Quantum Inf Process (2018) 17:69
https://doi.org/10.1007/s11128-018-1838-5

Entanglement-assisted quantum MDS codes from negacyclic codes

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Received: 27 September 2017 / Accepted: 5 February 2018 / Published online: 14 February 2018

Abstract The entanglement-assisted formalism generalizes the standard stabilizer formalism, which can transform arbitrary classical linear codes into entanglement-assisted quantum error-correcting codes (EAQECCs) by using pre-shared entanglement between the sender and the receiver. In this work, we construct six classes of q-ary entanglement-assisted quantum MDS (EAQMDS) codes based on classical negacyclic MDS codes by exploiting two or more pre-shared maximally entangled states. We show that two of these six classes q-ary EAQMDS have minimum distance more larger than $q+1$. Most of these q-ary EAQMDS codes are new in the sense that their parameters are not covered by the codes available in the literature.

Keywords Entanglement-assisted quantum error-correcting codes (EAQECCs) · MDS codes · Negacyclic codes · Cyclotomic

1 Introduction

Quantum error-correcting codes (QECCs) were introduced to protect quantum information from decoherence during quantum computations [1]. The stabilizer formalism allows standard quantum codes to be constructed from dual-containing (or self-orthogonal) classical codes [2]. However, the dual-containing condition forms a barrier in the development of quantum coding theory. Recently, a breakthrough is the entanglement-assisted (EA) stabilizer formalism proposed by Brun et al. in Ref. [3]. They prove that if shared entanglement is available between the sender and receiver,

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non-dual-containing classical quaternary codes can be used to construct EAQECCs, this leads to a more general framework for construction of quantum codes [4–6]. Currently, many works have focused on the construction of binary EAQECCs based on classical binary or quaternary linear codes, see [7–14]. Just as in the classical error-correcting codes and QECCs, EAQECCs over higher alphabets have many wide applications, such as constructing easily decodable binary EAQECCs. However, little attention has been paid to non-binary EAQECCs, let alone EA-quantum MDS codes which can achieve entanglement-assisted quantum singleton bound[3].

Let q be a prime power. A q-ary $[[n, k, d; c]]$ EAQECC that encodes k information qubits into n channel qubits with the help of c pairs of maximally entangled Bell states (ebits) can correct up to $\lfloor \frac{d-1}{2} \rfloor$ errors, where d is the minimum distance of the code. If $c = 0$, then it is called a q-ary standard $[[n, k, d]]$ quantum code $\mathcal{Q}$. We denote a q-ary $[[n, k, d; c]]$ EAQECC by $[[n, k, d; c]]_q$, and q-ary $[[n, k, d]]$ QECC by $[[n, k, d]]_q$.

As in classical coding theory, one of the central tasks in quantum coding theory is to construct good quantum codes and EA-quantum codes.

Theory 1.1 [3](EA-Quantum Singleton Bound) *An* $[[n, k, d; c]]_q$ *EAQECC satisfies*

$$n + c - k \geq 2(d - 1),$$

where $0 \leq c \leq n - 1$.

A EAQECC achieving this bound is called a EA-quantum maximum-distance-separable (EAQMDS) code. If $c = 0$, then this bound is quantum singleton bound, and a code achieving the bound is called quantum maximum-distance-separable (QMDS) code. Just as in the classical linear codes, QMDS codes and EAQMDS codes form an important family of quantum codes. Constructing QMDS codes and EAQMDS codes had become a central topic for quantum error correction codes in rent years. Many classes of QMDS codes have been constructed by different methods, in particular the constructions obtained from constacyclic codes or negacyclic codes containing their Hermitian dual over F_{q^2} [15–23]. According to the MDS conjecture in [24], the maximum-distance-separable (MDS) code cannot exceed $q^2 + 1$. Many QMDS codes with lengths between $q + 1$ and $q^2 + 1$ have been constructed [16–23,25]. However, the problem of constructing QMDS codes with length n larger than $q + 1$ is much more difficult.

It seems that there is a barrier for constructing more QMDS codes with distance larger than $q + 1$. For larger distance than $q + 1$ of code length $n \leq q^2 + 1$, one need to construct a EAQMSD code.

The following Proposition is one of the most frequently used construction methods.

Proposition 1.2 [3,5] *If* $\mathcal{C} = [n, k, d]_{q^2}$ *is a classical code over* F_{q^2} *and* H *is its parity check matrix , then* $\mathcal{C}^{\perp_h}$ *EA stabilizes an* $[[n, 2k - n + c, d; c]]_q$ *EAQECC, where* $c =$*rank*$(HH^{\dagger})$ *is the number of maximally entangled states required and* $H^{\dagger}$ *is the conjugate matrix of* H *over* F_{q^2}.

Until now, little attention has been paid to q-ary EA-quantum MDS codes. In [26], Fan et al. proposed several constructions of q-ary EAQMDS codes with minimum distance greater than $q+1$ based on classical MDS codes.

In this paper, we propose a concept of decomposition of the defining set of negacyclic codes. Recently, Chen et al. [30] proposed a same concept at the very same moment. Based on the concept, they construct some EA-quantum MDS codes which are all different with the codes in this paper. More precisely, based on concept of decomposition of the defining set of negacyclic codes, we construct several classes of EA-quantum MDS codes as follows:

(1) Let q be an odd prime power of the form $q = atm + 1$, a be even, or a be odd and t be even, then there exists a q-ary $[[\frac{q^2-1}{at}, \frac{q^2-1}{at} - 2d + 2, d; 0]]$ EA-quantum MDS codes, where $2 \le d \le (\frac{at}{2} + 1)m + 1$; there exists a q-ary $[[\frac{q^2-1}{at}, \frac{q^2-1}{at} - 2d + 4, d; 2]]$- EA-quantum MDS codes, where $(\frac{at}{2} + 1)m + 2 \le d \le (\frac{at}{2} + 2)m + 1$; and there exists a q-ary $[[\frac{q^2-1}{at}, \frac{q^2-1}{at} - 2d + 6, d; 4]]$-EA-quantum MDS codes, where $(\frac{at}{2} + 2)m + 2 \le d \le (\frac{at}{2} + 3)m + 1$.

(2) Let q be an odd prime power of the form $q = 30m + 11$, then there exists a q-ary $[[\frac{q^2-1}{30}, \frac{q^2-1}{30} - 2d + 4, d; 2]]$- EA-quantum MDS codes, where $8m + 4 \le d \le 11m + 5$; and there exists a q-ary $[[\frac{q^2-1}{30}, \frac{q^2-1}{30} - 2d + 6, d; 4]]$- EA-quantum MDS codes, where $11m + 6 \le d \le 14m + 7$.

(3) Let q be an odd prime power of the form $q = 30m + 19$, then there exists a q-ary $[[\frac{q^2-1}{30}, \frac{q^2-1}{30} - 2d + 4, d; 2]]$- EA-quantum MDS codes, where $8m + 6 \le d \le 11m + 7$; there exists a q-ary $[[\frac{q^2-1}{30}, \frac{q^2-1}{30} - 2d + 6, d; 4]]$- EA-quantum MDS codes, where $11m + 8 \le d \le 13m + 8$; and there exists a q-ary $[[\frac{q^2-1}{30}, \frac{q^2-1}{30} - 2d + 8, d; 6]]$- EA-quantum MDS codes, where $13m + 9 \le d \le 16m + 10$.

(4) Let q be an odd prime power of the form $q = 12m + 5$, then there exists a q-ary $[[\frac{q^2-1}{12}, \frac{q^2-1}{12} - 2d + 4, d; 2]]$- EA-quantum MDS codes, where $5m + 3 \le d \le 7m + 3$; and there exists a q-ary $[[\frac{q^2-1}{12}, \frac{q^2-1}{12} - 2d + 6, d; 4]]$- EA-quantum MDS codes, where $7m + 4 \le d \le 8m + 3$.

(5) Let q be an odd prime power of the form $q = 10m + 3$. (a) If $m = 2t + 1$ is odd, then there exists a q-$[[\frac{q^2+1}{5}, \frac{q^2+1}{5} - 2d + 6, d; 4]]$- EA-quantum MDS codes, where $4m + 3 \le d \le 6m + 1$ is odd and $6m + 4 \le d \le 10m + 4$ is even. (b) If $m = 2t$ is even, then there exists a q-ary $[[\frac{q^2+1}{5}, \frac{q^2+1}{5} - 2d + 3, d; 1]]$-EA-quantum MDS codes, where $2 \le d \le 8m + 1$ is even; there exists a q-ary $[[\frac{q^2+1}{5}, \frac{q^2+1}{5} - 2d + 6, d; 4]]$-EA-quantum MDS codes, where $4m + 3 \le d \le 6m + 1$ is odd; and there exists a q-ary $[[\frac{q^2+1}{5}, \frac{q^2+1}{5} - 2d + 7, d; 5]]$-EA-quantum MDS codes, where $8m + 4 \le d \le 12m + 4$ is even.

(6) Let q be an odd prime power of the form $q = 10m + 7$. (a)If $m = 2t + 1$ is odd, then there exists a q-$[[\frac{q^2+1}{5}, \frac{q^2+1}{5} - 2d + 6, d; 4]]$- EA-quantum MDS codes, where $8m + 7 \le d \le 14m + 11$ is odd; and $6m + 6 \le d \le 10m + 8$ is even.(b) If $m = 2t$ is even, then there exists a q-$[[\frac{q^2+1}{5}, \frac{q^2+1}{5} - 2d + 3, d; 1]]$-EA-quantum MDS codes, where $2 \le d \le 8m + 6$ is even; there exists a q-

$[[\frac{q^2+1}{5}, \frac{q^2+1}{5} - 2d + 6, d; 4]]$- EA-quantum MDS codes, where $8m + 7 \leq d \leq 14m + 11$ is odd; and there exists a q-$[[\frac{q^2+1}{5}, \frac{q^2+1}{5} - 2d + 7, d; 5]]$- EA-quantum MDS codes, where $8m + 8 \leq d \leq 12m + 8$ is even.

The first class of EAQMDS codes has minimum distance upper limit greater than $\frac{q}{2} + 1$ by consuming a few ebits. EAQMDS codes in (2)–(4) have minimum distance upper limit closed to $\frac{q}{2} + 1$. EAQMDS codes in (5) and (6) have minimum distance upper limit more greater than $q + 1$. Briefly, most of these EAQMDS codes are new in the sense that their parameters are not covered by the codes available in the literature.

This paper is organized as follows. In Sect. 2, we introduce some basic notations and definitions of classical negacyclic codes and EAQECCs. In Sect. 3, we give some new classes of EA-quantum MDS codes. The conclusion is given in Sect. 4.

2 Preliminaries

In this section, we review some basic results on negacyclic codes, BCH codes, decomposition of defining sets of codes and EAQECCs for the purpose of this paper. Details on BCH codes and negacyclic codes can be found in standard textbook on coding theory [27], and for EAQECCs please see Refs. [3–9].

Let p be a prime number and q a power of p, i.e., $q = p^l$ for some $l > 0$. F_{q^2} denotes the finite field with q^2 elements. For any $\alpha \in F_{q^2}$, the conjugation of α is denoted by $\overline{\alpha} = \alpha^q$. Given two vectors $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ and $\mathbf{y} = (y_1, y_2, \ldots, y_n) \in F_{q^2}^n$, their Hermitian inner product is defined as

$$(\mathbf{x}, \mathbf{y})_h = \sum \overline{x_i} y_i = \overline{x_1} y_1 + \overline{x_2} y_2 + \cdots + \overline{x_n} y_n.$$

For a linear code $\mathcal{C}$ over F_{q^2} of length n, the Hermitian dual code $\mathcal{C}^{\perp_h}$ is defined as

$$\mathcal{C}^{\perp_h} = \{x \in F_{q^2}^n | (x, y)_h = 0, \forall y \in \mathcal{C}\}$$

If $\mathcal{C} \subseteq \mathcal{C}^{\perp_h}$, then $\mathcal{C}$ is called a Hermitian dual-containing code, and $\mathcal{C}^{\perp_h}$ is called a Hermitian self-orthogonal code.

We now recall some results about classical negacyclic codes. For any vector $(c_0, c_1, \ldots, c_{n-1}) \in F_{q^2}^n$, if a q^2-ary linear code $\mathcal{C}$ of length n is invariant under the permeation of F_{q^2}, i.e.,

$$\lambda(c_0, c_1, \ldots, c_{n-1}) = (\lambda c_{n-1}, c_0, \ldots, c_{n-2}),$$

where λ is a nonzero element of F_{q^2}, then $\mathcal{C}$ is a constacyclic code. If $\lambda = 1$, then $\mathcal{C}$ is called a cyclic code , and if $\lambda = -1$, then $\mathcal{C}$ is called a negacyclic code.

For a negacyclic code $\mathcal{C}$, each codeword $c = (c_0, c_1, \ldots, c_{n-1})$ is customarily represented in its polynomial form: $c(x) = c_0 + c_1 x + \cdots + c_{n-1} x_{n-1}$, and the code $\mathcal{C}$ is in turn identified with the set of all polynomial representations of its codewords. The proper context for studying negacyclic codes is the residue class ring $\mathcal{R}_n =$

$\mathbb{F}_q[x]/(x^n+1)$. $xc(x)$ corresponds to a negacyclic shift of $c(x)$ in the ring $\mathcal{R}_n$. As we all know, a linear code $\mathcal{C}$ of length n over F_{q^2} is negacyclic if and only if C is an ideal of the quotient ring $\mathcal{R}_n=\mathbb{F}_q[x]/(x^n+1)$. It follows that $\mathcal{C}$ is generated by monic factors of (x^n+1), i.e., $\mathcal{C}=\langle f(x)\rangle$ and $f(x)|(x^n+1)$. The $f(x)$ is called the generator polynomial of $\mathcal{C}_n$.

Let $gcd(n,q)=1$ and m be the multiplicative order of q^2 modulo $2n$. Let $\beta\in F_{q^{2m}}$ be a primitive $2n$-th root of unity. Then, ξ be a primitive $2n$-th root of unity such that $\xi=\beta^2\in F_{q^{2m}}$. Hence, $x^n+1=\prod_{i=0}^{n-1}(x-\beta^{2i+1})$. Let Z_{2n} denote the set of odd integers from 1 to $2n$, i.e., $Z_{2n}=\{1,3,\ldots,2n-1\}$. For each $i\in Z_{2n}$, let s be an integer with $0\le s<2n$, the q^2-cyclotomic coset modulo $2n$ that contains s is defined by the set $C_s=\{s,sq^2,sq^{2\cdot 2},\ldots,sq^{2(k-1)}\}$ (mod $2n$), where k is the smallest positive integer such that $xq^{2k}\equiv x$ (mod $2n$).

The defining set of a negacyclic code $\mathcal{C}=<g(x)>$ of length n is the set $T=\{i\in Z_{2n}|\beta^i$ is a root of $g(x)\}$. We can see that the defining set T is a union of some q^2-cyclotomic cosets module $2n$ and $dim(\mathcal{C})=n-|T|$.

Lemma 2.1 [18] *Let $\mathcal{C}$ be a q^2-ary negacyclic code of length n with defining set T. Then, $\mathcal{C}$ contains its Hermitian dual code if and only if $T\bigcap T^{-q}=\emptyset$, where T^{-q} denotes the set $Z^{-q}=\{-qz(mod\ 2n)|z\in T\}$.*

Let $\mathcal{C}$ be a negacyclic code with a defining set $T=\bigcup_{s\in S}C_s$. Denoting $T^{-q}=\{2n-qs|s\in T\}$, then we can deduce that the defining set of $\mathcal{C}^{\perp_h}$ is $T^{\perp_h}=\mathbb{Z}_n\backslash T^{-q}$, see Ref. [16].

Since there is a striking similarity between cyclic codes and negacyclic code, we give a correspondence defining of skew-symmetric and skew-asymmetric as follows.

A cyclotomic coset C_s is *skew-symmetric* if $2n-qs$ mod $2n\in C_s$, and otherwise is skew-asymmetric. *Skew-asymmetric cosets* C_s and C_{2n-qs} come in pair, and we use (C_s,C_{2n-qs}) to denote such a pair.

Thus, one has the following lemma.

Lemma 2.2 [28] *If $\mathcal{C}$ is a negacyclic code of length n over F_{q^2} with defining set T, then $\mathcal{C}^{\perp_h}\subseteq\mathcal{C}$ if and only if one of the following holds:*

(1) *$T\cap T^{-q}=\emptyset$, where $T^{-q}=\{2n-qs\mid s\in T\}$.*
(2) *If $i,j,k\in T$, then C_i is a skew-asymmetric coset and (C_j,C_k) is not a skew-asymmetric cosets pair.*

Using above-mentioned Lemma 2.2, one can get that $\mathcal{C}^{\perp_h}\subseteq\mathcal{C}$ can be described by the relationship of its cyclotomic coset C_s. Firstly, we introduce a fundamental definition.

Definition 2.3 [12] Let $\mathcal{C}$ be a negacyclic code of length n with defining set T. Denote $T_{ss}=T\cap T^{-q}$ and $T_{sas}=T\setminus T_{ss}$, where $T^{-q}=\{2n-qx|x\in T\}$. $T=T_{ss}\cup T_{sas}$ is called decomposition of the defining set of $\mathcal{C}$.

To determine T_{ss} and T_{sas}, we give the following lemma to characterize them.

Lemma 2.4 [12] *Let $\mathcal{C}$ be a negacyclic code of length n over F_{q^2} with defining set T, $T=T_{ss}\cup T_{sas}$ be decomposition of T.*

(1) *If* $i, j \in T_{sas}$, *then* C_i *is skew-asymmetric coset, and* C_i *and* C_j *cannot form a skew-asymmetric cosets pair.*
(2) *If* $l \in T_{ss}$, *then either* C_l *is a skew-symmetric coset, or* C_l *is a skew-asymmetric coset and there is a* $p \in T$ *such that* C_l *and* C_p *form a skew-asymmetric cosets pair.*

To determine T_{ss} and T_{sas}, we give the following lemma to characterize them.

Lemma 2.5 [12,18,29] *Let* $gcd(q, n) = 1$, $ord_{2n}(q^2) = m$, $0 \leq x, y, z \leq n-1$.

(1) C_x *is skew-symmetric if and only if there is a* $t \leq \lfloor \frac{m}{2} \rfloor$ *such that* $x \equiv xq^{2t+1}(mod\ 2n)$.
(2) *If* $C_y \neq C_z$, (C_y, C_z) *form a skew-asymmetric pair if and only if there is a* $t \leq \lfloor \frac{m}{2} \rfloor$ *such that* $y \equiv zq^{2t+1}$ *(mod* n*) or* $z \equiv yq^{2t+1}(mod\ 2n)$.

Using decomposition of the defining set T of a negacyclic code $\mathcal{C}$, one can give a decomposition of $\mathcal{C}^{\perp_h}$ as follow.

Lemma 2.6 [12] *Let* $\mathcal{C}$ *be a negacyclic code with defining set* T, $T = T_{ss} \cup T_{sas}$ *be decomposition of* T. *Denote the negacyclic codes with defining set* T_{sas} *and* T_{ss} *be* $\mathcal{C}_R$ *and* $\mathcal{C}_E$, *respectively. Then,* $\mathcal{C}_R^{\perp_h} \subseteq \mathcal{C}_R$, $\mathcal{C}_E \cap \mathcal{C}_E^{\perp_h} = \{0\}$, $\mathcal{C}_R^{\perp_h} \subset \mathcal{C}_E$, $\mathcal{C}_R \cap \mathcal{C}_E = \mathcal{C}$ *and* $\mathcal{C}_R^{\perp_h} + \mathcal{C}_E^{\perp_h} = \mathcal{C}^{\perp_h}$.

Lemma 2.7 [12] *Let* T *be a defining set of a negacyclic code* $\mathcal{C}$, $T = T_{ss} \cup T_{sas}$ *be decomposition of* T. *Using* $\mathcal{C}^{\perp_h}$ *as EA stabilizer, the optimal number of needed ebits is* $c = \mid T_{ss} \mid$.

Proof According to Definition 2.3, we denote the defining sets of negacyclic codes $\mathcal{C}_1$ and $\mathcal{C}_2$ into T_{ss} and T_{sas}, respectively. The parity check matrix of $\mathcal{C}_1$ and $\mathcal{C}_2$ is H_1 and H_2, respectively. Let $H = \begin{pmatrix} H_1 \\ H_2 \end{pmatrix}$ be the parity check matrix of $\mathcal{C}$. Then,

$$HH^{\dagger} = \begin{pmatrix} H_1 H_1^{\dagger} & H_1 H_2^{\dagger} \\ H_2 H_1^{\dagger} & H_2 H_2^{\dagger} \end{pmatrix}.$$

Since H_2 is the parity check matrix of $\mathcal{C}_2$ with defining set of T_{sas}, $H_2 H_2^{\dagger} = 0$. Because of $\mathcal{C}_1^{\perp h} \subseteq \mathcal{C}_2$, $H_1 H_2^{\dagger} = 0$ and $H_2 H_1^{\dagger} = 0$. Therefore,

$$HH^{\dagger} = \begin{pmatrix} H_1 H_1^{\dagger} & 0 \\ 0 & 0 \end{pmatrix}.$$

According to Refs. [3–5], one obtains that $c = rank(HH^{\dagger}) = rank(H_1 H_1^{\dagger})$. Since H_1 is the parity check matrix of $\mathcal{C}_2$ with defining set of T_{ss}, H_1 is a full-rank matrix. Hence, $c = rank(H_1 H_1^{\dagger}) = |T_{ss}|$. □

Lemma 2.8 *Let* $\mathcal{C}$ *be an* $[n, k, d]_{q^2}$ *negacyclic code with defining set* T, *and the decomposition of* T *be* $T = T_{ss} \cup T_{sas}$. *Then,* $\mathcal{C}^{\perp_h}$ *EA stabilizes an* q*-ary* $[[n, n - 2|T| + |T_{ss}|, d \geq \delta; |T_{ss}|]]$ *EAQECC.*

Proof The dimension of $\mathcal{C}$ is $k = n - |T|$. From Proposition 1 and Lemma 2.6, we know $\mathcal{C}^{\perp_h}$ EA stabilizes an EAQECC with parameters $[[n, 2k - n + c, d; c]] = [[n, n - 2|T| + |T_{ss}|, d; |T_{ss}|]]$.

If $\mathcal{C}$ is a negacyclic BCH code $BCH(n, \delta)$, denote its defining set T as $T = T(\delta)$, the decomposition of T as $T(\delta) = T_{ss}(\delta) \cup T_{sas}(\delta)$. According to Lemma 2.7, $\mathcal{C}^{\perp_h}$ EA stabilizes an EAQECC with parameters $[[n, k^{ea}, d; c]] = [[n, n - 2|T(\delta)| + |T_{ss}(\delta)|, d \geq \delta; |T_{ss}|]]$. In the following two sections, we will discuss how to determine $|T_{ss}(\delta)|, |T(\delta)|$. □

3 New EA-quantum MDS codes

3.1 New EA-quantum MDS codes of length $n = \frac{q^2-1}{at}$

In this subsection, we construct some classes of q-ary EA-quantum MDS codes of length $n = \frac{q^2-1}{at}$, where q be an odd prime power of the form $q = atm + 1$, a be a even number, or a be an odd number and t be a even number. Since $2n|q^2 - 1$, then for each odd x in the range $1 \leq x \leq 2n$, the q^2-cyclotomic coset C_x modulo $2n$ is $C_x = x$.

Lemma 3.1 *Let q be an odd prime power of the form $q = atm+1$, a be a even number, or a be an odd number and t be a even number, $n = \frac{q^2-1}{at}$. If $\mathcal{C}$ is a q^2-ary negacyclic code of length n with defining set $T = \bigcup_{j=0}^{s} C_{1+2j}$, where $0 \leq s \leq (\frac{at}{2} + 1)m - 1$, then $\mathcal{C}^{\perp h} \subseteq \mathcal{C}$.*

Proof For $0 \leq s \leq (\frac{at}{2}+1)m - 1$, it is sufficient to prove $T \cap (-qT) = \emptyset$. According to Lemma 2.2 and Definition 2.3, one obtains that $\mathcal{C}^{\perp_h} \subseteq \mathcal{C}$ if and only if there is no skew-symmetric cyclotomic coset and any two cyclotomic cosets do not form a skew-asymmetric pair in the defining set T. Suppose there exist integers $0 \leq x \leq y \leq (at + 2)m - 1$ such that $C_x = -qC_y$, that is $x \equiv -qy \bmod 2n$. In other words $x + qy \equiv 0 \bmod 2n$. Since $q = atm + 1$, $2n = 2m(q + 1)$.

If $1 \leq x \leq y \leq 2m-1$, then $0 \leq x+qy \leq (2m-1)(q+1) \leq 2m(q+1)-(q+1) < 2n$, a contradiction.

Similarly, we have for $1 \leq i \leq \frac{at}{2}$, if $[at + 2 - 2(i - 1)]m + 1 \leq x < y \leq (at + 2 - 2i)m - 1$, then $(i - 1)2n < [(at - 2i)(m + 1)](q + 1) \leq x + qy \leq [(at + 2 - 2i)m - 1](q + 1) \leq 2im(q + 1) - (q + 1) < i2n$, a contradiction.

Hence, for $0 \leq s \leq (\frac{at}{2} + 1)m - 1$, there are no skew-symmetric cyclotomic cosets and skew-asymmetric cosets pairs in defining set $T = \bigcup_{j=0}^{s} C_{1+2j}$. It means that $T_{ss} = \emptyset$. $\mathcal{C}^{\perp h} \subseteq \mathcal{C}$ holds. □

Theory 3.2 *Let q be an odd prime power of the form $q = atm + 1$, a be even, or a be odd and t be even, $n = \frac{q^2-1}{at}$. Then, there exists a q-ary $[[\frac{q^2-1}{at}, \frac{q^2-1}{at} - 2d + 2, d]]$ quantum MDS codes, where $2 \leq d \leq (\frac{at}{2} + 1)m + 1$.*

Proof Consider the negacyclic codes over F_{q^2} of length $n = \frac{q^2-1}{at}$ with defining set $T = \bigcup_{i=0}^{s} C_{1+2i}$, where $0 \leq s \leq (\frac{at}{2} + 1)m - 1$ for q be an odd prime power of the form $q = atm + 1$, a be even, or a be odd and t be even. By Lemma 3.1, there is $\mathcal{C}^{\perp_h} \subseteq \mathcal{C}$. Since every q^2-cyclotomic coset C_x has exactly one element and x must be odd number, we can obtain that T consists of $s+1$ integers $\{1, 3, \dots, 1+2s\}$. It implies that $\mathcal{C}$ has minimum distance at least $s+2$. Hence, $\mathcal{C}$ is a q^2-ary negacyclic code with parameters $[n, n-(s+1), \geq s+2]$. Combining the Hermitian construction with quantum singleton bound, we can obtain a quantum MDS code with parameters $[[\frac{q^2-1}{at}, \frac{q^2-1}{at} - 2d + 2, d]]$, where $2 \leq d \leq (\frac{at}{2}+1)m + 1$, q be an odd prime power of the form $q = atm + 1$, a be even, or a be odd and t be even. □

Example 1 Let $a = 3, t = 2$. Then, $q = 19$, $n = 60$ applying Theory 3.2 produces quantum MDS codes with parameters $[[60, 58, 2]]_{19}$, $[[60, 56, 3]]_{19}$, $[[60, 54, 4]]_{19}$, $[[60, 52, 5]]_{19}$,$[[60, 50, 6]]_{19}$, $[[60, 48, 7]]_{19}$, $[[60, 46, 8]]_{19}$, $[[60, 44, 9]]_{19}$, $[[60, 42, 10]]_{19}$, $[[60, 40, 11]]_{19}$, $[[60, 38, 12]]_{19}$, $[[60, 36, 13]]_{19}$.

Lemma 3.3 *Let q be an odd prime power of the form $q = atm + 1$, a be a even number, or a be an odd number and t be a even number, $n = \frac{q^2-1}{at}$.*

(i) *For $1 \leq i \leq 3$, $(C_{1+(at+2i)m}, C_{1+(2im-2)})$ forms a skew-asymmetric pair.*
(ii)

$$|T_{ss}(\delta)| = \begin{cases} 0, & \text{if } 2 \leq \delta \leq (\frac{at}{2}+1)m + 1; \\ 2, & \text{if } (\frac{at}{2}+1)m + 2 \leq \delta \leq (\frac{at}{2}+2)m + 1; \\ 4, & \text{if } (\frac{at}{2}+2)m + 2 \leq \delta \leq (\frac{at}{2}+3)m + 1. \end{cases}$$

Proof (i) For $q = atm + 1$, then $n = \frac{q^2-1}{at} = m(q+1)$. $2n - q(1 + (2im-2)) = 2m(q+1) - [2mi(q+1) - 2mi - q] \equiv 1 + (at+2i)m \bmod 2m(q+1)$. Hence, for $1 \leq i \leq 3$, $(C_{1+(at+2i)m}, C_{1+(2im-2)})$ forms a skew-asymmetric pair.

(ii) According to Lemma 3.1, for $2 \leq \delta \leq (\frac{at}{2}+1)m + 1$ in the defining set $T(\delta) = \bigcup_{j=0}^{\delta-1} C_{1+2j}$, we have $T_{ss}(\delta) = \emptyset$.

□

For $(\frac{at}{2}+1)m + 2 \leq \delta \leq (\frac{at}{2}+2)m + 1$, according to Lemma 2.3 and Lemma 2.4, one determine the set T_{ss} and T_{sas} by decomposition of the defining set T. In T_{sas}, there is no skew-symmetric cyclotomic coset and any two cyclotomic cosets do not form a skew-asymmetric pair; and in T_{ss}, there are skew-symmetric cyclotomic cosets or there exist skew-asymmetric pairs. Suppose there exist integers $y \in [(at+2)m + 3, (at+4)m - 1]$ and $x \in [0, (at+2)m - 1] \cup [(at+2)m + 3, (at+4)m - 1]$ such that $x \equiv -qy \bmod 2n$. We find a contradiction as follows.

If $(at+2)m + 1 \leq x < y \leq (at+4)m - 1$, then $(\frac{at}{2}+1)2n < (\frac{at}{2}+1)2m(q+1) + 3(q+1) = [(at+2)m+3](q+1) \leq x + yq \leq [(at+4)m-1](q+1) = (\frac{at}{2}+2)2m(q+1) - (q+1) < (\frac{at}{2}+2)2n$, that means that $(\frac{at}{2}+1)2n < x + yq < (\frac{at}{2}+2)2n$, a contradiction. And if $(at+2)m + 3 \leq y \leq (at+4)m - 1, 0 \leq y \leq (at+2)m - 1$, then $(\frac{at}{2}+1)2n < (\frac{at}{2}+1)2m(q+1) + 2atm + 2m + 3 = [(at+2)m+3]q \leq x + yq \leq (at+2)m - 1 + [(at+4)m - 1]q = (\frac{at}{2}+2)2n - q < (\frac{at}{2}+2)2n$, a contradiction.

Similarly, for $(\frac{at}{2}+2)m + 2 \leq \delta \leq (\frac{at}{2}+3)m + 1$, suppose there exist integers $y \in [(at+4)m + 3, (at+6)m - 1]$ and $x \in [0, (at+2)m - 1] \cup [(at+2)m +$

$3, (at+4)m-1] \cup [(at+4)m+3, (at+6)m-1]$ such that $x \equiv -qy \bmod 2n$. Using the same methods, we can deduce that $(\frac{at}{2}+2)2n < x+qy < (\frac{at}{2}+3)2n$, a contradiction.

Theory 3.4 *Let q be an odd prime power of the form $q = atm+1$, a be even, or a be odd and t be even, $n = \frac{q^2-1}{at}$. Then, there exists a q-ary $[[\frac{q^2-1}{at}, \frac{q^2-1}{at}-2d+4, d; 2]]$-EA-quantum MDS codes, where $(\frac{at}{2}+1)m+2 \le d \le (\frac{at}{2}+2)m+1$; and there exists a q-ary $[[\frac{q^2-1}{at}, \frac{q^2-1}{at}-2d+6, d; 4]]$- EA-quantum MDS codes, where $(\frac{at}{2}+2)m+2 \le d \le (\frac{at}{2}+3)m+1$.*

Proof Consider the negacyclic codes over F_{q^2} of length $n = \frac{q^2-1}{at}$ with defining set $T = \bigcup_{i=0}^{s} C_{1+2i}$, where $0 \le s \le (\frac{at}{2}+3)m-1$ for q be an odd prime power of the form $q = atm+1$, a be even, or a be odd and t be even. By Lemma 3.3, there is $c = |T_{ss}(\delta)| = 2$ if $(\frac{at}{2}+1)m \le s \le (\frac{at}{2}+2)m-1$ and $c = |T_{ss}(\delta)| = 4$ if $(\frac{at}{2}+2)m \le s \le (\frac{at}{2}+3)m-1$. Since every q^2-cyclotomic coset C_x has exactly one element and x must be odd number, we can obtain that T consists of $s+1$ integers $\{1, 3, \ldots, 1+2s\}$. It implies that $\mathcal{C}$ has minimum distance at least $s+2$. Hence, $\mathcal{C}$ is a q^2-ary negacyclic code with parameters $[n, n-(s+1), \ge s+2]$. Combining Lemma 2.8 with EA-quantum singleton bound, we can obtain a quantum MDS code with parameters $[[\frac{q^2-1}{at}, \frac{q^2-1}{at}-2d+4, d; 2]]$, where $(\frac{at}{2}+1)m+2 \le d \le (\frac{at}{2}+2)m+1$; $[[\frac{q^2-1}{at}, \frac{q^2-1}{at}-2d+6, d; 4]]$, where $(\frac{at}{2}+2)m+2 \le d \le (\frac{at}{2}+3)m+1$, for q be an odd prime power of the form $q = atm+1$, a be even, or a be odd and t be even. □

Example 2 Let $a = 3, t = 2$. Then, $q = 19, n = 60$ applying Theory 3.4 produces:

(1) new $2-ebits$ EA-quantum MDS codes with parameters $[[60, 36, 14; 2]]_{19}$, $[[60, 34, 15; 2]]_{19}$, $[[60, 32, 16; 2]]_{19}$.

(2) new 4-ebits EA-quantum MDS codes with parameters $[[60, 32, 17; 4]]_{19}$, $[[60, 30, 18; 4]]_{19}$, $[[60, 28, 19; 4]]_{19}$.

3.2 New EA-quantum MDS codes of length $n = \frac{q^2-1}{2s_1s_2}$

In this subsection, we construct some classes of q-ary EA-quantum MDS codes of length $n = \frac{q^2-1}{2s_1s_2}$, where q be an odd prime power, $2s_1|(q-1)$, $s_2|(q+1)$ and s_2 is an odd integer. Let $n = \frac{q^2-1}{2s_1s_2}$, $r = 2$. Since $2n|q^2-1$, then for each odd x in the range $1 \le x \le 2n$, the q^2-cyclotomic coset C_x modulo $2n$ is $C_x = x$.

Lemma 3.5 *Let q be an odd prime power of the form $q = 30m+11$, $n = \frac{q^2-1}{30}$.*

(i) $(C_{1+2(10m+3)}, C_{1+2(5m+1)})$, $(C_{1+2(13m+4)}, C_{1+2(8m+4)})$ *and* $(C_{1+2(16m+5)}, C_{1+2(11m+3)})$ *form skew-asymmetric pairs.*

(ii)

$$|T_{ss}(\delta)| = \begin{cases} 0, & \text{if } 2 \le \delta \le 8m+3; \\ 2, & \text{if } 8m+4 \le \delta \le 11m+5; \\ 4, & \text{if } 11m+6 \le \delta \le 14m+7. \end{cases}$$

Proof (i) Let $q = 30m + 11$ and $2n = 2\frac{q^2-1}{30} = 2(3m + 1)(10m + 4)$. Since $[1+2(5m+1)]q = (10m+4-1)(30m+10+1) = 5\cdot 2(10m+4)(3m+1) - [1+2(10m+3)]$, $-[1+2(5m+1)]q \equiv 1+2(10m+3) \bmod 2n$.
Since $[1+2(8m+4)]q = (10m+4+6m+1)(30m+10+1) = 8\cdot 2(10m+4)(3m+1) - [1+2(13m+4)]$, $-(16m+5)q \equiv 1+2(13m+4) \bmod 2n$.
Since $[1+2(11m+3)]q = (20m+8+2m-7)(30m+10+1) = 11\cdot 2(10m+4)(3m+1) - [1+2(16m+5)]$, $-(22m+7)q \equiv 1+2(16m+5) \bmod 2n$.

(ii) According to Lemma 17 in Ref. 22, if the defining set $T = \bigcup_{j=2m+1}^{l} C_{1+2j}$, where $2m+1 \le l \le 10m+2$, then $T_{ss}(\delta) = \emptyset$ for $2 \le \delta \le 8m+3$.
Let $T_1 = \bigcup_{j=2m+1}^{10m+2} C_{1+2j}$. If the defining set $T = \bigcup_{j=10m+4}^{l} C_{1+2j} \bigcup T_1$, where $10m+4 \le l \le 13m+3$, then $8m+4 \le \delta \le 11m+5$ holds.
Let $I_0 = [1+2(2m+1), 1+2(10m+2)]$, $I_1 = [1+2(10m+4), 1+2(13m+3)]$ and $I_2 = [1+2(13m+5), 1+2(16m+4)]$.

□

According to Lemmas 2.4 and 2.1 in [12], one needs to testify that for $x, y \in I_0 \cup I_1 \cup I_2$, there $x + yq \not\equiv 0 \bmod 2n$ holds.

For $8m+4 \le \delta \le 11m+5$, suppose there exist integers $y \in I_1$, $x \in I_0 \cup I_1$, such that $x \equiv -qy \bmod 2n$. We find a contradiction in the following.

We divided I_1 into three parts such as $[1+2(10m+4), 1+2(10m+4)+2(m-1)] \cup [1+2(10m+4)+2m, 1+2(10m+4)+4(m-1)] \cup [1+2(10m+4)+4m-2, 1+2(13m+3)]$.

If $x, y \in [1+2(10m+4), 1+2(10m+4)+2(m-1)]$, then $10(2n) < 10(2n) + 70m + 28 = (20m+19)(30m+12) \le y(q+1) \le (22m+7)(30m+12) < 11(2n)$; if $x, y \in [1+2(10m+4)+2m, 1+2(10m+4)+4(m-1)]$, then $11(2n) < (22m+9)(30m+12) \le y(q+1) \le (24m+5)(30m+12) < 12(2n)$; and if $x, y \in [1+2(10m+4)+4m-2, 1+2(13m+3)]$, then $12(2n) < (24m+7)(30m+12) \le y(q+1) \le (26m+7)(30m+12) = 13(2n) - 50m - 20 < 13(2n)$, a contradiction.

Similarly, for $11m+6 \le \delta \le 14m+7$, suppose there exist integers $y \in I_2$, $x \in I_0 \cup I_1 \cup I_2$, such that $x \equiv -qy \bmod 2n$. Using the same method, one also finds a contradiction.

Theory 3.6 *Let q be an odd prime power of the form $q = 30m + 11$, $n = \frac{q^2-1}{30}$. There exists a q-ary $[[\frac{q^2-1}{30}, \frac{q^2-1}{30} - 2d + 4, d; 2]]$- EA-quantum MDS codes, where $8m+4 \le d \le 11m+5$; and there exists a q-ary $[[\frac{q^2-1}{30}, \frac{q^2-1}{30} - 2d + 6, d; 4]]$- EA-quantum MDS codes, where $11m+6 \le d \le 14m+7$.*

Proof Consider the negacyclic codes over F_{q^2} of length $n = \frac{q^2-1}{30}$ with defining set $T = \bigcup_{i=2m+1}^{s} C_{1+2i}$, where $2m+1 \le s \le 16m+4$ for q be an odd prime power of the form $q = 30m+11$, m is integer number. By Lemma 3.5, there is $c = |T_{ss}(\delta)| = 2$ if $10m+3 \le s \le 13m+3$ and $c = |T_{ss}(\delta)| = 4$ if $13m+4 \le s \le 16m+4$. Since every q^2-cyclotomic coset C_x has exactly one element and x must be odd number, we can obtain that T consists of $s+1$ integers $\{1+2(2m+1), 1+2(2m+2), \ldots, 1+2s\}$. It implies that $\mathcal{C}$ has minimum distance at least $s-(2m+1)+1$. Hence, $\mathcal{C}$ is a q^2-ary negacyclic code with parameters $[n, n-(s-(2m+1)+1), \ge s-(2m+1)+2]$.

Combining Lemma 2.8 with EA-quantum singleton bound, we can obtain a quantum MDS code with parameters $[[\frac{q^2-1}{30}, \frac{q^2-1}{30} - 2d + 4, d; 2]]$, where $8m + 4 \leq d \leq 11m + 5$; $[[\frac{q^2-1}{30}, \frac{q^2-1}{30} - 2d + 6, d; 4]]$, where $11m + 6 \leq d \leq 14m + 7$, for q be an odd prime power of the form $q = 30m + 11$. □

Example 3 Let $q = 43$, applying Theory 3.6 produces:

(1) new 2-ebits EA-quantum MDS codes with parameters $[[56, 36, 12; 2]]_{43}$, $[[56, 34, 13; 2]]_{43}$, $[[56, 32, 14; 2]]_{43}$, $[[56, 30, 15; 2]]_{43}$, $[[56, 28, 16; 2]]_{43}$.
(2) new 4-ebits EA-quantum MDS codes with parameters $[[56, 28, 17; 4]]_{43}$, $[[56, 26, 18; 4]]_{43}$, $[[56, 24, 19; 4]]_{43}$, $[[56, 22, 20; 4]]_{43}$, $[[56, 20, 21; 4]]_{43}$.

Lemma 3.7 *Let q be an odd prime power of the form $q = 30m + 19$, $n = \frac{q^2-1}{30}$.*

(i) $(C_{1+2(9m+5)}, C_{1+2(6m+3)})$, $(C_{1+2(12m+7)}, C_{1+2(3m+1)})$, $(C_{1+2(14m+8)}, C_{1+2(11m+6)})$ *and* $(C_{1+2(16m+10)}, C_{1+2(8m+4)})$ *form skew-asymmetric pairs.*
(ii)

$$|T_{ss}| = \begin{cases} 0, & \textit{if } 2 \leq \delta \leq 8m + 5; \\ 2, & \textit{if } 8m + 6 \leq \delta \leq 11m + 7; \\ 4, & \textit{if } 11m + 8 \leq \delta \leq 13m + 8; \\ 6, & \textit{if } 13m + 9 \leq \delta \leq 16m + 10. \end{cases}$$

Proof (i) Let $q = 30m + 19$ and $2n = 2\frac{q^2-1}{30} = 2(3m + 1)(10m + 6)$. Since $[1 + 2(6m + 3)]q = (12m + 8 - 1)(30m + 18 + 1) = 6 \cdot 2(10m + 6)(3m + 2) - [1 + 2(9m + 5)]$, $-[1 + 2(6m + 3)]q \equiv 1 + 2(9m + 5) \bmod 2n$.
Since $[1 + 2(3m + 1)]q = (6m + 4)(30m + 18 + 1) = 3 \cdot 2(3m + 2)(10m + 6) - [1 + 2(12m + 7)]$, $-[1 + 2(3m + 1)]q \equiv [1 + 2(12m + 7)] \bmod 2n$.
Since $[1 + 2(11m + 6)]q = (21m + 14 + m - 1)(30m + 18 + 1) = 12 \cdot 2(3m + 2)(10m + 6) - [1 + 2(14m + 8)]$, $-[1 + 2(11m + 6)]q \equiv [1 + 2(14m + 8)] \bmod 2n$.
Since $[1 + 2(8m + 4)]q = (15m + 14 + m - 1)(30m + 18 + 1) = 12 \cdot 2(3m + 2)(10m + 6) - [1 + 2(17m + 10)]$, $-[1 + 2(17m + 10)]q \equiv [1 + 2(8m + 4)] \bmod 2n$.
(ii) According to Lemma 17 in Ref. [22], if the defining set $T = \bigcup_{j=m+1}^{l} C_{1+2j}$, where $m + 1 \leq l \leq 9m + 4$, then $T_{ss}(\delta) = \emptyset$ for $2 \leq \delta \leq 8m + 5$.
Let $I_0 = [1 + 2(m + 1), 1 + 2(9m + 4)]$, $I_1 = [1 + 2(9m + 6), 1 + 2(12m + 6)]$, $I_2 = [1+2(12m+8), 1+2(14m+7)]$, and $I_3 = [1+2(14m+9), 1+2(17m+9)]$. □

According to Lemmas 2.4 and 2.1 in [12], one needs to testify that for $x, y \in I_0 \cup I_1 \cup I_2 \cup I_3$, there $x + yq \not\equiv 0 \bmod 2n$ holds.

Let $T_1 = \bigcup_{j=m+1}^{10m+2} C_{1+2j}$. If the defining set $T = \bigcup_{j=9m+5}^{l} C_{1+2j} \bigcup T_1$, where $9m + 5 \leq l \leq 12m + 6$, then $8m + 6 \leq \delta \leq 11m + 7$ holds.

For $8m + 6 \leq \delta \leq 11m + 7$, suppose there exist integers $y \in I_1$, $x \in I_0 \cup I_1$, such that $x \equiv -qy \bmod 2n$. We find a contradiction in the following.

We divided I_1 into three parts such as $[1+2(9m+6), 1+2(10m+4)+2(m-1)] \cup [1+2(10m+4)+2m, 1+2(10m+4)+4(m-1)] \cup [1+2(10m+4)+4m-2, 1+2(13m+3)]$.

If $x, y \in [1+2(10m+4), 1+2(10m+4)+2(m-1)]$, then $10(2n) < 10(2n) + 70m + 28 = (20m+19)(30m+12) \le y(q+1) \le (22m+7)(30m+12) < 11(2n)$; if $x, y \in [1+2(10m+4)+2m, 1+2(10m+4)+4(m-1)]$, then $11(2n) < (22m+9)(30m+12) \le y(q+1) \le (24m+5)(30m+12) < 12(2n)$; and if $x, y \in [1+2(10m+4)+4m-2, 1+2(13m+3)]$, then $12(2n) < (24m+7)(30m+12) \le y(q+1) \le (26m+7)(30m+12) = 13(2n) - 50m - 20 < 13(2n)$, a contradiction.

Similarly, for $\delta \in [8m+6, 11m+7] \cup [11m+8, 13m+8] \cup [13m+9, 16m+10]$, using the same method, the lemma holds.

Theory 3.8 *Let q be an odd prime power of the form $q = 30m + 19$, $n = \frac{q^2-1}{30}$. There exists a q-ary $[[\frac{q^2-1}{30}, \frac{q^2-1}{30} - 2d + 4, d; 2]]$- EA-quantum MDS codes, where $8m + 6 \le d \le 11m + 7$; there exists a q-ary $[[\frac{q^2-1}{30}, \frac{q^2-1}{30} - 2d + 6, d; 4]]$- EA-quantum MDS codes, where $11m + 8 \le d \le 13m + 8$; and there exists a q-ary $[[\frac{q^2-1}{30}, \frac{q^2-1}{30} - 2d + 8, d; 6]]$- EA-quantum MDS codes, where $13m + 9 \le d \le 16m + 10$.*

Proof Consider the negacyclic codes over F_{q^2} of length $n = \frac{q^2-1}{30}$ with defining set $T = \bigcup_{i=m+1}^{s} C_{1+2i}$, where $m + 1 \le s \le 16m + 9$ for q be an odd prime power of the form $q = 30m + 19$, m is integer number. By Lemma 3.7, there is $c = |T_{ss}(\delta)| = 2$ if $9m + 5 \le s \le 12m + 6$, $c = |T_{ss}(\delta)| = 4$ if $12m + 7 \le s \le 14m + 7$, and $c = |T_{ss}(\delta)| = 6$ if $14m + 8 \le s \le 17m + 9$. Since every q^2-cyclotomic coset C_x has exactly one element and x must be odd number, we can obtain that T consists of $s + 1$ integers $\{1 + 2(m + 1), 1 + 2(m + 2), \ldots, 1 + 2s\}$. It implies that $\mathcal{C}$ has minimum distance at least $s - (m + 1) + 1$. Hence, $\mathcal{C}$ is a q^2-ary negacyclic code with parameters $[n, n - (s - (m + 1) + 1), \ge s - (m + 1) + 2]$. Combining Lemma 2.8 with EA-quantum singleton bound, we can obtain a EA-quantum MDS code with parameters $[[\frac{q^2-1}{30}, \frac{q^2-1}{30} - 2d + 4, d; 2]]_q$ where $8m + 6 \le d \le 11m + 7$; $[[\frac{q^2-1}{30}, \frac{q^2-1}{30} - 2d + 6, d; 4]]_q$s, where $11m + 8 \le d \le 13m + 8$; and $[[\frac{q^2-1}{30}, \frac{q^2-1}{30} - 2d + 8, d; 6]]_q$, where $13m + 9 \le d \le 16m + 10$, for q be an odd prime power of the form $q = 30m + 19$. □

Example 4 Let $q = 49$, applying Theory 3.8 produces:

(1) 2-ebits EA-quantum MDS codes with parameters $[[80, 56, 14; 2]]_{49}$, $[[80, 54, 15; 2]]_{49}$, $[[80, 52, 16; 2]]_{49}$, $[[80, 50, 17; 2]]_{49}$.
(2) 4-ebits EA-quantum MDS codes with parameters $[[80, 48, 19; 4]]_{49}$, $[[80, 46, 20; 4]]_{49}$, $[[80, 44, 21; 4]]_{49}$.
(3) 6-ebits EA-quantum MDS codes with parameters $[[80, 44, 22; 6]]_{49}$, $[[80, 42, 23; 6]]_{49}$, $[[80, 40, 24; 6]]_{49}$, $[[80, 38, 25; 6]]_{49}$, $[[80, 36, 26; 6]]_{49}$.

Lemma 3.9 *Let q be an odd prime power of the form $q = 12m + 5$, $n = \frac{q^2-1}{12}$.*

(i) *$(C_{1+2(7m+2)}, C_{1+2(5m+1)})$, $(C_{1+2(9m+3)}, C_{1+6m})$ and $(C_{1+2(10m+3)}, C_{1+2(8m+2)})$ form skew-asymmetric pairs.*

(ii)
$$|T_{ss}| = \begin{cases} 0, & if\ 2 \le \delta \le 5m+2; \\ 2, & if\ 5m+3 \le \delta \le 7m+3; \\ 4, & if\ 7m+4 \le \delta \le 8m+3. \end{cases}$$

Proof (i) Let $q = 12m+5$ and $2n = 2\frac{q^2-1}{30} = 4(2m+1)(3m+1)$. Since $[1+2(5m+1)]q = (10m+5-2)(12m+4+1) = 5 \cdot 4(2m+1)(3m+1) - [1+2(7m+2)]$, $-[1+2(7m+2)]q \equiv 1+2(5m+1) \bmod 2n$.
Since $(6m+1)q = (6m+3-2)(12m+4+1) = 3 \cdot 4(2m+1)(3m+1) - [1+2(9m+3)]$, $-[1+2(9m+3)]q \equiv 6m+1 \bmod 2n$.
Since $(6m+1)q = (6m+3-2)(12m+4+1) = 3 \cdot 4(2m+1)(3m+1) - [1+2(9m+3)]$, $-[1+2(9m+3)]q \equiv 6m+1 \bmod 2n$.
Since $[1+2(8m+2)]q = (16m+8-3)(12m+4+1) = 8 \cdot 4(2m+1)(3m+1) - [1+2(10m+3)]$, $-[1+2(10m+3)]q \equiv 1+2(8m+2) \bmod 2n$.

(ii) According to Lemma 17 in Ref. 22, if the defining set $T = \bigcup_{j=2m+1}^{l} C_{1+2j}$, where $2m+1 \le l \le 7m+2$, then $T_{ss}(\delta) = \emptyset$ for $2 \le \delta \le 5m+2$.
Let $I_0 = [1+2(m+1), 1+2(7m+1)]$, $I_1 = [1+2(7m+3), 1+2(9m+2)]$ and $I_2 = [1+2(9m+4), 1+2(10m+2)]$.

□

According to Lemmas 2.4 and 2.1 in [12], one needs to testify that for $x, y \in I_0 \cup I_1 \cup I_2$, there $x + yq \not\equiv 0 \bmod 2n$ holds.

Let $T_0 = \bigcup_{j=m+1}^{7m+2} C_{1+2j}$. If the defining set $T = \bigcup_{j=7m+3}^{l} C_{1+2j} \bigcup T_0$, where $7m+3 \le l \le 9m+3$, then $5m+3 \le \delta \le 7m+3$ holds.

For $5m+3 \le \delta \le 7m+3$, suppose there exist integers $y \in I_1$, $x \in I_0 \cup I_1$, such that $x \equiv -qy \bmod 2n$. We find a contradiction in the following.

We divided I_1 into two parts such as $[1+2(7m+3), 1+2(8m+2)] \cup [1+2(8m+3), 1+2(9m+2)]$.

If $x, y \in [1+2(7m+3), 1+2(8m+2)]$, then $7(2n) < 7(2n) + 28m + 14 = (14m+7)(12m+6) \le y(q+1) \le (16m+5)(12m+6) = 8(3m+1)(8m+4) - (4m+2) < 8(2n)$; and if $x, y \in [1+2(8m+3), 1+2(9m+2)]$, then $8(2n) < (16m+7)(12m+6) \le y(q+1) \le (18m+5)(12m+6) < 9(2n)$, a contradiction.

If $x \in I_0$, $y \in [1+2(7m+3), 1+2(8m+2)]$, since $7(2n) + 14m + 7 = (14m+7)(12m+5) \le yq \le (16m+5)(12m+5) = 7(2n) + 24m^2 - 3$, then $2n - yq > x \bmod 2n$;

and if $x \in I_0$, $y \in [1+2(8m+3), 1+2(9m+2)]$, since $8(2n) + 4m + 3 = (16m+7)(12m+5) \le yq \le (18m+5)(12m+5) = 9(2n) - 30m^2 - 11$, then $2n - yq > x \bmod 2n$, a contradiction.

Similarly, for $\delta \in [7m+4, 8m+3]$, using the same method, the lemma holds.

Theory 3.10 *Let q be an odd prime power of the form $q = 12m+5$, $n = \frac{q^2-1}{12}$. There exists a q-ary $[[\frac{q^2-1}{12}, \frac{q^2-1}{12} - 2d + 4, d; 2]]$- EA-quantum MDS codes, where $5m+3 \le d \le 7m+3$; and there exists a q-ary $[[\frac{q^2-1}{12}, \frac{q^2-1}{12} - 2d + 6, d; 4]]$- EA-quantum MDS codes, where $7m+4 \le d \le 8m+3$.*

Proof Consider the negacyclic codes over F_{q^2} of length $n = \frac{q^2-1}{12}$ with defining set $T = \bigcup_{i=2m+1}^{s} C_{1+2i}$, where $2m+1 \le s \le 10m+2$ for q be an odd prime power of the form $q = 12m+5$, m is integer number. By Lemma 3.9, there is $c = |T_{ss}(\delta)| = 2$ if $7m+2 \le s \le 9m+2$ and $c = |T_{ss}(\delta)| = 4$ if $9m+3 \le s \le 10m+2$. Since every q^2-cyclotomic coset C_x has exactly one element and x must be odd number, we can obtain that T consists of $s+1$ integers $\{1+2(2m+1), 1+2(2m+2), \cdots, 1+2s\}$. It implies that $\mathcal{C}$ has minimum distance at least $s-(2m+1)+1$. Hence, $\mathcal{C}$ is a q^2-ary negacyclic code with parameters $[n, n-(s-(2m+1)+1), \ge s-(2m+1)+2]$. Combining Lemma 2.8 with EA-quantum singleton bound, we can obtain a quantum MDS code with parameters $[[\frac{q^2-1}{12}, \frac{q^2-1}{12} - 2d + 4, d; 2]]_q$, where $5m+3 \le d \le 7m+3$; and $[[\frac{q^2-1}{12}, \frac{q^2-1}{12} - 2d + 6, d; 4]]_q$, where $7m+4 \le d \le 8m+3$; for q be an odd prime power of the form $q = 12m+5$. □

Example 5 Let $q = 29$, applying Theory 3.10 produces:

(1) 2-ebits EA-quantum MDS codes with parameters $[[70, 48, 13; 2]]_{29}$, $[[70, 46, 14; 2]]_{29}$, $[[70, 44, 15; 2]]_{29}$, $[[70, 42, 16; 2]]_{29}$, $[[70, 40, 17; 2]]_{29}$.
(2) 4-ebits EA-quantum MDS codes with parameters $[[70, 40, 18; 4]]_{29}$, $[[70, 38, 19; 4]]_{29}$.

3.3 New EA-quantum MDS codes of length $n = \frac{q^2+1}{5}$

In this section, let q be an odd prime power of the $q = 10m+3$ or $q = 10m+7$, where m is a positive integer. Let $n = \frac{q^2+1}{5}$, $r = 2$ and $\eta \in F_{q^2}$ be a primitive rth root of unity. Since 5 and 2 are two factors of q^2+1 and $2n|q^4-1$, then for each odd x in the range $1 \le x \le n$, the q^2-cyclotomic coset C_x modulo $2n$ is $C_x = \{x, n-x\}$. Then, we discuss negacyclic codes of length n over F_{q^2} to construct EA-quantum MDS codes.

Lemma 3.11 *Let q be an odd prime power of the form $q = 10m+3$, $n = \frac{q^2+1}{5}$, $s = \frac{n}{2}$.*

(1) If $1 \le x \le 12m+3$, then (C_{1+4m}, C_{1+2m}) and $(C_{1+2(6m+1)}, C_{1+2(3m+1)})$ form skew-asymmetric pairs, respectively;

$$|T_{ss}(\delta)| = \begin{cases} 0, & \text{if } 3 \le \delta \le 4m+1, \quad \text{for } \delta \text{ is odd;} \\ 4, & \text{if } 4m+3 \le \delta \le 12m+3, \quad \text{for } \delta \text{ is odd.} \end{cases}$$

(2) If $m = 2t+1$ is odd, and $s \le x \le s+10m+4$, then (C_{s+6m+2}, C_{s+2m}), $(C_{s+10m+4}, C_{s+10m+2})$ form skew-asymmetric pairs, respectively;

$$|T_{ss}(\delta)| = \begin{cases} 0, & \text{if } 2 \le \delta \le 6m+2, \quad \text{for } \delta \text{ is even;} \\ 4, & \text{if } 6m+4 \le \delta \le 10m+4, \quad \text{for } \delta \text{ is even.} \end{cases}$$

If $m = 2t$ is even, and $s \le x \le s+12m+4$, then C_s is skew-symmetric, and (C_{s+8m+2}, C_{s+4m+2}), $(C_{s+12m+4}, C_{s+4m})$ form skew-asymmetric pairs, respectively;

$$|T_{ss}(\delta)| = \begin{cases} 1, & \text{if } 2 \le \delta \le 8m+2, \quad \text{for } \delta \text{ is even;} \\ 5, & \text{if } 8m+4 \le \delta \le 12m+4, \quad \text{for } \delta \text{ is even.} \end{cases}$$

Proof Let $q = 10m + 3$. Since $2n = 40m^2 + 24m + 4$.

(1) (i) Let $1 \le x \le 12m + 2$. Since $(4m + 1)q = 40m^2 + 24m + 3 - (2m + 1)$, (C_{1+4m}, C_{1+2m}) form skew-asymmetric pairs. Since $(12m + 3)q = 120m^2 + 66m + 9 = 4n + 2n - (6m + 3) \equiv 2n - (6m + 3) \bmod 2n$, $(C_{1+2(6m+1)}, C_{1+2(3m+1)})$ form skew-asymmetric pairs.

(ii) (a) If the defining set $T = \bigcup_{j=1}^{l} C_{1+2j}$, where $1 \le l \le 2m - 1$, we testify that $T_{ss}(\delta) = \emptyset$ for $2 \le \delta \le 4m + 1$. According to Lemmas 2.4 and 2.1 in [12], one needs to testify that for $x, y \in [1, 4m - 1]$, there $x + yq \not\equiv 0 \bmod 2n$ holds.

If $x, y \in [1, 4m - 1]$, $1 < y(q + 1) < (4m - 1)(q + 1) = 40m^2 + 6m - 4 < 2n$. Hence, if the defining set $T = \bigcup_{j=1}^{l} C_{1+2j}$, where $1 \le l \le 2m - 1$, the $T_{ss}(\delta) = \emptyset$ for $2 \le \delta \le 6m + 2$.

(b) Let $I_0 = [1, 4m - 1]$, $I_1 = [4m + 3, 12m + 1]$.

According to Lemmas 2.4 and 2.1 in [12], one needs to testify that for $x, y \in I_0 \cup I_1$, there $x + yq \not\equiv 0 \bmod 2n$ holds.

Let $T_0 = \bigcup_{1}^{j=4m-1} C_{1+2j}$, and the defining set $T = \bigcup_{j=2m}^{l} C_{1+2j} \bigcup T_0$, where $2m \le l \le 6m$.

For $4m + 3 \le \delta \le 12m + 3$, suppose there exist integers $y \in I_1$, $x \in I_0 \cup I_1$, such that $x \equiv -qy \bmod 2n$. We find a contradiction in the following.

We divided I_1 into four parts such as $[4m+3, 6m+1] \cup [6m+3, 8m+1] \cup [8m+3, 10m+1] \cup [10m+3, 12m+1]$.

If $x, y \in [4m + 3, 6m + 1]$, then $(2n) < 40m^2 + 24m + 4 + 22m + 8 = (4m + 3)(q + 1) \le y(q + 1) \le (6m + 1)(q + 1) = 60m^2 + 34m + 4 < 2(2n)$; if $x, y \in [6m+3, 8m+1]$, $2(2n) < (6m+3)(q+1) \le y(q+1) \le (8m+1)(q+1) < 4(2n)$; if $x, y \in [8m+3, 10m+1]$, $4(2n) < (8m+3)(q+1) \le y(q+1) \le (8m+1)(q+1) < 6(2n)$; if $x, y \in [10m + 3, 12m + 1]$, $6(2n) < (10m + 3)(q + 1) \le y(q + 1) \le (12m + 1)(q + 1) < 8(2n)$, a contradiction.

If $x \in I_0$, $y \in [4m + 3, 6m + 1]$, since $2n < 2n + 18m + 5 = (4m + 3)q \le yq \le (6m + 1)q = 2(2n) - 20m^2 - 20m - 3$, then $2n - yq > x \bmod 2n$, a contradiction. Similarly, if $x \in I_0$, $y \in [8m + 3, 10m + 1]$ and $x \in I_0$, $y \in [10m + 3, 12m + 1]$, using the same method, one can deduce a contradiction.

(2) (i) If $m = 2t + 1$, $s \le x \le s + 10m + 4$, $(s + 6m + 2)q = (\frac{n}{2} + 6m + 2)q = 5mn + 4n + \frac{n}{2} + 2m$. Since $m = 2t + 1$, $(s + 6m + 2)q \equiv \frac{3}{2}n + 2m \bmod 2n$. $-(s + 6m + 2)q \equiv \frac{n}{2} + 2m \bmod 2n$.

Since $(s+10m+4)q = (\frac{n}{2}+10m+4)q = 5mn+6n+\frac{1}{2}n+10m+2$, $-(s+10m+4)q \equiv \frac{1}{2}n+10m+2 \bmod 2n$. Hence, (C_{s+6m+2}, C_{s+2m}), $(C_{s+10m+4}, C_{s+10m+2})$ form skew-asymmetric pairs, respectively;

If the defining set $T = \bigcup_{j=1}^{l} C_j$, where $s \le l \le s + 6m$, we testify that $T_{ss}(\delta) = \emptyset$ for $2 \le \delta \le 6m + 2$. According to Lemma 2.4 and Lemma 2.1 in [12], one needs to testify that for $x, y \in [s, s + 6m]$, there $x + yq \not\equiv 0 \bmod 2n$ holds.

If $x, y \in [s, s + 6m]$, $1 < y(q + 1) < (4m - 1)(q + 1) = 40m^2 + 6m - 4 < 2n$. Hence, if the defining set $T = \bigcup_{j=1}^{l} C_{1+2j}$, where $1 \le l \le 2m - 1$, the $T_{ss}(\delta) = \emptyset$ for $2 \le \delta \le 6m + 2$.

Let $I_0 = [s, s + 6m]$, $I_1 = [s + 6m + 4, s + 10m + 4]$.

According to Lemmas 2.4 and 2.1 in [12], one needs to testify that for $x, y \in I_0 \cup I_1$, there $x + yq \not\equiv 0 \bmod 2n$ holds.

Let $T_0 = \bigcup_{j=0}^{3m} C_{s+2j}$, and the defining set $T = \bigcup_{j=3m+2}^{l} C_{s+2j} \bigcup T_0$, where $3m+4 \le l \le 5m+2$.

For $6m+4 \le \delta \le 10m+4$, suppose there exist integers $y \in I_1, x \in I_0 \cup I_1$, such that $x \equiv -qy \bmod 2n$. We find a contradiction in the following.

We divided I_1 into four parts such as $[4m+3, 6m+1] \cup [6m+3, 8m+1] \cup [8m+3, 10m+1] \cup [10m+3, 12m+1]$.

If $x, y \in [4m+3, 6m+1]$, then $(2n) < 40m^2+24m+4+22m+8 = (4m+3)(q+1) \le y(q+1) \le (6m+1)(q+1) = 60m^2+34m+4 < 2(2n)$; if $x, y \in [6m+3, 8m+1], 2(2n) < (6m+3)(q+1) \le y(q+1) \le (8m+1)(q+1) < 4(2n)$; if $x, y \in [8m+3, 10m+1], 4(2n) < (8m+3)(q+1) \le y(q+1) \le (8m+1)(q+1) < 6(2n)$; if $x, y \in [10m+3, 12m+1]$, $6(2n) < (10m+3)(q+1) \le y(q+1) \le (12m+1)(q+1) < 8(2n)$, a contradiction.

(ii) For $m = 2t$ is even, since $2n - sq = 2n - (5mn + n + \frac{n}{2}) \equiv s \bmod 2n$, C_s is skew-symmetric. $2n - (s+8m+2)q \equiv s+4m+2 \bmod 2n$ and $2n-(s+12m+4)q \equiv s+4m \bmod 2n$.

If the defining set $T = \bigcup_{1}^{j=l} C_{s+2j}$, where $1 \le l \le 4m$, we testify that $T_{ss}(\delta) = \{C_s\}$ for $2 \le \delta \le 8m+2$. Since C_s is skew-symmetric, according to Lemma 2.4 and Lemma 2.1 in [12], one needs to testify that for $x, y \in [s+2, s+8m]$, there $x + yq \not\equiv 0 \bmod 2n$ holds.

If $x, y \in [s+2, s+8m]$, $5mn+2n+20m+8 = (s+2)(q+1) < y(q+1) < (4m-1)(q+1) = 5mn+2n+4n-16m-8$. For $x, y \in [s+2, s+8m]$, there is no skew-symmetric cyclotomic coset and any two cyclotomic cosets are not skew-asymmetric cosets. Hence, if the defining set $T = \bigcup_{1}^{j=l} C_{s+2j}$, where $1 \le l \le 4m$, the $T_{ss}(\delta) = \{C_s\}$ for $2 \le \delta \le 8m+2$.

Let $T_0 = \bigcup_{2}^{j=4m} C_{s+2j}$, and the defining set $T = \bigcup_{4m+2}^{j=l} C_{s+2j} \bigcup T_0$, where $4m+2 \le l \le 6m+1$. We testify that for $x, y \in T$, there is no skew-symmetric cyclotomic coset and any two cyclotomic cosets are not skew-asymmetric cosets.

Let $I_0 = [s+2, s+8m]$, $I_1 = [s+8m+4, s+12m+2]$. According to Lemmas 2.4 and 2.1 in [12], one needs to testify that for $x, y \in I_0 \cup I_1$, there $x + yq \not\equiv 0 \bmod 2n$ holds. Using the same above-mentioned method, one can easily testify that the lemma holds. □

Theory 3.12 *Let q be an odd prime power of the form $q = 10m + 3$.*

(1) *If $m = 2t+1$ is odd, then there exists a q-$[[\frac{q^2+1}{5}, \frac{q^2+1}{5} - 2d + 6, d; 4]]$- EA-quantum MDS codes, where $4m+3 \le d \le 6m+1$ be odd and $6m+4 \le d \le 10m+4$ be even.*

(2) *If $m = 2t$ is even, then there exists a q-ary $[[\frac{q^2+1}{5}, \frac{q^2+1}{5} - 2d + 3, d; 1]]$-EA-quantum MDS codes, where $2 \le d \le 8m+1$ be even; there exists a q-ary $[[\frac{q^2+1}{5}, \frac{q^2+1}{5} - 2d + 6, d; 4]]$-EA-quantum MDS codes, where $4m+3 \le d \le 6m+1$ be odd; and there exists a q-ary $[[\frac{q^2+1}{5}, \frac{q^2+1}{5} - 2d + 7, d; 5]]$-EA-quantum MDS codes, where $8m+4 \le d \le 12m+4$ be even.*

Proof (a) Consider the negacyclic codes over F_{q^2} of length $n = \frac{q^2+1}{5}$ with defining set $T = \bigcup_{i=0}^{s} C_{1+2i}$, where $0 \le s \le 6m$ for q be an odd prime power of the form $q = 10m+3$. If m is odd, by Lemma 3.11 (i), there is $c = |T_{ss}(\delta)| = 4$

if $2m \leq s \leq 6m$. Since every q^2-cyclotomic coset $C_x = \{x, n-x\}$ and x must be odd number, we can obtain that T consists of $2s+1$ integers $\{n-(1+2s), \ldots, n-1, 1, 3, \ldots, 1+2s\}$. It implies that $\mathcal{C}$ has minimum distance at least $2s+2$. Hence, $\mathcal{C}$ is a q^2-ary negacyclic code with parameters $[n, n-2(s+1)+1, \geq 2s+2]$. Combining Lemma 2.8 with EA-quantum singleton bound, we can obtain a EA-quantum MDS code with parameters $[[\frac{q^2+1}{5}, \frac{q^2+1}{5} - 2d + 6, d; 4]]_q$, where $4m+3 \leq d \leq 6m+1$ be odd. If m is even, using the same method, one can obtain that $[[\frac{q^2+1}{5}, \frac{q^2+1}{5} - 2d + 6, d; 4]]_q$, where $4m+3 \leq d \leq 6m+1$ be odd.

(b) Consider the negacyclic codes over F_{q^2} of length $n = \frac{q^2+1}{5}$ with defining set $T = \bigcup_{i=0}^{k} C_{s+2i}$, where $0 \leq k \leq 5m+1$, $s = \frac{n}{2}$ for q be an odd prime power of the form $q = 10m+3$. If m is odd, by Lemma 3.11 (ii), there is $c = |T_{ss}(\delta)| = 4$ if $3m+1 \leq s \leq 5m+1$. Since every q^2-cyclotomic coset $C_x = \{x, n-x\}$ and x must be odd number, we can obtain that T consists of $2k+1$ integers $\{n-(s+2k), \ldots, n-s, s, s+2, \ldots, s+2k\}$. It implies that $\mathcal{C}$ has minimum distance at least $2k+2$. Hence, $\mathcal{C}$ is a q^2-ary negacyclic code with parameters $[n, n-2(s+1)+1, \geq 2s+2]$. Combining Lemma 2.8 with EA-quantum singleton bound, we can obtain a EA-quantum MDS code with parameters $[[\frac{q^2+1}{5}, \frac{q^2+1}{5} - 2d + 6, d; 4]]_q$, $6m+4 \leq d \leq 10m+4$ be even.
If m is even, by Lemma 3.11 (ii), there is $c = |T_{ss}(\delta)| = 1$ if $1 \leq k \leq 4m$ and $c = |T_{ss}(\delta)| = 5$ if $4m+1 \leq k \leq 6m+1$. Since every q^2-cyclotomic coset $C_x = \{x, n-x\}$ and x must be odd number, we can obtain that T consists of $2k+1$ integers $\{n-(s+2k), \ldots, n-s, s, s+2, \ldots, s+2k\}$. It implies that $\mathcal{C}$ has minimum distance at least $2k+2$. Hence, $\mathcal{C}$ is a q^2-ary negacyclic code with parameters $[n, n-2(s+1)+1, \geq 2s+2]$. Combining Lemma 2.8 with EA-quantum singleton bound, we can obtain a EA-quantum MDS code with parameters $[[\frac{q^2+1}{5}, \frac{q^2+1}{5} - 2d + 6, d; 1]]_q$, where $2 \leq d \leq 8m+1$ be even; and $[[\frac{q^2+1}{5}, \frac{q^2+1}{5} - 2d + 7, d; 5]]_q$ codes, where $8m+4 \leq d \leq 12m+4$ be even.

□

Example 6 Let $m = 1$, $q = 13$, applying Theory 3.12 (1) produces 4-*ebits* EA-quantum MDS codes with parameters $[[34, 26, 7; 4]]_{13}$ for d odd; and $[[34, 20, 10; 4]]_{13}$, $[[34, 16, 12; 4]]_{13}$, $[[34, 12, 14; 4]]_{13}$ for d even.

Example 7 Let $m = 2$, $q = 23$, applying Theory 3.12 (2) produces 1-*ebits* EA-quantum MDS codes with parameters $[[106, 105, 2; 1]]_{23}$, $[[106, 101, 4; 1]]_{23}$, $[[106, 97, 6; 1]]_{23}$, $[[106, 93, 8; 1]]_{23}$, $[[106, 89, 10; 1]]_{23}$, $[[106, 85, 12; 1]]_{23}$, $[[106, 81, 14; 1]]_{23}$, $[[106, 77, 16; 1]]_{23}$ for d even; 4-*ebits* EA-quantum MDS codes with parameters $[[106, 90, 11; 4]]_{23}$, $[[106, 86, 13; 4]]_{23}$ for d odd; and 5-*ebits* EA-quantum MDS codes with parameters $[[106, 73, 20; 5]]_{23}$, $[[106, 69, 22; 5]]_{23}$, $[[106, 65, 24; 5]]_{23}$, $[[106, 61, 26; 5]]_{23}$, $[[106, 57, 28; 5]]_{23}$ for d even.

Lemma 3.13 *Let* $q = 10m + 7$, $n = \frac{q^2+1}{5}$, $s = \frac{n}{2}$.

(1) If $1 \le x \le 14m+9$*, then* $(C_{8m+5}, C_{6m+5}), (C_{14m+11}, C_{12m+7})$ *form skew-asymmetric pairs, respectively.*

$$|T_{ss}(\delta)| = \begin{cases} 0, & \textit{if } 3 \le \delta \le 8m+5, \quad \textit{for } \delta \textit{ is odd}; \\ 4, & \textit{if } 8m+7 \le \delta \le 14m+11, \quad \textit{for } \delta \textit{ is odd}. \end{cases}$$

(2) If $m = 2t+1$ *is odd, and* $s \le x \le s+10m+8$*, then* (C_{s+6m+4}, C_{s+2m+2})*,* $(C_{s+10m+8}, C_{s+10m+6})$ *form skew-asymmetric pairs;*

$$|T_{ss}(\delta)| = \begin{cases} 0, & \textit{if } 2 \le \delta \le 6m+4, \quad \textit{for } \delta \textit{ is even}; \\ 4, & \textit{if } 6m+6 \le \delta \le 10m+8, \quad \textit{for } \delta \textit{ is even}. \end{cases}$$

If $m = 2t$ *is even,and* $s \le x \le s+12m+8$*, then* C_s *is skew-symmetric, and* (C_{s+8m+6}, C_{s+4m+2})*,* $(C_{s+12m+8}, C_{s+4m+4})$ *form skew-asymmetric pairs;*

$$|T_{ss}(\delta)| = \begin{cases} 1, \textit{ if } 2 \le \delta \le 8m+6, & \textit{for } \delta \textit{ is even}; \\ 5, \textit{ if } 8m+8 \le \delta \le 12m+8, & \textit{for } \delta \textit{ is even}. \end{cases}$$

Proof Let $q = 10m+7$. Since $2n = 40m^2+56m+20$.

(1) Let $1 \le x \le 14m+10$. Since $(8m+5)q = 40m^2+56m+20-(6m+5)$, (C_{8m+5}, C_{6m+5}) form skew-asymmetric pairs.
Since $(8m+5)q = 40m^2+56m+20-(6m+5)$, (C_{8m+5}, C_{6m+5}) form skew-asymmetric pairs.
  (a) If the defining set $T = \bigcup_{j=1}^{l} C_{1+2j}$, where $1 \le l \le 4m+1$, we testify that $T_{ss}(\delta) = \emptyset$ for $3 \le \delta \le 8m+5$. According to Lemmas 2.4 and 2.1 in [12], one needs to testify that for $x, y \in [1, 8m+3]$, there $x+yq \not\equiv 0 \bmod 2n$ holds.
  We divided $[1, 8m+3]$ into $[1, 4m+1] \cup [4m+3, 8m+3]$. If $x, y \in [1, 4m+1]$, $1 < y(q+1) < (4m+1)(q+1) = 40m^2+42m+8 < 2n$; if $x, y \in [4m+3, 8m+3]$, $2n < 4m^2+62m+24 = (4m+1)(q+1) < y(q+1) < (8m+3)(q+1) = 80m^2+94m+24 < 4n$. Hence, if the defining set $T = \bigcup_{j=1}^{l} C_{1+2j}$, where $1 \le l \le 4m+1$, the $T_{ss}(\delta) = \emptyset$ for $3 \le \delta \le 8m+5$.
  (b) Let $I_0 = [1, 8m+3]$, $I_1 = [8m+7, 14m+9]$.
  According to Lemmas 2.4 and 2.1 in [12], one needs to testify that for $x, y \in I_0 \cup I_1$, there $x+yq \not\equiv 0 \bmod 2n$ holds.
  Let $T_0 = \bigcup_{j=1}^{4m+1} C_{1+2j}$, and the defining set $T = \bigcup_{j=4m+3}^{l} C_{1+2j} \bigcup T_0$, where $4m+3 \le l \le 7m+4$.
  For $8m+7 \le \delta \le 14m+11$, suppose there exist integers $y \in I_1$, $x \in I_0 \cup I_1$, such that $x \equiv -qy \bmod 2n$. We find a contradiction in the following.
  We divided I_1 into four parts such as $[8m+7, 12m+7] \cup [12m+9, 14m+9]$. If $x, y \in [8m+7, 12m+7]$, $2 \cdot 2n < 80m^2+134m+56 = (8m+7)(10m+8) \le y(q+1) \le (12m+7)(10m+8) = 120m^2+166m+56 < 3 \cdot 2n$; if $x, y \in [12m+9, 14m+9]$, $3 \cdot 2n < 120m^2+186m+72 = (12m+9)(q+1) \le y(q+1) \le (14m+9)(q+1) = 140m^2+202m+72 < 4 \cdot 2n$.

Hence, if the defining set $T = \bigcup_{j=1}^{l} C_{1+2j}$, where $1 \le l \le 7m+4$, besides (C_{s+8m+6}, C_{s+4m+2}), there is no skew-symmetric cyclotomic coset and any two cyclotomic cosets are not skew-asymmetric cosets in T.

(2) For $m = 2t$ is even, since $2n - sq = 2n - (5mn + 3n + \frac{n}{2}) \equiv s \bmod 2n$, C_s is skew-symmetric. $2n - (s+8m+6)q \equiv s+4m+2 \bmod 2n$ and $2n-(s+12m+8)q \equiv s+4m+4 \bmod 2n$.

(a) Let $T_0 = \bigcup_{j=s}^{4m+3} C_{1+2j}$, and the defining set $T = \bigcup_{j=4m+5}^{l} C_{1+2j} \bigcup T_0$, where $4m+3 \le l \le 7m+4$.
If the defining set $T = T_0$, we testify that $T_{ss}(\delta) = \{C_s\}$ for $2 \le \delta \le 8m+6$ and δ is even. Since C_s is skew-symmetric cyclotomic coset, and according to Lemma 2.4 and Lemma 2.1 in [12], one needs to testify that for $x, y \in [s+2, s+8m+4]$, there $x + yq \not\equiv 0 \bmod 2n$ holds.
We divided $[s+2, s+8m+4]$ into three parts such as $[s+2, s+3m+2] \cup [s+3m+4, s+7m+4] \cup [s+7m+6, s+8m+4]$.
If $x, y \in [s+2, s+3m+2]$, $(5t+2)\cdot 2n < (5m+4)n + 20m + 16 = (s+2)(10m+8) \le y(q+1) \le (s+3m+2)(10m+8) < (5t+3)\cdot 2n$; if $x, y \in [s+3m+4, s+7m+4]$, $(5t+3)\cdot 2n < (s+3m+4)(q+1) \le y(q+1) \le (s+7m+4)(q+1) < (5t+4)\cdot 2n$. If $x, y \in [s+7m+6, s+8m+4]$, $(5t+5)\cdot 2n < (s+7m+6)(q+1) \le y(q+1) \le (s+8m+4)(q+1) < (5t+5)\cdot 2n$.
Hence, if the defining set $T = \bigcup_{j=s}^{l} C_{s+2j}$, where $1 \le l \le 4m+2$, besides C_s, there is no skew-symmetric cyclotomic coset and any two cyclotomic cosets are not skew-asymmetric cosets in the defining set T.

(b) We divided $[s+8m+8, s+12m+6]$ into two parts such as $[s+8m+8, s+11m+6] \cup [s+11m+8, s+12m+6]$.
If $x, y \in [s+8m+8, s+11m+6]$, $(5t+4)\cdot 2n < (5m+8)n + 32m + 24 = (s+8m+8)(10m+8) \le y(q+1) \le (s+11m+6)(10m+8) = (5m+8)n + 30m^2 + 36m + 8 < (5t+8)\cdot 2n + 2n$; if $x, y \in [s+11m+8, s+12m+6]$, $(5t+9)\cdot 2n < (s+11m+8)(q+1) \le y(q+1) \le (s+12m+6)(q+1) < (5t+9)\cdot 2n + 2n$.

Hence, using the same method, one can obtain that if the defining set $T = \bigcup_{j=s}^{l} C_{s+2j}$, where $1 \le l \le 6m+3$, besides C_s and (C_{s+8m+6}, C_{s+4m+2}), there is no skew-symmetric cyclotomic coset and any two cyclotomic cosets are not skew-asymmetric cosets in the defining set T.

□

Theory 3.14 *Let q be an odd prime power of the form $q = 10m+7$.*

If $m = 2t+1$ is odd, then there exists a q-$[[\frac{q^2+1}{5}, \frac{q^2+1}{5} - 2d + 6, d; 4]]$- EA-quantum MDS codes, where $8m+7 \le d \le 14m+11$ be odd; and $6m+6 \le d \le 10m+8$ be even.

If $m = 2t$ is even, then there exists a q-$[[\frac{q^2+1}{5}, \frac{q^2+1}{5} - 2d + 3, d; 1]]$- EA-quantum MDS codes, where $2 \le d \le 8m+6$ be even; there exists a q-$[[\frac{q^2+1}{5}, \frac{q^2+1}{5} - 2d + 6, d; 4]]$- EA-quantum MDS codes, where $8m+7 \le d \le 14m+11$ be odd; and there exists a q-$[[\frac{q^2+1}{5}, \frac{q^2+1}{5} - 2d + 7, d; 5]]$- EA-quantum MDS codes, where $8m+8 \le d \le 12m+8$ be even.

Proof (a) Consider the negacyclic codes over F_{q^2} of length $n = \frac{q^2+1}{5}$ with defining set $T = \bigcup_{i=0}^{s} C_{1+2i}$, where $0 \leq s \leq 7m+4$ for q be an odd prime power of the form $q = 10m+7$. By Lemma 3.13 (i), there is $c = |T_{ss}(\delta)| = 4$ if $4m+2 \leq s \leq 7m+4$. Since every q^2-cyclotomic coset $C_x = \{x, n-x\}$ and x must be odd number, we can obtain that T consists of $2s+1$ integers $\{n-(1+2s), \ldots, n-1, 1, 3, \ldots, 1+2s\}$. It implies that $\mathcal{C}$ has minimum distance at least $2s+2$. Hence, $\mathcal{C}$ is a q^2-ary negacyclic code with parameters $[n, n-2(s+1)+1, \geq 2s+2]$. Combining Lemma 2.8 with EA-quantum singleton bound, we can obtain a EA-quantum MDS code with parameters $[[\frac{q^2+1}{5}, \frac{q^2+1}{5}-2d+6, d; 4]]_q$, $8m+7 \leq d \leq 14m+11$ be odd.

(b) If m is odd, consider the negacyclic codes over F_{q^2} of length $n = \frac{q^2+1}{5}$ with defining set $T = \bigcup_{i=0}^{k} C_{s+2i}$, where $0 \leq k \leq 5m+4$, $s = \frac{n}{2}$ for q be an odd prime power of the form $q = 10m+7$. By Lemma 3.11 (ii), there is $c = |T_{ss}(\delta)| = 4$ if $3m+2 \leq s \leq 5m+4$. Since every q^2-cyclotomic coset $C_x = \{x, n-x\}$ and x must be odd number, we can obtain that T consists of $2k+1$ integers $\{n-(s+2k), \ldots, n-s, s, s+2, \ldots, s+2k\}$. It implies that $\mathcal{C}$ has minimum distance at least $2k+2$. Hence, $\mathcal{C}$ is a q^2-ary negacyclic code with parameters $[n, n-2(s+1)+1, \geq 2s+2]$. Combining Lemma 2.8 with EA-quantum singleton bound, we can obtain a EA-quantum MDS code with parameters $[[\frac{q^2+1}{5}, \frac{q^2+1}{5} - 2d+6, d; 4]]_q$, $6m+6 \leq d \leq 10m+8$ be even.

If m is even, consider the negacyclic codes over F_{q^2} of length $n = \frac{q^2+1}{5}$ with defining set $T = \bigcup_{i=0}^{k} C_{s+2i}$, where $0 \leq k \leq 6m+3$, $s = \frac{n}{2}$ for q be an odd prime power of the form $q = 10m+7$. By Lemma 3.11 (ii), there is $c = |T_{ss}(\delta)| = 1$ if $1 \leq k \leq 4m+2$ and $c = |T_{ss}(\delta)| = 5$ if $4m+3 \leq k \leq 6m+3$. Since every q^2-cyclotomic coset $C_x = \{x, n-x\}$ and x must be odd number, we can obtain that T consists of $2k+1$ integers $\{n-(s+2k), \ldots, n-s, s, s+2, \ldots, s+2k\}$. It implies that $\mathcal{C}$ has minimum distance at least $2k+2$. Hence, $\mathcal{C}$ is a q^2-ary negacyclic code with parameters $[n, n-2(s+1)+1, \geq 2s+2]$. Combining Lemma 2.8 with EA-quantum singleton bound, we can obtain a EA-quantum MDS code with parameters $[[\frac{q^2+1}{5}, \frac{q^2+1}{5}-2d+6, d; 1]]_q$, $2 \leq d \leq 8m+6$ be even; and $[[\frac{q^2+1}{5}, \frac{q^2+1}{5}-2d+7, d; 5]]_q$ codes, where $8m+8 \leq d \leq 12m+8$ be even. □

Example 8 Let $m = 1$, $q = 17$, applying Theory 3.14 (1) produces 4-*ebits* EA-quantum MDS codes with parameters $[[58, 34, 15; 4]]_{17}$, $[[58, 30, 17; 4]]_{17}$, $[[58, 26, 19; 4]]_{17}$, $[[58, 22, 21; 4]]_{17}$, $[[58, 18, 23; 4]]_{17}$, $[[58, 14, 25; 4]]_{17}$ for d odd; and $[[58, 40, 12; 4]]_{17}$, $[[58, 36, 14; 4]]_{17}$, $[[58, 32, 16; 4]]_{17}$, $[[58, 28, 18; 4]]_{17}$ for d even.

Example 9 Let $m = 2$, $q = 27$, applying Theory 3.14 (2) produces 1-*ebits* EA-quantum MDS codes with parameters $[[146, 145, 2; 1]]_{27}$, $[[146, 141, 4; 1]]_{27}$, $[[146, 137, 6; 1]]_{27}$, $[[146, 133, 8; 1]]_{27}$, $[[146, 129, 10; 1]]_{27}$, $[[146, 125, 12; 1]]_{27}$, $[[146, 121, 14; 1]]_{27}$, $[[146, 117, 16; 1]]_{27}$, $[[146, 113, 18; 1]]_{27}$, $[[146, 109, 20; 1]]_{27}$, $[[146, 105, 22; 1]]_{27}$ for d even; 4-*ebits* EA-quantum MDS codes with parameters $[[146, 106, 23; 4]]_{27}$, $[[146, 102, 25; 4]]_{27}$, $[[146, 98, 27; 4]]_{27}$, $[[146, 94, 29; 4]]_{27}$, $[[146, 90, 31; 4]]_{27}$, $[[146, 86, 33; 4]]_{27}$, $[[146, 82, 35; 4]]_{27}$, $[[146, 78, 37; 4]]_{27}$,

$[[146, 74, 39; 4]]_{27}$ for d odd; and 5-*ebits* EA-quantum MDS codes with parameters $[[146, 105, 24; 5]]_{27}$, $[[146, 101, 26; 5]]_{27}$, $[[146, 97, 28; 5]]_{27}$, $[[146, 93, 30; 5]]_{27}$, $[[146, 89, 32; 5]]_{27}$ for d even.

4 Discussion and Conclusion

In this paper, we have constructed six families of entanglement-assisted quantum MDS (EAQMDS) codes based on classical negacyclic MDS codes. Two of these six classes q-ary EAQMDS have minimum distance more larger than $q+1$. Most of these q-ary EAQMDS codes are new in the sense that their parameters are not covered by the codes available in the literature. In Table 1, we list the EA-quantum MDS codes constructed in this paper. The results show that using entanglement, EAQMDS codes have a larger

Table 1 New EA-Quantum MDS codes

Class	q	Code	Distance
1	q=atm+1,	$[[\frac{q^2-1}{at}, \frac{q^2-1}{at}-2d+2, d; 0]]$	$2 \le d \le (\frac{at}{2}+1)m+1$
	a even,		
	or	$[[\frac{q^2-1}{at}, \frac{q^2-1}{at}-2d+4, d; 2]]$	$(\frac{at}{2}+1)m+2 \le d \le (\frac{at}{2}+2)m+1$
	a odd, t even,		
		$[[\frac{q^2-1}{at}, \frac{q^2-1}{at}-2d+6, d; 4]]$	$(\frac{at}{2}+2)m+2 \le d \le (\frac{at}{2}+3)m+1$
2	$q=30m+11$	$[[\frac{q^2-1}{30}, \frac{q^2-1}{30}-2d+4, d; 2]]$	$8m+4 \le d \le 11m+5$
		$[[\frac{q^2-1}{30}, \frac{q^2-1}{30}-2d+6, d; 4]]$	$11m+6 \le d \le 14m+7$
3	$q=30m+19$	$[[\frac{q^2-1}{30}, \frac{q^2-1}{30}-2d+4, d; 2]]$	$8m+6 \le d \le 11m+7$
		$[[\frac{q^2-1}{30}, \frac{q^2-1}{30}-2d+6, d; 4]]$	$11m+8 \le d \le 13m+8$
		$[[\frac{q^2-1}{30}, \frac{q^2-1}{30}-2d+8, d; 6]]$	$13m+9 \le d \le 16m+10$
4	$q=12m+5$	$[[\frac{q^2-1}{12}, \frac{q^2-1}{12}-2d+4, d; 2]]$	$5m+3 \le d \le 7m+3$
		$[[\frac{q^2-1}{12}, \frac{q^2-1}{12}-2d+6, d; 4]]$	$7m+4 \le d \le 8m+3$
5	$q=10m+3$,	$[[\frac{q^2+1}{5}, \frac{q^2+1}{5}-2d+6, d; 4]]$	$4m+3 \le d \le 6m+1$ is odd
	m odd		$6m+6 \le d \le 10m+8$ is even.
	m even	$[[\frac{q^2+1}{5}, \frac{q^2+1}{5}-2d+3, d; 1]]$	$2 \le d \le 8m+1$ is even
		$[[\frac{q^2+1}{5}, \frac{q^2+1}{5}-2d+6, d; 4]]$	$4m+3 \le d \le 6m+1$ is odd
		$[[\frac{q^2+1}{5}, \frac{q^2+1}{5}-2d+7, d; 5]]$	$8m+4 \le d \le 12m+4$ is even
6	$q=10m+7$,	$[[\frac{q^2+1}{5}, \frac{q^2+1}{5}-2d+6, d; 4]]$	$8m+7 \le d \le 14m+11$ is odd
	m odd		$6m+6 \le d \le 10m+8$ is even
	m even	$[[\frac{q^2+1}{5}, \frac{q^2+1}{5}-2d+3, d; 1]]$	$2 \le d \le 8m+6$ is even
		$[[\frac{q^2+1}{5}, \frac{q^2+1}{5}-2d+6, d; 4]]$	$8m+7 \le d \le 14m+11$ is odd
		$[[\frac{q^2+1}{5}, \frac{q^2+1}{5}-2d+7, d; 5]]$	$8m+8 \le d \le 12m+8$ is even

minimum distance than QMDS codes. We look forward to seeing that some special types of $[[n, k, d; c]]$ EAQMDS codes that better perform than $[[n + c, k, d]]$ QMDS codes even if these $[[n, k, d; c]]$ EAQMDS codes are equivalent to those $[[n+c, k, d]]$ QMDS codes.

Acknowledgements This work is supported by National Natural Science Foundations of China under Grant No. 11471011 and the Natural Science Foundation of Shaanxi under Grant No. 2017JQ1032. We are grateful to the two anonymous reviewers for their helpful and constructive comments.

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