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Unambiguous discrimination between linearly dependent equidistant states with multiple copies

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Abstract Linearly independent quantum states can be unambiguously discriminated, but linearly dependent ones cannot. For linearly dependent quantum states, however, if C copies of the single states are available, then they may form linearly independent states, and can be unambiguously discriminated. We consider unambiguous discrimination among $N = D+1$ linearly dependent states given that C copies are available and that the single copies span a D-dimensional space with equal inner products. The maximum unambiguous discrimination probability is derived for all C with equal a priori probabilities. For this classification of the linearly dependent equidistant states, our result shows that if C is even then adding a further copy fails to increase the maximum discrimination probability.

Keywords Minimum-error discrimination · Unambiguous discrimination · Maximum discrimination probability

1 Introduction

Orthogonal quantum states can be perfectly distinguished in quantum mechanics. However, if quantum states are nonorthogonal, they cannot be perfectly discriminated. Quantum state discrimination (QSD) [1–3] is a fundamental issue of quantum information theory. Up to date, a vast of researches has been focused on this problem and has developed rapidly recently.

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QSD can be described as follows: discrimination of a set of quantum states $\{|\rho_i\rangle\}_{i=1}^{N}$ with a prior probability η_i for $\sum_{i=1}^{N}\eta_i = 1$. In QSD, there are two fundamental strategies: One strategy is minimum-error (ME) [4] discrimination and the other is unambiguous discrimination (UD) [5–8]. In ME, the measurement of the initial states produces the correct probability, along with the error probability. One criterion is to minimize the error probability. A number of investigations have dedicated to the problems of finding the ME measurements [9–13]. The UD strategy gives no error in the identification of the initial states at the expense of producing an inconclusive result with some nonzero probability. The simplest case is to distinguish two nonorthogonal states for equal prior probability [5–7] and for arbitrary prior probability [8]. When $N \geq 3$, some special cases were given the optimal solutions [14–16]. There are other state discrimination strategies [17–19]. Minimum-error discrimination and unambiguous discrimination have been realized in experiment recently [20–22].

The UD strategy is possible if and only if quantum states to be discriminated are linearly independent [23]. A set of linearly dependent quantum states cannot be discriminated unambiguously. Suppose that we are given a set of linearly dependent quantum states with $C \geq 2$ copies of each single state. If using a single-by-single measurement on the copies, it is obvious that these linearly dependent initial states cannot be discriminated unambiguously. If, however, the quantum states with C copies become linearly independent ones, then unambiguous discrimination will be carried out by a collective (many-by-many) measurement on the C copies.

In Ref. [24], the author considered three linearly dependent states in 2-dimensional Hilbert space, i.e., the trine state, and derived the maximum discrimination probability. In this paper, we investigate $N = D + 1$ linearly dependent states in D dimensions. The inner products of linearly dependent states are equal, i.e., the linearly dependent equidistant state $\{|\psi_i\rangle\}$ with their inner product $\langle\psi_i \mid \psi_j\rangle = \frac{1}{D}e^{i\pi}$ for $i \neq j$. If we are given $C \geq 2$ copies of these states, then they will form a set of linearly independent states, i.e., $\{|\Psi_i\rangle = |\psi_i\rangle^{\otimes C}\}$ for $i = 1, 2, \ldots, D+1$, and the state $\{|\Psi_i\rangle\}$ can be unambiguously discriminated. Note that even the states $\{|\Psi_i\rangle\}$ are linearly independent, if using separate measurements on the single copies, unambiguous discrimination is also impossible because the states $\{|\psi_i\rangle\}$ are linearly dependent. However, if we perform a collective measurement on the ensemble, the states $\{|\Psi_i\rangle\}$ can then be unambiguously discriminated. We first carry a unitary transformation on $C \geq 2$ copies of these states, and then measure the output. When C is even, if adding another copy to an even number of copies, the maximum discrimination probability does not increase. Our result covers the contributions in Ref. [24]. Furthermore, our method of introducing a unitary transformation provides a physical realization of a collective measurement.

The paper is organized as follows. In Sect. 2, we introduce the linearly dependent equidistant states, and give the explicit forms of the trine and tetrad states. We derive the maximum discrimination probability in Sect. 3. The paper ends with a summary.

2 Linearly dependent equidistant states

Suppose that quantum states $|\psi_i\rangle$ for $i = 1, 2, \ldots, N$ span a D-dimensional Hilbert space $\mathscr{H}$ it is obvious that if $D < N$, then the N states are linearly dependent. In Ref. [24], the authors defined three states in 2-dimensional Hilbert space, i.e., the trine state:

$$|\psi_1\rangle = |0\rangle\,, \quad |\psi_2\rangle = -\frac{1}{2}\left(|0\rangle + \sqrt{3}\,|1\rangle\right), \quad |\psi_3\rangle = -\frac{1}{2}\left(|0\rangle - \sqrt{3}\,|1\rangle\right), \tag{1}$$

where the state $\{|0\rangle\,, |1\rangle\}$ is an orthonormal basis. Obviously, the trine states are linearly dependent, and their inner products are equal, i.e., $\langle\psi_i \mid \psi_j\rangle = s_{ij}e^{i\varphi_{ij}} = \frac{1}{2}e^{i\pi}$ for $i \neq j$. For given $C \geq 2$ copies of the trine states, the states $\left\{|\psi_i\rangle^{\otimes C}\right\}$ are linearly independent and then can be unambiguously discriminated.

The states defined by Eq. (1) may be defined as the three linearly dependent equidistant states. We next define N linearly dependent equidistant states. For $N = D+1$ states in D-dimensional Hilbert space, it is of course that they are linearly dependent. Suppose that these states have a property that their inner products are equal, we then define them as the equidistant states. We will determine the values of the inner products among them. If the states are linearly dependent, then the determinant formed by the inner products of the equidistant states must be zero, i.e., $\det[\left|a_{ij} = \langle\psi_i \mid \psi_j\rangle\right|_{N\times N}] = 0$. The matrix has the following form

$$\left|a_{ij}\right|_{N\times N} = \begin{vmatrix} 1 & s & s & \cdots & s \\ s & 1 & s & \cdots & s \\ s & s & 1 & \cdots & s \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ s & s & s & \cdots & 1 \end{vmatrix}. \tag{2}$$

It is straightforward to calculate the value of the determinant

$$\det\left[\left|a_{ij}\right|_{N\times N}\right] = [1 + (N-1)\,s]\,(1-s)^{N-1}\,. \tag{3}$$

Since $s \in (-1, 1)$, we therefore obtain the value of the inner products of linearly dependent equidistant states as

$$s = -\frac{1}{N-1} = \frac{1}{N-1}e^{i\pi} = \frac{1}{D}e^{i\pi} \tag{4}$$

If $\det[\left|a_{ij}\right|] > 0$, on the other hand, the states will be linearly independent. For $C \geq 2$ copies of each of $N = D+1$ linearly dependent equidistant states, clearly, the states $\left\{|\psi_i\rangle^{\otimes C}\right\}$ are linearly independent, since $\det[\left|a_{ij} = \langle\psi_i \mid \psi_j\rangle^{\otimes C}\right|_{N\times N}] > 0$.

For this classification of the linearly dependent equidistant states discussed above, the tetrad states in a 3-dimensional Hilbert space may be defined as follows

$$\begin{aligned} |\psi_1\rangle &= |0\rangle, |\psi_2\rangle = -\frac{1}{3}|0\rangle + \frac{\sqrt{8}}{3}|1\rangle, \\ |\psi_3\rangle &= -\frac{1}{3}|0\rangle - \frac{\sqrt{2}}{3}|1\rangle + \sqrt{\frac{2}{3}}|2\rangle, \\ |\psi_4\rangle &= -\frac{1}{3}|0\rangle - \frac{\sqrt{2}}{3}|1\rangle - \sqrt{\frac{2}{3}}|2\rangle. \end{aligned} \tag{5}$$

The inner product among the tetrad states is $s = \frac{1}{3}e^{i\pi}$.

In Ref. [24], the authors studied unambiguous discrimination of $C \geq 2$ copies of the trine states defined by Eq. (1), and presented the maximum success discrimination probability:

$$P\text{max}\left(|\psi_i\rangle^{\otimes C}\right) = \begin{cases} 1 - 2^{-C} & \text{even } C \\ 1 - 2^{-(C-1)} & \text{odd } C \end{cases} \tag{6}$$

In the next section, we generalize this situation to D-dimensional Hilbert space and derive the maximum success discrimination probability. Our result covers the contributions in Ref. [24].

3 Unambiguous discrimination among linearly dependent equidistant states with multiple copies

Due to linear dependency of the states $\{|\psi_i\rangle\}$, they cannot be unambiguously discriminated at level of one copy. Even for the linearly independent equidistant states $\left\{|\psi_i\rangle^{\otimes C}\right\}$, they cannot also be unambiguously discriminated by using separate measurement on each single copy. Therefore, we need perform a collective measurement on the set $\left\{|\psi_i\rangle^{\otimes C}\right\}$. We first act a unitary transformation on the initial set, and then measure the output states for the derivation of the minimum failure probability.

The unitary transformation U acting on the input states $|\psi_i\rangle^{\otimes C}$ for discriminating the states $|\psi_i\rangle$ and producing the output states for measurement, is defined as

$$U|\psi_i\rangle^{\otimes C} \to \sqrt{p_i}\,|\Pi_i\rangle\,|\Psi_i\rangle + \sqrt{q_i}\,|\Xi_i\rangle\,|\Psi_?\rangle. \tag{7}$$

It is obvious that $p_i + q_i = 1$. The measurement states,$|\Psi_i\rangle$ and $|\Psi_?\rangle$ for $i = 1, 2, \ldots, N$, are orthonormal states as the measurement operators; the discrimination state $|\Pi_i\rangle$ and the failure state $|\Xi_i\rangle$ for $i = 1, 2, \ldots, N$, are normalized, but are unnecessary orthogonal. After the unitary transformation acting on the initial states, we can measure the measurement states $|\Psi_i\rangle$ and $|\Psi_?\rangle$. If the state $|\Psi_i\rangle$ is measured, the probability of successfully detecting the state $|\psi_i\rangle$ is $\eta_i p_i$. Measurement of the state $|\Psi_?\rangle$ will give an inconclusive answer, so the failure probability is $\eta_i q_i$. Therefore, the average success and average failure probabilities are conventionally defined, respectively,

$$P = \sum_{i=1}^{N} \eta_i p_i, \quad Q = \sum_{i=1}^{N} \eta_i q_i, \tag{8}$$

From Eq. (8), it is obvious that $P + Q = 1$.

The unitary transformation defined by (7) yields an inner-product-preserving condition

$$D^{-C} e^{iC\pi} = \sqrt{q_i q_j} \left\langle \Xi_i \mid \Xi_j \right\rangle = \sqrt{q_i q_j} \tilde{s}_{ij} e^{i\varphi_{ij}}, \tag{9}$$

where $\tilde{s}_{ij} e^{i\varphi_{ij}} = \left\langle \Xi_i \mid \Xi_j \right\rangle$. For the UD scheme, the task is to maximize the success probability, or to minimize the failure probability. Note that unambiguous discrimination is possible only for linearly independent states [23]. For the failure states $|\Xi_k\rangle$ for $k = 1, 2, \ldots, N$, they must be linearly dependent. If not so, after the state $|\Psi_?\rangle$ is measured, the output state will be collapsed to the failure states $|\Xi_k\rangle$ with a probability q_k, and then the failure states $|\Xi_k\rangle$ can also be unambiguously discriminated successively. Until they become linearly dependent states, the failure states can no longer be unambiguously discriminated. Generally speaking, for arbitrary a *prior* probability, it is difficult to calculate the minimum failure probability under the condition given by Eq. (9). For convenience, we consider the case of equal a prior probability, i.e., $\eta_i = 1/N$. Our task is to minimize the failure probability

$$Q = \frac{1}{N} \sum_{i=1}^{N} q_i. \tag{10}$$

When $C \geq 2$ is even, the inner product of the states $\left\{ |\psi_i\rangle^{\otimes C} \right\}$ becomes $s = D^{-C}$. Equation (9) is reduced to

$$D^{-C} = \sqrt{q_i q_j} \tilde{s}_{ij} e^{i\varphi_{ij}}. \tag{11}$$

From Eq. (11), it is clear that the phase factor must be $\varphi_{ij} = 0$. We then determine the real value of $\tilde{s}_{ij}$. Using inequality of arithmetic and geometric means, Eq. (10) can be rewritten

$$Q = \frac{1}{N} \sum_{i=1}^{N} q_i \geq \left(\prod_{i=1}^{N} q_i \right)^{1/N}, \tag{12}$$

where the equality holds if and only if all the failure probabilities q_i are equal, i.e., $q_i = q$. This condition implies that $Q = q$. Therefore, we get the value of the failure probability $Q = q = D^{-C} \big/ \tilde{s}_{ij}$. Obviously, the larger the value of the inner products,

$\langle \Xi_i \mid \Xi_j \rangle$, of the failure states has, the smaller the value of the failure probability reaches. Taking $\tilde{s}_{ij} = 1$, the minimum failure probability is arrived at

$$Q_{\min} = D^{-C} \quad \text{for even } C. \tag{13}$$

In the case of $C \geq 2$ being even, the inner products of the failure states are $\langle \Xi_i \mid \Xi_j \rangle = 1$, which implies that $|\Xi_i\rangle = |\Xi_j\rangle$ and that the failure states are linearly dependent.

When $C \geq 2$ is odd, the inner products of the states $\{|\psi_i\rangle^{\otimes C}\}$ become $\langle \psi_i \mid \psi_j \rangle^{\otimes C} = D^{-C} e^{i\pi}$. Equation (9) is then reduced to

$$D^{-C} e^{i\pi} = \sqrt{q_i q_j} \tilde{s}_{ij} e^{i\varphi_{ij}}. \tag{14}$$

This equation gives two conditions, i.e., $\varphi_{ij} = \pi$ and $D^{-C} = \sqrt{q_i q_j} \tilde{s}_{ij}$. Since the failure states are linearly dependent, from Eq. (4) we must take $\tilde{s}_{ij} = D^{-1}$ when $\varphi_{ij} = \pi$. We then derive the minimum failure probability

$$Q_{\min} = D^{-(C-1)} \quad \text{for odd } C \tag{15}$$

We can also obtain Eq. (15) without the condition that the failure states must be linearly dependent, which comes from the result given by Ref. [23]. For the equidistant states with the inner product $\tilde{s}_{ij} e^{i\pi}$, when $\tilde{s}_{ij} \in [0, D^{-1})$ they are linearly independent, and when $\tilde{s}_{ij} = D^{-1}$ they are linearly dependent. From the equation $D^{-C} = \sqrt{q_i q_j} \tilde{s}_{ij}$, we have $q = q_i = D^{-C} / \tilde{s}_{ij}$. Obviously, the larger the value of $\tilde{s}_{ij}$ has, the smaller the value of q reaches. So we must take $\tilde{s}_{ij} = D^{-1}$, but do not take $\tilde{s}_{ij} \in [0, D^{-1})$. The value $\tilde{s}_{ij} = D^{-1}$ implies that the failure states are linearly dependent.

We here derive the minimum failure probability, which covers the contributions in Ref. [24]. Intuitively, if we have more copies of the state, we can discriminate it better. This is indeed the case for linearly independent quantum states. From our results, it can be seen that the probability is the same for the numbers of even C and odd $C + 1$ copies. In [24], the authors proposed one of collective measurements, i.e., a two-by-two measurement. The success probability for all C copies by their method is optimal $P_{\max} = 1 - \left[Q_{\min}\left(|\psi_i\rangle^{\otimes 2}\right)\right]^{C/2}$. By exploiting their collective measurement, all C copies must be measured. From our method of introducing unitary transformation performed on the initial states, we merely need to measure the output states $|\Psi_i\rangle$ and $|\Psi_?\rangle$, formed only by a small quality of particles defined by Eq. [7]. This measurement for optimum discrimination is clearly convenient from a practical perspective.

4 Summary

In this paper, we have investigated unambiguous discrimination among linearly dependent states with multiple copies. Unambiguous discrimination is impossible if using separate measurements on the single copies of linearly dependent equidistant states. If the C copy states are linearly independent, then unambiguous discrimination will be possible by carrying out a collective measurement on a small quality of particles. Meanwhile, our method of introducing a unitary transformation provides a physical realization of a collective measurement.

We have studied a classification of linearly dependent equidistant states. If $C \geq 2$ copies of these states are available, then they construct a set of linearly independent states, and can be unambiguously discriminated. We have derived the optimal maximum success probability, which increase as the number of the copies increase. For the numbers of even C and odd $C + 1$ copies, however, the probability is the same, which is deferent from unambiguous discrimination of linearly independent quantum states.

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