Quantum Information Processing (2018) 17:249
https://doi.org/10.1007/s11128-018-2016-5

Analysing nonlocality robustness in multiqubit systems under noisy conditions and weak measurements

Parvinder Singh[1] · **Atul Kumar**[1]

Received: 13 February 2018 / Accepted: 2 August 2018 / Published online: 13 August 2018

Abstract
We analyse robustness of nonlocal correlation in multiqubit entangled states—three- and four-qubit GHZ class and three-qubit W class—useful for quantum information and computation, under noisy conditions and weak measurements. For this, we use a Bell-type inequality whose violation is considered as a signature for confirming the presence of genuine nonlocal correlations between the qubits. In order to demonstrate the effects of noise and weak measurements, an analytical relation is established between the maximum expectation value of three and four-qubit Svetlichny operators for the systems under study, noise parameter and strengths of weak measurements. Our results show that for a set of three- and four-qubit GHZ class states, maximal nonlocality does not coincide with maximum entanglement for a given noise parameter and a certain range of weak measurement parameter. Our analysis further shows an excellent agreement between the analytical and numerical results.

Keywords Robust nonlocal correlations · GHZ and W class of states · Weak measurement

1 Introduction

The nonlocal correlations existing between particles quantify the fundamental differences between quantum and classical systems [1–5]. Due to the advantages offered by nonlocal correlations, theoretical as well as experimental characterization of nonlocality has been at the centre of research in foundations of quantum mechanics and quantum information [6–24]. Such correlations have also taken the centre stage for many efficient and potential applications in quantum information and computation [25–29]. Therefore, the analysis of nonlocality not only satisfies the fundamental quest to verify the foundations of quantum mechanics, but it also leads to secure and

✉ Atul Kumar
atulk@iitj.ac.in

1 Indian Institute of Technology Jodhpur, Rajasthan 342037, India

optimal quantum information and communication protocols. Initially, the description of quantum correlations for bipartite and multiqubit systems was mainly associated to entanglement and nonlocality. However, with the advent of discord, one started raising questions of usefulness of bipartite separable states as well for quantum information processing [30–33]. Moreover, for multiqubit systems, the characterization of nonlocality is much more complex due to the increased complexity of the system [34–49]. For example, in case of three-qubit systems one needs to distinguish between bi-separable versus genuine tripartite nonlocality, and within the class of genuinely entangled three-qubit states, one needs a way to identify all the entangled states exhibiting genuine quantum correlations [47–51]. In order to confirm the presence of genuine long-range quantum correlations between three and four qubits, one can use the Svetlichny inequality whose violation is a signature of genuine three or four-qubit correlations [34].

In general, entangled resources violating three- or four-qubit Bell-type inequalities are considered to be useful resources for quantum information and computation. These resources, however, suffer from decoherence under real experimental set-ups, and such degradation of nonlocal correlations may lead to nonviolation of Bell-type inequalities [52–61]- questioning their usefulness for quantum information and computation. Moreover, Acin et al. [62] have shown that robustness of nonlocality against decoherence, though relevant, is not a genuine measure of nonlocality. Contrary to the general belief that the maximally entangled states would give rise to maximum nonlocality, it was further analysed that nonmaximally entangled states may be more nonlocal than the maximally entangled states for different nonlocality measures [62,63]. Clearly, this anomaly suggests that entanglement and nonlocality can be treated as distinct resources. This fact of non-coincidence of maximal nonlocality and entanglement has received a great deal of attention [64–71]. On the other hand, considering the numerical value of a Bell-type function as a witness rather than a quantifier of nonlocality, Fonseca et al. [64] have analysed a measure of nonlocality and found no discrepancy between states with maximal entangled and maximal nonlocality, at least for a pair of entangled qutrits and for entangled systems comprising of two four-level subsystems. More recently, Rosier et al. [71] have presented a numerical analysis based on linear programming to analyse violations of local realism by different classes of multipartite states, and shown that the probability of violation of local realism is a witness of genuine multipartite entanglement. Furthermore, a lot of studies have also been devoted to analyse nonlocality robustness in different multiqubit entangled classes [72–75]. For the analysis presented in this article, the distinction of nonlocality robustness as against a genuine nonlocality quantifier is valid throughout the article.

Another important aspect of multiqubit entanglement and nonlocality is to protect entanglement and nonlocality from noise by defining mechanisms to optimize entanglement and nonlocal correlations in the presence of noise. For this, several decoherence models have been proposed and studied [76–86]. Recently, one of the approaches to protect entanglement and nonlocality is developed in terms of weak measurements, i.e., partial collapse measurement operators [87–94]. The basic concept behind the working principle of weak measurement and its reversal lies in the factual possibility of reversing any partial collapse measurement. The process of performing weak measurements and its reversal on individual qubits has been found to be a very useful technique to protect and enhance correlations under noisy condition [95–102]. Moreover, the strategies to use weak measurement and its reversal operations have been experimentally demonstrated in many quantum systems [88–91,103–105]. Hence, the analysis of robustness of nonlocal properties in multiqubit systems under real conditions is very important to understand the complex nature of multiqubit entanglement and nonlocality, and to identify the set of states relevant for quantum information processing.

In this article, we readdress the question of three and four-qubit nonlocality robustness under real experimental or noisy conditions. For this, we consider different classes of three-qubit entangled systems which are shown to be useful for quantum information and computation, e.g., Greenberger–Horne–Zeilinger (GHZ) and W class states [106]. The analysis of robustness of nonlocal correlations in these systems under real conditions allows us to establish an analytical relation between the maximum expectation value of the Svetlichny operator for a given system, noise parameter, and strengths of weak measurement and its reversal operations. As an example of noisy channel, we use the interaction between the principal system and the environment through an amplitude damping channel. The analytical results obtained in this article are in complete agreement with the numerical results as well. Interestingly, for generalized GHZ class, our results indicate that for certain values of weak measurement strengths and range of τ of the initially prepared state, the violation of Svetlichny inequality is more if one starts with a nonmaximally entangled state instead of a maximally entangled GHZ state, i.e., nonmaximally entangled states are more robust against noise in comparison with maximally entangled states. Apart from GHZ and W class of states, we also characterize nonlocal properties in W_n type of states [107–109], considering its importance in quantum information processing. In addition, we further study nonlocality robustness in four-qubit GHZ class states by establishing an analytical relation between the maximum expectation value of the four-qubit Svetlichny operator, noise parameter and strengths of weak measurement and its reversal operations. We believe that the results obtained in this article will be of significant importance since the states considered here for the analysis of nonlocal correlations are experimentally accessible [40,41,110–114].

2 Three-qubit GHZ and W states

Three-qubit states can be classified into two different inequivalent classes, i.e., GHZ class and W class [106]. The states of both the classes are shown to be useful for quantum information and computation. The degree of entanglement in the GHZ class is quantified in terms of residual entanglement, i.e., three-tangle τ [115] given by

$$\tau = C^2_{i(jk)} - C^2_{ij} - C^2_{ik} \tag{1}$$

where C_{ij} represents concurrence and quantifies the bipartite entanglement between qubits i and j, and $C_{i(jk)}$ quantifies the entanglement between qubit i and the joint state of qubits j and k [116]. On the other hand, the three-tangle τ fails to capture the genuine entanglement in W class states as the states in W class satisfy

$$C_{i(jk)}^2 = C_{ij}^2 + C_{ik}^2 \tag{2}$$

Alternately, one can use σ [117] or sum of concurrences of the three reduced bipartite density operators obtained from a W class of state as an entanglement monotone for W class states [106]. In this article, we use the following two GHZ class states, generalized GHZ states

$$\left|\Psi_g\right\rangle = \cos\theta\,|000\rangle + \sin\theta\,|111\rangle \tag{3}$$

and Slice states [118]

$$|\Psi_{ms}\rangle = \frac{1}{\sqrt{2}}\left[|000\rangle + |11\rangle\left\{\cos\theta_3\,|0\rangle + \sin\theta_3\,|1\rangle\right\}\right] \tag{4}$$

Here, θ and θ_3 are state parameters. The maximally entangled GHZ state for ($\theta = \pi/4, \theta_3 = \pi/2$), has been used for deterministic transfer of information in many theoretical protocols [119–124]. On the similar lines, we consider to use two different W class states as well, namely

$$|\Psi_W\rangle = x\,|001\rangle + y\,|010\rangle + z\,|100\rangle \tag{5}$$

where x, y, and z are real, and

$$\left|\Psi_{W_n}\right\rangle = \frac{1}{\sqrt{2+2n}}\left[|100\rangle + \sqrt{n}e^{i\delta}\,|010\rangle + \sqrt{n+1}e^{i\zeta}\,|001\rangle\right] \tag{6}$$

where n is a positive integer and δ and ζ are relative phases. Unlike the maximally entangled GHZ states, the standard W states cannot be used for deterministic information transfer [124,125]. On the other hand, $\left|\Psi_{W_n}\right\rangle$ states [107] can be used as resources for deterministic teleportation and dense coding. The price one needs to pay for the deterministic information transfer using $\left|\Psi_{W_n}\right\rangle$ is in terms of joint three-qubit measurements. The use of standard single-qubit and two-qubit measurements instead of three-qubit joint measurements leads to significant reduction in the efficiency of $\left|\Psi_{W_n}\right\rangle$ states [108]. The special class of W states has been subsequently generalized for a case of N qubits [109]. Considering the importance of these states for quantum information, it is imperative to characterize nonlocal properties in these states. Such study will certainly provide an idea regarding the usefulness of these resources in real conditions.

3 Robustness of nonlocality in the generalized GHZ class under noisy conditions

In order to characterize the genuine tripartite nonlocality, we use the Svetlichny inequality (SI) [34], S_v, such that

$$S_v(\rho) \equiv |\langle\psi|\,S_v\,|\psi\rangle| \leq 4 \tag{7}$$

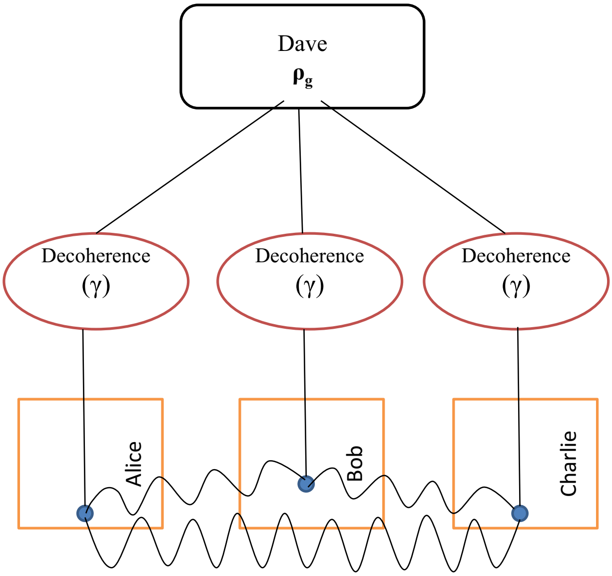

Fig. 1 A scenario to analyse the effect of decoherence on maximum expectation value of the Svetlichny operator

where the Svetlichny operator S_v is given by

$$S_v = A\left(BC + BC' + B'C - B'C'\right) + A'\left(BC - BC' - B'C - B'C'\right) \tag{8}$$

and measurements $A = \vec{a}.\vec{\sigma}_1$, and $A' = \vec{a}'.\vec{\sigma}_1$ are performed on the first qubit. Here $\vec{a}$, and $\vec{a}'$ are unit vector, and $\vec{\sigma}_i$'s are spin projection operators. The measurements B or B', and C or C' are defined in a similar fashion and are performed on qubits 2 and 3, respectively. The above inequality is satisfied by all the separable and bi-separable states, and hence, the violation of Svetlichny inequality confirms the presence of genuine tripartite nonlocality in the underlying system.

We now proceed to investigate the effect of decoherence on the violation of Svetlichny inequality for three-qubit GHZ states by establishing an analytical relation between the maximum expectation value of the Svetlichny operator, and noise parameter. For this, we consider a scenario where Dave prepares a three-qubit pure GHZ state $\left|\Psi_g\right\rangle = \cos\theta\,|000\rangle + \sin\theta\,|111\rangle$ and sends one qubit each to Alice, Bob and Charlie through amplitude damping channels (Fig. 1). For the mathematical convenience and simplicity, we consider identical decoherence parameter for all the three channels.

3.1 Amplitude damping channel

The single-qubit amplitude damping channels can be described by the following Kraus operators [126] ,

$$E_0 = \begin{pmatrix} 1 & 0 \\ 0 & \sqrt{1-\gamma} \end{pmatrix}, \quad E_1 = \begin{pmatrix} 0 & \sqrt{\gamma} \\ 0 & 0 \end{pmatrix} \tag{9}$$

where γ represents the magnitude of noise parameter. The three-qubit state of the quantum system after an amplitude damping noise is given by

$$\rho^{\gamma} = \sum_{k,l,m} (E_k \otimes E_l \otimes E_m)\, \rho \left(E_k^{\dagger} \otimes E_l^{\dagger} \otimes E_m^{\dagger}\right) \tag{10}$$

where $k, l, m = (0, 1)$.

Considering that the shared three-qubit state evolves as ρ_g^{γ}, the maximum expectation value for the Svetlichny operator defined in Eq. (8) can be obtained by defining

$$\begin{aligned} a &= (\sin\theta_a \cos\phi_a, \sin\theta_a \sin\phi_a, \cos\theta_a) \\ b &= (\sin\theta_b \cos\phi_b, \sin\theta_b \sin\phi_b, \cos\theta_b) \\ c &= (\sin\theta_c \cos\phi_c, \sin\theta_c \sin\phi_c, \cos\theta_c) \end{aligned} \tag{11}$$

where a', b' and c' can be defined in similar fashion with primes on required angles. Moreover, the expression for S_v can be further simplified by defining a pair of mutually orthogonal unit vectors $R = \vec{r} \cdot \vec{\sigma_2}$ and $R' = \vec{r'} \cdot \vec{\sigma_2}$ such that $\vec{b} + \vec{b'} = 2\cos\chi \cdot \vec{r}$, and $\vec{b} - \vec{b'} = 2\sin\chi \cdot \vec{r'}$, which leads to

$$\vec{r} \cdot \vec{r'} = \cos\theta_r \cos\theta_{r'} + \sin\theta_r \sin\theta_{r'} \cos(\phi_r - \phi_{r'}) = 0 \tag{12}$$

Therefore, Eq. (8) can be re-expressed as

$$\begin{aligned} S_v = 2| &\langle ARC\rangle \cos\chi + \langle AR'C'\rangle \sin\chi \\ &+ \langle A'R'C\rangle \sin\chi - \langle A'RC'\rangle \cos\chi | \end{aligned} \tag{13}$$

Equation (13) when maximized with respect to χ gives

$$\begin{aligned} S_v &\leq 2|\sqrt{\langle ARC\rangle^2 + \langle AR'C'\rangle^2} \\ &\quad + \sqrt{\langle A'R'C\rangle^2 + \langle A'RC'\rangle^2}| \\ &= M + M' \end{aligned} \tag{14}$$

where M and M' are Mermin's operators [127] . Here, we have used the fact that

$$u\cos\theta_1 + v\sin\theta_1 \leq (u^2 + v^2)^{\frac{1}{2}} \tag{15}$$

with the equality resulting when $\tan\theta_1 = \frac{v}{u}$. For evaluating the maximum expectation value of $S_v(\rho_g^{\gamma})$, we first consider calculating $\langle ARC\rangle$ corresponding to the first term in Eq. (14), such that

$$\langle ARC\rangle_{\rho_g^\gamma} = \Big[\Big(\cos^2\theta + (2\gamma-1)^3\sin^2\theta\Big)\cos\theta_a\cos\theta_r\cos\theta_c + \sin 2\theta\,(1-\gamma)^{\frac{3}{2}}\sin\theta_a\sin\theta_r\sin\theta_c\cos(\phi_{arc})\Big] \quad (16)$$

The expectation value $\langle ARC\rangle_{\rho_g^\gamma}$ can be maximized with respect to θ_c, i.e.,

$$\langle ARC\rangle_{\max} = \Big[\Big(\cos^2\theta + (2\gamma-1)^3\sin^2\theta\Big)^2\cos^2\theta_a\cos^2\theta_r + \sin^2 2\theta(1-\gamma)^3\sin^2\theta_a\sin^2\theta_r\Big]^{\frac{1}{2}} \quad (17)$$

where $\cos^2(\phi_{arc}) = \cos^2(\phi_a+\phi_r+\phi_c) = 1$. Similarly $\langle AR'C'\rangle_{max}$ can be given as

$$\langle AR'C'\rangle_{\max} = \Big[\Big(\cos^2\theta\ + (2\gamma-1)^3\sin^2\theta\Big)^2\cos^2\theta_a\cos^2\theta_{r'} + \sin^2 2\theta(1-\gamma)^3\sin^2\theta_a\sin^2\theta_{r'}\Big]^{\frac{1}{2}} \quad (18)$$

The maximum values of the operators $\langle A'R'C\rangle$ and $\langle A'RC'\rangle$ can also be defined in a similar way with primes on required angles. Therefore, from Eq. (14), we have

$$\begin{aligned} S_v\left(\rho_g^\gamma\right)_{\max} \le 2\Big[&\Big(\cos^2\theta + (2\gamma-1)^3\sin^2\theta\Big)^2\cos^2\theta_a\Big(\cos^2\theta_r + \cos^2\theta_{r'}\Big) + \sin^2 2\theta(1-\gamma)^3\sin^2\theta_a\Big(\sin^2\theta_r+\sin^2\theta_{r'}\Big)\Big]^{\frac{1}{2}} \\ +2\Big[&\Big(\cos^2\theta\ + (2\gamma-1)^3\sin^2\theta\Big)^2\cos^2\theta_{a'}\Big(\cos^2\theta_r + \cos^2\theta_{r'}\Big) + \sin^2 2\theta(1-\gamma)^3\sin^2\theta_{a'}\Big(\sin^2\theta_r+\sin^2\theta_{r'}\Big)\Big]^{\frac{1}{2}} \end{aligned} \quad (19)$$

In order to optimize the expectation value of the Svetlichny operator, we use the fact that the maximum of $\cos^2\theta_r + \cos^2\theta_{r'}$ is 1, while the maximum of $\sin^2\theta_r + \sin^2\theta_{r'}$ is 2. Further, we know that

$$u\cos^2\theta_1 + v\sin^2\theta_1 \le \begin{cases} u, & u \ge v \\ v, & u \le v \end{cases} \quad (20)$$

where the first inequality is realized when $\theta_1 = 0$ or π and the second inequality is realized when $\theta_1 = \frac{\pi}{2}$, and hence Eq. (19) can be rewritten as

$$\begin{aligned} S_v\left(\rho_g^\gamma\right)_{\text{opt}} \le\ & 2\Big[\left(\cos^2\theta\ + (2\gamma-1)^3\sin^2\theta\right)^2\cos^2\theta_a + 2\sin^2 2\theta(1-\gamma)^3\sin^2\theta_a\Big]^{\frac{1}{2}} \\ &+2\Big[\left(\cos^2\theta\ + (2\gamma-1)^3\sin^2\theta\right)^2\cos^2\theta_{a'} + 2\sin^2 2\theta(1-\gamma)^3\sin^2\theta_{a'}\Big]^{\frac{1}{2}} \end{aligned}$$

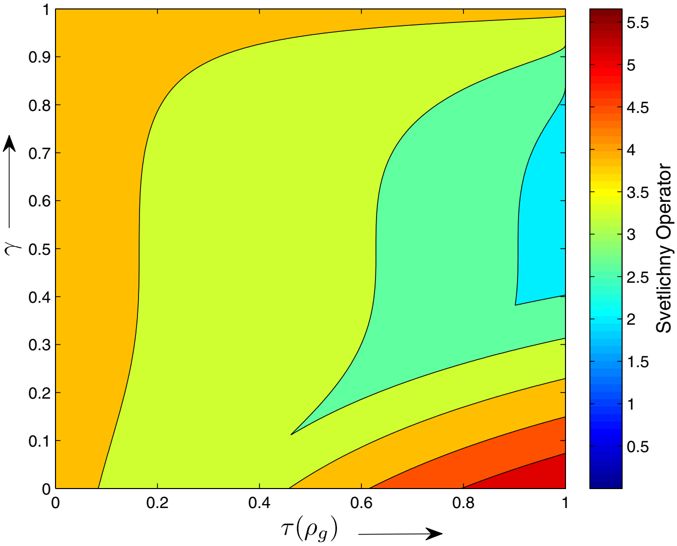

Fig. 2 Maximum expectation value of $S_v(\rho_g^\gamma)_{\text{opt}}$ with respect to three-tangle ($\tau(\rho_g)$) of the initial three-qubit GHZ state and decoherence parameter γ

$$= \begin{cases} 4\left(\cos^2\theta\ + (2\gamma-1)^3\sin^2\theta\right), & \left(\cos^2\theta\ + (2\gamma-1)^3\sin^2\theta\right)^2 \geq 2\sin^2 2\theta(1-\gamma)^3 \\ 4\sqrt{2(1-\gamma)^3\sin^2 2\theta}, & \left(\cos^2\theta\ + (2\gamma-1)^3\sin^2\theta\right)^2 \leq 2\sin^2 2\theta(1-\gamma)^3 \end{cases} \tag{21}$$

In case there are no environmental interactions, i.e., Dave sends all the three qubits through perfect channels such that $\gamma = 0$, then using $\tau(\Psi_g) = \sin^2 2\theta$, the maximum expectation value of Svetlichny operator is

$$S_v\left(\Psi_g\right)_{\text{opt}} \leq \begin{cases} 4\sqrt{1-\tau(\Psi_g)}, & \tau(\Psi_g) \leq \frac{1}{3} \\ 4\sqrt{2\tau(\Psi_g)}, & \tau(\Psi_g) \geq \frac{1}{3} \end{cases} \tag{22}$$

The inequality expressed in Eq. (22) is same as the one given in [49] as it should be for transmission through an ideal quantum channel. The analytical result obtained here is in complete agreement with the numerical optimization of the Svetlichny operator for the generalized GHZ state in the presence of amplitude damping channels.

Figure 2 clearly describes that the violation of Svetlichny inequality decreases very fast even for small values of noise parameters. Moreover, it also depicts that for noiseless channels and $\tau > \frac{1}{2}$, finally shared state always violates the Svetlichny inequality. The range of violation, however, decreases with increase in the value of decoherence parameter, e.g, see Fig. 3. Further, in the legends of Fig. 3, A and N stand for analytical and numerical results, respectively—these abbreviations are valid for all subsequent figures in this article. We now move forward to analyse the effect of weak measurement and quantum measurement reversal operations to reduce the effect of decoherence on nonlocal correlations. For this, we start with a scenario

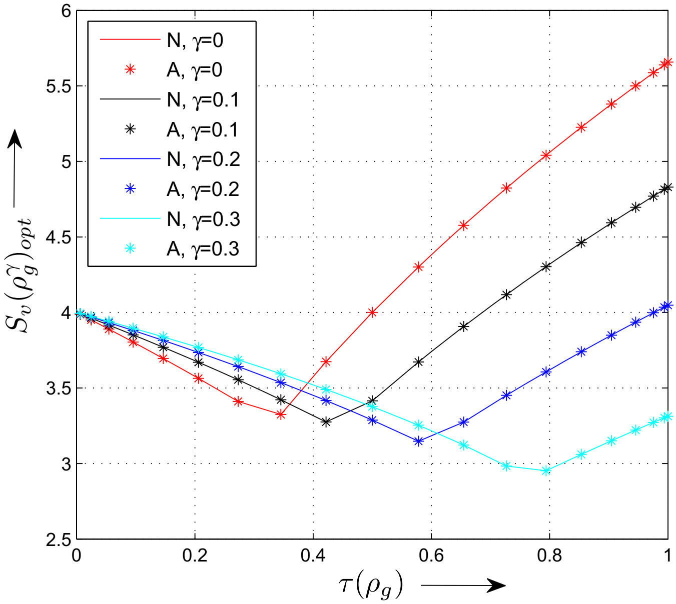

Fig. 3 Plot of $S_v(\rho_g^\gamma)_{\text{opt}}$ with respect to three-tangle ($\tau(\rho_g)$) of the initial three-qubit GHZ state for four different values of decoherence parameter: *represents analytical result (A) and solid line represents numerical optimization (N)

represented in Fig. 4 where Dave prepares a three-qubit pure generalized GHZ state $|\Psi_g\rangle = \cos\theta\,|000\rangle + \sin\theta\,|111\rangle$, and performs weak measurements on each qubit before distributing the qubits through amplitude damping channels. After receiving the qubits, Alice, Bob and Charlie perform reverse quantum weak measurements on their respective qubits. Again for the mathematical convenience and simplicity, we assume same weak measurement strengths for all the channels. The weak measurement Λ^{wk} and reverse weak measurement Λ^{wkr} operations can be described as

$$\Lambda^{wk} = \begin{pmatrix} 1 & 0 \\ 0 & \sqrt{1-\eta} \end{pmatrix}, \quad \Lambda^{wkr} = \begin{pmatrix} \sqrt{1-\eta_r} & 0 \\ 0 & 1 \end{pmatrix} \tag{23}$$

where η and η_r are the strengths of weak measurement and weak measurement reversal operations, respectively [89]. The optimal weak measurement reversal strength [87,88] is defined by $\eta_r = \eta + \gamma\,(1-\eta)$. Assuming that the strength of weak measurement reversal operation is optimal, the expectation value $\langle ARC\rangle$ with respect to the finally shared state is given as

$$\langle ARC\rangle_{\rho_g^{wk}} = \frac{1}{N}\Big[\Big(\cos^2\theta - (1+\gamma(\eta-1))^3\sin^2\theta\Big)\cos\theta_a\cos\theta_r\cos\theta_c + \sin 2\theta\cos(\phi_a+\phi_r+\phi_c)\sin\theta_a\sin\theta_r\sin\theta_c\Big] \tag{24}$$

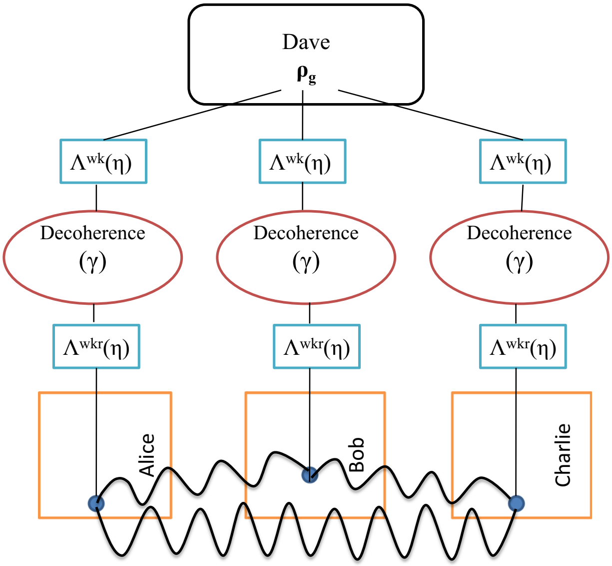

Fig. 4 A scenario to analyse the effects of decoherence and weak measurement and its reversal operations on the robustness of three-qubit nonlocal correlations

where $N = \left(\cos^2\theta + (1+\gamma(1-\eta))^3 \sin^2\theta\right)$. Similar to the discussion on amplitude damping channel, one can show that the optimum expectation value of the Svetlichny operator for the finally shared state (after implementing the protocol based on weak measurements and quantum measurement reversal operations) is given as

$$S_v\left(\rho_g^{wk}\right)_{\text{opt}} = \begin{cases} \frac{4}{N}\left(\cos^2\theta - (1+\gamma(\eta-1))^3\sin^2\theta\right), & \left(\cos^2\theta - (1+\gamma(\eta-1))^3\sin^2\theta\right)^2 \geq 2\sin^2 2\theta \\ \frac{4}{N}\sqrt{2\sin^2 2\theta}, & \left(\cos^2\theta - (1+\gamma(\eta-1))^3\sin^2\theta\right)^2 \leq 2\sin^2 2\theta \end{cases} \tag{25}$$

For $\eta = 1$, the expression in Eq. (25) will again be the same as for a pure three-qubit GHZ state [49]. From Figs. 2 and 3, in the absence of weak measurement, the Svetlichny inequality will not be violated for $\gamma \geq 0.3$ even if the initial shared state is a maximally entangled GHZ state. The effect of weak measurement strengths on the maximum expectation value of Svetlichny operator for a decoherence value of $\gamma = 0.5$ is depicted in Figs. 5 and 6. Clearly, for a given decoherence parameter the violation of Svetlichny inequality increases with the increase in weak measurement strength, i.e., robustness of nonlocal correlations against noise increases with the increase in weak measurement strength. Interestingly, Fig. 6 shows that for certain values of weak measurement strengths (except $\eta = 1$) and range of τ of the initially prepared state, the violation of Svetlichny inequality is more if one starts with nonmaximally entangled pure states instead of a maximally entangle GHZ state, suggesting that nonmaximally entangled states are more robust against the noise in comparison to maximally entangled states under the application of weak measurements. For example,

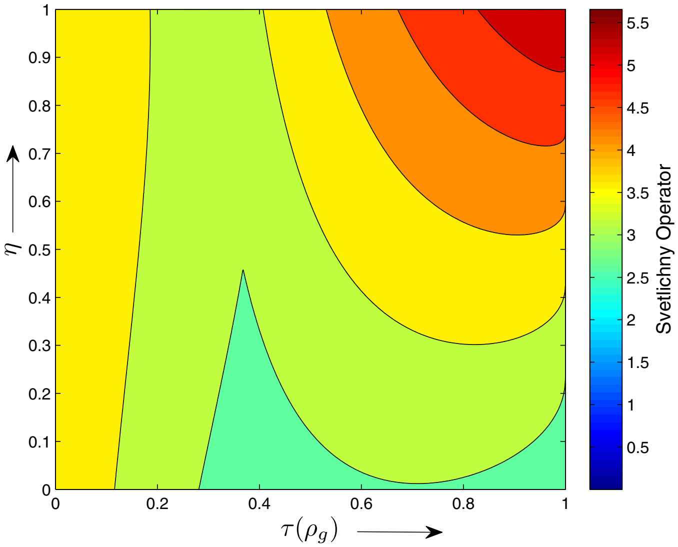

Fig. 5 Maximum expectation value of $S_v(\rho_g^{wk})_{\text{opt}}$ versus three-tangle $(\tau(\rho_g))$ of the initial three-qubit GHZ state and weak measurement strength parameter η, considering $\gamma = 0.5$

if the value of weak measurement parameter is 0.7 then the maximal nonlocality does not coincide with maximum entanglement, further confirming the existence of nonlocality anomalies in multiqubit entangled states.

For a maximally entangled initial GHZ state, Fig. 7 describes the effects of noise parameter and weak measurement strength on the maximum expectation value of Svetlichny operator. Clearly, the use of weak measurement and quantum measurement reversal operations is a win–win situation for increasing the robustness of tripartite nonlocality against noise.

4 Analysing the robustness of nonlocality in the generalized GHZ class

In the previous section, we discussed nonlocality robustness in the generalized GHZ class states under noisy conditions. Here, we analyse the effects of amplitude damping and weak measurement strength on the following class of states,

$$\left|\Psi_{gs}\right\rangle = \cos\theta\,|000\rangle + \sin\theta\,|11\rangle\,[\cos\theta_3\,|0\rangle + \sin\theta_3\,|1\rangle] \tag{26}$$

where θ and θ_3 are state parameters. For $\theta_3 = \pi/2$ and $\theta = \pi/4$, the set of states in Eq. (26) correspond to the set of states in Eqs. (3) and (4), respectively. In this case, we assume that Charlie prepares a three-qubit state as defined in Eq. (26) and sends qubit 1 to Alice and qubit 2 to Bob. Before distributing the qubits through amplitude damping channels, Charlie performs weak measurements on qubits 1 and 2. Similarly, Alice and Bob also perform reverse quantum measurements on their respective qubits

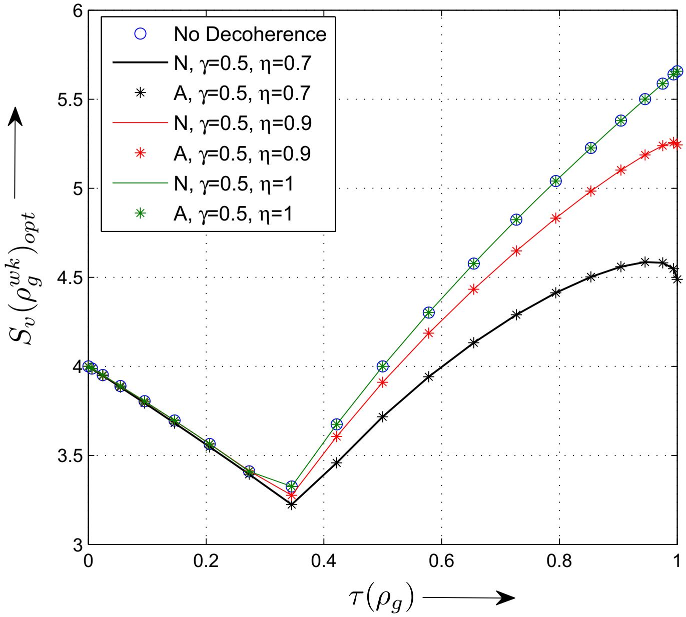

Fig. 6 Plot of $S_v(\rho_g^{wk})_{\rm opt}$ versus three-tangle (τ) of the initial three-qubit GHZ state for different weak measurement strengths, considering $\gamma = 0.5$

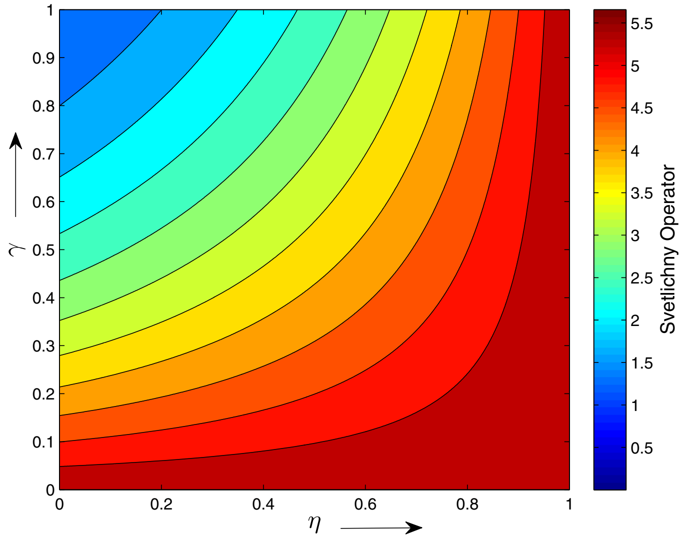

Fig. 7 Effect of weak measurement on the maximum expectation value of the three-qubit Svetlichny operator as a function of noise parameter γ, for a maximally entangled input state

once they receive it from Charlie. In order to find the maximum expectation value of the Svetlichny operator in the evolved three-qubit mixed state ρ_{gs}^{wk}, the first term $\langle ARC\rangle$ in Eq. (14) can be expressed as

$$\begin{aligned}\langle ARC\rangle_{\rho_{gs}^{wk}} = \frac{1}{N'}\,[(\alpha\cos\theta_c + \beta\cos\phi_c\sin\theta_c)\cos\theta_a\cos\theta_r \\ + \sin 2\theta\,(\cos\theta_3\cos\theta_c\cos\phi_{ar} \\ + \sin\theta_3\cos\phi_{arc}\sin\theta_c)\sin\theta_a\sin\theta_r]\end{aligned} \tag{27}$$

where $N' = \left[\left(\cos^2\theta + (1+\gamma(1-\eta))^2\sin^2\theta\right)\right]$, $\alpha = \left[\cos^2\theta + (1+\gamma(\eta-1))^2\sin^2\theta\cos 2\theta_3\right]$, $\beta = \left[(1+\gamma(\eta-1))^2\sin^2\theta\sin 2\theta_3\right]$, $\cos\phi_{arc} = \cos(\phi_a+\phi_r+\phi_c)$ and $\cos\phi_{ar} = \cos(\phi_a+\phi_r)$. Equation (27) can be further maximized using Eq. (12) with respect to $(\phi_r - \phi_{r'})$ by considering $\theta_{r'}$, ϕ_r, and $(\phi_r - \phi_{r'})$ to be independent variables. Thus, one can easily deduce that $(\phi_r - \phi_{r'}) = 0$ and $\theta_r = \frac{\pi}{2}$. The sequential optimization of the Mermin operator in Eq. (14) is summarized below as [128]

$$\begin{aligned}M\left(\rho_{gs}^{wk}\right) &= 2\sqrt{\langle ARC\rangle^2 + \langle AR'C'\rangle^2} \\ &\le \frac{2}{N'}\Big[\sin^2\theta_a\sin^2 2\theta\Big\{(\cos\theta_3\cos\phi_{ar}\cos\theta_c + \sin\theta_3\cos\phi_{arc}\sin\theta_c)^2 \\ &\quad + (\cos\theta_3\sin\phi_{ar}\cos\theta_{c'} + \sin\theta_3\sin\phi_{arc'}\sin\theta_{c'})^2\Big\} \\ &\quad + \cos^2\theta_a\,(\alpha\cos\theta_{c'} + \beta\cos\phi_{c'}\sin\theta_{c'})^2\Big]^{\frac{1}{2}}\end{aligned} \tag{28}$$

$$\le \begin{cases}\frac{2}{N'}\sin 2\theta\left[(\cos\theta_3\cos\phi_{ar}\cos\theta_c + \sin\theta_3\cos\phi_{arc}\sin\theta_c)^2 \right. \\ \qquad\left. + (\cos\theta_3\sin\phi_{ar}\cos\theta_{c'} + \sin\theta_3\sin\phi_{arc'}\sin\theta_{c'})^2\right]^{\frac{1}{2}} \\ \frac{2}{N'}(\alpha\cos\theta_{c'} + \beta\cos\phi_{c'}\sin\theta_{c'})\end{cases} \tag{29}$$

$$\le \begin{cases}\frac{2}{N'}\sin 2\theta\left[\left(\cos^2\theta_3\cos^2\phi_{ar} + \sin^2\theta_3\cos^2\phi_{arc}\right)\right. \\ \qquad\left. + \left(\cos^2\theta_3\sin^2\phi_{ar} + \sin^2\theta_3\sin^2\phi_{arc'}\right)\right]^{\frac{1}{2}} \\ \frac{2}{N'}\left(\alpha^2 + \beta^2\cos^2\phi_{c'}\right)^{\frac{1}{2}}\end{cases} \tag{30}$$

$$\le \begin{cases}\frac{2}{N'}\sin 2\theta\sqrt{1+\sin^2\theta_3} \\ \frac{2}{N'}\sqrt{\alpha^2+\beta^2} = \frac{2}{N'}\Big[\left(\cos^2\theta + (1+\gamma(\eta-1))^2\sin^2\theta\right)^2 \\ \qquad - (1+\gamma(\eta-1))^2\sin^2 2\theta\sin^2\theta_3\Big]^{\frac{1}{2}}\end{cases} \tag{31}$$

where we have used inequalities (15) and (20). In the above equations, the maximization is performed with respect to $\theta_{r'}$ in Eq. (28), θ_a in Eq. (29), and θ_c and $\theta_{c'}$ in Eq. (30). Furthermore, in Eq. (31) we assumed that $\phi_{ar} = 0$, $\phi_{arc} = 0$, and $\phi_{arc'} = \frac{\pi}{2}$. Similarly, the optimized value for the operator M' turns out to be the same as in Eq. (31) by considering $A \to A'$ and $C \to C'$. Moreover, three-tangle and residual concurrence for the states $|\Psi_{gs}\rangle$ are given by $\tau\left(|\Psi_{gs}\rangle\right) = \sin^2 2\theta\sin^2\theta_3$

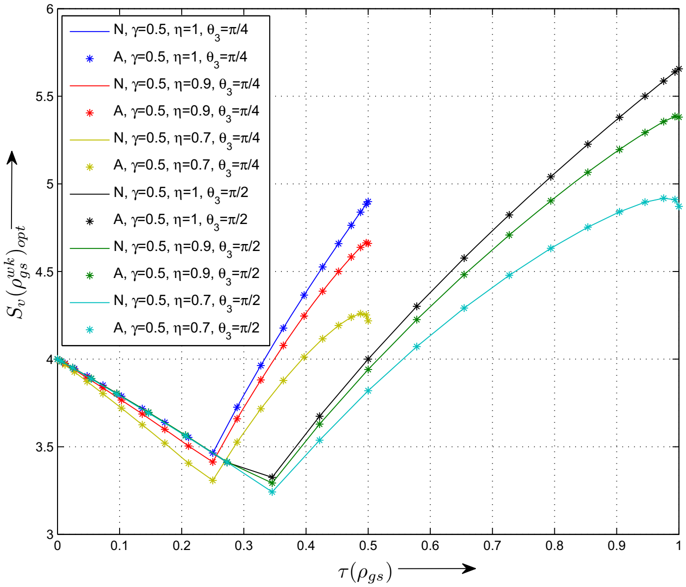

Fig. 8 Maximum expectation value of $S_v(\rho_{gs}^{wk})_{\text{opt}}$ with respect to three-tangle $(\tau(\rho_{gs}))$ of the initial state given in Eq. (26) for different weak measurement strengths, considering $\gamma = 0.5$, for two different values of $\theta_3 = (\frac{\pi}{4}, \frac{\pi}{2})$

and $C_{12}^2\left(\left|\Psi_{gs}\right\rangle\right) = \sin^2 2\theta \cos^2\theta_3$, respectively. Therefore, using these expressions for three-tangle and residual concurrence of the input state, the optimum value of Svetlichny operator can be expressed as

$$S_v\left(\rho_{gs}^{wk}\right)_{\text{opt}} \leq \begin{cases} \frac{4}{N'} \sin 2\theta\sqrt{1+\sin^2\theta_3} \\ \frac{4}{N'}\Big[\left(\cos^2\theta + (1+\gamma(\eta-1))^2\sin^2\theta\right)^2 \\ \qquad -(1+\gamma(\eta-1))^2 \sin^2 2\theta \sin^2\theta_3\Big]^{\frac{1}{2}} \end{cases} \tag{32}$$

$$\leq \begin{cases} \frac{4}{N'}\sqrt{2\tau + C_{12}^2}, \\ \qquad \left[2+(1+\gamma(\eta-1))^2\right]\tau + C_{12}^2 \geq \left[\cos^2\theta + (1+\gamma(\eta-1))^2\sin^2\theta\right]^2 \\ \frac{4}{N'}\left[\left(\cos^2\theta + (1+\gamma(\eta-1))^2\sin^2\theta\right)^2 - (1+\gamma(\eta-1))^2\tau\right]^{\frac{1}{2}}, \\ \qquad \left[2+(1+\gamma(\eta-1))^2\right]\tau + C_{12}^2 \leq \left[\cos^2\theta + (1+\gamma(\eta-1))^2\sin^2\theta\right]^2 \end{cases} \tag{33}$$

Clearly, for perfect channels, i.e., for $\gamma = 0$, the maximum expectation value of Svetlichny operator is same as given in [128]. Similarly, for strong weak measurement strength, i.e., $\eta = 1$, the effect of decoherence fully vanishes and the optimum value of Svetlichny operator is again the same as given in [128].

Figure 8 shows the relationship between $S_v\left(\rho_{gs}^{wk}\right)_{\text{opt}}$ and three-tangle (τ) of the input state, for different values of weak measurement strength considering noise parameter $\gamma = 0.5$, for two different sets of GHZ states, i.e., for $\theta_3 = \frac{\pi}{2}$, and $\theta_3 = \frac{\pi}{4}$.

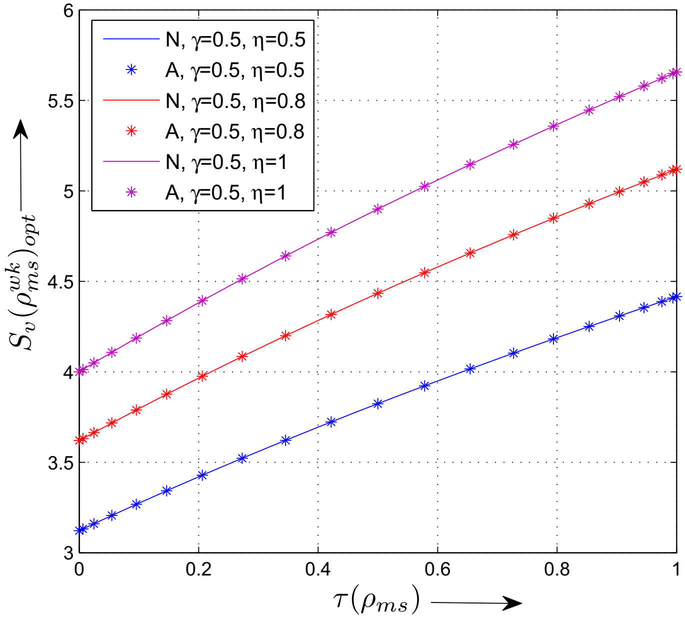

Fig. 9 Plot of $S_v(\rho_{gs}^{wk})_{\text{opt}}$ with respect to three-tangle ($\tau(\rho_{gs})$) of the initial state given in Eq. (26) for different weak measurement strengths, considering $\gamma = 0.5$ and $\theta = \frac{\pi}{4}$

Similar to Fig. 6, one can clearly observe nonlocality anomalies for $\eta = 0.7$, considering $\theta_3 = \frac{\pi}{2}$, and $\theta_3 = \frac{\pi}{4}$. Moreover in Fig. 9, we show the relation between $S_v\left(\rho_{gs}^{wk}\right)_{\text{opt}}$ and three-tangle (τ) of the initial input state, i.e., MS state ($\theta = \frac{\pi}{4}$), for different values of weak measurement strength considering noise parameter $\gamma = 0.5$. Evidently, for $\eta = 1$ the effect of decoherence is fully suppressed as the finally shared state always violates the Svetlichny inequality; and for lower value of weak measurement strength, finally shared state still violates the Svetlichny inequality for a considerable range of three-tangle of initial input state. Surprisingly for MS states, our analysis does not find any discrepancy between maximally entangled and maximally nonlocal states. The results obtained in this section further confirm the importance of weak measurements for improving three-qubit nonlocality robustness in the generalized GHZ class in the presence of noise.

5 Analysis of robustness of nonlocal correlations in the W class and W_n-type states

We now proceed to analyse another important class of three-qubit states, i.e., W states as represented in Eq. (5). Ajoy and Rungta [128] have shown that the Svetlichny inequality is more suitable to identify the tripartite nonlocality in W class of states—the inequality, though, is violated only when the sum of concurrences of three bipartite reduced states exceeds a certain threshold.

In order to analyse nonlocality robustness of W class of states in a similar communication scenario as described in the previous section, we first calculate the expectation value of the first term, i.e., $\langle ABC\rangle$ in Eq. (8) in the evolved three-qubit state ρ_W^{wk} such that

$$\begin{aligned}\langle ABC\rangle_{\rho_W^{wk}} &= \cos\theta_b\,(C_{31}\sin\theta_a\sin\theta_c\cos\phi_{ac} - \Delta\cos\theta_a\cos\theta_c)\\ &\quad + \sin\theta_b\,(C_{12}\cos\theta_a\sin\theta_c\cos\phi_{bc} + C_{23}\sin\theta_a\cos\theta_c\cos\phi_{ab})\end{aligned} \tag{34}$$

where $\Delta = \frac{(x^2+(1+\gamma(\eta-1))(y^2+z^2))}{(x^2+(1+\gamma(1-\eta))(y^2+z^2))}$, $\cos\phi_{ab} = \cos(\phi_a - \phi_b)$, $C_{12} = \frac{2xy}{(x^2+(1+\gamma(1-\eta))(y^2+z^2))}$, $C_{23} = \frac{2yz}{(x^2+(1+\gamma(1-\eta))(y^2+z^2))}$, and $C_{31} = \frac{2xz}{(x^2+(1+\gamma(1-\eta))(y^2+z^2))}$. The details of further calculation for maximizing the expectation value of Svetlichny operator are given in "Appendix A". Therefore, the optimum expectation value of Svetlichny operator for W class of states, as represented in Eq. (A.3), can be give as

$$S_v\left(\rho_W^{wk}\right)_{\text{opt}} \equiv 4\,(s_1\Delta + s_2C_{12} + s_3C_{23} + s_4C_{31}) \tag{35}$$

Following Eq. (35), the finally shared tripartite entangled states will violate the Svetlichny inequality iff $(s_1\Delta + s_2C_{12} + s_3C_{23} + s_4C_{31}) > 1$. Clearly, for weak measurement strength $\eta = 1$, the effect of decoherence fully vanishes, and a general tripartite entangled W state violates the SI when $\left(s_1 + s_2C'_{12} + s_3C'_{23} + s_4C'_{31}\right) > 1$, where $C'_{12} = 2xy$, $C'_{23} = 2yz$, and $C'_{31} = 2xz$ are the concurrences of the three reduced states of input state $|\Psi_W\rangle$. Moreover, the optimum value of Svetlichny operator is 4.354 and occurs when $\eta = 1$, $C'_{12} = C'_{23} = C'_{31} = 2/3$, and $\theta'_a = \theta'_b = \theta'_c = 54.736°$. The violation of Svetlichny operator with respect to the varying sum of the concurrences of the three reduced states of ρ_W^{wk} is depicted in Fig. 10. Here, we have used sum of concurrences of three bipartite reduced states as a quantifier for three-qubit entanglement with a condition that $\min\left(C_{12}^2, C_{13}^2, C_{23}^2\right) > 0$, which is the condition for W class entanglement to be nonzero [106].

Furthermore, Pati and Agrawal [107] have shown that there is a special class of W states, Eq. (6), which can be used for deterministic teleportation and dense coding. Considering the importance of such a class in quantum information and computation, we characterize the nonlocal properties of these states under noisy conditions. For this, we consider the set of states given in Eq. (6) and assume the phase vectors δ and ζ to be 0 for mathematical convenience. The first term $\langle ABC\rangle$ in Eq. (8) for the evolved three-qubit state $\rho_{W_n}^{wk}$ can be represented as

$$\begin{aligned}\langle ABC\rangle_{\rho_{W_n}^{wk}} &= \cos\theta_b\left(C''_{31}\sin\theta_a\sin\theta_c\cos\phi_{ac} - \Delta'\cos\theta_a\cos\theta_c\right)\\ &\quad + \sin\theta_b\left(C''_{12}\cos\theta_a\sin\theta_c\cos\phi_{bc} + C''_{23}\sin\theta_a\cos\theta_c\cos\phi_{ab}\right)\end{aligned} \tag{36}$$

where $\Delta' = \frac{(2+\gamma(\eta-1))}{(2-\gamma(\eta-1))}$, $\cos\phi_{ab} = \cos(\phi_a - \phi_b)$, and $C''_{12} = \frac{2\sqrt{n}}{\sqrt{n+1}(2-\gamma(\eta-1))}$, $C''_{23} = \frac{2\sqrt{n}}{(n+1)(2-\gamma(\eta-1))}$, and $C''_{31} = \frac{2}{\sqrt{n+1}(2-\gamma(\eta-1))}$ are the concurrences of three reduced

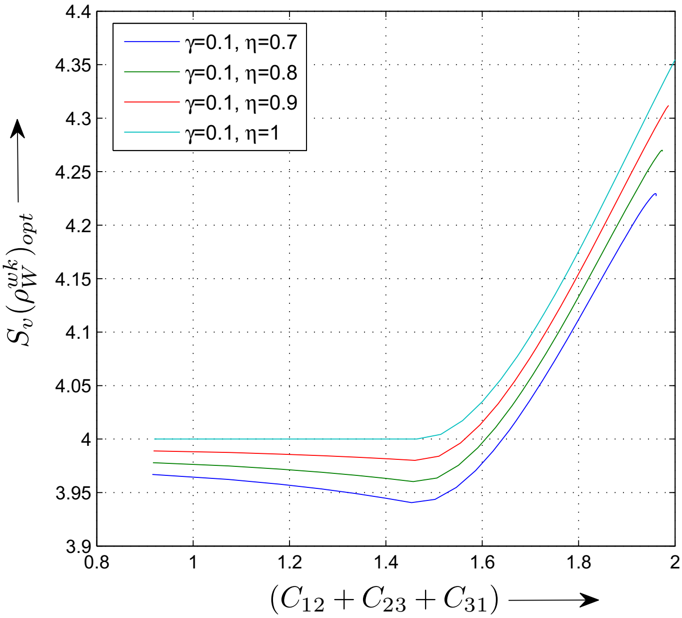

Fig. 10 A plot $S_v\left(\rho_W^{wk}\right)_{\text{opt}}$ with respect to the varying sum of concurrences $(C_{12}+C_{23}+C_{31})$, for different weak measurement strengths, considering $\gamma=0.1$ and $C_{12}=\frac{2}{3}$

states corresponding to finally shared state $\rho_{W_n}^{wk}$. The optimized value of the Svetlichny operator for the state $\rho_{W_n}^{wk}$ can be calculated in a similar fashion as in the case of ρ_W^{wk}, and can be given as

$$S_v\left(\rho_{W_n}^{wk}\right)_{\text{opt}} = 4\left(s_1\Delta' + s_2C''_{12} + s_3C''_{23} + s_4C''_{31}\right) \tag{37}$$

The violation of Svetlichny inequality for W_n class states is confirmed, when $\left(s_1\Delta' + s_2C''_{12} + s_3C''_{23} + s_4C''_{31}\right) > 1$. Figures 11 and 12 describe the effects of weak measurement strength η on the robustness of nonlocality, against the noise parameter $\gamma = 0.1$, in term of optimum expectation value of the Svetlichny operator with respect to the varying sum of the concurrences of the three reduced states of $\rho_{W_n}^{wk}$, and the state parameter n for W_n states, respectively.

6 Nonlocality robustness in four-qubit GHZ class of states

The Svetlichny inequality for a four-qubit system can be expressed as

$$S'_v(\rho) \equiv \left|\langle\psi|\, S'_v\, |\psi\rangle\right| \leq 8 \tag{38}$$

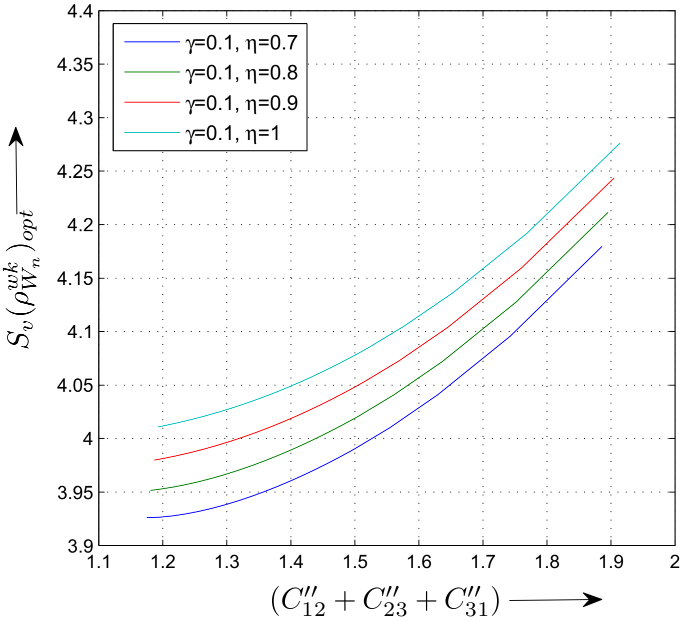

Fig. 11 $S_v(\rho^{wk}_{W_n})_{\text{opt}}$ with respect to the varying sum of concurrences $\left(C''_{12} + C''_{23} + C''_{31}\right)$, for different weak measurement strengths, considering $\gamma = 0.1$

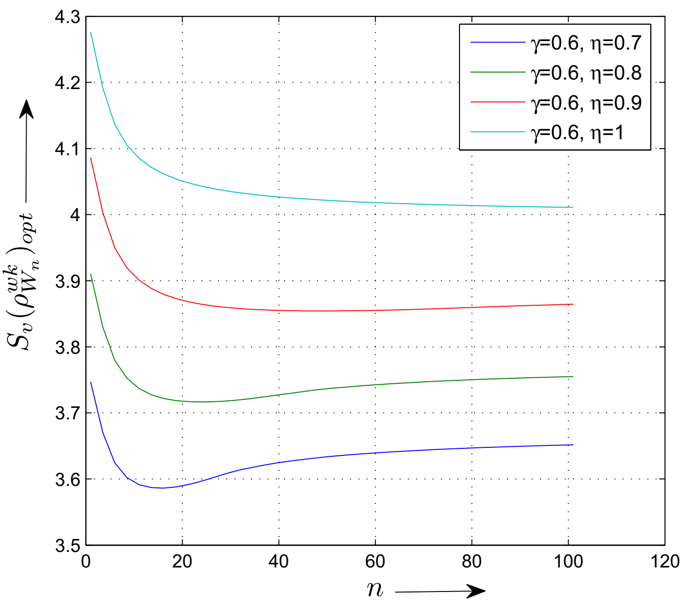

Fig. 12 Maximum expectation value of $S_v(\rho^{wk}_{W_n})_{\text{opt}}$ versus n for W_n states at different weak measurement strengths, considering $\gamma = 0.1$

where the Svetlichny operator S'_v is given by

$$S'_v = (AB - A'B')K + (AB' + A'B)K' \tag{39}$$

where $K = C(D - D') - C'(D + D')$, $K' = C'(D' - D) - C(D + D')$, and measurement operators (A, B, C, D etc.) are defined in a similar fashion as for three-qubit systems. The violation of Svetlichny inequality in Eq. (38) confirms the presence of nonlocal correlations in the underlying state. In this section, we analyse the effects of decoherence and weak measurements on robustness of nonlocality of the generalized four-qubit GHZ states, i.e.,

$$|\Psi_G\rangle = \cos\theta\,|0000\rangle + \sin\theta\,|1111\rangle \tag{40}$$

For this, we again establish an analytical relation between the optimum expectation value of the four-qubit Svetlichny operator for finally shared state with four-qubit entanglement measure of the initial state $|\Psi_G\rangle$, noise parameter and the weak measurement strengths.

In order to analyse the effect of decoherence and weak measurements, we consider a scenario where Alice prepares a four-qubit pure GHZ state $|\Psi_G\rangle$, performs weak measurements on qubits 2, 3 and 4 and then sends second qubit to Bob, third to Charlie and fourth to Dave through amplitude damping channels. Similarly, after receiving the qubits, Bob, Charlie and Dave perform reverse quantum measurements on their respective qubits. For simplicity, we consider identical decoherence parameters and identical weak measurement strengths for every channel. Here, the amplitude damping channel and weak measurements can be defined by Eqs. (9) and (23), respectively. Therefore, the expectation value $\langle ABCD\rangle$ in Eq. (39) with respect to the finally shared state ρ_G^{wk} is

$$\begin{aligned}\langle ABCD\rangle_{\rho_G^{wk}} = \frac{1}{N''}\,[&\kappa\cos\theta_a\cos\theta_b\cos\theta_c\cos\theta_d \\ &+ \sin 2\theta\cos\phi_{abcd}\sin\theta_a\sin\theta_b\sin\theta_c\sin\theta_d]\end{aligned} \tag{41}$$

where $\kappa = \left(\cos^2\theta + (1+\gamma(\eta-1))^3\sin^2\theta\right)$, $N'' = \left(\cos^2\theta + (1+\gamma(1-\eta))^3 \sin^2\theta\right)$ and $\phi_{abcd} = (\phi_a + \phi_b + \phi_c + \phi_d)$. Similarly, one can evaluate other terms in Eq. (39), and rearrange it as a sum of two terms so that

$$S'_v\left(\rho_G^{wk}\right) = \frac{1}{N''}|\kappa\cdot t_1 + \sin 2\theta\cdot t_2| \tag{42}$$

Here, t_1 and t_2 are defined as

$$\begin{aligned}t_1 = {}&\cos\theta_a\cos\theta_b G_Q + \cos\theta_a\cos\theta_{b'}G_{Q'} \\ &+ \cos\theta_{a'}\cos\theta_b G_{Q'} - \cos\theta_{a'}\cos\theta_{b'}G_Q\end{aligned} \tag{43}$$

$$\begin{aligned}t_2 = {}&\sin\theta_a\sin\theta_b f_{ab} + \sin\theta_a\sin\theta_{b'}f_{ab'} \\ &+ \sin\theta_{a'}\sin\theta_b f_{a'b} - \sin\theta_{a'}\sin\theta_{b'}f_{a'b'}\end{aligned} \tag{44}$$

where

$$G_Q = \cos\theta_c(\cos\theta_d - \cos\theta_{d'}) - \cos\theta_{c'}(\cos\theta_d + \cos\theta_{d'}) \tag{45}$$

$$G_{Q'} = \cos\theta_{c'}(\cos\theta_{d'} - \cos\theta_d) - \cos\theta_c(\cos\theta_d + \cos\theta_{d'}) \tag{46}$$

and

$$\begin{aligned} f_{ab} &= \sin\theta_c(\sin\theta_d\cos\phi_{abcd} - \sin\theta_{d'}\cos\phi_{abcd'}) \\ &\quad - \sin\theta_{c'}(\sin\theta_d\cos\phi_{abc'd} + \sin\theta_{d'}\cos\phi_{abc'd'}) \end{aligned} \tag{47}$$

$$\begin{aligned} f_{ab'} &= \sin\theta_{c'}(\sin\theta_{d'}\cos\phi_{ab'c'd'} - \sin\theta_d\cos\phi_{ab'c'd}) \\ &\quad - \sin\theta_c(\sin\theta_d\cos\phi_{ab'cd} + \sin\theta_{d'}\cos\phi_{ab'cd'}) \end{aligned} \tag{48}$$

$$\begin{aligned} f_{a'b} &= \sin\theta_{c'}(\sin\theta_{d'}\cos\phi_{a'bc'd'} - \sin\theta_d\cos\phi_{a'bc'd}) \\ &\quad - \sin\theta_c(\sin\theta_d\cos\phi_{a'bcd} + \sin\theta_{d'}\cos\phi_{a'bcd'}) \end{aligned} \tag{49}$$

$$\begin{aligned} f_{a'b'} &= \sin\theta_c(\sin\theta_d\cos\phi_{a'b'cd} - \sin\theta_{d'}\cos\phi_{a'b'cd'}) \\ &\quad - \sin\theta_{c'}(\sin\theta_d\cos\phi_{a'b'c'd} + \sin\theta_{d'}\cos\phi_{a'b'c'd'}) \end{aligned} \tag{50}$$

In "Appendix B", we provide a detailed calculation to evaluate a relationship between t1 and t2, such that (Eq. B.25),

$$|t_2| \leq 8\sqrt{2} - 2\sqrt{2}|t_1| \tag{51}$$

Using Eq. (51), the optimum value of Svetlichny operator in Eq. (42) can now be given as

$$S'_v\left(\rho_G^{wk}\right)_{\text{opt}} \leq \frac{1}{N''}\left[8\sqrt{2}\sin 2\theta + (\kappa - 2\sqrt{2}\sin 2\theta)|t_1|\right] \tag{52}$$

Clearly, for states lying in the range

$$(\kappa - 2\sqrt{2}\sin 2\theta) \geq 0, \tag{53}$$

the bound of Svetlichny operator in Eq. (52) is maximized when t_1 attains its maximum value of 4, and hence, the corresponding value of $|t_2| = 0$. Therefore, the expression for the maximum expectation value of Svetlichny operator for the states in range given by Eq. (52) is

$$S'_v\left(\rho_G^{wk}\right)_{\text{opt}} = \frac{4\kappa}{N''} \tag{54}$$

Similarly, for the states lying in the range

$$(\kappa - 2\sqrt{2}\sin 2\theta) \leq 0 \tag{55}$$

the bound of Svetlichny operator is maximized when $|t_1| = 0$, and $|t_2|$ is $8\sqrt{2}$. Therefore, the expression for maximum expectation value of Svetlichny operator for the states in range given by Eq. (55) is

$$S'_v\left(\rho_G^{wk}\right)_{\text{opt}} = \frac{1}{N''}\left[8\sqrt{2}\sin 2\theta\right] \tag{56}$$

Hence, the optimum expectation value of the Svetlichny operator for the finally shared state is now given by

$$S'_v\left(\rho_G^{wk}\right)_{\text{opt}} \leq \begin{cases} \frac{4\kappa}{N''}, & 2\sqrt{2}\sin 2\theta \leq \kappa \\ \frac{8}{N''}\sqrt{2\sin^2 2\theta}, & 2\sqrt{2}\sin 2\theta \geq \kappa \end{cases} \tag{57}$$

Equation (57) can be re-expressed in form of the four-qubit entanglement of the initially shared generalized GHZ state, given by $\tau_4 = \sin^2 2\theta$, such that

$$S'_v\left(\rho_G^{wk}\right)_{\text{opt}} \leq \begin{cases} \frac{4\kappa}{N''}, & \tau_4 \leq \frac{\kappa}{8} \\ \frac{8}{N''}\sqrt{2\tau_4}, & \tau_4 \geq \frac{\kappa}{8} \end{cases} \tag{58}$$

For perfect channels, i.e., with no noise or amplitude damping ($\gamma = 0$), one can deduce that the maximum expectation value of Svetlichny operator is

$$S'_v\left(\rho_G^{wk}\right) \leq \begin{cases} 4, & \tau_4 \leq \frac{1}{8} \\ 8\sqrt{2\tau_4}, & \tau_4 \geq \frac{1}{8} \end{cases} \tag{59}$$

which is the same as in [39]. Similarly, the effect of amplitude damping channel completely vanishes when $\eta = 1$, i.e., the finally shared state becomes a pure four-qubit state. Therefore, the expression for optimized expectation value of Svetlichny operator will be the same as in Eq. (59).

In the absence of weak measurement and its reversal operations, i.e., if the system is only subjected to amplitude damping noise then the expectation value of $\langle ABCD\rangle$ (Eq. 39) with respect to the finally shared state ρ_G^{γ} can be given as

$$\begin{aligned} \langle ABCD\rangle_{\rho_G^{\gamma}} = \big[&\kappa' \cos\theta_a \cos\theta_b \cos\theta_c \cos\theta_d \\ &+ (1-\gamma)^{3/2} \sin 2\theta \cos\phi_{abcd} \sin\theta_a \sin\theta_b \sin\theta_c \sin\theta_d\big] \end{aligned} \tag{60}$$

where $\kappa' = \left(\cos^2\theta + (1-2\gamma)^3 \sin^2\theta\right)$, and $\phi_{abcd} = (\phi_a + \phi_b + \phi_c + \phi_d)$. Therefore, the optimum expectation value of the Svetlichny operator for the finally shared state in this scenario can be evaluated as

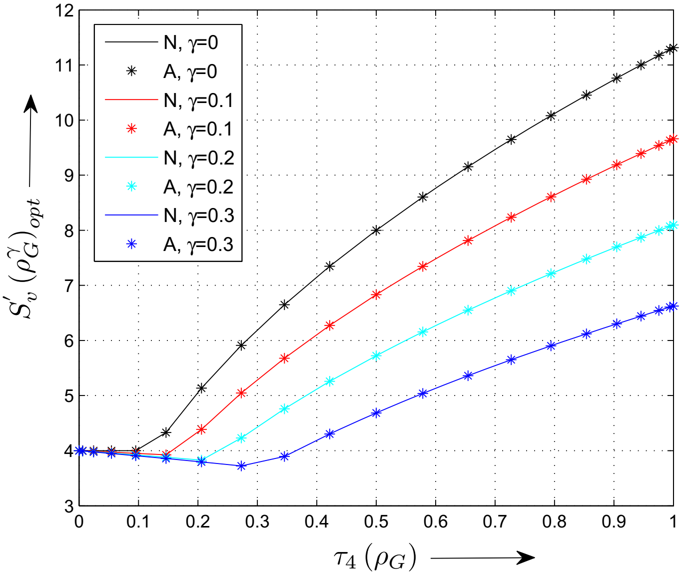

Fig. 13 A plot of $S'_v(\rho_G^{\gamma})_{\text{opt}}$ with respect to four-qubit entanglement measure (τ_4) of the initial four-qubit GHZ state considering four different values of decoherence parameter γ

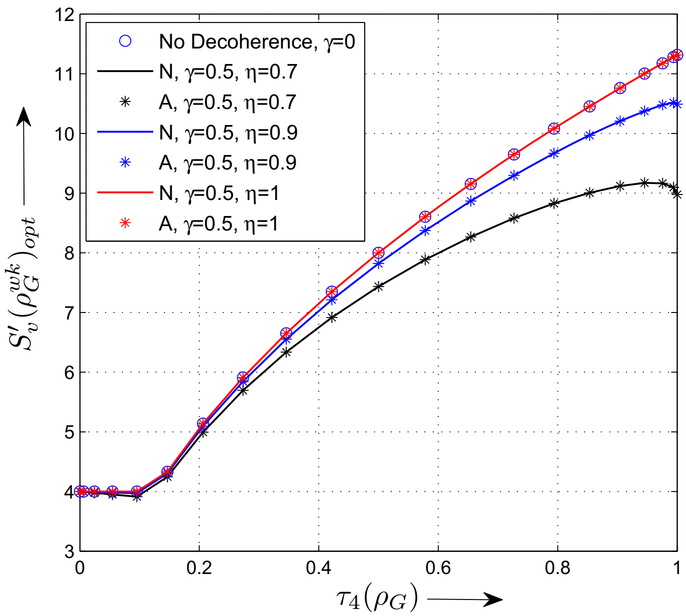

Fig. 14 Effect of weak measurement on $S'_v(\rho_G^{wk})_{\text{opt}}$ with respect to four-qubit entanglement measure (τ_4) of the initial four-qubit GHZ state, considering $\gamma = 0.5$

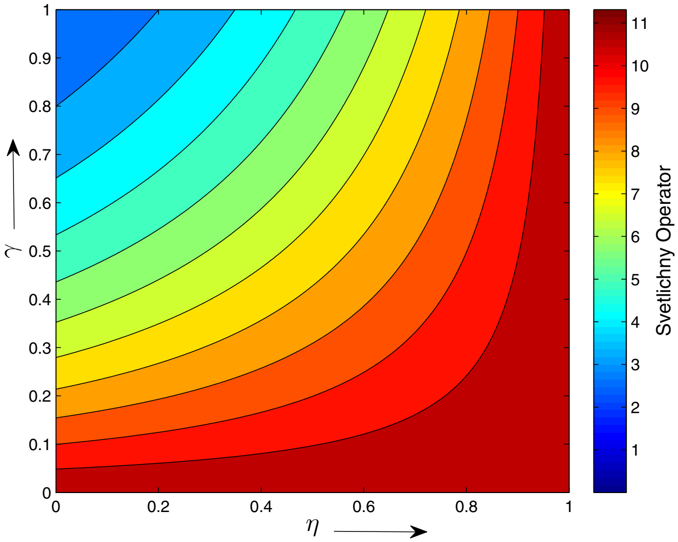

Fig. 15 Effect of weak measurement on the maximum expectation value of the four-qubit Svetlichny operator as a function of noise parameter γ, for a maximally entangled input state

$$S'_v\left(\rho_G^{\gamma}\right)_{\rm opt} = \begin{cases} 4\left(\cos^2\theta + (1-2\gamma)^3\sin^2\theta\right) \\ \qquad\qquad \left(2\,(1-\gamma)^{3/2}\sqrt{2}\sin 2\theta\right) \le \kappa' \\ 8\sqrt{2\,(1-\gamma)^3\sin^2 2\theta}, \\ \qquad\qquad \left(2\,(1-\gamma)^{3/2}\sqrt{2}\sin 2\theta\right) \ge \kappa' \end{cases} \tag{61}$$

Based on our results, in Fig. 13, we demonstrate the effect of noise parameter γ on the expectation value of the Svetlichny operator against the entanglement of the initially shared four-qubit GGHZ state. Similarly, Fig. 14 demonstrates the effect of weak measurement strength η on the expectation value of the Svetlichny operator against the entanglement of the initially shared four-qubit GGHZ state, considering noise parameter $\gamma = 0.5$. Similar to the case of three-qubit GHZ states, our analysis suggests that for four-qubit GHZ states as well, performing weak measurements on all the qubits strengthens the robustness of nonlocal correlations against noise in the finally shared state. Moreover, for certain ranges of τ_4 and η, nonlocality anomalies are also observed, i.e., nonmaximally entangled four-qubit initial states are found to be more robust against noise in comparison with a maximally entangled four-qubit GHZ state. For example, the lower curve corresponding to $\eta = 0.7$ in Fig. 14 indicates that maximum entanglement and maximal nonlocality do not coincide. The analytical results obtained here completely agree with the numerical results obtained for analysing nonlocality robustness using violation of the four-qubit Svetlichny inequality. For a maximally entangled initial state, Fig. 15 describes the effects of noise parameter γ and weak measurement strength η on the maximum expectation value of the Svetlichny operator.

7 Conclusion

In summary, we have analysed the effects of decoherence using an amplitude damping channel and weak measurement and its reversal operations on robustness of three and four-qubit nonlocality. We found that for three- and four-qubit GHZ class states, the maximal nonlocality does not coincide with the maximum entanglement for a given range of weak measurement strength under noisy conditions leading to nonlocality anomalies. However, for three-qubit MS states, our analysis did not reveal any discrepancy between maximum entanglement and maximal nonlocality. Our analysis for generalized three and four-qubit GHZ states, three-qubit W class and W_n states further allowed us to characterize the multiqubit nonlocal correlations in terms of noise parameters and strengths of weak measurements. The results obtained here clearly suggest that effects of amplitude damping on multiqubit nonlocality robustness can be reduced or completely removed depending on the strengths of weak measurement and its reversal operations. We have further shown that the analytical results obtained in all the cases are in excellent agreement with the numerical results. In future, it will be interesting to investigate the usefulness of finally shared three-qubit and four-qubit mixed states for quantum information and computation.

Acknowledgements The authors thank MHRD and IIT Jodhpur for providing the research facility.

Appendix

A Maximization of the expectation value of the three-qubit Svetlichny operator for W states

For maximizing the value of Svetlichny operator in Eq. (34) described in Sec. V, we assume $\phi_i = 0$ [128], and then add the first four terms in Eq. (8) to get

$$
\begin{aligned}
\langle M \rangle = \frac{1}{4} [& (-\Delta - C_{12} - C_{23} - C_{31}) \{\cos(\theta_a + \theta_b + \theta_{c'}) \\
& + \cos(\theta_{a'} + \theta_b + \theta_c) + \cos(\theta_a + \theta_{b'} + \theta_c) - \cos(\theta_{a'} + \theta_{b'} + \theta_{c'})\} \\
& + (-\Delta + C_{12} - C_{23} + C_{31}) \{\cos(\theta_a + \theta_b - \theta_{c'}) \\
& + \cos(\theta_{a'} + \theta_b - \theta_c) + \cos(\theta_a + \theta_{b'} - \theta_c) - \cos(\theta_{a'} + \theta_{b'} - \theta_{c'})\} \\
& + (-\Delta + C_{12} + C_{23} - C_{31}) \{\cos(\theta_a - \theta_b + \theta_{c'}) \\
& + \cos(\theta_{a'} - \theta_b + \theta_c) + \cos(\theta_a - \theta_{b'} + \theta_c) - \cos(\theta_{a'} - \theta_{b'} + \theta_{c'})\} \\
& + (-\Delta - C_{12} + C_{23} + C_{31}) \{\cos(\theta_a - \theta_b - \theta_{c'}) \\
& + \cos(\theta_{a'} - \theta_b - \theta_c) + \cos(\theta_a - \theta_{b'} - \theta_c) - \cos(\theta_{a'} - \theta_{b'} - \theta_{c'})\}]
\end{aligned}
\tag{A.1}
$$

The expression for M' can be written in a similar fashion. For simplicity and mathematical convenience, let us define $\Sigma = \left(\theta'_a + \theta'_b + \theta'_c\right)$, $\Sigma_k = \Sigma - 2\theta'_k$, $\tilde{\theta_k} = (\theta_k + \theta_{k'})/2$, and $\theta'_k = (\theta_{k'} - \theta_k)/2$ where $k \epsilon \{a, b, c\}$ such that

$$
\begin{aligned}
S_v\left(\rho_W^{wk}\right)_{\text{opt}} &= \frac{1}{2}\left[(-\Delta - C_{12} - C_{23} - C_{31})\right.\\
&\quad \times \sin\left(\tilde{\theta}_a + \tilde{\theta}_b + \tilde{\theta}_c\right)\left\{K - 2\sin\left(\theta'_a - \theta'_b - \theta'_c\right)\right\}\\
&\quad + (-\Delta + C_{12} - C_{23} + C_{31})\\
&\quad \times \sin\left(\tilde{\theta}_a + \tilde{\theta}_b - \tilde{\theta}_c\right)\left\{K - 2\sin\left(\theta'_a - \theta'_b + \theta'_c\right)\right\}\\
&\quad + (-\Delta + C_{12} + C_{23} - C_{31})\\
&\quad \times \sin\left(\tilde{\theta}_a - \tilde{\theta}_b + \tilde{\theta}_c\right)\left\{K - 2\sin\left(\theta'_a + \theta'_b - \theta'_c\right)\right\}\\
&\quad + (-\Delta - C_{12} + C_{23} + C_{31})\\
&\quad \left.\times \sin\left(\tilde{\theta}_a - \tilde{\theta}_b - \tilde{\theta}_c\right)\left\{K - 2\sin\left(\theta'_a + \theta'_b + \theta'_c\right)\right\}\right]\\
&= \Delta\left\{\sin\Sigma_a + \sin\Sigma_b + \sin\Sigma_c - \sin\Sigma\right\}\\
&\quad + C_{12}\left\{\sin\Sigma - \sin\Sigma_a + \sin\Sigma_b + \sin\Sigma_c\right\}\\
&\quad + C_{23}\left\{\sin\Sigma + \sin\Sigma_a + \sin\Sigma_b - \sin\Sigma_c\right\}\\
&\quad + C_{31}\left\{\sin\Sigma + \sin\Sigma_a - \sin\Sigma_b + \sin\Sigma_c\right\} \qquad \text{(A.2)}\\
&\equiv 4\left(s_1\Delta + s_2 C_{12} + s_3 C_{23} + s_4 C_{31}\right) \qquad \text{(A.3)}
\end{aligned}
$$

where

$$
\begin{aligned}
K = &\left\{\sin\left(\theta'_a + \theta'_b + \theta'_c\right) + \sin\left(\theta'_a - \theta'_b + \theta'_c\right)\right.\\
&\left. + \sin\left(\theta'_a + \theta'_b - \theta'_c\right)\sin\left(\theta'_a - \theta'_b - \theta'_c\right)\right\} \qquad \text{(A.4)}
\end{aligned}
$$

and the equality in Eq. (A.2) can be achieved by considering $\tilde{\theta}_a = \tilde{\theta}_b = \tilde{\theta}_c = \pi/2$.

B Derivation for the relationship between t_1 and t_2

For evaluating the relationship between t_1 and t_2, we further consider two unit vectors p and p' where $\vec{b} + \vec{b'} = 2\vec{p}\cos\theta_1$ and $\vec{b} - \vec{b'} = 2\vec{p'}\sin\theta_1$ such that

$$
\vec{p}.\vec{p'} = \cos\theta_p\cos\theta_{p'} + \sin\theta_p\sin\theta_{p'}\cos(\phi_p - \phi_{p'}) = 0 \qquad \text{(B.1)}
$$

Therefore, t_1 and t_2 can be re-expressed as

$$
\begin{aligned}
|t_1| = &\,|l_{ap'cd}\sin\theta_1 - l_{apcd'}\cos\theta_1 - l_{apc'd}\cos\theta_1\\
&- l_{ap'c'd'}\sin\theta_1 + l_{a'pc'd'}\cos\theta_1\\
&- l_{a'p'c'd}\sin\theta_1 - l_{a'pcd}\cos\theta_1 - l_{a'p'cd'}\sin\theta_1| \qquad \text{(B.2)}
\end{aligned}
$$

and

$$
\begin{aligned}
|t_2| = &\,|s_{ap'cd}\sin\theta_1 - s_{apcd'}\cos\theta_1 - s_{apc'd}\cos\theta_1\\
&- s_{ap'c'd'}\sin\theta_1 + s_{a'pc'd'}\cos\theta_1\\
&- s_{a'p'c'd}\sin\theta_1 - s_{a'pcd}\cos\theta_1 - s_{a'p'cd'}\sin\theta_1| \qquad \text{(B.3)}
\end{aligned}
$$

where

$$l_{ap'cd} = 2\cos\theta_a \cos\theta_{p'} \cos\theta_c \cos\theta_d \tag{B.4}$$

$$s_{ap'cd} = 2\sin\theta_a \sin\theta_{p'} \sin\theta_c \sin\theta_d \cos\phi_{ap'cd} \tag{B.5}$$

The other coefficients $l_{apcd'}$, $s_{apcd'}$ *etc.* can be defined in a similar fashion with prime on different angles. In order to simplify and optimize the expressions further, we assume $\theta_c = \theta_{c'}$, and define two unit vectors q and q' such that $\vec{d} + \vec{d'} = 2\vec{q}\cos\theta_2$ and $\vec{d} - \vec{d'} = 2\vec{q'}\sin\theta_2$, i.e.,

$$\vec{q}\cdot\vec{q'} = \cos\theta_q \cos\theta_{q'} + \sin\theta_q \sin\theta_{q'} \cos(\phi_q - \phi_{q'}) = 0 \tag{B.6}$$

This allows us to re-express Eqs. (B.2) and (B.3) as

$$|t_1| = |l'_{ap'cq'} \sin\theta_1 \sin\theta_2 - l'_{apcq} \cos\theta_1 \cos\theta_2 \\ - l'_{a'pcq'} \cos\theta_1 \sin\theta_2 - l'_{a'p'cq} \sin\theta_1 \cos\theta_2| \tag{B.7}$$

and

$$|t_2| = |s'_{ap'cq'} \sin\theta_1 \sin\theta_2 - s'_{apcq} \cos\theta_1 \cos\theta_2 \\ - s'_{a'pcq'} \cos\theta_1 \sin\theta_2 - s'_{a'p'cq} \sin\theta_1 \cos\theta_2| \tag{B.8}$$

where

$$l'_{apcq} = 4\cos\theta_a \cos\theta_p \cos\theta_c \cos\theta_q \tag{B.9}$$

$$s'_{apcq} = 4\sin\theta_a \sin\theta_p \sin\theta_c \sin\theta_q \cos\phi_{apcq} \tag{B.10}$$

From Eqs. (B.7) and (B.8), one can get

$$|t_1| \le |l'_{ap'cq'}||\sin\theta_1||\sin\theta_2| + |l'_{apcq}||\cos\theta_1||\cos\theta_2| \\ + |l'_{a'pcq'}||\cos\theta_1||\sin\theta_2| + |l'_{a'p'cq}||\sin\theta_1||\cos\theta_2| \tag{B.11}$$

$$|t_2| \le |s'_{ap'cq'}||\sin\theta_1||\sin\theta_2| + |s'_{apcq}||\cos\theta_1||\cos\theta_2| \\ + |s'_{a'pcq'}||\cos\theta_1||\sin\theta_2| + |s'_{a'p'cq}||\sin\theta_1||\cos\theta_2| \tag{B.12}$$

Using these inequalities, the iterative maximization of Eq. (39) can be summarized below as

$$2\sqrt{2}\kappa|t_1| + |t_2| \le \Bigg[\Bigg\{(2\sqrt{2}\kappa|l'_{apcq}| + |s'_{apcq}|)|\cos\theta_1| \\ + (2\sqrt{2}\kappa|l'_{a'p'cq}| + |s'_{a'p'cq}|)|\sin\theta_1|\Bigg\}|\cos\theta_2| \\ + \Bigg\{(2\sqrt{2}\kappa|l'_{a'pcq'}| + |s'_{a'pcq'}|)|\cos\theta_1| \\ + (2\sqrt{2}\kappa|l'_{ap'cq'}| + |s'_{ap'cq'}|)|\sin\theta_1|\Bigg\}|\sin\theta_2|\Bigg] \tag{B.13}$$

$$
\begin{aligned}
2\sqrt{2}\kappa|t_1| + |t_2| \leq \Big[&\Big\{(2\sqrt{2}\kappa|l'_{apcq}| + |s'_{apcq}|)|\cos\theta_1| \\
&+ (2\sqrt{2}\kappa|l'_{a'p'cq}| + |s'_{a'p'cq}|)|\sin\theta_1|\Big\}^2 \\
&+ \Big\{(2\sqrt{2}\kappa|l'_{a'pcq'}| + |s'_{a'pcq'}|)|\cos\theta_1| \\
&+ (2\sqrt{2}\kappa|l'_{ap'cq'}| + |s'_{ap'cq'}|)|\sin\theta_1|\Big\}^2\Big]^{\frac{1}{2}} \qquad \text{(B.14)}
\end{aligned}
$$

$$
\begin{aligned}
2\sqrt{2}\kappa|t_1| + |t_2| \leq \Big[&(2\sqrt{2}\kappa|l'_{apcq}| + |s'_{apcq}|)^2 + (2\sqrt{2}\kappa|l'_{a'p'cq}| + |s'_{a'p'cq}|)^2 \\
&+ (2\sqrt{2}\kappa|l'_{a'pcq'}| + |s'_{a'pcq'}|)^2 + (2\sqrt{2}\kappa|l'_{ap'cq'}| \\
&+ |s'_{ap'cq'}|)^2\Big]^{\frac{1}{2}} \qquad \text{(B.15)}
\end{aligned}
$$

where Eq. (B.13) is maximized with respect to θ_2, and the first and second terms in Eq. (B.14) are maximized separately with respect to θ_1. To simplify and optimize Eq. (B.15), we use Eqs. (B.9) and (B.10), such that

$$
\begin{aligned}
(2\sqrt{2}\kappa|l'_{apcq}| + |s'_{apcq}|) = 8\sqrt{2}\kappa|\cos\theta_a\cos\theta_p\cos\theta_c\cos\theta_q| \\
+ 4|\sin\theta_a\sin\theta_p\sin\theta_c\sin\theta_q\cos\phi_{apcq}| \qquad \text{(B.16)}
\end{aligned}
$$

Equation (B.16) when maximized with respect to θ_a, where we have used the inequality (15), gives

$$
\begin{aligned}
(2\sqrt{2}\kappa|l'_{apcq}| + |s'_{apcq}|) \leq 4\Big[&8\kappa^2\cos^2\theta_c\cos^2\theta_p\cos^2\theta_q \\
&+ \sin^2\theta_c\sin^2\theta_p\sin^2\theta_q\cos^2\phi_{apcq}\Big]^{\frac{1}{2}} \qquad \text{(B.17)}
\end{aligned}
$$

Similarly, the other terms in Eq. (B-15) can be evaluated as

$$
\begin{aligned}
(2\sqrt{2}\kappa|l'_{ap'cq'}| + |s'_{ap'cq'}|) \leq 4\Big[&8\kappa^2\cos^2\theta_c\cos^2\theta_{p'}\cos^2\theta_{q'} \\
&+ \sin^2\theta_c\sin^2\theta_{p'}\sin^2\theta_{q'}\cos^2\phi_{ap'cq'}\Big]^{\frac{1}{2}} \qquad \text{(B.18)} \\
(2\sqrt{2}\kappa|l'_{a'p'cq}| + |s'_{a'p'cq}|) \leq 4\Big[&8\kappa^2\cos^2\theta_c\cos^2\theta_{p'}\cos^2\theta_q \\
&+ \sin^2\theta_c\sin^2\theta_{p'}\sin^2\theta_q\cos^2\phi_{a'p'cq}\Big]^{\frac{1}{2}} \qquad \text{(B.19)} \\
(2\sqrt{2}\kappa|l'_{a'pcq'}| + |s'_{a'pcq'}|) \leq 4\Big[&8\kappa^2\cos^2\theta_c\cos^2\theta_p\cos^2\theta_{q'} \\
&+ \sin^2\theta_c\sin^2\theta_p\sin^2\theta_{q'}\cos^2\phi_{a'pcq'}\Big]^{\frac{1}{2}} \qquad \text{(B.20)}
\end{aligned}
$$

where, for optimization, we consider $\cos^2\phi_{apcq} = \cos^2\phi_{ap'cq'} = \cos^2\phi_{a'p'cq} = \cos^2\phi_{a'pcq'} = 1$. Therefore, using Eqs. (B.17–B.20), Eq. (B.15) can be re-expressed as

$$
\begin{aligned}
2\sqrt{2}\kappa|t_2| + |t_1| \leq 4\Big[&8\kappa^2\cos^2\theta_c\cos^2\theta_q(\cos^2\theta_p + \cos^2\theta_{p'}) \\
&+ \sin^2\theta_c\sin^2\theta_q(\sin^2\theta_p + \sin^2\theta_{p'}) \\
&+ 8\kappa^2\cos^2\theta_c\cos^2\theta_{q'}(\cos^2\theta_p + \cos^2\theta_{p'}) \\
&+ \sin^2\theta_c\sin^2\theta_{q'}(\sin^2\theta_p + \sin^2\theta_{p'})\Big]^{\frac{1}{2}}
\end{aligned} \tag{B.21}
$$

Considering the orthogonality of unit vectors $\vec{p}$ and $\vec{p}'$, the maximum value of $(\sin^2\theta_p + \sin^2\theta_{p'})$ is 2 and maximum value of $(\cos^2\theta_p + \cos^2\theta_{p'})$ is 1, i.e.,

$$
\begin{aligned}
2\sqrt{2}\kappa|t_1| + |t_2| \leq 4\Big[&8\kappa^2\cos^2\theta_c(\cos^2\theta_q + \cos^2\theta_{q'}) \\
&+ 2\sin^2\theta_c(\sin^2\theta_q + \sin^2\theta_{q'})\Big]^{\frac{1}{2}}
\end{aligned} \tag{B.22}
$$

Similarly from the orthogonality of unit vectors $\vec{q}$ and $\vec{q}'$, Eq. (B.22) can be further optimized as

$$
2\sqrt{2}\kappa|t_1| + |t_2| \leq 4\left[8\kappa^2\cos^2\theta_c + 4\sin^2\theta_c\right]^{\frac{1}{2}} \tag{B.23}
$$

A further maximization on the parameter κ gives

$$
2\sqrt{2}|t_1| + |t_2| \leq 8\left[1 + \cos^2\theta_c\right]^{\frac{1}{2}} \leq 8\sqrt{2} \tag{B.24}
$$

Therefore, the relationship between t_1 and t_2 can be defined as

$$
|t_2| \leq 8\sqrt{2} - 2\sqrt{2}|t_1| \tag{B.25}
$$

References

1. Einstein, A.: Can quantum-mechanical description of physical reality be considered complete? Phys. Rev. **47**, 777–780 (1935)
2. Bohm, D., Aharanov, Y.: Discussion of experimental proof for the paradox of Einstein, Rosen, and Podolsky. Phys. Rev. **108**, 1070–1076 (1957)
3. Bell, J.S.: On the Einstein Podolsky Rosen paradox. Physics **1**, 195–200 (1964)
4. Clauser, J.F., Horne, M.A., Shimony, A., Holt, R.A.: Proposed experiment to test local hidden-variable theories. Phys. Rev. Lett. **23**, 880–884 (1969)
5. Gisin, N.: Bell's inequality holds for all non-product states. Phys. Lett. A **154**, 201–202 (1991)
6. Peres, A.: Incompatible results of quantum measurements. Phys. Lett. A **151**, 107–108 (1990)
7. Mermin, N.D.: What's wrong with these elements of reality? Phys. Today **43**, 9–11 (1990)
8. Home, D., Selleri, F.: Bells theorem and the EPR paradox. Riv. Nuovo Cimento **14**, 1 (1991)
9. Batle, J., Plastino, A.R., Casas, M., Plastino, A.: Conditional q-entropies and quantum separability: a numerical exploration. J. Phys. A Math. Gen. **35**, 10311 (2002)
10. Batle, J., Casas, M.: Nonlocality and entanglement in qubit systems. J. Phys. A Math. Gen. **44**, 445304 (2011)
11. Ozdemir, S.K., Bartkiewicz, K., Liu, Y.X., Miranowicz, A.: Teleportation of qubit states through dissipative channels: conditions for surpassing the no-cloning limit. Phys. Rev. A **76**, 042325 (2007)

12. Bartkiewicz, K., Lemr, K., Cernoch, A., Miranowicz, A.: Bell nonlocality and fully entangled fraction measured in an entanglement-swapping device without quantum state tomography. Phys. Rev. A **95**, 030102 (2017)
13. Khalfin, L., Tsirelson, B.: Quantum/classical correspondence in the light of Bell's inequalities found. Physics **22**, 879–948 (1992)
14. Mermin, N.D.: Hidden variables and the two theorems of John Bell. Rev. Mod. Phys. **65**, 803–815 (1993)
15. Horodecki, R., Horodecki, M., Horodecki, P.: Teleportation, Bell's inequalities and inseparability. Phys. Lett. A **222**, 21–25 (1996)
16. Zeilinger, A.: Experiment and the foundations of quantum physics. Rev. Mod. Phys. **71**, S288–S297 (1999)
17. Genovese, M.: Research on hidden variable theories: a review of recent progresses. Phys. Rep. **413**, 319–396 (2005)
18. Brunner, N., et al.: Bell nonlocality. Rev. Mod. Phys. **86**, 419–478 (2014)
19. Tittel, W., Brendel, J., Zbinden, H., Gisin, N.: Violation of Bell inequalities by photons more than 10 km apart. Phys. Rev. Lett. **81**, 3563–3566 (1998)
20. Mair, A., Vaziri, A., Weihs, G., Zeilinger, A.: Entanglement of the orbital angular momentum states of photons. Nature **412**, 313–316 (2001)
21. Kwiat, P.: New high-intensity source of polarization-entangled photon pairs. Phys. Rev. Lett. **75**, 4337–4341 (1995)
22. Kwiat, P.: Hyper-entangled states. J. Mod. Opt. **44**, 2173–2184 (1997)
23. Babichev, S.A., Appel, J., Lvovsky, A.I.: Homodyne tomography characterization and nonlocality of a dual-mode optical qubit. Phys. Rev. Lett. **92**, 193601 (2004)
24. Thew, R., Acn, A., Zbinden, H., Gisin, N.: Bell-type test of energy-time entangled qutrits. Phys. Rev. Lett. **93**, 010503 (2004)
25. Bennett, C.H., Wiesner, S.: Communication via one-and two-particle operators on Einstein–Podolsky–Rosen states. Phys. Rev. Lett. **69**, 2881–2884 (1992)
26. Bennett, C.H.: Teleporting an unknown quantum state via dual classical and Einstein–Podolsky–Rosen channels. Phys. Rev. Lett. **70**, 1895–1899 (1993)
27. Bostrom, K., Felbinger, T.: Deterministic secure direct communication using entanglement. Phys. Rev. Lett. **89**, 187902 (2002)
28. Gisin, N., et al.: Quantum cryptography. Rev. Mod. Phys. **74**, 145–195 (2002)
29. Zukowski, M., Zeilinger, A., Horne, M.A., Eckert, A.: "Event-ready-detectors" Bell experiment via entanglement swapping. Phys. Rev. Lett. **71**, 4287–4290 (1993)
30. Knill, E., Laflamme, R.: Power of one bit of quantum information. Phys. Rev. Lett. **81**, 5672–5675 (1998)
31. Luo, S.: Quantum discord for two-qubit systems. Phys. Rev. A **77**, 042303 (2008)
32. Datta, A., Shaji, A., Caves, C.M.: Quantum discord and the power of one qubit. Phys. Rev. Lett. **100**, 050502 (2008)
33. Dakic, B., et al.: Necessary and sufficient condition for nonzero quantum discord. Phys. Rev. Lett. **105**, 190502 (2010)
34. Svetlichny, G.: Distinguishing three-body from two-body nonseparability by a Bell-type inequality. Phys. Rev. D **35**, 3066–3069 (1987)
35. Seevinck, M., Svetlichny, G.: Bell-type inequalities for partial separability in N-particle systems and quantum mechanical violations. Phys. Rev. Lett. **89**, 060401 (2002)
36. Collins, D., Gisin, N., Linden, N., Massar, S., Popescu, S.: Bell inequalities for arbitrarily high-dimensional systems. Phys. Rev. Lett. **88**, 040404 (2002)
37. Bancal, J.D., Gisin, N., Pironio, S.: Looking for symmetric Bell inequalities. J. Phys. A **43**, 385303 (2010)
38. Barrett, J., Colbeck, R., Kent, A.: Memory attacks on device-independent quantum cryptography. Phys. Rev. Lett. **110**, 010503 (2013)
39. Ghose, S., Debnath, S., Sinclair, N., Kabra, A., Stock, R.: Multiqubit nonlocality in families of 3-and 4-qubit entangled states. J. Phys. A Math. Theor. **43**, 445301 (2010)
40. Pan, J.W., Bouwmeester, D., Daniell, M., Weinfurter, H., Zeilinger, A.: Experimental test of quantum nonlocality in three-photon Greenberger–Horne–Zeilinger entanglement. Nature **403**, 515–519 (2000)

41. Eibl, M., Kiesel, N., Bourennane, M., Kurtsiefer, C., Weinfurter, H.: Experimental realization of a three-qubit entangled W state. Phys. Rev. Lett. **92**, 077901 (2004)
42. Zhao, Z., Yang, T., Chen, Y.-A., Zhang, A.-N., Zukowski, M., Pan, J.W.: Experimental violation of local realism by four-photon Greenberger–Horne–Zeilinger entanglement. Phys. Rev. Lett. **91**, 180401 (2003)
43. Eibl, M., Gaertner, S., Bourennane, M., Kurtsiefer, C., Zukowski, M., Weinfurter, H.: Experimental observation of four-photon entanglement from parametric down-conversion. Phys. Rev. Lett. **90**, 200403 (2003)
44. Walther, P., Aspelmeyer, M., Resch, K.J., Zeilinger, A.: Experimental violation of a cluster state Bell inequality. Phys. Rev. Lett. **95**, 020403 (2005)
45. Lavoie, J., Kaltenbaek, R., Resch, K.: Experimental violation of Svetlichny's inequality. New J. Phys. **11**, 073051 (2009)
46. Zhao, J.Q., Cao, L.Z., Wang, X.Q., Lu, H.X.: Experimental investigation of tripartite entanglement and nonlocality in three-qubit generalized Greenberger–Horne–Zeilinger states. Phys. Lett. A **376**, 2377–2380 (2012)
47. Collins, D., Gisin, N., Popescu, S., Roberts, D., Scarani, V.: Bell-type inequalities to detect true n-body nonseparability. Phys. Rev. Lett. **88**, 170405 (2002)
48. Cereceda, J.L.: Three-particle entanglement versus three-particle nonlocality. Phys. Rev. A **66**, 024102 (2002)
49. Ghose, S., Sinclair, N., Debnath, S., Rungta, P., Stock, R.: Tripartite entanglement versus tripartite nonlocality in three-qubit Greenberger–Horne–Zeilinger-class states. Phys. Rev. Lett. **102**, 250404 (2009)
50. Lu, H.X., Zhao, J.Q., Wang, X.Q., Cao, L.Z.: Experimental demonstration of tripartite entanglement versus tripartite nonlocality in three-qubit Greenberger–Horne–Zeilinger-class states. Phys. Rev. A **84**, 012111 (2011)
51. Zhang, C.: Experimental test of genuine multipartite nonlocality under the no-signalling principle. Sci. Rep. **6**, 39327 (2016). https://doi.org/10.1038/srep39327
52. Zurek, W.H.: Decoherence, einselection, and the quantum origins of the classical. Mod. Phys. Rev. **75**, 715–775 (2003)
53. Carvalho, A.R.R., Mintert, F., Buchleitner, A.: Decoherence and multipartite entanglement. Phys. Rev. Lett. **93**, 230501 (2004)
54. Hein, M., Dr, W., Briegel, H.J.: Entanglement properties of multipartite entangled states under the influence of decoherence. Phys. Rev. A **71**, 032350 (2005)
55. Mintert, F., Ku, M., Buchleitner, A.: Concurrence of mixed multipartite quantum states. Phys. Rev. Lett. **95**, 260502 (2005)
56. Bandyopadhyay, S., Lidar, D.A.: Robustness of multiqubit entanglement in the independent decoherence model. Phys. Rev. A **72**, 042339 (2005)
57. Almeida, M.P., et al.: Environment-induced sudden death of entanglement. Science **316**, 579–582 (2007)
58. Frowis, F., Dr, W.: Stable macroscopic quantum superpositions. Phys. Rev. Lett. **106**, 110402 (2011)
59. Mahdiana, M., Yousefjani, R., Salimi, S.: Quantum discord evolution of three-qubit states under noisy channels. Eur. Phys. J. D **66**, 133 (2012)
60. Ramzan, M.: Decoherence dynamics of discord for multipartite quantum systems. Eur. Phys. J. D **67**, 170 (2013)
61. Tchoffo, M.: Quantum correlations dynamics and decoherence of a three-qubit system subject to classical environmental noise. Eur. Phys. J. Plus **131**, 380 (2016)
62. Acin, A., Durt, T., Gisin, N., Latorre, J.I.: Quantum nonlocality in two three-level systems. Phys. Rev. A **65**, 052325 (2002)
63. Methot, A.A., Scarani, V.: An anomaly of non-locality. Quantum Inf. Comput. **7**, 157–170 (2007)
64. Fonseca, E.A., Parisio, F.: Measure of nonlocality which is maximal for maximally entangled qutrits. Phys. Rev. A **92**, 030101 (2015)
65. Lee, S.W., Jaksch, D.: Maximal violation of tight Bell inequalities for maximal high-dimensional entanglement. Phys. Rev. A **80**, 010103(R) (2009)
66. Gunge, M., Palazuelos, C.: Large violation of Bell inequalities with low entanglement. Commun. Math. Phys. **306**, 695 (2011)
67. Hiesmayr, B.C.: Nonlocality and entanglement in a strange system. Eur. Phys. J. C **50**, 73–79 (2007)
68. Vidick, T., Wehner, S.: More nonlocality with less entanglement. Phys. Rev. A **83**, 052310 (2011)

69. Ji, S.W., Lee, J., Lim, J., Nagata, K., Lee, H.W.: Multisetting Bell inequality for qudits. Phys. Rev. A **78**, 052103 (2008)
70. Brunner, N.: New perspectives on quantum correlations. Phys. E **42**, 354–358 (2010)
71. de Rosier, A., Gruca, J., Parisio, F., Vrtesi, T., Laskowski, W.: Multipartite nonlocality and random measurements. Phys. Rev. A **96**, 012101 (2017)
72. Chaves, R., Acn, A., Aolita, L., Cavalcanti, D.: Detecting nonlocality of noisy multipartite states with the Clauser–Horne–Shimony–Holt inequality. Phys. Rev. A **89**, 042106 (2014)
73. Laskowski, W., Vertesi, T., Wiesniak, M.: Highly noise resistant multiqubit quantum correlations. J. Phys. A **48**, 465301 (2015)
74. Sohbi, A., Zaquine, I., Diamanti, E., Markham, D.: Decoherence effects on the nonlocality of symmetric states. Phys. Rev. A **91**, 022101 (2015)
75. Divianszky, P., Trencsenyi, R., Bene, E., Vertesi, T.: Bounding the persistency of the nonlocality of W states. Phys. Rev. A **93**, 042113 (2016)
76. Bennett, C.H., et al.: Purification of noisy entanglement and faithful teleportation via noisy channels. Phys. Rev. Lett. **76**, 722–725 (1996)
77. Bennett, C.H., Bernstein, H.J., Popescu, S., Schumacher, B.: Concentrating partial entanglement by local operations. Phys. Rev. A **53**, 2046–2052 (1996)
78. Pan, J.W., Gasparoni, S., Ursin, R., Weihs, G., Zeilinger, A.: Experimental entanglement purification of arbitrary unknown states. Nature **423**, 417–422 (2003)
79. Kwiat, P.G., Berglund, A.J., Alterpeter, J.B., White, A.G.: Experimental verification of decoherence-free subspaces. Science **290**, 498–501 (2000)
80. Lidar, D.A., Chuang, I.L., Whaley, K.B.: Decoherence-free subspaces for quantum computation. Phys. Rev. Lett. **81**, 2594–2597 (1998)
81. Shor, P.W.: Scheme for reducing decoherence in quantum computer memory. Phys. Rev. A **52**, R2493–R2496 (1995)
82. Calderbank, A.R., Shor, P.W.: Good quantum error-correcting codes exist. Phys. Rev. A **54**, 1098–1105 (1996)
83. Knill, E., Laflamme, R.: Theory of quantum error-correcting codes. Phys. Rev. A **55**, 900–911 (1997)
84. Steane, A.M.: Error correcting codes in quantum theory. Phys. Rev. Lett. **77**, 793–797 (1996)
85. Facchi, P., Lidar, D.A., Pascazio, S.: Unification of dynamical decoupling and the quantum Zeno effect. Phys. Rev. A **69**, 032314 (2004)
86. Maniscalco, S., Francica, F., Zaffino, R.L., Gullo, N.L., Plastina, F.: Protecting entanglement via the quantum Zeno effect. Phys. Rev. Lett. **100**, 090503 (2008)
87. Korotkov, A.N., Keane, K.: Decoherence suppression by quantum measurement reversal. Phys. Rev. A **81**, 040103 (2010)
88. Lee, J.C., et al.: Experimental demonstration of decoherence suppression via quantum measurement reversal. Opt. Express **19**, 16309–16316 (2011)
89. Kim, Y.S.: Protecting entanglement from decoherence using weak measurement and quantum measurement reversal. Nat. Phys. **8**, 117–120 (2012)
90. Kim, Y.S., Cho, Y.W., Ra, Y.S., Kim, Y.H.: Reversing the weak quantum measurement for a photonic qubit. Opt. Express **17**, 11978–11985 (2009)
91. Korotkov, A.N., Jordan, A.N.: Undoing a weak quantum measurement of a solid-state qubit. Phys. Rev. Lett. **97**, 166805 (2006)
92. Xiao, X., Li, Y.L.: Protecting qutrit-qutrit entanglement by weak measurement and reversal. Eur. Phys. J. D **67**, 204 (2013)
93. Cheong, Y.W., Lee, S.W.: Balance between information gain and reversibility in weak measurement. Phys. Rev. Lett. **109**, 150402 (2012)
94. Sun, Q., Al-Amri, M., Suhail Zubairy, M.: Reversing the weak measurement of an arbitrary field with finite photon number. Phys. Rev. A **80**, 033838 (2009)
95. Paraoanu, G.S.: Interaction-free measurements with superconducting qubits. Phys. Rev. Lett. **97**, 180406 (2006)
96. Bellomo, B., Lo Franco, R., Maniscalco, S., Compagno, G.: Entanglement trapping in structured environments. Phys. Rev. A **78**, 060302(R) (2008)
97. Barreiro, J.T., et al.: Experimental multiparticle entanglement dynamics induced by decoherence. Nat. Phys. **6**, 943–946 (2010)
98. Lo Franco, R., Bellomo, B., Andersson, E., Compagno, G.: Revival of quantum correlations without system-environment back-action. Phys. Rev. A **85**, 032318 (2012)

99. Katz, N., et al.: Reversal of the weak measurement of a quantum state in a superconducting phase qubit. Phys. Rev. Lett. **101**, 200401 (2008)
100. Sun, Q., Al-Amri, M., Davidovich, L., Zubairy, M.S.: Reversing entanglement change by a weak measurement. Phys. Rev. A **82**, 052323 (2010)
101. Paraoanu, G.S.: Extraction of information from a single quantum. Phys. Rev. A **83**, 044101 (2011)
102. Singh, P., Kumar, A.: Correlations, nonlocality and usefulness of an efficient class of two-qubit mixed entangled states. Z. Naturforsch. A **73**, 191–206 (2018)
103. Xu, X.Y., et al.: Phase estimation with weak measurement using a white light source. Phys. Rev. Lett. **111**, 033604 (2013)
104. Katz, N., et al.: Coherent state evolution in a superconducting qubit from partial-collapse measurement. Science **312**, 1498–1500 (2006)
105. Groen, J.P., et al.: Partial-measurement backaction and nonclassical weak values in a superconducting circuit. Phys. Rev. Lett. **111**, 090506 (2013)
106. Dur, W., Vidal, G., Cirac, J.I.: Three qubits can be entangled in two inequivalent ways. Phys. Rev. A **62**, 062314 (2000)
107. Agarwal, P., Pati, A.K.: Perfect teleportation and superdense coding with W states. Phys. Rev. A **74**, 062320 (2006)
108. Adhikari, S., Gangopadhyay, S.: A study of the efficiency of the class of W-states as a quantum channel. IJTP **48**, 403–408 (2009)
109. Singh, P., Adhikari, S., Kumar, A.: Usefulness of multiqubit W-type states in quantum information processing. JTEP **123**, 572–581 (2016)
110. Laflamme, R., Knill, E., Zurek, W., Catasti, P., Mariappan, S.V.S.: NMR Greenberger–Horne–Zeilinger states. Philos. Trans. R. Soc. A **356**, 1941–1948 (1998)
111. Bouwmeester, D., Pan, J.W., Daniell, M., Weinfurter, H., Zeilinger, A.: Observation of three-photon Greenberger–Horne–Zeilinger entanglement. Phys. Rev. Lett. **82**, 1345–1349 (1999)
112. Pan, J.W., Daniell, M., Gasparoni, S., Weihs, G., Zeilinger, A.: Experimental demonstration of four-photon entanglement and high-fidelity teleportation. Phys. Rev. Lett. **86**, 4435–4438 (2001)
113. Dogra, S., Dorai, K.: Arvind: experimental construction of generic three-qubit states and their reconstruction from two-party reduced states on an NMR quantum information processor. Phys. Rev. A **91**, 022312 (2015)
114. Dong, L., et al.: Nearly deterministic preparation of the perfect W state with weak cross-Kerr nonlinearities. Phys. Rev. A **93**, 012308 (2016)
115. Coffman, V., Kundu, J., Wootters, W.K.: Distributed entanglement. Phys. Rev. A **61**, 052306 (2000)
116. Wootters, W.K.: Entanglement of formation of an arbitrary state of two qubits. Phys. Rev. Lett. **80**, 2245–2248 (1998)
117. Emary, C., Beenakker, C.W.J.: Relation between entanglement measures and Bell inequalities for three qubits. Phys. Rev. A **69**, 032317 (2004)
118. Carteret, H.A., Sudbery, A.: Local symmetry properties of pure three-qubit states. J. Phys. A **33**, 4981 (2000)
119. Gottesman, D., Chuang, I.L.: Demonstrating the viability of universal quantum computation using teleportation and single-qubit operations. Nature **402**, 390–393 (1999)
120. Raussendorf, R., Browne, D.E., Briegel, H.J.: Measurement-based quantum computation on cluster states. Phys. Rev. A **68**, 022312 (2003)
121. Browne, D.E., Rudolf, T.: Resource-efficient linear optical quantum computation. Phys. Rev. Lett. **95**, 010501 (2005)
122. Hillery, M., Buzek, V., Berthiaume, A.: Quantum secret sharing. Phys. Rev. A **59**, 1829–1834 (1999)
123. Lee, H., Lim, J., Yang, H.: Quantum direct communication with authentication. Phys. Rev. A **73**, 042305 (2006)
124. Karlsson, A., Bourennane, M.: Quantum teleportation using three-particle entanglement. Phys. Rev. A **58**, 4394–4400 (1998)
125. Shi, B.S., Tomita, A.: Teleportation of an unknown state by W state. Phys. Lett. A **296**, 161–164 (2002)
126. Nielsen, M.A., Chuang, I.L.: Quantum Computation and Quantum Information. Cambridge University Press, Cambridge (2000)

127. Mermin, N.D.: Extreme quantum entanglement in a superposition of macroscopically distinct states. Phys. Rev. Lett. **65**, 1838–1840 (1990)
128. Ajoy, A., Rungta, P.: Svetlichnys inequality and genuine tripartite nonlocality in three-qubit pure states. Phys. Rev. A **81**, 052334 (2010)