Quantum Information Processing (2019) 18:270
https://doi.org/10.1007/s11128-019-2384-5

Quantum codes from $(1-2u_1-2u_2-\cdots-2u_m)$-skew constacyclic codes over the ring $F_q+u_1F_q+\cdots+u_{2m}F_q$

Tushar Bag[1] · **Mohammad Ashraf**[2] · **Ghulam Mohammad**[2] · **Ashish K. Upadhyay**[1]

Received: 13 November 2018 / Accepted: 16 July 2019 / Published online: 24 July 2019

Abstract
In this paper, we study a class of skew constacyclic codes over the ring $R = F_q + u_1F_q + \cdots + u_{2m}F_q$, where $u_i^2 = u_i, u_iu_j = u_ju_i = 0$, for $i, j = 1, 2, \ldots, 2m$, $i \neq j$ and $q = p^s$, and derive the generator polynomials of this class of codes over R. Also, by using Calderbank–Shor–Steane construction, some new non-binary quantum codes have been obtained. Moreover, new quantum codes $[[225, 201, 5]]_9$, $[[351, 333, 4]]_9$, $[[405, 393, 3]]_9$, $[[405, 381, 5]]_9$ have been constructed.

Keywords Linear codes · Gray map · Skew constacyclic codes · Quantum codes

Introduction

Linear codes over finite rings have recently raised a great interest for their new role in algebraic coding theory and for their successful application in combined coding and modulation. Recent developments have contributed toward achieving the reliability required by today's high-speed digital systems, and the use of coding for error control has become an integral part in the design of modern communication systems and digital computers.

Constacyclic codes consist of an algebraically rich family of error-correcting codes and are generalizations of cyclic and negacyclic codes. These codes can be easily

✉ Ghulam Mohammad
mohdghulam202@gmail.com

Tushar Bag
tushar.pma16@iitp.ac.in

Mohammad Ashraf
mashraf80@hotmail.com

Ashish K. Upadhyay
upadhyay@iitp.ac.in

1 Department of Mathematics, Indian Institute of Technology Patna, Patna 801103, India

2 Department of Mathematics, Aligarh Muslim University, Aligarh 202002, India

encoded using shift registers and can be easily decoded due to their rich algebraic structure, which justify their preferred role from engineering perspective.

Boucher et al. [11] generalized the notion of cyclic codes to skew cyclic codes by using generator polynomials in (non-commutative) skew polynomial rings. Since skew polynomial rings are left and right Euclidean, the obtained codes share most properties of cyclic codes. Since there are much more skew cyclic codes, this new class of codes allows to systematically search for codes with good properties.

Jitman et al. [22] defined skew constacyclic codes by defining the skew polynomial ring with coefficients from finite chain rings, especially the ring $F_{p^m} + uF_{p^m}$ where $u^2 = 0$. Further, the structural properties of skew cyclic codes through the decomposition method over $F_q + vF_q$, where $v^2 = v$ and $q = p^m$, were given by Gursoy et al. [18]. Ashraf and Mohammad [1] studied the skew cyclic codes over the ring $F_q + vF_q$ with $v^2 = 1$ by taking the automorphism as $\theta : v \mapsto -v$. Recently, AL-Ashker and Abu-Jafar [6] investigated the structure of skew constacyclic codes over the ring $F_p + vF_p$ with $v^2 = v$. Later on, Ashraf and Mohammad [2] gave the construction of skew constacyclic codes over the ring $F_q + vF_q + v^2F_q$, where $v^3 = v$. Motivated by this study, in this paper, we study $(1 - 2u_1 - 2u_2 - \cdots - 2u_m)$-skew constacyclic codes over the ring $F_q + u_1F_q + \cdots + u_{2m}F_q$, where $u_i^2 = u_i$, $u_iu_j = u_ju_i = 0$, for $i, j = 1, 2, \ldots, 2m$; $i \neq j$ and $q = p^s$.

Quantum error-correcting codes play a prominent role in both quantum communication and quantum computation. Quantum error-correcting codes provide an efficient way to overcome decoherence. Shor discovered the first quantum error-correcting code in [31]. Later on, a method to obtain quantum error-correcting codes from classical error-correcting codes was given by Calderbank et al. [14]. Recently, the theory of quantum error-correcting codes has been developed rapidly. Many good quantum error-correcting codes have been constructed by using classical cyclic codes over finite field F_q (q is a power of prime number) with self-orthogonal or dual containing properties (for references see [7,19–21,24–26,32]).

The construction for finding quantum codes from cyclic codes of odd length over the finite chain ring $F_2 + uF_2$ with $u^2 = 0$ was first given by Qian et al. [27]. Later on, Kai and Zhu [23] gave a construction for obtaining quantum codes from cyclic codes of odd length over the finite chain ring $F_4 + uF_4$ with $u^2 = 0$. Further, Qian [28] provided a new method of constructing quantum error-correcting codes from cyclic codes over the finite non-chain ring $F_2 + vF_2$ with $v^2 = v$ of arbitrary length. Motivated by this study, Ashraf and Mohammad [3–5] obtained non-binary quantum codes from cyclic codes over different types of finite rings. A lot of work has been done in this direction (see [9,15–17,29]). In this paper, we obtain quantum codes over F_q from skew constacyclic codes over the ring $R = F_q + u_1F_q + \cdots + u_{2m}F_q$ using decomposition method.

1 Preliminaries

Let R be the ring $F_q + u_1F_q + \cdots + u_{2m}F_q$, where $u_i^2 = u_i$, $u_iu_j = u_ju_i = 0$, for $i, j = 1, 2, \ldots, 2m$; $i \neq j$ and $q = p^s$. It is a commutative ring with q^{2m+1} elements. Here, F_q denotes the finite field with q elements.

Recall that a linear code C of length n over R is a R-submodule of R^n. Elements of C are called code words. Let $x=(x_0,x_1,\ldots,x_{n-1})$ and $y=(y_0,y_1,\ldots,y_{n-1})$ be two elements of R^n. Then, the Euclidean inner product of x and y is defined as $x\cdot y=x_0y_0+x_1y_1+\cdots+x_{n-1}y_{n-1}$. The dual code $C^\perp$ of C is defined as $C^\perp=\{x\in R^n \mid x\cdot y=0,\ \forall\, y\in C\}$. A code C is called self-orthogonal if $C\subseteq C^\perp$ and self-dual if $C=C^\perp$.

Any element of R can be written as $a_0+u_1a_1+u_2a_2+\cdots+u_{2m}a_{2m}=a_0(1-u_1-u_2-\cdots-u_{2m})+u_1(a_0+a_1)+u_2(a_0+a_2)+\cdots+u_{2m}(a_0+a_{2m})$.

Let

$$\eta_0=1-u_1-u_2-\cdots-u_{2m},$$
$$\eta_1=u_1,\quad \eta_2=u_2,\ldots,\ \eta_{2m}=u_{2m}.$$

It is easy to see

$$\sum_{i=0}^{2m}\eta_i=1,\quad \eta_i^2=\eta_i\quad\text{and}\quad \eta_i.\eta_j=0\quad\text{for}\quad i,j=0,1,2,\ldots,2m\quad\text{and}\quad i\neq j.$$

Thus, $R=\eta_0R\oplus\eta_1R\oplus\cdots\oplus\eta_{2m}R$. Therefore, any arbitrary element of R can be uniquely expressed as $x=\eta_0a_0+\eta_1a_1+\cdots+\eta_{2m}a_{2m}$, where $a_0,a_1,\ldots,a_{2m}\in F_q$. Now, we define a gray map Φ from R to F_q^{2m+1} defined as

$$\Phi(x)=(a_0,a_1,\ldots,a_{2m}).$$

It is easy to see that this is a linear map and can be extended component-wise. For any element $x=\eta_0a_0+\eta_1a_1+\cdots+\eta_{2m}a_{2m}\in R$, we define the Lee weight of x as $w_{\mathrm{L}}(x)=w_{\mathrm{H}}(\Phi(x))$, where $w_{\mathrm{H}}(\Phi(x))$ denotes the Hamming weight of $\Phi(x)$ over F_q, where the Hamming weight of any elements is defined as the number of nonzero components. We define the Lee weight of x as $w_{\mathrm{L}}(x)=\sum_{i=0}^{n-1}w_{\mathrm{L}}(x_i)$. The Lee distance between $x=(x_0,x_1,\ldots,x_{n-1})$ and $y=(y_0,y_1,\ldots,y_{n-1})\in R^n$ is defined by $d_{\mathrm{L}}(x,y)=w_{\mathrm{L}}(x-y)=\sum_{i=0}^{n-1}w_{\mathrm{L}}(x_i-y_i)$. The Lee distance of C is defined as $d_{\mathrm{L}}(C)=\min d_{\mathrm{L}}(x,y)$ for any $x\neq y$.

Let $A_i;\ i=0,1,2,\ldots,2m$ be code over R. We denote $A_0\oplus A_1\oplus\cdots\oplus A_{2m}=\{a_0+a_1+\cdots+a_{2m}\mid a_i\in A_i,\ i=0,1,2,\ldots,2m\}$ and $A_0\otimes A_1\otimes\cdots\otimes A_{2m}=\{(a_0,a_1,\ldots,a_{2m})\mid a_i\in A_i,\ i=0,1,2,\ldots,2m\}$. For a linear code C of length n over R, define

$$C_0=\{a_0\in F_q^n\mid \eta_0a_0+\eta_1a_1+\cdots+\eta_{2m}a_{2m}\in C,\ \ a_i\in F_q^n,\ i=1,2,\ldots,2m\},$$
$$C_1=\{a_1\in F_q^n\mid \eta_0a_0+\eta_1a_1+\cdots+\eta_{2m}a_{2m}\in C,\ \ a_i\in F_q^n,\ i=0,2,\ldots,2m\},$$
$$\cdots\cdots\cdots$$
$$C_{2m}=\{a_{2m}\in F_q^n\mid \eta_0a_0+\eta_1a_1+\cdots+\eta_{2m}a_{2m}\in C,\ \ a_i\in F_q^n,\ i=0,1,2,\ldots,2m-1\}.$$

Here, C_i are linear codes over F_q^n for $i = 0, 1, 2, \ldots, 2m$. Then, C_i are q-ary linear codes of length n. Hence, a linear code C of length n over R can be uniquely expressed as $C = \eta_0 C_0 \oplus \eta_1 C_1 \oplus \cdots \oplus \eta_{2m} C_{2m}$ and $|C| = |C_0||C_1| \cdots |C_{2m}|$ and $d_H(C) = \min\{d_H(C_i),\ i = 0, 1, 2, \ldots, 2m\}$.

A matrix is called generator matrix of C if the rows generate C. If M_i are the generator matrices of q-ary linear codes $C_i,\ i = 0, 1, \ldots, 2m$, respectively, then the generator matrix of C is

$$M = \begin{pmatrix} \eta_0 M_0 \\ \eta_1 M_1 \\ \cdots \\ \eta_{2m} M_{2m} \end{pmatrix}$$

and the generator matrix of $\Phi(C)$ is

$$\Phi(M) = \begin{pmatrix} \Phi(\eta_0 M_0) \\ \Phi(\eta_1 M_1) \\ \cdots \\ \Phi(\eta_{2m} M_{2m}) \end{pmatrix}.$$

Here, we define an automorphism on R as

$$\theta_t : R \longrightarrow R$$

defined by

$$\theta_t(a_0 + ua_1 + u_2 a_2 + \cdots + u_{2m} a_{2m}) = (a_0^{p^t} + u_1 a_1^{p^t} + u_2 a_2^{p^t} + \cdots + u_{2m} a_{2m}^{p^t})$$

for all $a_0, a_1, \ldots, a_{2m} \in F_q$. Also, the automorphism θ_t acts on F_q as follows:

$$\theta_t : F_q \longrightarrow F_q$$
$$\theta_t(a) = a^{p^t}.$$

The order of this automorphism is $|\langle\theta_t\rangle| = s/t$.

Definition 1.1 The set $R[x, \theta_t] = \{a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n |\ a_i \in R, n \geq 0\}$ forms a non-commutative ring under usual addition of polynomials, and multiplication is defined by the rule $(ax^i)(bx^j) = a\theta_t^i(b)x^{i+j}$. This ring is called skew polynomial ring. This ring was first introduced by Ore. The ring has no nonzero divisors; the units of this ring are exactly the units of the ring R.

Definition 1.2 A subset C of R^n is called a λ-skew constacyclic code of length n if

(1) C forms an R-submodule of R^n and
(2) If $c = (c_0, c_1, \ldots, c_{n-1}) \in C$, then $\sigma_\lambda(c) := (\theta_t(\lambda c_{n-1}), \theta_t(c_0), \ldots, \theta_t(c_{n-2})) \in C$.

Defining a map, $\psi : R^n \longrightarrow R_n = R[x,\theta_t]/\langle x^n-\lambda\rangle$ as $c=(c_0,c_1,\ldots,c_{n-1}) \longrightarrow c_0+c_1x+\cdots+c_{n-1}x^{n-1}$. Each code word $c=(c_0,c_1,\ldots,c_{n-1}) \in R^n$ can be identified with the polynomial $c(x)=c_0+c_1x+\cdots+c_{n-1}x^{n-1}$ in R_n. This map is a R-module isomorphism. By this identification, C is a λ-skew constacyclic code over R_n if and only if C is a left $R[x,\theta_t]$-submodule of $R[x,\theta_t]/\langle x^n-\lambda\rangle$.

2 Results on gray map and linear codes over R

Proposition 2.1 *Let Φ be the gray map defined in the preliminary section. Then,*

(1) *Φ is a F_q-linear distance preserving map from R^n (Lee distance) to $F_q^{(2m+1)n}$ (Hamming distance).*
(2) *If C is a $[n,k,d_L]$ linear code over R, then $\Phi(C)$ is a $[(2m+1)n,k,d_H]$ linear code over F_q, where $d_L=d_H$.*

Proof Let $x_1=a_0\eta_0+a_1\eta_1+\cdots+a_{2m}\eta_{2m}$ and $x_2=b_0\eta_0+b_1\eta_1+\cdots+b_{2m}\eta_{2m} \in R^n$. $\Phi(x_1+x_2)=(a_0+b_0,a_1+b_1,\ldots,a_{2m}+b_{2m})=(a_0,a_1,\ldots,a_{2m})+(b_0,b_1,\ldots,b_{2m})=\Phi(x_1)+\Phi(x_2)$ and $\Phi(cx_1)=(ca_0,ca_1,\ldots,ca_{2m})=c\Phi(x)$ for $c\in F_q$. Therefore, Φ is a F_q-linear map.
Since $d_L(c_1,c_2)=w_L(c_1-c_2)=w_H(\Phi(c_1-c_2))=w_H(\Phi(c_1)-\Phi(c_2))=d_H(\Phi(c_1),\Phi(c_2))$, Φ is a F_q-linear distance preserving map.

By the first part, Φ is a distance preserving linear map so $d_L=d_H$. Also, as Φ is bijection, therefore $|C|=|\Phi(C)|=q^k$. Hence, the result follows. □

Proposition 2.2 *Let C be a linear code of length n over R.*

(1) *Then, C is self-orthogonal, if and only if C_j are self-orthogonal over F_q, for $j=0,1,2,\ldots,2m$.*
(2) *If C is a self-orthogonal, then $\Phi(C)$ is also self-orthogonal.*

Proof (1) Let C be a self-orthogonal linear code over R and $x=a_0\eta_0+a_1\eta_1+\cdots+a_{2m}\eta_{2m}$ be any element of C, where $a_j\in C_j$ for $j=0,1,2,\ldots,2m$. Since C is self orthogonal,

$$\begin{aligned} x\cdot x &= (a_0\eta_0+a_1\eta_1+\cdots+a_{2m}\eta_{2m})\cdot(a_0\eta_0+a_1\eta_1+\cdots+a_{2m}\eta_{2m}) \\ &= a_0^2\eta_0+a_1^2\eta_1+\cdots+a_{2m}^2\eta_{2m}=0. \end{aligned}$$

This implies $a_0^2=a_1^2=\cdots=a_{2m}^2=0$. Hence, $a_j\in C_j^{\perp}$, for $j=0,1,2,\ldots,2m$. Therefore, C_j are self orthogonal over F_q, for $j=0,1,2,\ldots,2m$.

(2) Let $x_1=a_0\eta_0+a_1\eta_1+\cdots+a_{2m}\eta_{2m}$ and $x_2=b_0\eta_0+b_1\eta_1+\cdots+b_{2m}\eta_{2m}\in C$, where $a_j,b_j\in F_q$, for $j=0,1,2,\ldots,2m$. Now by inner product of x_1 and x_2, we have $x_1\cdot x_2=a_0b_0\eta_0+a_1b_1\eta_1+\cdots+a_{2m}b_{2m}\eta_{2m}$. Since C is self-orthogonal, $a_0b_0=a_1b_1=\cdots=a_{2m}b_{2m}=0$.
On the other hand, $\Phi(x_1)\cdot\Phi(x_2)=(a_0,a_1,\ldots,a_{2m})\cdot(b_0,b_1,\ldots,b_{2m})=a_0b_0+a_1b_1+\cdots+a_{2m}b_{2m}=0$. Hence, $\Phi(C)$ is self-orthogonal. □

Theorem 2.1 *Let C be a linear code of length n over R. Then, $\Phi(C)=C_0\otimes C_1\otimes\cdots C_{2m}$.*

3 Properties of skew constacyclic codes over R

Theorem 3.1 *Let $C = \oplus_j \eta_j C_j$ be a linear code of length n over R for $j = 0, 1, 2, \ldots, 2m$.*

(1) *Then, C is a $(1-2u_1-2u_2-\cdots-2u_m)$-skew constacyclic code of length n over R if and only if $C_0, C_{m+1}, C_{m+2}, \ldots, C_{2m}$ are skew cyclic and $C_1, C_2, C_3, \ldots, C_m$ are skew negacyclic codes of length n over F_q.*

(2) *Let the order of the automorphism divides n. If C is a $(1-2u_1-2u_2-\cdots-2u_m)$-skew constacyclic code of length n over R, then the dual $C^{\perp} = \oplus_j \eta_j C_j^{\perp}$ is a $(1-2u_1-2u_2-\cdots-2u_m)$-skew constacyclic code of length n over R, for $j = 0, 1, \ldots, 2m$.*

Proof (1) Let C be a $(1-2u_1-2u_2-\cdots-2u_m)$-skew constacyclic code of length n over R and let $c = (c_0, c_1, \ldots, c_{n-1}) \in C$, where $c_i = c_i^0\eta_0 + c_i^1\eta_1 + \cdots + c_i^{2m}\eta_{2m}$, $c_i^j \in F_q$ for $i = 0, 1, \ldots, n-1$ and $j = 0, 1, 2, \ldots, 2m$. So $(c_0^j, c_1^j, \ldots, c_{n-1}^j) \in C_j$. Since C is a $(1-2u_1-2u_2-\cdots-2u_m)$-skew constacyclic code of length n over R, we have

$$\sigma_\lambda(c) = (\theta_t(1-2u_1-2u_2-\cdots-2u_m)c_{n-1}), \theta_t(c_0), \ldots, \theta_t(c_{n-2})) \in C.$$

Note that

$$(1-2u_1-2u_2-\cdots-2u_m)\eta_l = -\eta_l \quad \text{for} \quad l = 1, 2, \ldots, m$$

and

$$(1-2u_1-2u_2-\cdots-2u_m)\eta_k = \eta_k \quad \text{for} \quad k = 0, m+1, \ldots, 2m.$$

Therefore,

$$\begin{aligned}\sigma_\lambda(c) = {} & \eta_0(\theta_t(c_{n-1}^0), \theta_t(c_0^0), \ldots, \theta_t(c_{n-2}^0)) \\ & + \sum_{j=1}^{m} \eta_j(-\theta_t(c_{n-1}^j), \theta_t(c_1^j), \ldots, \theta_t(c_{n-2}^j)) \\ & + \sum_{j=m+1}^{2m} \eta_j(\theta_t(c_{n-1}^j), \theta_t(c_0^j), \ldots, \theta_t(c_{n-2}^j)).\end{aligned}$$

Hence,

$$(-\theta_t(c_{n-1}^j), \theta_t(c_1^j), \ldots, \theta_t(c_{n-2}^j)) \in C_j \quad \text{for} \quad j = 1, 2, \ldots, m,$$

and

$$(\theta_t(c_{n-1}^j), \theta_t(c_1^j), \ldots, \theta_t(c_{n-2}^j)) \in C_j \quad \text{for} \quad j = 0, m, m+1, m+2, \ldots, 2m.$$

Therefore, $C_0, C_{m+1}, C_{m+2}, \ldots, C_{2m}$ are skew cyclic and $C_1, C_2, C_3, \ldots, C_m$ are skew negacyclic codes of length n over F_q.

Conversely, suppose that $C_0, C_{m+1}, C_{m+2}, \ldots, C_{2m}$ are skew cyclic and $C_1, C_2, C_3, \ldots, C_m$ are skew negacyclic codes of length n over F_q. Then, considering the above notations,

$$(-\theta_t(c^j_{n-1}), \theta_t(c^j_1), \ldots, \theta_t(c^j_{n-2})) \in C_j \quad \text{for} \quad j = 1, 2, \ldots, m,$$

and

$$(\theta_t(c^j_{n-1}), \theta_t(c^j_1), \ldots, \theta_t(c^j_{n-2})) \in C_j \quad \text{for} \quad j = 0, m, m+1, m+2, \ldots, 2m.$$

Since

$$\sigma_\lambda(c) = (\theta_t(c^0_{n-1}), \theta_t(c^0_0), \ldots, \theta_t(c^0_{n-2})) + \sum_{j=1}^{m}(-\theta_t(c^j_{n-1}), \theta_t(c^j_1), \ldots, \theta_t(c^j_{n-2}))$$
$$+ \sum_{j=m+1}^{2m} (\theta_t(c^j_{n-1}), \theta_t(c^j_0), \ldots, \theta_t(c^j_{n-2})) \in C,$$

we find that C is a $(1-2u_1-2u_2-\cdots-2u_m)$-skew constacyclic code of length n over R.

(2) We have $C^\perp = \oplus_j \eta_j C_j^\perp$ for $j = 0, 1, \ldots, 2m$. As the order of the automorphism divides n, the dual code of every skew constacyclic code over F_q is also skew constacyclic [12,13], by (1) of this theorem, $C^\perp$ is a $(1-2u_1-2u_2-\cdots-2u_m)$-skew constacyclic code. □

Theorem 3.2 *Let $C = \oplus_i \eta_i C_i$ be a $(1-2u_1-2u_2-\cdots-2u_m)$-skew constacyclic code of length n over R. Suppose f_i are the monic generator polynomials of C_i for $i = 0, 1, 2, \ldots, 2m$.*

1. *Then, $C = \langle \eta_0 f_0, \eta_1 f_1, \ldots, \eta_{2m} f_{2m}\rangle$ and $|C| = q^{(2m+1)n-(\sum_{i=0}^{2m} deg(f_i))}$.*
2. *There exists a polynomial $f(x) \in R[x, \theta_t]$ such that $C =< f(x) >$, where $f(x) = \eta_0 f_0(x) + \eta_1 f_2(x) + \cdots + \eta_{2m} f_{2m}(x)$ and $f(x)|(x^n - (1-2u_1-2u_2-\cdots-2u_m))$.*

Proof (1) Let C be a $(1-2u_1-2u_2-\cdots-2u_m)$-skew constacyclic code of length n over R. Then, by Theorem 3.1 we get, $C_0, C_{m+1}, C_{m+2}, \ldots, C_{2m}$ are skew cyclic and $C_1, C_2, C_3, \ldots, C_m$ are skew negacyclic codes of length n over F_q. So we can write $C_k = (f_k(x)) \subseteq F_q[x, \theta_t]/(x^n - 1)$, for $k = 0, m+1, m+2, \ldots, 2m$, and $C_l = (f_l(x)) \subseteq F_q[x, \theta_t]/(x^n + 1)$, for $l = 1, 2, \ldots, m$. Also as $C = \oplus_i \eta_i C_i$, we can write C as

$$C = \{f(x) \ : \ f(x) = \eta_0 f_0(x) + \eta_1 f_1(x) + \cdots + \eta_{2m} f_{2m}(x),$$
$$where \ f_i(x) \in C_i \quad \text{for} \quad i = 0, 1, \ldots, 2m\}.$$

This implies $C \subseteq \, < \eta_0 f_0, \eta_1 f_1, \ldots, \eta_{2m} f_{2m} > \, \subseteq \ R[x, \theta_t]/(x^n - (1 - 2u_1 - 2u_2 - \cdots - 2u_m))$.
On the other hand, let $\sum_{i=0}^{2m} \eta_i f_i(x) g_i(x) \in < \eta_0 f_0, \eta_1 f_1, \ldots, \eta_{2m} f_{2m} >$, where $g_i(x)$ are elements of $R[x, \theta_t]/(x^n - (1-2u_1-2u_2-\cdots-2u_m))$. Then, there exists

$$r_k(x) \in F_q[x, \theta_t]/(x^n - 1) \quad \text{for} \quad k = 0, m+1, m+2, \ldots, 2m$$

and

$$r_l(x) \in \ F_q[x, \theta_t]/(x^n + 1) \quad \text{for} \quad l = 1, 2, \ldots, m$$

such that $\eta_i g_i(x) = \eta_i r_i(x)$ for $i = 0, 1, \ldots, 2m$. Therefore, $< \eta_0 f_0, \eta_1 f_1, \ldots, \eta_{2m} f_{2m} > \, \subseteq \ C$. Hence, $C = < \eta_0 f_0, \eta_1 f_1, \ldots, \eta_{2m} f_{2m} >$.

For the other part, it is worth noting that $|\Phi(C)| = |C|$, and hence

$$|C| = |C_0||C_1| \cdots |C_{2m}| = q^{n-\deg(f_0)} q^{n-\deg(f_1)} \cdots q^{n-\deg(f_{2m})}$$
$$= q^{(2m+1)n - \left(\sum_{i=0}^{2m} \deg(f_i)\right)}.$$

(2) Let C be a $(1-2u_1-2u_2-\cdots-2u_m)$-skew constacyclic code of length n over R and suppose f_i are monic generator polynomials of C_i for $i = 0, 1, \ldots, 2m$. Then, we can write $C = < \eta_0 f_0, \eta_1 f_1, \ldots, \eta_{2m} f_{2m} >$. Suppose $C' = < \eta_0 f_0(x) + \eta_1 f_1(x) + \cdots + \eta_{2m} f_{2m}(x) >$, then it is obvious that $C' \subseteq C$. As $\eta_i(\eta_0 f_0(x) + \eta_1 f_1(x) + \cdots + \eta_{2m} f_{2m}(x)) = \eta_i f_i$ for $i = 0, 1, \ldots, 2m$, this implies $C \subseteq C'$. Therefore, $C = C'$, and $C = < f(x) >$, where $f(x) = \eta_0 f_0(x) + \eta_1 f_1(x) + \cdots + \eta_{2m} f_{2m}(x)$.

Now suppose f_i is the monic generator polynomial of C_i for $i = 0, 1, \ldots, 2m$. Then, f_k divides $x^n - 1$ for $k = 0, m+1, \ldots, 2m$ and f_l divides $x^n + 1$ for $l = 1, \ldots, m$.

$$x^n - (1 - 2u_1 - 2u_2 - \cdots - 2u_m) = \left(\sum_{i=0}^{2m} \eta_i g_i(x)\right)\left(\sum_{i=0}^{2m} \eta_i f_i(x)\right)$$

for $g_i \in C_i$, for $i = 0, 1, \ldots, 2m$. Therefore, $x^n - (1 - 2u_1 - 2u_2 - \cdots - 2u_m) = (\eta_0 g_0(x) + \eta_1 g_1(x) + \cdots + \eta_{2m} g_{2m}(x)) f(x)$. Hence, $f(x) | (x^n - (1 - 2u_1 - 2u_2 - \cdots - 2u_m))$. □

Corollary 3.1 *Let the order of the automorphism divides n and* $C = \oplus_i \eta_i C_i$ *be a* $(1 - 2u_1 - 2u_2 - \cdots - 2u_m)$*-skew constacyclic code of length n over R and suppose* f_i *are the generator polynomials of* C_i *for* $i = 0, 1, 2, \ldots, 2m$*. Then,*

(1) $C^{\perp} = < \eta_0 h_0^*, \eta_1 h_1^*, \ldots, \eta_{2m} h_{2m}^* >$ *and* $|C^{\perp}| = q^{(\sum_{i=0}^{2m} deg(f_i))}$
(2) *There exists a polynomial* $h^*(x)$ *such that* $C^{\perp} = < h^*(x) >$ *where* $h^*(x) = \eta_0 h_0^*(x) + \eta_1 h_1^*(x) + \cdots + \eta_{2m} h_{2m}^*(x)$.

The polynomial $h_i(x)$ and $h_i^*(x)$ are defined as $x^n-1=h_i(x)f_i(x)$, where $f_i, h_i \in F_q[x,\theta_t]$. If $f_i(x)=a_0+a_1x+\cdots+a_sx^s$ and $h(x)=b_0+b_1x+\cdots+b_{n-s}x^{n-s}$, then the dual code of C is generated by $h_i^*(x)$, where $h_i^*(x)=b_{n-s}+\theta_t(b_{n-s-1})x+\cdots+\theta_t^{n-s}(b_0)x^{n-s}$.

4 Necessary and sufficient condition of self-dual skew cyclic and skew negacyclic codes over *R*

By Theorem 3.1, any $(1-2u_1-2u_2-\cdots-2u_m)$-skew constacyclic code over R is a direct product of skew cyclic and skew negacyclic codes over F_q. Here, in this section, we study the necessary and sufficient condition for skew cyclic codes to contain its dual.

In [30], it has shown that if $\gcd(n,k)=1$, then a skew cyclic (skew negacyclic) code C of length n over any finite field is equivalent to a cyclic (negacyclic) code, where k is the order of the automorphism of the finite field. Further, this result has been extended over a finite ring in [8]. Therefore, if $\gcd(n,k)=1$, then a skew cyclic (skew negacyclic) code C of length n over R is equivalent to a cyclic (negacyclic) code of length n over R.

Lemma 4.1 [14] *Let C be a linear cyclic or negacyclic code with generator polynomial $g(x)$ over F_p. Then, C contains its dual code if and only if*

$$x^n-\lambda \equiv 0(mod\, g(x)g^*(x)),$$

where $g^*(x)$ is the reciprocal polynomial of $g(x)$ and $\lambda=\pm 1$.

Theorem 4.1 *Let $C=\oplus_i \eta_i C_i$ be a $(1-2u_1-2u_2-\cdots-2u_m)$-skew constacyclic code of length n over R, $\gcd(n,k)=1$ and $C_i=\langle g_i(x)\rangle$ for $i=0,1,\ldots,2m$, where k is the order of the automorphism θ_t. Then, $C^\perp\subseteq C$ if and only if $x^n-1\equiv 0\,(mod\, f_jf_j^*)$, $x^n+1\equiv 0\,(mod\, f_lf_l^*)$, where $j=0,m+1,m+2,\ldots,2m$ and $l=1,2,\ldots,m$.*

Proof Let $x^n-1\equiv 0\,(\text{mod } f_jf_j^*)$, $x^n+1\equiv 0\,(\text{mod } f_lf_l^*)$, where $j=0,m+1,m+2,\ldots,2m$ and $l=1,2,\ldots,m$. Since $\gcd(n,k)=1$, each C_j are equivalent to cyclic code for $j=0,m+1,m+2,\ldots,2m$ and C_l are equivalent to negacyclic code $l=1,2,\ldots,m$ Now, using Lemma 4.1, we have $C_i^\perp\subseteq C_i,\ i=0,1,\ldots,2m$. Therefore, $\eta_iC_i^\perp\subseteq\eta_iC_i,\ i=0,1,2,\ldots,2m$. Thus, $\oplus_i\eta_iC_i^\perp\subseteq\oplus_i\eta_iC_i$. Hence, $C^\perp\subseteq C$.

Conversely, as $\gcd(n,k)=1$, a λ-skew constacyclic code C of length n over R is equivalent to a λ-constacyclic code of length n over R. Suppose that $C^\perp\subseteq C$. Then, $\oplus_i\eta_iC_i^\perp\subseteq\oplus_i\eta_iC_i$. Since C_i are linear codes over F_q such that $\eta_iC_i\equiv C(\text{mod } \eta_i)$, for $i=0,1,2,\ldots,2m$, $C_i^\perp\subseteq C_i$ for $i=0,1,2,\ldots,2m$. Therefore, $x^n-1\equiv 0\,(\text{mod } f_jf_j^*)$, $x^n+1\equiv 0\,(\text{mod } f_lf_l^*)$, where $j=0,m+1,m+2,\ldots,2m$ and $l=1,2,\ldots,m$. □

Corollary 4.1 *Let $C=\oplus_i\eta_iC_i$ be a $(1-2u_1-2u_2-\cdots-2u_m)$-skew constacyclic code of length n over R and $\gcd(n,k)=1$, where k is the order of the automorphism θ_t. Then, $C^\perp\subseteq C$ if and only if $C_i^\perp\subseteq C_i,\ i=0,1,\ldots,2m$.*

5 Quantum codes from $(1-2u_1-2u_2-\cdots-2u_m)$-skew constacyclic codes over R

Let H be a q-dimensional Hilbert space over the complex numbers $\mathbb{C}$. Define $H^{\otimes n}$ to be n-fold tensor product of the Hilbert space H, that is, $H^{\otimes n} = H \otimes H \otimes \cdots \otimes H$($n$-times). Then, $H^{\otimes n}$ is a Hilbert space of q^n dimension. A quantum code of length n and dimension k over F_q is defined to be the Hilbert subspace of $H^{\otimes n}$. A quantum code with length n, dimension k and minimum distance d over F_q is denoted by $[[n, k, d]]_q$.

Theorem 5.1 [14] (CSS Construction) *Let C_1 and C_2 be $[n, k_1, d_1]_q$ and $[n, k_2, d_2]_q$ linear codes over F_q, respectively, with $C_2^\perp \subseteq C_1$. Furthermore, let $d = min\{d_1, d_2\}$. Then, there exists a quantum error-correcting code C with parameters $[[n, k_1 + k_2 - n, d]]_q$. In particular, if $C_1^\perp \subseteq C_1$, then there exists a quantum error-correcting code C with parameters $[[n, 2k_1 - n, d_1]]$.*

Theorem 5.2 *Let $C = \oplus_i \eta_i C_i$ be a $(1-2u_1-2u_2-\cdots-2u_m)$-skew constacyclic code of length n over R and $gcd(n, k) = 1$. If $C_i^\perp \subseteq C_i,\ i = 0, 1, \ldots, 2m$, then $C^\perp \subseteq C$ and there exists a quantum error-correcting code with parameters $[[(2m+1)n,\ 2k-(2m+1)n,\ d_L]]$, where d_L denotes the minimum Lee weight of the code C and k denotes the dimension of the code $\Phi(C)$.*

Proof Let $C_i^\perp \subseteq C_i$ for $i = 0, 1, \ldots, 2m$. Then, by the Corollary 4.1, we get $C^\perp \subseteq C$. Now let $x \in \Phi(C^\perp) = \Phi(C)^\perp$, then there exists $y \in C^\perp$ such that $x = \Phi(y)$, where $y \cdot y' = 0$ for all $y' \in C$. Since $C^\perp \subseteq C$ and $y \in C^\perp$, we have $y \in C$. Hence, $x = \Phi(y) \in \Phi(C)$. Therefore, $\Phi(C)^\perp \subseteq \Phi(C)$. As $\Phi(C)$ is a $[(2m+1)n, k, d_L]$ linear code over F_q. Then, by CSS Construction, there exists a quantum error-correcting code with parameters $[[(2m+1)n,\ 2k-(2m+1)n,\ d_L]]$. □

Example 5.1 Let $R = F_9 + u_1F_9 + u_2F_9$ and $\lambda = (1 - 2u_1)$. Let $\theta(a) = a^3$ for $a \in F_9$.

$$x^{33} - 1 = (2+2x+x^2+2x^3+x^5)^3(2+x^2+2x^3+x^4+x^5)^3(2+x)^3 \in F_9[x, \theta].$$

Let $f_i(x) = (2+x^2+2x^3+x^4+x^5)^2$. Then $C_i = \langle f_i(x) \rangle$ are skew cyclic codes over F_9 with parameters $[33, 23, 3]$. As $gcd(33, 2) = 1$ so C_i are equivalent to cyclic codes, also as $f_i(x) f_i^*(x)$ divide $x^{33} - 1$, by Lemma 4.1, $C_i^\perp \subseteq C_i$, where $i = 0, 2$.

$$x^{33} + 1 = (1+2x+2x^2+2x^3+x^5)^3(1+2x^2+2x^3+2x^4+x^5)^3(1+x)^3 \in F_9[x, \theta].$$

Let $f_1(x) = (1+2x^2+2x^3+2x^4+x^5)^2$. Then $C_1 = \langle f_1(x) \rangle$ is a skew negacyclic code over F_9 with parameter $[33, 23, 3]$. As $gcd(33, 2) = 1$ so C_1 is equivalent to negacyclic code, also as $f_1(x) f_1^*(x)$ divides $x^{33}+1$, by Lemma 4.1, we get $C_1^\perp \subseteq C_1$.
Then

$$C = \langle \gamma_0 f_0(x) \rangle, \gamma_1 f_1(x), \gamma_2 f_2(x) \rangle$$

is a λ-skew constacyclic code of length 33 over R. Thus, $\Phi(C)$ is a code over F_9 with parameters [99, 69, 3]. As $C_i^{\perp} \subseteq C_i$ for $i = 0, 1, 2$, we get $C^{\perp} \subseteq C$. Now using Theorem 5.2, we get a quantum code with parameter [[99, 39, 3]].

Example 5.2 Let $R = F_{81} + u_1 F_{81} + u_2 F_{81}$ and $\lambda = (1-2u_1)$. Let $\theta(a) = a^3$ for $a \in F_{81}$.

$$x^{13} - 1 = (2+x)(2+x^2+x^3)(2+2x+x^3)(2+2x+2x^2+x^3)$$
$$(2+x+x^2+x^3) \in F_{81}[x,\theta].$$

Let $f_0(x) = (2+2x+x^3)(2+2x+2x^2+x^3)$ and $f_2(x) = (2+x^2+x^3)(2+x+x^2+x^3)$. Then $C_i = \langle f_i(x) \rangle$ are skew cyclic codes of length 13 over F_{81} with parameters [13, 7, 4]. As gcd(13, 4) = 1, so C_i are equivalent to cyclic codes of length 13 and $f_i(x) f_i^*(x)$ divide $x^{13} - 1$, by Lemma 4.1, we get $C_i^{\perp} \subseteq C_i$, for $i = 0, 2$.

$$x^{13} + 1 = (1+x)(1+2x^2+x^3)(1+2x+x^3)(1+2x+x^2+x^3)$$
$$(1+x+2x^2+x^3) \in F_{81}[x,\theta].$$

Let $f_1(x) = (1+2x^2+x^3)(1+2x+x^2+x^3)$. Then $C_1 = \langle f_1(x) \rangle$ is skew negacyclic code over F_{81} with parameter [13, 7, 4]. As $gcd(13, 4) = 1$, so C_1 is equivalent to negacyclic code of length 13 and as $f_1(x) f_1^*(x)$ divides $x^{13} + 1$. Therefore, by Lemma 4.1, we get $C_1^{\perp} \subseteq C_1$. Thus,

Then

$$C = \langle \gamma_0 f_0(x), \gamma_1 f_1(x), \gamma_2 f_2(x) \rangle$$

is a λ-skew constacyclic code of length 13 over R. Thus, $\Phi(C)$ is a code over F_{25} with parameters [39, 21, 4]. As $C_i^{\perp} \subseteq C_i$ for $i = 0, 1, 2$, we get $C^{\perp} \subseteq C$. Now, using Theorem 5.2, we get a quantum code with parameter [[39, 3, 4]]. This quantum code is new in the literature.

Example 5.3 Let $R = F_{27} + u_1 F_{27} + u_2 F_{27} + u_3 F_{27} + u_4 F_{27}$ and $\lambda = (1-2u_1-2u_2)$. Let $\theta(a) = a^3$ for $a \in F_{27}$.

$$x^{11} - 1 = (2+x)(2+x^2+2x^3+x^4+x^5)(2+2x+x^2+2x^3+x^5) \in F_{27}[x,\theta].$$

Let $f_i(x) = (2+x^2+2x^3+x^4+x^5)$. Then $C_i = \langle f_i(x) \rangle$ are skew cyclic codes over F_{27} with parameters [11, 6, 5]. As $gcd(11, 3) = 1$ so C_i are equivalent to cyclic codes, also as $f_i(x) f_i^*(x)$ divide $x^{11} - 1$, by Lemma 4.1, we get $C_i^{\perp} \subseteq C_i$, where $i = 0, 3, 4$.

$$x^{11} + 1 = (1+x)(1+2x^2+2x^3+2x^4+x^5)(1+2x+2x^2+2x^3+x^5)$$
$$\in F_{27}[x,\theta].$$

Let $f_j(x) = (1+2x+2x^2+2x^3+x^5)$. Then $C_j = \langle f_j(x) \rangle$ are skew negacyclic codes over F_{27} with parameters [11, 6, 5]. As $gcd(11, 3) = 1$ so C_j are equivalent to negacyclic codes, also as $f_j(x)f_j^*(x)$ divide $x^{11}+1$, by Lemma 4.1, we get $C_j^\perp \subseteq C_j$, where $j = 1, 2$.

Then

$$C = \langle \gamma_0 f_0(x)\rangle, \gamma_1 f_1(x), \cdots, \gamma_4 f_4(x)\rangle$$

is a λ-skew constacyclic code of length 11 over R. Thus, $\Phi(C)$ is a code over F_{27} with parameters [55, 30, 5]. As $C_i^\perp \subseteq C_i$ for $i = 0, 1, \ldots, 4$, we get $C^\perp \subseteq C$. Now using Theorem 5.2, we get a quantum code with parameter [[55, 5, 5]]. This quantum code is new in the literature.

Example 5.4 Let $R = F_{49} + u_1 F_{49} + \cdots + u_6 F_{49}$ and $\lambda = (1 - 2u_1 - 2u_2 - 2u_3)$. Let $\theta(a) = a^7$ for $a \in F_{49}$.

$$x^{15} - 1 = (3 + x)(5 + x)(6 + x)(1 + x + x^2 + x^3 + x^4)(4 + x + 2x^2 + 4x^3 + x^4) \times (2 + x + 4x^2 + 2x^3 + x^4) \in F_{49}[x, \theta].$$

Let $f_i(x) = (4 + x + 2x^2 + 4x^3 + x^4)(5 + x)$. Then $C_i = \langle f_i(x)\rangle$ are skew cyclic codes over F_{49} with parameters [15, 10, 3]. As $gcd(15, 2) = 1$ so C_i are equivalent to cyclic codes, also as $f_i(x)f_i^*(x)$ divide $x^{15} - 1$, by Lemma 4.1, we get $C_i^\perp \subseteq C_i$, where $i = 0, 4, 5, 6$.

$$x^{15} + 1 = (1 + x)(2 + x)(4 + x)(1 + 6x + x^2 + 6x^3 + x^4) \times (4 + 6x + 2x^2 + 3x^3 + x^4)(2 + 6x + 4x^2 + 5x^3 + x^4) \in F_{49}[x, \theta].$$

Let $f_j(x) = (2 + 6x + 4x^2 + 5x^3 + x^4)(4 + x)$. Then $C_j = \langle f_j(x)\rangle$ are skew negacyclic codes over F_{49} with parameters [15, 10, 3]. As $gcd(15, 2) = 1$ so C_j are equivalent to negacyclic codes, also as $f_j(x)f_j^*(x)$ divide $x^{15} + 1$, by Lemma 4.1, we get $C_j^\perp \subseteq C_j$, where $j = 1, 2, 3$.

Then

$$C = \langle \gamma_0 f_0(x)\rangle, \gamma_1 f_1(x), \cdots, \gamma_6 f_6(x)\rangle$$

is a λ-skew constacyclic code of length 15 over R. Thus, $\Phi(C)$ is a code over F_{49} with parameters [105, 70, 3]. As $C_i^\perp \subseteq C_i$ for $i = 0, 1, \ldots, 6$, we get $C^\perp \subseteq C$. Now using Theorem 5.2, we get a quantum code with parameter [[105, 35, 3]].

Example 5.5 Let $R = F_{121} + u_1 F_{121} + \cdots + u_{10} F_{121}$ and $\lambda = (1 - 2u_1 - \cdots - 2u_5)$. Let $\theta(a) = a^{11}$ for $a \in F_{121}$.

$$x^{15} - 1 = (2 + x)(6 + x)(7 + x)(8 + x)(10 + x)(9 + 3x + x^2)(5 + 4x + x^2) \times (4 + 9x + x^2)(3 + 5x + x^2)(1 + x + x^2) \in F_{121}[x, \theta].$$

Let $f_i(x) = (4 + 9x + x^2)(5 + 4x + x^2)(6 + x)(8 + x)$. Then $C_i = \langle f_i(x)\rangle$ are skew cyclic codes over F_{49} with parameters [15, 9, 5]. As $gcd(15, 2) = 1$ so C_i are equivalent to cyclic codes, also as $f_i(x)f_i^*(x)$ divide $x^{15}-1$, by Lemma 4.1, we get $C_i^\perp \subseteq C_i$, where $i=0,6,7,8,9,10$.

$$x^{15}+1=(1+x)(3+x)(4+x)(5+x)(9+x)(9+8x+x^2)(5+7x+x^2) \times(4+2x+x^2)(3+6x+x^2)(1+10x+x^2)\in F_{121}[x,\theta].$$

Let $f_j(x)=(5+7x+x^2)(3+x)(4+2x+x^2)(5+x)$. Then $C_j=\langle f_j(x)\rangle$ are skew negacyclic codes over F_{121} with parameters [15, 9, 5]. As $gcd(15,2)=1$ so C_j are equivalent to negacyclic codes, also as $f_j(x)f_j^*(x)$ divide $x^{15}+1$, by Lemma 4.1, we get $C_j^\perp\subseteq C_j$, where $j=1,2,3,4,5$.

Then

$$C=\langle\gamma_0 f_0(x)\rangle,\gamma_1 f_1(x),\cdots,\gamma_{10}f_{10}(x)\rangle$$

is a λ-skew constacyclic code of length 15 over R. Thus, $\Phi(C)$ is a code over F_{121} with parameters [165, 99, 5]. As $C_i^\perp\subseteq C_i$ for $i=0,1,\ldots,10$, we get $C^\perp\subseteq C$. Now, using Theorem 5.2, we get a quantum code with parameter [[165, 33, 5]]. This quantum code is new in the literature.

The following table contains some new quantum error-correcting codes over F_9. Let $R=F_9+u_1F_9+u_2F_9$, first column of the table denotes the length of $(1-2u_1)$-skew constacyclic codes over R, second column denotes the generator polynomials of skew cyclic codes C_i for $i=0,2$, column third denotes the generator polynomial of skew negacyclic code C_1, column four denotes the parameters of the gray images of $(1-2u_1)$-skew constacyclic codes over R, and the last column denotes the parameters of the associated quantum codes. We write coefficients of generator polynomials in descending order, for example, the polynomial $x^8+\alpha^2x^6+x^5+\alpha x^4+\alpha x^3+x^2+\alpha^2x+\alpha^5$ is represented by $10\alpha^2 1\alpha\alpha 1\alpha^2\alpha^5$.

n	$f_0(x)=f_2(x)$	$f_1(x)$	$\Phi(C)$	$[[n,k,d]]$
27	111	121	[81, 75, 3]	[[81, 69, 3]]
75	11111	12121	[225, 213, 5]	[[225, 201, 5]]
91	1112	1211	[273, 264, 4]	[[273, 255, 4]]
99	102122	102221	[297, 282, 5]	[[297, 267, 5]]
117	1112	1121	[351, 342, 4]	[[351, 333, 4]]
135	111	121	[405, 399, 3]	[[405, 393, 3]]
135	11111	12121	[405, 393, 5]	[[405, 381, 5]]

Comparison: Compared to previously known quantum error-correcting codes in the references [10,19,20], some of our quantum error-correcting codes $[[(2m+1)n,\ 2k-(2m+1)n,\ d_L]]$ are new. In the above table, our quantum codes [[225, 201, 5]], [[351, 333, 4]], [[405, 393, 3]] and [[405, 381, 5]] have better parameters than the known quantum codes [[224, 196, 5]], [[352, 329, 4]], [[401, 389, 3]] and [[401, 369, 5]], respectively, in [10].

6 Conclusion

In this paper, we have given the structural properties of $(1-2u_1-2u_2-\cdots-2u_m)$-skew constacyclic codes over the ring $R = F_q+u_1F_q+\cdots+u_{2m}F_q$. As an application of this class of codes over R, we have obtained some new quantum codes over the field F_q. For future work, it would be interesting to find quantum codes over F_q by taking another gray map over R.

Acknowledgements The authors are thankful to the anonymous referees for their careful reading of the paper and valuable comments. The first author is thankful to the University Grant Commission (UGC), Govt. of India, for financial support under Sr. No. 2061441025 with Ref No. 22/06/2014(i)EU-V.

References

1. Ashraf, M., Mohammad, G.: On skew cyclic codes over a semi-local ring. Discrete Math. Algorithms Appl. **7**(4), 1550042 (2015)
2. Ashraf, M., Mohammad, G.: Skew constacyclic codes over $F_q + vF_q + v^2F_q$. In: Algebra and Its Applications: Proceedings of the International Conference held at Aligarh Muslim University, pp. 25–36. De Gruyter, Berlin (2016). https://doi.org/10.1515/9783110542400-003
3. Ashraf, M., Mohammad, G.: Quantum codes from cyclic codes over $F_3 + vF_3$. Int. J. Quantum Inf. **12**(6), 1450042 (2014)
4. Ashraf, M., Mohammad, G.: Quantum codes from cyclic codes over $F_q+uF_q+vF_q+uvF_q$. Quantum Inf. Process. **15**(10), 4089–4098 (2016)
5. Ashraf, M., Mohammad, G.: Quantum codes over F_p from cyclic codes over $F_p[u, v]/\langle u^2-1, v^3-v, uv-vu\rangle$. Cryptogr. Commun. (2018). https://doi.org/10.1007/s12095-018-0299-0
6. Al-Ashker, M.M., Abu-Jafar, A.Q.M.: Skew constacyclic codes over $F_p + vF_p$. Palest. J. Math. **5**(2), 96–103 (2016)
7. Aly, S.A., Klappenecker, A., Sarvepalli, P.K.: On quantum and classical BCH codes. IEEE Trans. Inf. Theory **53**, 1183–1188 (1995)
8. Bag, T., Upadhyay, A.K.: Skew cyclic and skew constacyclic codes over the ring $F_p + u_1F_p + \cdots + u_{2m}F_p$. Asian Eur. J. Math. **12**(1), 1950083 (2019)
9. Bag, T., Upadhyay, A.K., Ashraf, M., Mohammad, G.: Quantum codes from cyclic codes over the ring $F_p[u]/ < u^3 - u >$. Asian Eur. J. Math. **13**(1), 2050008 (2020)
10. Bierbrauer, J., Edel, Y.: Quantum twisted codes. J. Comb. Des. **8**(3), 174–188 (2000)
11. Boucher, D., Geiselmann, W., Ulmer, F.: Skew cyclic codes. Appl. Algebra Eng. Commun. Comput. **18**(4), 379–389 (2007)
12. Boucher, D., Sole, P., Ulmer, F.: Skew constacyclic codes over Galois ring. Adv. Math. Commun. **2**(3), 273–292 (2008)
13. Boucher, D., Ulmer, F.: Coding with skew polynomial rings. J. Symb. Comput. **44**, 1644–1656 (2009)
14. Calderbank, A.R., Rains, E.M., Shor, P.M., Sloane, N.J.A.: Quantum error-correction via codes over GF(4). IEEE Trans. Inf. Theory **44**, 1369–1387 (1998)
15. Dertli, A., Cengellenmis, Y., Eren, S.: On quantum codes obtained from cyclic codes over A_2. Int. J. Quantum Inf. **13**(3), 1550031 (2015)
16. Dertli, A., Cengellenmis, Y., Eren, S.: Some results on the linear codes over the finite ring $F_2+v_1F_2+\cdots+v_rF_2$. Int. J. Quantum Inf. **14**(01), 1650012 (2016)
17. Gao, J., Wang, Y.: u-Constacyclic codes over $F_p + uF_p$ and their applications of constructing new non-binary quantum codes. Quantum Inf. Process. **17**, 4 (2018)
18. Gursoy, F., Siap, I., Yildiz, B.: Construction of skew cyclic codes over F_q+vF_q. Adv. Math. Commun. **8**, 313–322 (2014)
19. Grassl, M., Beth, T.: On optimal quantum codes. Int. J. Quantum Inf. **2**, 55–64 (2004)
20. Gaurdia, G., Palazzo Jr., R.: Constructions of new families of nonbinary CSS codes. Discrete Math. **310**, 2935–2945 (2010)
21. Gottesman, D.: An introduction to quantum error-correction. In: Proceedings of Symposia in Applied Mathematics, vol. 68. American Mathematical Society (2010)

22. Jitman, S., Ling, S., Udomkavanich, P.: Skew constacyclic codes over finite chain rings. Adv. Math. Commun. **6**, 29–63 (2012)
23. Kai, X., Zhu, S.: Quaternary construction of quantum codes from cyclic codes over $F_4 + uF_4$. Int. J. Quantum Inf. **9**, 689–700 (2011)
24. Ketkar, A., Klappenecker, A., Kumar, S., Sarvepalli, P.K.: Nonbinary quantum stabilizer codes over finite fields. IEEE Trans. Inf. Theory **52**, 4892–4914 (2006)
25. Li, R., Xu, Z., Li, X.: Binary construction of quantum codes of minimum distance three and four. IEEE Trans. Inf. Theory **50**, 1331–1335 (2004)
26. Li, R., Xu, Z.: Construction of $[[n, n-4, 3]]_q$ quantum codes for odd prime power q. Phys. Rev. A **82**, 1–4 (2010)
27. Qian, J., Ma, W., Gou, W.: Quantum codes from cyclic codes over finite ring. Int. J. Quantum Inf. **7**, 1277–1283 (2009)
28. Qian, J.: Quantum codes from cyclic codes over $F_2 + vF_2$. J. Inf. Comput. Sci. **10**, 1715–1722 (2013)
29. Sari, M., Siap, I.: On quantum codes from cyclic codes over a class of nonchain rings. Bull. Korean Math. Soc. **53**(6), 1617–1628 (2016)
30. Siap, I., Abualrub, T., Aydin, N., Seneviratne, P.: Skew cyclic codes of arbitrary length. Inf. Coding Theory **2**, 10–20 (2011)
31. Shor, P.W.: Scheme for reducing decoherence in quantum memory. Phys. Rev. A **52**, 2493–2496 (1995)
32. Steane, A.M.: Simple quantum error-correcting codes. Phys. Rev. A **54**, 4741–4751 (1996)