pi_{ac}(U)\Psi_0\|^2 \mathrm{d}v,$$

where δ_0 is the Dirac measure for the point 0 and $E_{V_J^+}(\cdot)$ is the spectral measure of V_J^+. Then, it follows that

$$\lim_{t\to\infty} \mathbb{E}[e^{i\xi Q(t)/t}] = \mathbb{E}[e^{i\xi V}], \quad \xi \in \mathbb{R}.$$

Proof The proof is quite similar to [37, Corollary 2.4]. We omit the proof. □

6 Conclusion

In this paper, we derived the weak limit theorem for a class of discrete time quantum walks which partially include long-range types. Recall that the support of limit distribution of C_0 is determined by $|a|$ the absolute value of a diagonal component of C_0. For any $x \in \mathbb{Z}$, it is seen that

$$\begin{aligned} C(x) &= \begin{bmatrix} e^{-i\xi(x)} & 0 \\ 0 & e^{i\xi(x)} \end{bmatrix} \begin{bmatrix} ae^{i\alpha} & be^{i\beta} \\ -be^{-i\beta+i\delta} & a^* e^{-i\alpha+i\delta} \end{bmatrix} \\ &= \begin{bmatrix} ae^{-i\xi(x)}e^{i\alpha} & be^{-i\xi(x)}e^{i\beta} \\ -be^{i\xi(x)}e^{-i\beta+i\delta} & a^* e^{i\xi(x)}e^{-i\alpha+i\delta} \end{bmatrix}. \end{aligned}$$

Thus, absolute values of diagonal components of $C(x)$ are position independent. It is expected that the shape of limit distribution of $U = SC$ is quite similar to that of $U_0 = SC_0$. It may be an interesting problem to reveal a relationship between a difference of two coin operators and a difference of a shape of limit distribution.

In this paper, we could not treat a following type of coin operator:

$$C'(x) = \frac{1}{\sqrt{2}} \begin{bmatrix} \sqrt{1 + \langle x \rangle^{-\epsilon}} & \sqrt{1 - \langle x \rangle^{-\epsilon}} \\ \sqrt{1 - \langle x \rangle^{-\epsilon}} & -\sqrt{1 + \langle x \rangle^{-\epsilon}} \end{bmatrix}, \quad x \in \mathbb{Z}.$$

$C'(x)$ converges to C_H as $x \to \pm\infty$ the Hadamard coin which is a quantum version of symmetric classical random walks:

$$C_H = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}.$$

Since absolute values of diagonal components of C' are position dependent, it is expected that long-time behavior of a quantum walker under $U' = SC'$ is more complicated. Our next problem is to establish a method of construction of a modifier for $U' = SC'$ to derive the weak limit theorem.

Acknowledgements The author would like to thank A. Suzuki for various comments and constant encouragements. The author would also like to thank H. Ohno and S. Richard for helpful comments. This work was supported by the Research Institute of Mathematical Sciences, an International Joint Usage/Research Center located in Kyoto University.

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