Quantum Information Processing (2019) 18:45
https://doi.org/10.1007/s11128-018-2149-6

Pairwise nonclassical correlations for superposition of Dicke states via local quantum uncertainty and trace distance discord

Youssef Khedif[1] · **Mohammed Daoud**[1,2]

Received: 24 July 2018 / Accepted: 6 December 2018 / Published online: 2 January 2019
© Springer Science+Business Media, LLC, part of Springer Nature 2019

Abstract
The pairwise nonclassical correlations for two-qubit states, extracted from multi-qubit system with exchange symmetry and parity, are quantified by local quantum uncertainty and trace distance discord. The explicit expressions of local quantum uncertainty and geometric trace distance discord for Dicke states and their superpositions are given. A comparison between the two quantum correlations quantifiers is discussed.

Keywords Pairwise nonclassical correlations · Local quantum uncertainty · Trace distance discord · Dicke states

1 Introduction

The concept of entanglement is at the heart of the violation of Bell inequalities [1], and it is a very useful ingredient in the field of quantum information [2–4]. Entanglement can be used to perform several quantum tasks such as superdense coding [5], quantum teleportation [6] and quantum states merging [7,8]. The characterization of quantum correlations is one of the central issues in quantum information. In this sense, there are still many open questions regarding the entanglement properties of two and more quantum systems. Various entanglement measures are reported in the literature [3,9] to quantify entanglement and to distinguish between entangled and separable (not entangled) states. But, nontrivial nonclassical correlations may exist even in separable states [10]. Thereby, the general nonclassical correlations are not quantitatively

✉ Youssef Khedif
youssef.khedif@gmail.com

Mohammed Daoud
m_daoud@hotmail.com

1 Laboratory of High Energy and Condensed Matter Physics, Department of Physics, Faculty of Sciences Aïn Chock, University Hassan II, P.O. Box 5366, Maarif, Casablanca 20100, Morocco

2 Abdus Salam International Centre for Theoretical Physics, Strada Costiera 11, 34151 Trieste, Italy

characterized by quantum entanglement. In this respect, quantum discord [11–13] (a measure of quantum correlation beyond entanglement) has been widely studied in the literature as an alternative way to characterize nonclassical correlations even in separable mixed states. The investigation of this information-theoretic measure of non-classical correlations has generated an increasing interest during the last two decades [14]. The computational complexity of quantum discord led Dakić et al. to introduce a geometric version of quantum discord [14] using Hilbert–Schmidt norm. Another interesting geometrical approach to describe quantum correlation is the so-called trace distance discord (TDD). It measures, by means of the Schatten 1-norm (trace norm), the distance between the state under consideration and its closest zero discord states [15,16]. The explicit form of trace distance discord for two-qubit X states was derived in [17].
Besides TDD measure, the local quantum uncertainty (LQU) has been proposed as another prominent quantifier of nonclassical correlations in multipartite systems. It was introduced by Girolami et al. [18] in order to quantify the uncertainty which can arise in a given quantum state due to its noncommutativity with a measuring single local observable. Furthermore, LQU has a relevant application in the field of quantum metrology. It is also connected to quantum Fisher information [19] due to the link between this quantum precision measurement and skew information [20,21] on which is based local quantum uncertainty. Its closed form is available for qubit–qudit system [22]. For a generic family of two-qubit X states, the analytical expression of local quantum uncertainty was derived in [23].

In investigating quantum correlations, multipartite states have received a special attention. The pairwise entanglement in symmetric multi-qubit systems was studied in [24]. The quantum discord, based on the Hilbert–Schmidt distance, was also employed to determine the pairwise quantum correlations in such systems [25]. However, it has been pointed out that the Hilbert–Schmidt quantum discord not a good measure of quantum correlations [26] since it may increase under local reversible operation acting on the un-measured qubit. This violates the property according to which every quantum correlations quantifier, constructed on the basis of a distance between two states, should be nonincreasing under any given completely positive trace preserving map [27].

Thus, in this paper, we shall reconsider the evaluation of the pairwise quantum correlations for a pair of two-qubit extracted from a symmetric of N two-level systems [24]. We shall employ the local quantum uncertainty and geometric quantum discord based on the trace distance (the Schatten 1-norm). We focus essentially on Dicke states and their superpositions [28]. They provide a rich resource for quantum information tasks (see Ref. [29] for a comprehensive review), and they can easily be converted to W states [30] or GHZ states [31]. Furthermore, Dicke states have attracted a lot of attention and become the subject of both experimental [32] as well as theoretical studies [25,29,32]. In this paper, we give the analytical expressions of local quantum uncertainty and trace distance discord for two-qubit X-states type extracted from multi-qubit Dicke states and some of their special superpositions such as generalized W states and generalized GHZ states [4,30,31] and even and odd spin coherent states [33,34].

This paper is organized as follows: In Sect. 2, we give the expressions of local quantum uncertainty and trace distance discord for two-qubit systems extracted from symmetric multi-qubit system. In Sect. 3, we investigate the pairwise quantum correlations in Dicke states and their superpositions (generalized W states and GHZ, the superpositions of two Dicke states and even and odd spin coherent states). Concluding remarks are given in Sect. 4.

2 Nonclassical correlations quantifiers for two-qubit X states with exchange and parity symmetries

2.1 Two-qubit density matrix

In the present work, we consider an ensemble of N spin$-\frac{1}{2}$ particles with the states $|0\rangle$ and $|1\rangle$. This system has exchange symmetry, and its properties can be described by the collective spin operators $J_\mu(\mu = x,\ y,\ z)$ defined as the sum over all elementary spin operators (Pauli matrices) $\sigma_\mu^{(k)}$:

$$J_\mu = \frac{1}{2}\sum_{k=1}^{N}\sigma_\mu^{(k)}. \tag{1}$$

In order to compute the TDD and LQU for this system, we consider two-qubit states which are extracted from the whole ensemble by tracing out the remaining $(N-2)$ qubits. Specifically, we focus on the states with exchange symmetry and parity. In this case, the two-qubit reduced density matrix in the standard computational basis $\{|00\rangle, |01\rangle, |10\rangle, |11\rangle\}$ can be written as [24,25]

$$\rho_{AB} = \begin{pmatrix} \mu_+ & 0 & 0 & z^* \\ 0 & \eta & \eta & 0 \\ 0 & \eta & \eta & 0 \\ z & 0 & 0 & \mu_- \end{pmatrix}. \tag{2}$$

The density matrix ρ_{AB} satisfies the conditions: $\mathrm{Tr}\rho_{AB} = 1$ and $\mu_+\mu_- > |z|^2$ to ensure the positivity of the density matrix condition. The elements of the density matrix (2) can be expressed in terms of the expectations values of the collective spin operators as follows:

$$\begin{cases} \mu_\pm = \dfrac{N^2 - 2N + 4\langle J_z^2\rangle \pm 4\langle J_z\rangle(N-1)}{4N(N-1)}, \\ \eta = \dfrac{N^2 - 4\langle J_z^2\rangle}{4N(N-1)}, \\ z = \dfrac{\langle J_+^2\rangle}{N(N-1)} \end{cases} \tag{3}$$

where the ladder operators $J_\pm$ are defined by $J_\pm = J_x \pm iJ_y$.

2.2 Local quantum uncertainty

The concept of local quantum uncertainty is regarded as a faithful quantifier of quantum correlations. It quantifies the minimum quantum uncertainty in a quantum state due to a measurement of a single local observable. Let ρ_{AB} be a bipartite quantum state, and let $K := K_A \otimes \mathbb{1}_B$ denote a local maximally informative (with nondegenerate spectrum) observable associated with the subsystem A, with K_A being a Hermitian on A and $\mathbb{1}_B$ is the identity operator acting on the subsystem B. The LQU with respect to the subsystem A is defined as [18]

$$\mathcal{U}(\rho_{AB}) := \min_{\{K\}} \mathcal{I}\Big(\rho_{AB}, K_A \otimes \mathbb{1}_B\Big) \tag{4}$$

where $\mathcal{I}$ is the Wigner–Yanase *skew information* [20,21] defined by

$$\mathcal{I}(\varrho, K) := -\frac{1}{2}\mathrm{Tr}\left(\left[\sqrt{\varrho}, K\right]^2\right). \tag{5}$$

It serves as a measure of uncertainty of the observable K in the state ϱ. For pure states ($\varrho^2 = \varrho$), the skew information reduces indeed to the conventional variance given by

$$\mathrm{Var}\,(\varrho, K) = \mathrm{Tr}\left(\varrho K^2\right) - \Big(\mathrm{Tr}\big(\varrho K\big)\Big)^2. \tag{6}$$

The analytical evaluation of the local quantum uncertainty requires a minimization procedure over the set of all observable acting on the part A. A closed form for qubit–qudit systems was derived in [22]. In particular, for qubits (spin-$\frac{1}{2}$ particles), the local quantum uncertainty with respect to subsystem A is given by [18]

$$\mathcal{U}(\rho_{AB}) = 1 - \lambda_{\max}\left\{\mathcal{W}_{AB}\right\}, \tag{7}$$

where $\lambda_{\max}$ denotes the maximal eigenvalue of the 3×3 symmetric matrix $\mathcal{W}_{AB}$ whose elements are defined by

$$\Big(\mathcal{W}_{AB}\Big)_{ij} \equiv \mathrm{Tr}\left\{\sqrt{\rho_{AB}}\Big(\sigma_{Ai} \otimes \mathbb{1}_B\Big)\sqrt{\rho_{AB}}\Big(\sigma_{Aj} \otimes \mathbb{1}_B\Big)\right\} \tag{8}$$

and $\sigma_{Ai}\,(i = 1, 2, 3)$ being the standard Pauli matrices of the subsystem A. In what follows, we recall some interesting properties of local quantum uncertainty. It can be treated as discord-like quantifier. This means that LQU vanishes for all states which have zero discord. It has a geometrical significance in terms of Hellinger distance [18,35]. Besides, LQU is invariant under any local unitary operations. Based on this last property of LQU, the phase factor $\frac{z}{|z|} = e^{i\theta}$ [see Eq. (2)] can be removed. Indeed, by means of the following local unitary transformation

$$|0\rangle_A = \exp\Big(-\frac{i\theta}{2}\Big)|0\rangle_A, \qquad |0\rangle_B = \exp\Big(-\frac{i\theta}{2}\Big)|0\rangle_B,$$

the density matrix ρ_{AB} (2) becomes

$$\rho'_{AB} = \begin{pmatrix} \mu_+ & 0 & 0 & |z| \\ 0 & \eta & \eta & 0 \\ 0 & \eta & \eta & 0 \\ |z| & 0 & 0 & \mu_- \end{pmatrix}. \tag{9}$$

In the Fano-Bloch representation [36,37], the density matrix ρ'_{AB} writes as

$$\rho'_{AB} = \frac{1}{4} \sum_{\alpha,\beta=0}^{3} R_{\alpha\beta} \sigma_\alpha^A \otimes \sigma_\beta^B \tag{10}$$

where the nonvanishing correlations matrix elements $R_{\alpha\beta} = \mathrm{Tr}(\rho'_{AB}\, \sigma_\alpha^A \otimes \sigma_\beta^B)$ are given by

$$R_{00} = 1, \quad R_{03} = R_{30} = \mu_+ - \mu_-, \quad R_{11} = 2(\eta + |z|),$$
$$R_{22} = 2(\eta - |z|), \quad R_{33} = 1 - 4\eta, \tag{11}$$

and using the expressions (3), they write

$$R_{03} = \frac{2\langle J_z \rangle}{N}, \quad R_{11} = \frac{N^2 + 4\left(\langle J_+^2 \rangle - \langle J_z^2 \rangle\right)}{2N(N-1)}, \quad R_{22} = \frac{N^2 - 4\left(\langle J_+^2 \rangle + \langle J_z^2 \rangle\right)}{2N(N-1)},$$
$$R_{33} = -\frac{N - 4\langle J_z^2 \rangle}{N(N-1)}. \tag{12}$$

To determine the local quantum uncertainty $\mathcal{U}(\rho_{AB}) = \mathcal{U}(\rho'_{AB})$, defined by (7), one should compute the eigenvalues of matrix $\mathcal{W}_{AB}$ (8). Explicitly, they are given by [23,38]

$$\omega_\pm = \sqrt{\lambda}\left[\left(\sqrt{\lambda_+} + \sqrt{\lambda_-}\right) \pm \frac{(R_{11} - R_{22})}{2\left(\sqrt{\lambda_+} + \sqrt{\lambda_-}\right)}\right], \tag{13a}$$

$$\omega = \frac{1}{2}\left[\left(\sqrt{\lambda_+} + \sqrt{\lambda_-}\right)^2 + \frac{4R_{03}^2 - (R_{11} - R_{22})^2}{4\left(\sqrt{\lambda_+} + \sqrt{\lambda_-}\right)^2}\right] \tag{13b}$$

where $\lambda_\pm$ and λ denote the nonzero eigenvalues of the density matrix ρ_{AB}. They write in terms of the Fano-Bloch components (11) as follows:

$$\lambda_\pm = \frac{1}{2}\left[1 - \frac{R_{11} + R_{22}}{2}\right] \pm \sqrt{\left(\frac{R_{03}}{2}\right)^2 + \left(\frac{R_{11} - R_{22}}{4}\right)^2}, \quad \lambda = \frac{R_{11} + R_{22}}{2}. \tag{14}$$

It is must be noticed that $R_{11} \geq R_{22}$. This implies that $\omega_+ > \omega_-$. Hence, from Eq. (7), the local quantum uncertainty quantifying the pairwise quantum correlation in the state ρ_{AB} reads as

$$\mathcal{U}(\rho_{AB}) = 1 - \max\{\omega_+, \omega\}. \tag{15}$$

Hence, to determine the amount of quantum correlations by means of LQU, one must study the sign of the quantity $(\omega_+ - \omega)$.

2.3 Trace distance discord

The second quantifier of quantum correlations, we shall employ in this work, is the trace distance discord. It quantifies, via the Schatten 1-norm distance (trace norm), the quantum correlations between a given state ρ and the zero discord classical-quantum state $\rho_{\rm cq}$ [39]. It is defined by

$$D_{\rm T}(\rho) = \min_{\rho_{\rm cq}\in\Omega_0} \|\rho - \rho_{\rm cq}\|_1, \tag{16}$$

where Ω_0 being the set of classical-quantum states which have zero discord and $\|\mathcal{O}\|_1 = {\rm Tr}\left(\sqrt{\mathcal{O}^\dagger\mathcal{O}}\right)$ defines the trace norm of a generic operator $\mathcal{O}$. The classical-quantum states with zero discord $\rho_{\rm cq}$ can be expressed as [40]

$$\rho_{\rm cq} = \sum_{k=1}^{2} p_k |k\rangle\langle k|^A \otimes \rho_k^B, \tag{17}$$

where $\{p_k\}$ is a set of statistical probability distribution with $0 \le p_k \le 1$ and $\{|k\rangle\langle k|^A\}$ denotes a set of orthogonal projectors associated with the subsystem A, on which we assume that the measurement is performed, while ρ_k^B being a general reduced density operator for the subsystem B. In the two-qubit X state case (2), the analytical expression of TDD was recently derived in [17]. It can be written as

$$D_{\rm T}(\rho_{AB}) = \sqrt{\frac{R_{11}^2 R_{\rm max}^2 - R_{22}^2 R_{\rm min}^2}{R_{\rm max}^2 - R_{\rm min}^2 + R_{11}^2 - R_{22}^2}}, \tag{18}$$

where the quantities $R_{\rm max}^2$ and $R_{\rm min}^2$ are expressed in terms of the Fano-Bloch components as follows:

$$R_{\rm max}^2 = \max\{R_{33}^2, R_{22}^2 + R_{30}^2\} \qquad \text{and} \qquad R_{\rm min}^2 = \min\{R_{11}^2, R_{33}^2\}.$$

The nonzero correlation matrix elements $R_{\alpha\beta}$ are given by Eq. (12). Remark that, like LQU, the trace distance quantum discord is invariant under any local unitary transformation.

3 Local quantum uncertainty and trace discord for Dicke states and their superpositions

In this section, we give the explicit analytical expression of pairwise quantum correlation for some special instances of X-states. A particular emphasis is dedicated to

Dicke states and their superpositions [GHZ states, the superpositions of two Dicke states, even and odd coherent spin states (CSSs)].

3.1 Dicke states

Here, we introduce a class of permutationally symmetric states that is important from the perspective of quantum information. The N-qubit symmetric Dicke state with n excitations ($0 \leq n \leq N$) is defined by

$$|n\rangle_N \equiv |\frac{N}{2}, -\frac{N}{2}+n\rangle = \frac{1}{\sqrt{\binom{N}{n}}}\left[\sum_{j=1}^{\binom{N}{n}} \Pi_j\Big(|\overbrace{000\cdots00}^{N-n}\underbrace{111\cdots11}_{n}\rangle\Big)\right], \tag{19}$$

where $\{\Pi_j\}$ is the set of all possible distinct permutations of $(N-n)$ 0's and n 1's [41]. These states are simultaneous eigenstates of the two commuting collective spin operators $\hat{J}^2$ and its z-component $\hat{J}_z$. Indeed, we have

$$\hat{J}^2|n\rangle_N = \frac{N}{2}\left(\frac{N}{2}+1\right)|n\rangle_N, \qquad \hat{J}_z|n\rangle_N = \left(n-\frac{N}{2}\right)|n\rangle_N. \tag{20}$$

For later use, we give also the action of the raising $\hat{J}_+$ and lowering $\hat{J}_-$ operators on the state $|n\rangle_N$. They are given by

$$\hat{J}_+|n\rangle_N = \sqrt{(N-n)\,(n+1)}|n+1\rangle_N, \qquad \hat{J}_-|n\rangle_N = \sqrt{n\,(N-n+1)}|n-1\rangle_N. \tag{21}$$

From Eq. (19), the N-qubit ground state $|0\rangle_N$ and the excited state $|1\rangle_N$ write

$$\begin{aligned} |0\rangle_N &= |000\cdots00\rangle, \\ |1\rangle_N &= \frac{1}{\sqrt{N}}\Big(|000\cdots01\rangle + |000\cdots10\rangle + \cdots + |100\cdots00\rangle\Big). \end{aligned} \tag{22}$$

The state $|1\rangle_N$ is the so-called W state. To compute the local quantum uncertainty and trace quantum discord, we first calculate the expectation values of the global spin operators in the Dicke states (19)

$$\langle J_z\rangle = n-\frac{N}{2}, \qquad \langle J_z^2\rangle = \left(n-\frac{N}{2}\right)^2, \qquad \langle J_+^2\rangle = \langle J_-^2\rangle = 0. \tag{23}$$

Using Eqs. (12), (14) and (23), one can determine the eigenvalues of the matrix $\mathcal{W}_{AB}$ given by (13). After some algebra, one finds

$$\omega_{+}^{\rm DS} = \omega_{-}^{\rm DS} = \frac{\sqrt{2n\big(N-n\big)}}{N\big(N-1\big)} \left\{ \sqrt{n\big(n-1\big)} + \sqrt{\big(N-n\big)\big(\big(N-n\big)-1\big)} \right\} \quad (24)$$

$$\omega^{\rm DS} = 1 - \frac{2n(N-n)}{N(N-1)}. \quad (25)$$

Reporting Eqs. (24) and (25) in (15), one can evaluate the amount of pairwise local quantum uncertainty in Dicke states. It is given by

$$\mathcal{U}(\rho_{AB}^{|n\rangle_N}) = \min\left(1 - \frac{\sqrt{2n\big(N-n\big)}}{N\big(N-1\big)} \left\{ \sqrt{n\big(n-1\big)} + \sqrt{\big(N-n\big)\big(\big(N-n\big)-1\big)} \right\}, \frac{2n(N-n)}{N(N-1)}\right). \quad (26)$$

Similarly, using the expression of trace distance discord (18), the pairwise quantum correlation in Dicke states takes the explicit form

$$D_T(\rho_{AB}^{|n\rangle_N}) = \frac{2n(N-n)}{N(N-1)}. \quad (27)$$

It should be noted that the TDD expression (27) belongs to that of LQU (26). Therefore, due to minimization process, we can easily check that

$$D_T(\rho_{AB}^{|n\rangle_N}) \geq \mathcal{U}(\rho_{AB}^{|n\rangle_N}), \quad (28)$$

which reflects that the trace distance discord quantifier in Dicke states displays more quantum correlations amount than local quantum uncertainty.

On the other hand, it is clearly seen from (26) and (27) that both quantifiers in Dicke states are invariant under the change $n \to N-n$. It means that the pairwise nonclassical correlations, whether measured by local quantum uncertainty or by quantum trace norm, of the states $|n\rangle_N$ and $|N-n\rangle_N$ are the same. This can be explained by the fact that there is a local unitary transformation between these two states [25].

The pairwise local quantum uncertainty and trace quantum discord in Dicke states are functions of the total number of qubits N and the excitation number n. Therefore, it is interesting to analyze the variations of these two quantum correlation quantifiers in terms of N and n. We consider first the case where N is fixed. We observe that LQU and TDD vanish for $n = 0$ and $n = N$ corresponding, respectively, to states $|0\rangle_N$ and $|N\rangle_N$ which are separable. More precisely, when N is divisible by 4, the pairwise local quantum uncertainty is maximal for $n = N/4$ and $n = 3N/4$. The local quantum uncertainty increases in the interval $\{0,\ 1, \ldots, \frac{N}{4}\}$ and decreases when $n \in \{\frac{N}{4}, \ldots, \frac{N}{2}\}$. It increases after for $n \in \{\frac{N}{2}, \ldots, \frac{3N}{4}\}$ to reach a maximal value for $\frac{3N}{4}$ and decreases after to vanish for $n = N$. These variations are illustrated in Fig. 1a for $N = 20$ [1]. It must be noticed that the pairwise local quantum uncertainty presents

[1] For a fixed odd value of N, LQU and TDD vary in function the n according to curves similar to those of Fig. 1. The only difference lies in two concerned quantifiers extremes points. Indeed, when

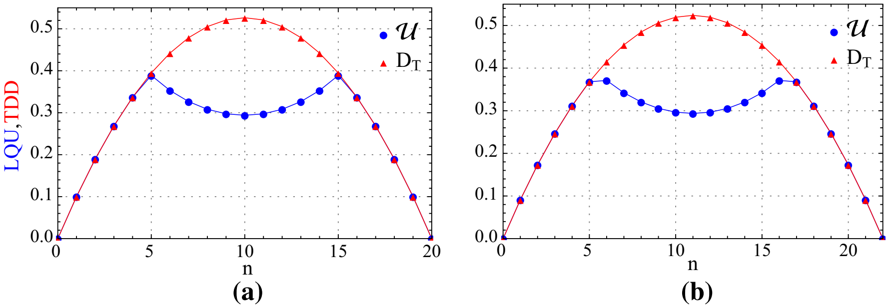

Fig. 1 (color online) Local quantum uncertainty (blue color) and trace distance discord (red color) of Dicke states versus excitations number n with the fixed spin number N (N even). In Fig. 1a, we consider $N = 20$. Nevertheless, when $N + 2$, which is divisible by 4 (for instance $N = 22$ as shown in Fig. 1b), then both quantifiers behave similarly to the first case except that the maximal values of LQU are obtained for $(N + 2)/4$ and $(3N - 2)/4$

a sudden change for $\frac{N}{4}$ and $\frac{3N}{4}$ which is not exhibited by the pairwise trace quantum discord. Besides, it turns out that, by means of (27) and numerical verification, one can find explicitly the pairwise local quantum uncertainty expression. Indeed, when a fixed particles number N is divisible by 4, Eq. (26) becomes

$$\mathcal{U}\left(\rho_{AB}^{|n\rangle_N}\right) = \begin{cases} 1 - \omega^{\mathrm{DS}}; & n \in \{0,\ 1,\ \cdots,\ \frac{N}{4}\} \bigcup \{\frac{3N}{4},\ \cdots,\ N\} \\ 1 - \omega_{+}^{\mathrm{DS}}; & n \in \{\frac{N}{4},\ \cdots,\ \frac{3N}{4}\}. \end{cases} \tag{29}$$

In analyzing the amount of pairwise quantum correlations measured by trace norm, we see that this quantifier increases to reach a maximal value for $\frac{N}{2}$ and, due to its spontaneous symmetry at $n = N/2$, decreases for states with $n \in \{\frac{N}{2},\ \ldots,\ N\}$.

It must be noticed that the two quantifiers are the same except that in a region where TDD is convex LQU is concave conversely. This typical difference between the two quantifiers comes back mainly from the LQU structure, as indicated in Eq. (26), on the one hand, and to the local unitary transformation effect on the other hand. For this last point, we can indeed notice that after the local unitary transformation, the eigenvalues of matrix $\mathcal{W}_{AB}$ given in (24) and (25) remain invariable (since the eigenvalues of a physical quantity are invariable under unitary transformation [25]). Consequently, the LQU in terms of $\mathcal{W}_{AB}$'s eigenvalues is also invariable. This conclusion leads to the symmetry of the LQU and also a local minimum at $n = \frac{N}{2}$ (because the increasing or decreasing of LQU under quantum operation on the measured party [18,42]).

Now, we consider the situation where n is fixed, and we compare the pairwise quantum correlations in states with different number of qubits and having the same excitation number n. We first analyze the case of W states corresponding to $n = 1$. In this case, the pairwise local quantum uncertainty is given by

Footnote 1 continued
$(N - 1)$ [Resp. $(N + 1)$] is a multiple of 4, the two local maxima of LQU are pointing in $n_- = (N + 1)/4$ [Resp. $(N - 1)/4$] and $n_+ = (3N - 1)/4$ [Resp. $(3N + 1)/4$], while its local minimum, which corresponds to TDD's local maximum, is pointed in $(N - 1)/2$ and $(N + 1)/2$.

$$\mathcal{U}\left(\rho_{AB}^{|1\rangle_N}\right) = \begin{cases} 1 \quad \text{if} \quad N = 2, \\ 1 - \frac{\sqrt{2(N-2)}}{N} \quad \text{if} \quad N \in \{3;\ 4\}, \\ 2/N \quad \text{if} \quad N \geq 4, \end{cases} \tag{30}$$

and the pairwise trace quantum discord (27) gives

$$D_T\left(\rho_{AB}^{|1\rangle_N}\right) = 2/N, \tag{31}$$

which is coincide with the so-called W's highest possible concurrence [24]. An important remark is that these results show that the behavior of LQU is similar to that of the TDD to large extent in W states. More precisely, both quantifiers in W states coincide in all points except that when $N = 3$ wherein TDD is greater.

In the next, our analysis will be focused on the case where the excitation number is $n = 2, 3, 4$. The LQU and TDD are shown in Fig. 2a, b for $n = 2$ and $n = 3$. They behave similarly. For each fixed n, both quantifiers coincide initially and increase from the same vanished value corresponding to $N = n$ wherein Dicke state reduces to product states $|n\rangle_n$. This reflects that there are no pairwise nonclassical correlations in the product states. This result can be verified analytically. Indeed, using the results (26) and (27) one can also straightforwardly check that

$$\mathcal{U}\left(\rho_{AB}^{|n\rangle_n}\right) = D_T\left(\rho_{AB}^{|n\rangle_n}\right) = 0. \tag{32}$$

When TDD reaches its maximum value corresponding to $(2n-1)$ and $2n$, it decreases monotonically with increasing N. Conversely, LQU has two local maxima, for $(n+1)$ and $4n$, and between them there exists a single local minimum obtained for $N = 2n$. After its second local maximum, the LQU quantifier coincides again with TDD and consequently decreases monotonically with increasing values of N. Besides, the quantifiers monotonic decay slowly for increasing values of excitation number n. Furthermore, when N becomes large enough, the pairwise nonclassical correlations tend to zero. On the other hand, the trace quantum norm and local quantum uncertainty disagree only when LQU exhibits sudden change phenomenon. Like the case when N is fixed, the pairwise local quantum uncertainty for a fixed value of excitation number n can be evaluated explicitly. From (26) and the plot results (Fig. 2), LQU takes the form

$$\mathcal{U}\left(\rho_{AB}^{|n\rangle_N}\right) = \begin{cases} 1 - \omega_+^{\mathrm{DS}}; & N \in \{n+1, \ldots,\ 4n\} \\ 1 - \omega^{\mathrm{DS}}; & \text{Otherwise.} \end{cases} \tag{33}$$

As the case where the particles number N is fixed, the amount of quantum correlations quantified by local quantum uncertainty and trace norm in Dicke states coincide except when LQU undergoes a sudden change behavior for a fixed value of excitation number n. Indeed, it is easily seen from the results (26) and (27) that LQU and TDD are coinciding only when the minimum of LQU (27) is given by the quantity $\frac{2n(N-n)}{N(N-1)}$.

It should be noted that the two quantifiers are the same except for the region where TDD is convex and conversely LQU is concave (see Figs. 1, 2). This typical difference between the two quantifiers arises mainly from the double sudden changes exhibited by the local quantum uncertainty. In fact, in evaluating local quantum uncertainty, we

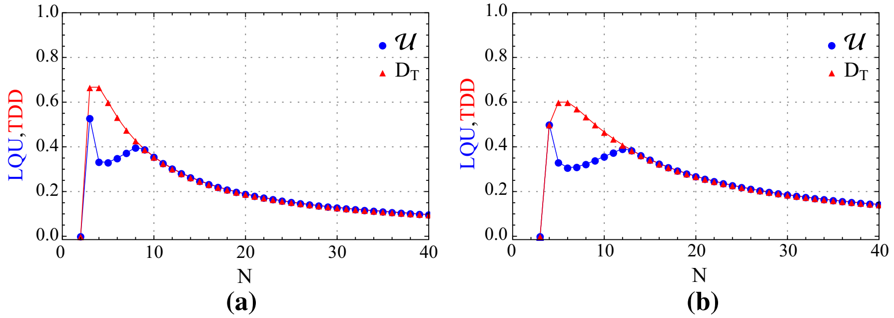

Fig. 2 (Color online) Local quantum uncertainty (blue color) and trace distance discord (red color) of Dicke states versus N with a fixed excitation number; $n = 2$ in Fig. 2a and $n = 3$ in Fig. 2b

need the expression given by Eq. (26). The optimization process shows that the local quantum uncertainty can exhibit two sudden changes. In particular, between the two points where the sudden changes occur, the local quantum uncertainty attains a minimal value when the trace distance discord is maximal (corresponding to $n = \frac{N}{2}$ for a fixed N and $N = 2n$ for a fixed n). It is clear that the trace geometric discord quantifier does not exhibit any behavior change in contrast to the local quantum uncertainty.

3.2 Generalized GHZ states

The generalized GHZ state is a special superposition of the Dicke states, $|0\rangle_N$ and $|N\rangle_N$ [31]

$$|\Psi\rangle_{\rm GHZ} = \cos\theta|0\rangle_N + e^{i\varphi}\sin\theta|N\rangle_N. \tag{34}$$

The spin expectation values of the global spin operators in the state $|\Psi\rangle_{\rm GHZ}$ are given by

$$\langle J_z\rangle = -\frac{N}{2}\cos 2\theta, \qquad \langle J_z^2\rangle = \frac{N^2}{4}, \qquad \langle J_+^2\rangle = 0. \tag{35}$$

Using Eq. (12), one can verify that $R_{11} = R_{22} = 0$. It follows the trace distance discord for the generalized GHZ states is zero

$$D_T\left(\rho_{AB}^{\rm GHZ}\right) = 0. \tag{36}$$

In addition, from Eqs. (12) and (35) we verify that $R_{03} = -\cos 2\theta$. Thus, by straightforward calculation, one finds that $\omega_\pm^{\rm GHZ} = 0$ and $\omega^{\rm GHZ} = 1$. Hence, one has

$$\mathcal{U}\left(\rho_{AB}^{\rm GHZ}\right) = 0. \tag{37}$$

These results show that there are no pairwise quantum correlations for the generalized GHZ states. This coincides with the results obtained in [25,43] where the pairwise quantum correlations were quantified by means of concurrence and Hilbert–Schmidt quantum discord. It is interesting to note that a GHZ state is maximally entangled, and its reduced density matrices are separable. Therefore, it seems that for GHZ states the trace procedure to get bipartite states destroys the nonclassical correlations in the system.

3.3 Superpositions of Dicke states

Now, we consider the evaluation of pairwise quantum correlations in the superpositions of Dicke states of type

$$|\Psi\rangle_{\mathrm{SD}} = \cos\theta|n\rangle_N + e^{i\varphi}\sin\theta|n+2\rangle_N, \qquad n = 0, \ldots, N-2, \tag{38}$$

with $\theta \in [0, \pi]$ and the relative phase $\varphi \in [0, 2\pi]$. It is straightforward to check that the corresponding spin expectation values are given by

$$\begin{aligned}\langle J_z\rangle = \left(n - \frac{N}{2}\right) + 2\sin^2\theta, \quad \langle J_z^2\rangle = \left(n - \frac{N}{2}\right)^2 \\ +4\left(n - \frac{N}{2} + 1\right)\sin^2\theta, \quad \langle J_+^2\rangle = \frac{1}{2}e^{i\varphi}\sqrt{\nu_n}\sin 2\theta\end{aligned} \tag{39}$$

where $\nu_n = (n+1)(n+2)(N-n)(N-n-1)$. For $\theta = 0$, one recovers the expectation values given by (23). Furthermore, the expectation values in (39) are φ-independents, and therefore, the local quantum uncertainty and trace discord are not functions of φ. This is due to the fact that LQU and TDD are invariant under local unitary transformations.

In the special case $n = 0$, the correlations matrix elements of a two-qubit state extracted from the state $|\Psi\rangle_{\mathrm{SD}}$ write

$$\begin{aligned} R_{11} &= \tfrac{4(N-2)}{N(N-1)}\sin^2\theta + \tfrac{2|\sin 2\theta|}{\sqrt{2N(N-1)}}, \\ R_{22} &= \tfrac{4(N-2)}{N(N-1)}\sin^2\theta - \tfrac{2|\sin 2\theta|}{\sqrt{2N(N-1)}}, \\ R_{33} &= \left[1 - \tfrac{8(N-2)}{N(N-1)}\sin^2\theta\right], \\ R_{03} &= \left[\tfrac{4}{N}\sin^2\theta - 1\right]. \end{aligned} \tag{40}$$

This serves to evaluate straightforwardly the local quantum uncertainty and trace distance discord using the results reported hereinabove. To obtain the explicit expression of local quantum uncertainty, one computes first the elements of the matrix $\mathcal{W}_{AB}$. Using Eq. (13), one gets

$$\omega_{\pm}^{\mathrm{SD}} = \frac{2\sqrt{(N-2)}}{N(N-1)}|\sin\theta|\zeta_{N,\theta} \pm 4\sqrt{\frac{2(N-2)}{N(N-1)}}\sin^2\theta|\cos\theta|\zeta_{N,\theta}^{-1} \tag{41}$$

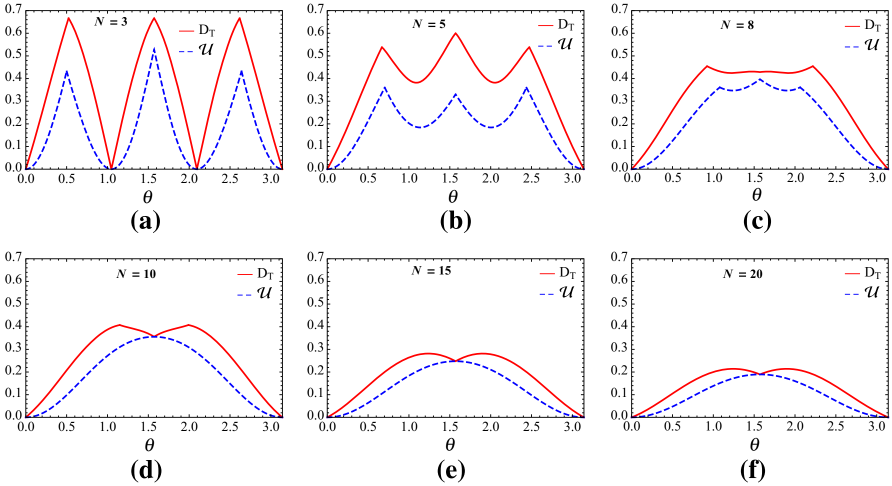

Fig. 3 (Color online) Comparison of LQU (blue dashed line) and TDD (red solid line) of superpositions of Dicke states (38) in function of θ within one period for a various system sizes when $n = 0$

and

$$\omega^{\text{SD}} = \frac{1}{2}\left[\left(N(N-1) + 8\left[\left(3 - \frac{2}{N}\right)\sin^2\theta - N\right]\sin^2\theta\right)\zeta_{N,\theta}^{-2} + \frac{1}{N(N-1)}\zeta_{N,\theta}^{2}\right] \tag{42}$$

where $\zeta_{N,\theta} = \sqrt{N(N-1) + \left(\sqrt{8(N-2)(N-3)} - 4(N-2)\right)\sin^2\theta}$ is a positive function. Having the eigenvalues of the matrix $\mathcal{W}_{AB}$ (13), one can determine the local quantum uncertainty. Let us first consider the case $N = 2$ where the superposition of Dicke state (38) reduces to so-called Bell-like (BL) states:

$$|\Psi\rangle_{\text{BL}} = \cos\theta|0\rangle_2 + e^{i\varphi}\sin\theta|2\rangle_2. \tag{43}$$

In this special case, one finds the following relation between local quantum uncertainty and trace quantum discord

$$D_T\left(\rho_{AB}^{BL}\right) = \sqrt{\mathcal{U}\left(\rho_{AB}^{BL}\right)} = |\sin(2\theta)|. \tag{44}$$

Next, we examine the variations of LQU and TDD as functions of the parameter θ for different values of N. The results are depicted in Fig. 3.
For $N = 3$, it is clearly shown from Fig. 3a that local quantum uncertainty and trace distance discord behave similarly and they are periodic functions of period π. It must be noticed that in addition to the edges, LQU and TDD are zero for $\theta = \frac{\pi}{3}, \frac{2\pi}{3}$. Besides, the trace distance discord is maximal for $\theta = \frac{\pi}{6}, \frac{\pi}{2}, \frac{5\pi}{6}$, while the LQU is maximal for $\theta = \left(\frac{\pi}{6} - \delta\theta\right), \frac{\pi}{2}, \left(\frac{5\pi}{6} + \delta\theta\right)$ where $\delta\theta \approx 0.023$.

Now we consider the behavior of pairwise quantum correlations in the case where $3 < N < 8$. It can be seen from Fig. 3b, for instance, that the LQU and TDD are

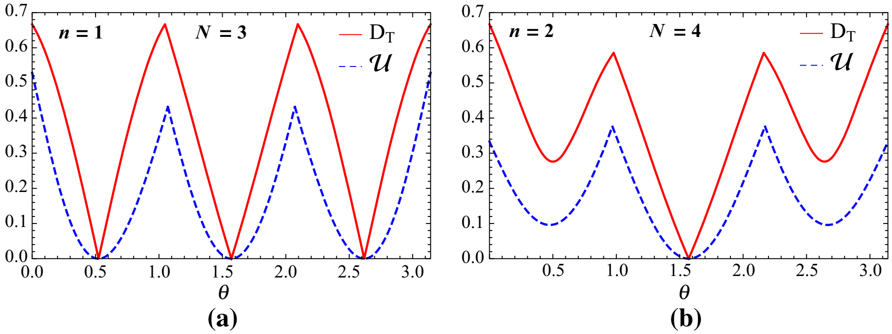

Fig. 4 (Color online) LQU (blue dashed line) and TDD (red solid line) for the superpositions of Dicke states (38) versus θ within one period; Fig. 4a $n = 1$, $N = 3$ and Fig. 4b when $n = 2$, $N = 4$

nonvanishing except for $\theta = 0$ and $\theta = \pi$. This implies that for $\theta \in]0, \pi[$, there exist pairwise quantum correlations in the system for $N = 5$ in contrast to the case where $N = 3$. The same results can be obtained for $N = 8$. This behavior changes for $N \geq 10$. Indeed, for instance for $N = 10$ the LQU is maximal for $\theta = \frac{\pi}{2}$ but the TDD is maximal for $\theta_1 \approx \frac{57\pi}{157}$ and $\theta_2 \approx \frac{100\pi}{157}$, and the maximal values of TDD are greater than the maximal value of LQU. We notice also that for $\theta = \frac{\pi}{2}$, the LQU coincides with TDD.
Let us now discuss the case where the particles number N takes large values. In this case, using Eq. (40) together with (18), one obtains

$$D_T\left(\rho_{AB}^{\mathrm{SD}}\right) = \frac{\sin^2\theta + |\sin 2\theta|}{N}. \tag{45}$$

The trace distance discord goes to zero for large values of N. On the other hand, it must be noticed that for a large particles number, we have $\xi_{N,\theta} \sim N$ so that

$$\mathcal{U}\left(\rho_{AB}^{\mathrm{SD}}\right) = 0. \tag{46}$$

This shows that there is no pairwise quantum correlations for higher values of N when $n = 0$.

Next, we consider the analysis of LQU and TDD in the superpositions of Dicke states with $n \neq 0$. The results are plotted in Fig. 4 for ($n = 1, N = 3$) and ($n = 2, N = 4$).
From Fig. 4, we have seen that the curves of the TDD and LQU behave almost identically and coincide only in the absence of any pairwise quantum correlations. This behavior is consistent with that observed in the case of a few nonexcited particles (see Fig. 4a). Furthermore, as illustrated in Fig. 4b, increasing of n with small value of the number of particles N reveals that both quantifiers are zero only for $\theta = \frac{\pi}{2}$. We investigated also the effect of excitation number n on the amounts of a pairwise nonclassical correlations. In this respect, we can see from Figs. 3a and 4a that each specific quantifier change an inverse trend with θ. Furthermore, one observes that

when ($n = 1,\ N = 3$) the quantifiers zeros, unlike the maximum values, reduce in comparison with ($n = 0,\ N = 3$) ones.

3.4 Even and odd spin coherent states

As another instance for states with exchange and parity, we consider the even and odd spin coherent states (SCSs) [34] and we examine the nonclassical correlations measured by local quantum uncertainty and trace quantum distance. A SCSs are the most quasi-classical pure quantum states of a symmetric ensemble of N elementary 1/2-spins (or N two-mode bosons) particles. They are obtained by a complex rotation of spin ground state $|0\rangle_N$, parameterized by a complex amplitude ξ. In the basis $(\{|n\rangle_N,\ n = 0, 1, \ldots, N\})$ generated by Dicke states $|n\rangle_N$, the SCSs are given by [33]

$$|\xi\rangle = (1 + |\xi|^2)^{-\frac{N}{2}} \sum_{n=0}^{N} \sqrt{\frac{N\,!}{n\,!(N-n)\,!}}\xi^n |n\rangle_N. \tag{47}$$

In what follows, we shall focus on the pairwise quantum correlations in two-qubit states extracted from even spin coherent states ($|\xi\rangle_+$) and odd spin coherent states ($|\xi\rangle_-$) defined by

$$|\xi\rangle_\pm = \frac{1}{\mathcal{N}_\pm}\Big(|\xi\rangle \pm |-\xi\rangle\Big) \tag{48}$$

where the normalization factors $\mathcal{N}_\pm$ are given by

$$\mathcal{N}_\pm = \sqrt{2(1 \pm p^N)} e^{i\phi} \qquad (\phi \in \mathbb{R}).$$

The quantity p denotes the overlap between the states $|-\xi\rangle$ and $|\xi\rangle$ [34]. It is given by

$$p = \frac{1 - |\xi|^2}{1 + |\xi|^2}. \tag{49}$$

The expectation values of the operators J_z, J_z^2 and J_+^2 on the even and odd coherent states (48) are given by

$$\langle J_z\rangle_\pm = -\frac{N}{2}\frac{p \pm p^{N-1}}{1 \pm p^N}, \quad \langle J_z^2\rangle_\pm = \frac{N^2}{4} \pm \frac{N(N-1)}{1 \pm p^N} q^\mp(p),$$
$$\langle J_+^2\rangle_\pm = \pm\frac{N(N-1)}{1 \pm p^N} q^\pm(p) \tag{50}$$

where

$$q^\pm(p) = \frac{(1-p^2)}{4}\Big(p^{N-2} \pm 1\Big).$$

The two-qubit matrix extracted from $|\xi\rangle_\pm$ can be obtained by tracing out the $(N-2)$ qubits. Due to the symmetric property of Dicke states, all two-qubits density matrices are identical. Therefore, it is sufficient to consider ϱ_{12} tracing out the qubit $3, 4, \ldots, N$ from the state $|\xi\rangle_+$ (or $|\xi\rangle_-$)

$$\varrho_{12}^{\pm} = \mathrm{Tr}_{3,4,\ldots,N}|\xi\rangle_\pm\langle\xi|. \tag{51}$$

After some algebra, one finds

$$\varrho_{12}^{\pm} = \frac{1}{|\mathcal{N}_\pm|^2}\begin{pmatrix} 2a^4(1\pm c) & 0 & 0 & 2a^2b^2(1\pm c) \\ 0 & 2a^2b^2(1\mp c) & 2a^2b^2(1\mp c) & 0 \\ 0 & 2a^2b^2(1\mp c) & 2a^2b^2(1\mp c) & 0 \\ 2a^2b^2(1\pm c) & 0 & 0 & 2b^4(1\pm c) \end{pmatrix}, \tag{52}$$

in the computational basis $\{|00\rangle, |01\rangle, |10\rangle, |11\rangle\}$, where $a = \sqrt{\frac{1-p}{2}}$, $b = \sqrt{\frac{1+p}{2}}$ and $c = p^{N-2}$. The two-qubit reduced density matrices of even and odd SCSs (52) are X-shaped. [See the density matrix (2).] Thus, the TDD as well as LQU of these states can be obtained as in the previous subsections. Indeed, the two-qubit states $\varrho_{12}^{\pm}$ can be written in the Fano-Bloch representation as in Eq. (10). The corresponding nonvanishing correlations matrix elements are

$$\begin{aligned} &R_{00}^{\pm} = 1; \quad R_{11}^{\pm} = \tfrac{1-p^2}{1\pm p^N}; \quad R_{22}^{\pm} = \mp\left(\tfrac{1-p^2}{1\pm p^N}\right)p^{N-2}; \\ &R_{33}^{\pm} = 1 - \left(\tfrac{1-p^2}{1\pm p^N}\right)\left(1\mp p^{N-2}\right); \quad R_{03}^{\pm} = -\tfrac{p\pm p^{N-1}}{1\pm p^N}. \end{aligned} \tag{53}$$

On the other hand, by using Eq. (13), one finds the eigenvalues of the matrix $\mathcal{W}_{\mathrm{AB}}$ (8). They are given for even and odd spin coherent states by

$$\begin{aligned} &\omega_{+}^{\pm} = \sqrt{\frac{1-p^2}{1+p^2}}\frac{\sqrt{1-p^{2N-4}}}{1\pm p^N}, \\ &\omega_{-}^{\pm} = p^2\sqrt{\frac{1-p^2}{1+p^2}}\frac{\sqrt{1-p^{2N-4}}}{1\pm p^N}, \quad \omega^{\pm} = \frac{2p^2}{1+p^2}\frac{1\pm p^{N-2}}{1\pm p^N}. \end{aligned} \tag{54}$$

Now, we have the necessary ingredients to quantify the pairwise nonclassical correlations, of both even and odd SCSs, using trace quantum distance and local quantum uncertainty. Figure 5 displays the behavior of TDD and LQU for even and odd SCSs pairwise as a function of the overlapping p for different spin values N.

We first start by comparing the nonclassical correlations behaviors in even and odd SCSs that are captured by TDD and LQU. For even SCSs, it is seen that TDD and LQU behave in terms of p almost identically. In even SCSs, the pairwise quantum correlations vanish for $p \to 1$ (see Fig. 5a, c). For odd SCSs, the trace quantum discord and local quantum uncertainty does not vanish for $p \to 1$ (see Fig. 5b, d). Note that for $p \to 0$, the pairwise quantum correlations is zero.

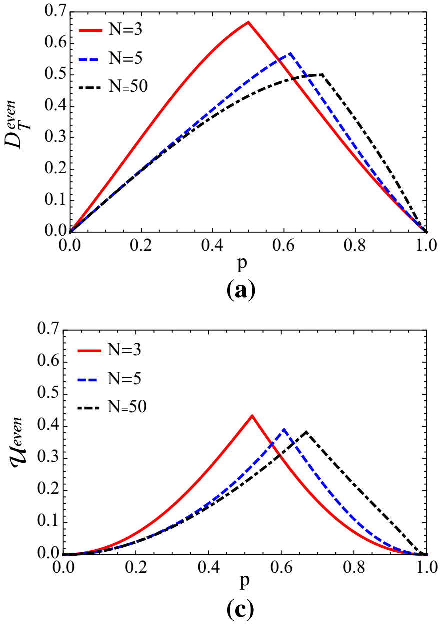

Fig. 5 (Color online) TDD (Fig. 5a, b) and LQU (Fig. 5c, d) for even and odd SCSs (48) versus the overlapping $p = \langle \xi \mid -\xi \rangle$ for $N = 3$ (solid red line), $N = 5$ (dashed blue line) and $N = 50$ (black dashed-dotted line)

Now, we examine some interesting special situations. We first consider the case where $N = 2$. In this case, the expressions (54) read as

$$\omega_+^+ = 0 \quad \omega_-^+ = 0 \quad \omega^+ = \frac{4p^2}{\left(1+p^2\right)^2}, \tag{55}$$

for even spin coherent states. For odd spin coherent states, it can be verified that Eq. (54) give

$$\omega_+^- = 0 \quad \omega_-^- = 0 \quad \omega^- = 0. \tag{56}$$

Thus, the local quantum uncertainty is given by

$$\mathcal{U}\left(\varrho_{12}^+\right) = \left(\frac{1-p^2}{1+p^2}\right)^2 \qquad \text{and} \qquad \mathcal{U}\left(\varrho_{12}^-\right) = 1 \tag{57}$$

while the trace quantum discord takes the values

$$D_T\left(\varrho_{12}^+\right) = \left(\frac{1-p^2}{1+p^2}\right) \qquad \text{and} \qquad D_T\left(\varrho_{12}^-\right) = 1, \tag{58}$$

for even and odd SCSs, respectively.

It is also simple to verify from Eq. (54) that in the limiting case where $p \rightarrow 1$,

$$\omega_+^+ = 0, \quad \omega_-^+ = 0, \quad \omega^+ = 1 \tag{59}$$

and

$$\omega_+^- = \frac{\sqrt{2(N-2)}}{N}, \quad \omega_-^- = \frac{\sqrt{2(N-2)}}{N}, \quad \omega^- = 1 - \frac{2}{N}, \tag{60}$$

so that

$$\mathcal{U}\left(\varrho_{12}^+\right) = 0 \qquad \text{and} \qquad \mathcal{U}\left(\varrho_{12}^-\right) = \begin{cases} 1 & \text{if} \quad N = 2, \\ 1 - \frac{\sqrt{2(N-2)}}{N} & \text{if} \quad N \in \{3;\ 4\}, \\ 2/N & \text{if} \quad N \geq 4, \end{cases} \tag{61}$$

similarly, one can check that the trace quantum discord in this limiting case writes

$$D_T\left(\varrho_{12}^+\right) = 0 \qquad \text{and} \qquad D_T\left(\varrho_{12}^-\right) = \frac{2}{N}. \tag{62}$$

It must be noticed that for $p \rightarrow 1$ (i.e., $\xi \rightarrow 0$), the even spin coherent states reduces to the separable Dicke state $|0\rangle_N$ and odd spin coherent states gives the so-called W state $|1\rangle_N$.
Another interesting limiting case concerns the situation where $p \rightarrow 0$. In this case, the spin coherent states $|\xi\rangle$ and $|-\xi\rangle$ become orthogonal. The even and odd spin coherent states reduce to states of GHZ type

$$|\xi \rightarrow 0\rangle_\pm \sim \frac{1}{\sqrt{2}}(|0\rangle \otimes |0\rangle \cdots \otimes |0\rangle \pm |1\rangle \otimes |1\rangle \cdots \otimes |1\rangle) \tag{63}$$

with $|0\rangle \equiv |-\xi \rightarrow 0\rangle$ and $|1\rangle \equiv |\xi \rightarrow 0\rangle$. From Eqs. (54), one obtains

$$\omega_+^\pm = 1, \quad \omega_-^\pm = 0, \quad \omega^\pm = 0 \tag{64}$$

and the pairwise local quantum uncertainty in the states (63) of GHZ type vanishes.

4 Concluding remarks

In this paper, we have derived the pairwise quantum correlations in two-qubit state extracted from multi-qubit states with exchange symmetry. We derived the nonclassical correlations in such system by employing the recently introduced concept of local quantum uncertainty. The obtained results are compared to those computed by using the trace norm (the geometric variant of quantum discord). In particular, for a variety of special symmetric multi-qubit states (Dicke states, their superpositions and even and odd spin coherent states), we have observed that the trace quantum discord is always upper bounded by local quantum uncertainty.

The measure of multipartite quantum correlations constitutes an important issue in the context of quantum information. Several attempts to provide a precise way to quantify and characterize the genuine multipartite correlations were discussed in the literature yielding different approaches [44–47]. In particular, Rulli and Sarandy [47] defined the multipartite measure of quantum correlations as the maximum of the quantum correlations existing between all possible bipartition of the multipartite quantum system. In a similar way, Ma et al. [44] suggested a slightly different definition to quantify the global multipartite quantum correlations. It is defined as the sum of the correlations in all possible bipartitions. In this sense, the global nonclassical correlations can be evaluated for the different bipartite subsystem. Two bipartitioning schemes can be considered. The first one can be obtained by decomposing the whole N system in a pure bipartite system where the first subsystem contains k qubits and the second subsystem is made by $(N-k)$ qubits $(k = 1, 2, \ldots, N-1)$. By a suitable mapping, the system can be viewed as two logical qubit (see for instance Daoud et al. [34]). The second partitioning scheme involves mixed states that one obtains by a trace procedure similar to one discussed in this paper. The question concerning whether the trace over some particles will destroy the quantum correlations in the whole system constitutes a challenging issue to characterize the genuine multipartite quantum correlations. Substantial efforts have been made to quantify multi-particle entanglement and to explore how multi-particle entanglement manifests itself under different partitioning schemes of the system.

The calculations reported in our work complete the results obtained with other quantum correlations quantifiers such as entanglement of formation, geometric quantum discord based on Hilbert–Schmidt distance and entropic quantum discord. Also, the results derived in this work can be adapted to deal with nonclassical pairwise correlations in collective spin systems such as the Dicke model [28] and the Lipkin–Meshkov–Glick model [48]. They can extended to all possible bipartitions (pure and mixed bipartite states) to deal with all pairwise quantum correlations. Clearly, this extension will involve several bipartitioning schemes which require laborious (but feasible) calculations especially for states with large number of qubits.

References

1. Bell, J.S.: On the Einstein Podolsky Rosen paradox. Physics **1**, 195 (1964)
2. Nielsen, M.A., Chuang, I.L.: Quantum Computation and Quantum Information. Cambridge University Press, Cambridge (2000)
3. Horodecki, R., Horodecki, P., Horodecki, M., Horodecki, K.: Quantum entanglement. Rev. Mod. Phys. **81**, 865 (2009)
4. Gühne, O., Tóth, G.: Entanglement detection. Phys. Rep. **474**, 1 (2009)
5. Bennett, C.H., Wiesner, S.J.: Communication via one- and two-particle operators on Einstein–Podolsky–Rosen states. Phys. Rev. Lett. **69**, 2881 (1992)
6. Bennett, C.H., Brassard, G., Crépeau, C., Jozsa, R., Peres, A., Wootters, W.K.: Teleporting an unknown quantum state via dual classical and Einstein–Podolsky–Rosen channels. Phys. Rev. Lett. **70**, 1895 (1993)
7. Horodecki, M., Oppenheim, J., Winter, A.: Partial quantum information. Nature **436**, 673 (2005)
8. Horodecki, M., Oppenheim, J., Winter, A.: Quantum state merging and negative information. Comm. Math. Phys. **269**, 107 (2007)
9. Plenio, M.B., Virmani, S.: An introduction to entanglement measures. Quant. Inf. Comput. **7**, 1 (2007)
10. Ferraro, A., Aolita, L., Cavalcanti, D., Cucchietti, F.M., Acín, A.: Almost all quantum states have nonclassical correlations. Phys. Rev. A **81**, 052318 (2010)

11. Henderson, L., Vedral, V.: Classical, quantum and total correlations. J. Phys. A: Math. Gen. **34**, 6899 (2001)
12. Ollivier, H., Zurek, W.H.: Quantum discord: a measure of the quantumness of correlations. Phys. Rev. Lett. **88**, 017901 (2001)
13. Modi, K., Brodutch, A., Cable, H., Paterek, T., Vedral, V.: The classical-quantum boundary for correlations: Discord and related measures. Rev. Mod. Phys. **84**, 1655 (2012)
14. Dakić, B., Vedral, V., Brukner, Č.: Necessary and sufficient condition for nonzero quantum discord. Phys. Rev. Lett. **105**, 190502 (2010)
15. Debarba, T., Maciel, T.O., Vianna, R.O.: Witnessed entanglement and the geometric measure of quantum discord. Phys. Rev. A **86**, 024302 (2012)
16. Montealegre, J.D., Paula, F.M., Saguia, A., Sarandy, M.S.: One-norm geometric quantum discord under decoherence. Phys. Rev. A **87**, 042115 (2013)
17. Ciccarello, F., Tufarelli, T., Giovannetti, V.: Toward computability of trace distance discord. New J. Phys. **16**, 013038 (2014)
18. Girolami, D., Tufarelli, T., Adesso, G.: Characterizing nonclassical correlations via local quantum uncertainty. Phys. Rev. Lett. **110**, 240402 (2013)
19. Paris, M.G.A.: Quantum estimation for quantum technology. Int. J. Quant. Inf. **7**, 125 (2009)
20. Wigner, E.P., Yanasse, M.M.: Information contents of distributions. Proc. Natl. Acad. Sci. USA **49**, 910 (1963)
21. Luo, S.: Wigner–Yanase Skew information and uncertainty relations. Phys. Rev. Lett. **91**, 180403 (2003)
22. Wang, S., Li, H., Lu, X., Long, G.-L.: Closed form of local quantum uncertainty and a sudden change of quantum correlations. quant-ph/ arXiv:1307.0576v2
23. Slaoui, A., Daoud, M., Ahl Laamara, R.: The dynamics of local quantum uncertainty and trace distance discord for two-qubit X states under decoherence: a comparative study. Quantum Inf. Process. **17**, 178 (2018)
24. Wang, X., Mølmer, K.: Pairwise entanglement in symmetric multi-qubit systems. Eur. Phys. J. D **18**, 385 (2002)
25. Yin, X., Xi, Z., Lu, X.-M., Sun, Z., Wang, X.: Geometric measure of quantum discord for superpositions of Dicke states. J. Phys. B: At. Mol. Opt. Phys. **44**, 245502 (2011)
26. Ozawa, M.: Entanglement measures and the Hilbert–Schmidt distance. Phys. Lett. A **268**, 158 (2000)
27. Vedral, V., Plenio, M.B., Rippin, M.A., Knight, P.L.: Quantifying entanglement. Phys. Rev. Lett. **78**, 2275 (1997)
28. Dicke, R.H.: Coherence in spontaneous radiation processes. Phys. Rev. **93**, 99 (1954)
29. Bergmann, M., Gühne, O.: Entanglement criteria for Dicke states. J. Phys. A: Math. Theor. **46**, 385304 (2013)
30. Dür, W., Vidal, G., Cirac, J.I.: Three qubits can be entangled in two inequivalent ways. Phys. Rev. A **62**, 062314 (2000)
31. Greenberger, D.M., Horne, M.A., Zeilinger, A.: Nonlocality of a single photon? Phys. Rev. Lett. **75**, 2064 (1995)
32. Kiesel, N., Schmid, C., Tóth, G., Solano, E., Weinfurter, H.: Experimental observation of four-photon entangled dicke state with high fidelity. Phys. Rev. Lett. **98**, 063604 (2007)
33. Radcliffe, J.M.: Some properties of coherent spin states. J. Phys. A: Gen. Phys. **4**, 313 (1971)
34. Daoud, M., Ahl Laamara, R., Kaydi, W.: Multipartite quantum correlations in even and odd spin coherent states. J. Phys. A: Math. Theor. **46**, 395302 (2013)
35. Luo, S., Zhang, Q.: Informational distance on quantum-state space. Phys. Rev. A **69**, 032106 (2004)
36. Bloch, F.: Nuclear Induction. Phys. Rev. **70**, 460 (1946)
37. Fano, U.: Pairs of two-level systems. Rev. Mod. Phys. **55**, 855 (1983)
38. Kedif, Y., Daoud, M.: Local quantum uncertainty and trace distance discord dynamics for two-qubit X states embedded in non-Markovian environment. Int. J. Mod. Phys. B **32**, 1850218 (2018)
39. Paula, F.M., de Oliveira, T.R., Sarandy, M.S.: Geometric quantum discord through the Schatten 1-norm. Phys. Rev. A **87**, 064101 (2013)
40. Luo, S.: Using measurement-induced disturbance to characterize correlations as classical or quantum. Phys. Rev. A **77**, 022301 (2008)
41. Li, D., Li, X., Huang, H.: Stochastic local operations and classical communication properties of the n-qubit symmetric Dicke states. Europhys. Lett. **87**, 20006 (2009)

42. Chang, L., Luo, S.: Remedying the local ancilla problem with geometric discord. Phys. Rev. A **87**, 062303 (2013)
43. Xi, Z.-J., Xiong, H.-N., Li, Y.-M., Wang, X.-G.: Pairwise quantum correlations for superpositions of dicke states. Commun. Theor. Phys. **57**, 771 (2012)
44. Ma, Z.-H., Chen, Z.-H., Fanchini, F.F.: Multipartite quantum correlations in open quantum systems. New J. Phys. **15**, 043023 (2013)
45. Okrasa, M., Walczak, Z.: Quantum discord and multipartite correlations. Europhys. Lett. **96**, 60003 (2011)
46. Chakrabarty, I., Agrawal, P., Pati, A.K.: Quantum dissension: generalizing quantum discord for three-qubit states. Eur. Phys. J. D **65**, 605 (2011)
47. Rulli, C.C., Sarandy, M.S.: Global quantum discord in multipartite systems. Phys. Rev. A **84**, 042109 (2011)
48. Lipkin, H.J., Meshkov, N., Glick, A.J. : Validity of many-body approximation methods for a solvable model. (I). Exact solutions and perturbation theory. Nucl. Phys. **62** 188 (1965)