

Steady states of continuous-time open quantum walks

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Abstract Continuous-time open quantum walks (CTOQW) are introduced as the formulation of quantum dynamical semigroups of trace-preserving and completely positive linear maps (or quantum Markov semigroups) on graphs. We show that a CTOQW always converges to a steady state regardless of the initial state when a graph is connected. When the graph is both connected and regular, it is shown that the steady state is the maximally mixed state. As shown by the examples in this article, the steady states of CTOQW can be very unusual and complicated even though the underlying graphs are simple. The examples demonstrate that the structure of a graph can affect quantum coherence in CTOQW through a long-time run. Precisely, the quantum coherence persists throughout the evolution of the CTOQW when the underlying topology is certain irregular graphs (such as a path or a star as shown in the examples). In contrast, the quantum coherence will eventually vanish from the open quantum system when the underlying topology is a regular graph (such as a cycle).

Keywords Semigroups on graphs · Continuous-time open quantum walks · Steady states

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1 Introduction

A standard starting point for the discussion of the dynamics of continuous-time open quantum systems is the Markovian quantum master equation [1–3]

$$\begin{aligned} \frac{d\rho(t)}{dt} = & i[\rho(t), H(t)] + \sum_k \gamma_k(t)[L_k(t)\rho(t)L_k^\dagger(t) \\ & - \frac{1}{2}\{L_k^\dagger(t)L_k(t), \rho(t)\}]. \end{aligned} \quad (1)$$

Here, $\rho(t)$ is the state of the system at time t , the self-adjoint operator $H(t)$ is the generator of the coherent part of the evolution, and the Lindblad operators $L_k(t)$ are the incoherent or dissipative system operators of the evolution with corresponding relaxation times $\gamma_k(t)$.

When $H(t)$, $L_k(t)$ and $\gamma_k(t)$ are independent of time t , the Markovian master equation describes a homogeneous evolution of continuous-time open quantum system. For simplicity, all $\gamma_k(t)$ can be assumed to be 1, then the above equation is written as

$$\begin{aligned} \frac{d\rho(t)}{dt} = & i[\rho(t), H] + \sum_k [L_k\rho(t)L_k^\dagger \\ & - \frac{1}{2}\{L_k^\dagger L_k, \rho(t)\}]. \end{aligned} \quad (2)$$

Introducing an operator \mathcal{L} as follows

$$\mathcal{L}\rho = i[\rho, H] + \sum_k [L_k\rho L_k^\dagger - \frac{1}{2}\{L_k^\dagger L_k, \rho\}], \quad (3)$$

one can rewrite Eq. (2) as

$$\frac{d}{dt}\rho(t) = \mathcal{L}\rho(t). \quad (4)$$

As we may know, $T_t = e^{t\mathcal{L}}$ forms a quantum dynamical semigroup of trace-preserving and completely positive linear maps, which is continuous in $t \in \mathbb{R}_+$. For any density matrix $\rho(0)$, $\rho(t) = T_t(\rho(0))$ fulfills Eq. (4). A detailed justification for these assertions can be found in literature, for example [4]. In the context of quantum dynamical semigroup, the operator \mathcal{L} is often called the generator or infinitesimal generator of the semigroup.

When dynamical processes take place on a graph, the topology of the graph often affects or even determines the behavior of the processes [5,6]. Our work here is concerned with a semigroup taking place on a graph; we are interested in determining the limiting states of the semigroup and knowing how the structure of the graph can determine the behavior of the semigroup. The study shows through the examples of this

article, among other things, that the long-time behavior of the semigroups can be very unusual and complicated even though the underlying graphs are simple. The aforesaid key attribute of the semigroups on graphs may be useful for quantum information processing and quantum computing [7].

To be more precise, we now consider a scenario that the quantum dynamical semigroup governed by Eq. (4) is confined to a graph $G = (V, E)$. Here, $\rho(t)$, the state of the system at time t , is a density matrix on the Hilbert space spanned by the orthonormal basis represented by the vertices of G . We interpret H , the generator of the coherent part, to be the *Laplacian* matrix of G , and L_k , the dissipative system operators, to be operators determined by the adjacency matrix of G (the precise description of these operators will be given in Sect. 2). This interpretation is not necessary. Nonetheless, it respects the topological structure of a graph G .

In this article, the formulation of the quantum dynamical semigroup of trace-preserving, completely positive linear maps on a graph G will be called *continuous-time open quantum walks* (abbreviated to CTOQW) [8].¹

We remark that a different approach to the notion of the CTOQW was recently proposed [9]. This type of continuous-time open quantum walks is obtained by taking a continuous-time limit of the discrete-time open quantum random walks [10, 11]. More recently, a second different species of CTOQW was presented [12]; it can be considered as the exact continuous-time version of the aforesaid discrete-time open quantum walks

It is noteworthy that there are two extreme cases of possible continuous-time transport processes on graphs that have been extensively studied. One is continuous-time random walks (CTRW) modeling a purely incoherent transport process, see, e.g., [6]. The other is continuous-time quantum walks (CTQW) modeling a purely coherent transport process [13]. The probability distribution for CTRW at time t is given by $p(t) = e^{tL} p(0)$ where $p(0)$ is the initial probability distribution. The transition amplitude for CTQW at time t is given by $|\psi\rangle = e^{-itL} |\psi(0)\rangle$ where $|\psi(0)\rangle$ is the initial amplitude of CTQW. In both cases described above, L can be chosen as the *Laplacian* matrix of the graph. The history and developments of continuous-time quantum walks, and their application to various sciences can be found in [6, 14, 15] and the references therein.

Coming back to CTOQW to be formulated, we may consider it as a mixture of the two aforesaid extreme cases of continuous-time transport processes on graphs to some extent. In addition to formulating CTOQW, we will also study properties of steady states of CTOQW.

This article is organized as follows. In Sect. 2, the formalism for continuous-time open quantum walks is presented. We then analyze the steady states of CTOQW and offer examples of CTOQW in Sect. 3. This article is closed in Sect. 4 with concluding remarks and related questions.

¹ We were not aware of the work [8] before we posted on arXiv the first version of the present article under the title *Continuous-time open quantum walks*. We acknowledge that [8] should be the very first paper concerning semigroups on graphs.

2 Formulation of continuous-time open quantum walks

2.1 Matrices associated with a graph

In this section, we will introduce continuous-time open quantum walks (CTOQW) as the formulation of homogeneous quantum Markov semigroups on graphs.

To prepare for the formulation, we will first recall some matrices associated with a graph. Given an undirected graph $G = (V, E)$ without multiple edges or self-loops, let A be the *adjacency matrix* of G , the $|V| \times |V|$ matrix with the elements,

$$A_{jk} = \begin{cases} 1 & \text{if } (j, k) \in E \\ 0 & \text{if } (j, k) \notin E \end{cases} \text{ for every pair } j, k \in V.$$

We denote by D the degree matrix with $D_{jj} = \text{deg}(j)$. Here, $\text{deg}(j)$ denotes the degree of vertex j for each vertex j in V . There are two more equally important matrices associated with the graph G : the *Laplacian* matrix of G and the (canonical) *matrix of transition probabilities* for the Markov chain generated by the graph G .

Their elements are defined as:

$$L_{jk} = \begin{cases} \text{deg}(j) & \text{if } j = k \\ -1 & \text{if } (j, k) \in E \\ 0 & \text{if } (j, k) \notin E \end{cases}$$

$$M_{jk} = \begin{cases} \frac{1}{\text{deg}(k)} & \text{if } (j, k) \in E \\ 0 & \text{Otherwise} \end{cases}$$

for every pair $j, k \in V$. By their definitions, it is evident that $L = D - A$, and $M = AD^{-1}$.

One can associate with every vertex of the graph G a basis vector in an $|V|$ -dimensional vector space. Now, these basis vectors form a complete orthonormal basis, which, for instance, is given by

$$|1\rangle = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, |2\rangle = \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix}, \dots, |N\rangle = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}. \tag{5}$$

Take as an example a cycle of three sites, then the matrices L and M read:

$$L = \begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix}, M = \begin{bmatrix} 0 & 1/2 & 1/2 \\ 1/2 & 0 & 1/2 \\ 1/2 & 1/2 & 0 \end{bmatrix}. \tag{6}$$

Writing M in a quantum mechanical fashion via the projection operators $|j\rangle\langle k|$ leads to

$$\begin{aligned}
 M &= \frac{1}{2} \sum_{j \neq k} |k\rangle\langle j| \\
 &= \frac{1}{2} [|2\rangle\langle 1| + |3\rangle\langle 1| + |1\rangle\langle 2| + |3\rangle\langle 2| + |1\rangle\langle 3| + |2\rangle\langle 3|] \tag{7}
 \end{aligned}$$

2.2 Definition of continuous-time open quantum walks

Having now specified a graph G through its associated matrices, we are in a position to define the generator of the semigroup $e^{t\mathcal{L}}$ on G . We choose H in Eq. (3) to be the *Laplacian* matrix of the graph G , Lindblad operators to be the swap operator $B_{jk} = \sqrt{M_{jk}}|j\rangle\langle k|$. The parameters M_{jk} specify the strength of swapping the state $|k\rangle$ for the state $|j\rangle$.

Then, the generator \mathcal{L} can be recast as

$$\mathcal{L}\rho = i[\rho, L] + \sum_{j,k} [B_{jk}\rho B_{jk}^\dagger - \frac{1}{2}\{B_{jk}^\dagger B_{jk}, \rho\}]. \tag{8}$$

For a given initial state $\rho(0)$ of the quantum system on G , the expression $\rho(t) = T_t(\rho(0)) = e^{t\mathcal{L}}\rho(0)$ is called the state of the continuous-time open quantum walk on G at time t .

It should be pointed out that the group of Lindblad operators given in Eq. (8) is just one of possible options; there are other options for Lindblad operators in Eq. (3). For example, (1) the Lindblad operators can be chosen to be e^{-iL} and \mathbb{I} where L is the *Laplacian* matrix of a graph; or (2) they can be chosen to be a projection operator P and $\mathbb{I} - P$ where $P = |1\rangle\langle 1| + \dots + |j\rangle\langle j| + \dots + |m\rangle\langle m|$ with j standing for the j th vertex of a graph. In what follows, unless specified, the Lindblad operators to be used are ones given by Eq. (8).

3 Steady states of continuous-times open quantum walks

In this section, we will study properties of steady states of CTOQW. In the theory of completely positive semigroups, a semigroup T_t is said to be relaxing if there exists a unique (steady) state ρ_∞ such that $T_t(\rho_\infty) = \rho_\infty$ for all t and $\lim_{t \rightarrow \infty} T_t(\rho) = \rho_\infty$ for an arbitrary state ρ .

It can be verified that ρ is a steady state of T_t for $t > 0$ if and only if ρ is in the kernel of \mathcal{L} , i.e., $T_t(\rho) = \rho$ for all $t > 0$ if and only if $\mathcal{L}(\rho) = 0$. The semigroup is obviously not relaxing if the equation $\mathcal{L}(\rho) = 0$ admits more than one solution.

It is desirable to have some condition on the structure of the graph which assures that the semigroup defining CTOQW is relaxing as well as a characterization of the steady states.

Here, we recall a very important result (please refer to [4, 16] and the references cited therein) as a sufficient condition for a semigroup being relaxing.

Lemma 1 Consider a quantum dynamical semigroup of trace-preserving, completely positive linear maps, $T_t = e^{\mathcal{L}t}$ with generator

$$\mathcal{L}\rho = i[\rho, H] + \sum_k [L_k \rho L_k^\dagger - \frac{1}{2}\{L_k^\dagger L_k, \rho\}], \tag{9}$$

for some set of indices I . Provided that the linear space spanned by the set of Lindblad operators L_k is Hermitian (this means that for every $X = \sum_k x_k L_k$ there are $y_k \in \mathbb{C}$ such that $X^\dagger = \sum_k y_k L_k$.) and the only operators commuting with all of them are proportional to the identity, which is expressed as $\{L_k, k \in I\}' = c\mathbb{I}$ (Note: the left-hand side of this equality is called the commutant of $\{L_k, k \in I\}$), then the semigroup T_t is relaxing and the steady state is a positive definite density matrix.

We now present our basic result which gives a sufficient condition for convergence of a continuous-time open quantum walk.

Theorem 1 If a graph G is connected, then CTOQW on the graph has a positive definite matrix (state) ρ_∞ such that for all density matrices ρ we have $\lim_{t \rightarrow \infty} e^{t\mathcal{L}}\rho = \rho_\infty$. If G is connected, then $\lim_{t \rightarrow \infty} e^{t\mathcal{L}}\rho = \frac{1}{|V|}\mathbb{I}$ for any initial state ρ if and only if the associated matrix of transition probabilities on G is doubly stochastic.

We call a square matrix with nonnegative entries doubly stochastic if the sum of the entries in each row and in each column is 1.

Proof Since G is an undirected graph, $M_{jk} \neq 0$ iff $M_{kj} \neq 0$ for any j and k . This implies that the linear space spanned by the set of Lindblad operators B_{jk} is Hermitian. Next, we prove that $\{B_{jk}, j, k \in I\}' = c\mathbb{I}$. Provided that $X = (X_{lm}) \in \{B_{jk}, j, k \in I\}'$, i.e., $B_{jk}X = XB_{jk}$ for all possible B_{jk} , then we claim $X = c\mathbb{I}$ for some constant c . To justify it, we will show that for each l_0 , $X_{l_0 m_0} = 0$ whenever $m_0 \neq l_0$. Now for the given l_0 , there exists k_0 such that $B_{k_0 l_0} \neq 0$ as G is connected. Please note that \square

$$\begin{aligned} B_{k_0 l_0} X &= \frac{1}{\sqrt{\deg(l_0)}} |k_0\rangle\langle l_0| \sum_{j,m} X_{jm} |j\rangle\langle m| \\ &= \frac{1}{\sqrt{\deg(l_0)}} \sum_m X_{l_0 m} |k_0\rangle\langle m|, \\ X B_{k_0 l_0} &= \frac{1}{\sqrt{\deg(l_0)}} \sum_{j,m} X_{jm} |j\rangle\langle m| k_0\rangle\langle l_0| \\ &= \frac{1}{\sqrt{\deg(l_0)}} \sum_j X_{j k_0} |j\rangle\langle l_0|. \end{aligned} \tag{10}$$

Then, $B_{k_0 l_0} X = X B_{k_0 l_0}$ implies that $X_{l_0 m_0} = 0$ whenever $m_0 \neq l_0$ and $X_{l_0 l_0} = X_{k_0 k_0}$. Repeated applying this reasoning leads to the claim that $X = c\mathbb{I}$ where c is a constant. Thus, $\{B_{jk}, j, k \in I\}' = c\mathbb{I}$. By Lemma 1, CTOQW on the graph has a positive definite matrix (steady state) ρ_∞ such that for all density matrices ρ we have $\lim_{t \rightarrow \infty} e^{t\mathcal{L}}\rho = \rho_\infty$.

To prove the second assertion, it suffices to show that $\mathcal{L}(\mathbb{I}) = 0$ if and only if $\sum_k M_{jk} = 1$ for all j (which means that M is doubly stochastic).

Since the sum of all entries on each column of M is one, i.e., $\sum_j M_{jk} = 1$ for all k , we have

$$\begin{aligned} \sum_j \sum_k B_{jk}^\dagger B_{jk} &= \sum_k \sum_j (\sqrt{M_{jk}})^2 |k\rangle\langle j|j\rangle\langle k| \\ &= \sum_k \left(\sum_j M_{jk} \right) |k\rangle\langle k| \\ &= \mathbb{I}. \end{aligned} \tag{11}$$

Observe that

$$\begin{aligned} \sum_j \sum_k B_{jk} B_{jk}^\dagger &= \sum_j \sum_k (\sqrt{M_{jk}})^2 |j\rangle\langle k|k\rangle\langle j| \\ &= \sum_j \left(\sum_k M_{jk} \right) |j\rangle\langle j|. \end{aligned} \tag{12}$$

Equations (11) and (12) together imply the following identities:

$$\begin{aligned} \mathcal{L}(\mathbb{I}) &= i[\mathbb{I}, L] + \sum_{j,k} [B_{jk}\mathbb{I}B_{jk}^\dagger - \frac{1}{2}\{B_{jk}^\dagger B_{jk}, \mathbb{I}\}] \\ &= \sum_j \left(\sum_k M_{jk} \right) |j\rangle\langle j| - \mathbb{I}. \end{aligned} \tag{13}$$

Note that $\mathcal{L}(\mathbb{I}) = 0$ iff $\sum_j (\sum_k M_{jk}) |j\rangle\langle j| = \mathbb{I}$ iff $\sum_k M_{jk} = 1$ for all j . Therefore, the result follows.

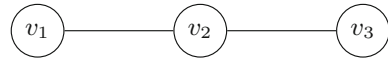
Since the matrix of transition probabilities associated with a regular graph is doubly stochastic, Theorem 1 has an immediate corollary, which concerns the convergence of CTOQW on a regular graph.

Corollary 1 *If graph G is regular, then $\lim_{t \rightarrow \infty} e^{t\mathcal{L}}\rho = \frac{1}{|V|}\mathbb{I}$ for any initial state ρ .*

In what follows, we consider three examples of CTOQW. The first example of CTOQW is the one on an n -cycle. Since the graph is regular, $\frac{1}{n}\mathbb{I}$ is the unique steady state such that $\lim_{t \rightarrow \infty} e^{t\mathcal{L}}\rho = \frac{1}{n}\mathbb{I}$ for any initial state ρ according to Corollary 1.

We then consider the CTOQW on a path with three sites (see the graph below). Meanwhile, we also consider the corresponding CTRW and CTQW on this graph to compare the long-time behaviors of these three distinct continuous-time dynamical processes in terms of steady states and limiting probability distributions (Fig. 1).

Fig. 1 Path with three sites



The Laplacian matrix and the matrix of transition probabilities associated with this graph are as follows:

$$L = \begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}, M = \begin{bmatrix} 0 & 1/2 & 0 \\ 1 & 0 & 1 \\ 0 & 1/2 & 0 \end{bmatrix} \tag{14}$$

The Lindblad operators/Swap operators are given by:

$$B_{12} = \sqrt{2}/2|v_1\rangle\langle v_2| = \begin{bmatrix} 0 & \sqrt{2}/2 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{15}$$

$$B_{32} = \sqrt{2}/2|v_3\rangle\langle v_2| = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & \sqrt{2}/2 & 0 \end{bmatrix} \tag{16}$$

$$B_{21} = |v_2\rangle\langle v_1| = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{17}$$

$$B_{23} = |v_2\rangle\langle v_3| = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \tag{18}$$

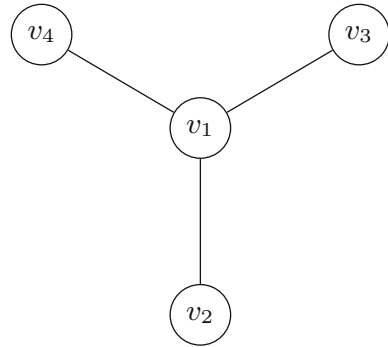
Since this path is a connected graph, CTOQW has a unique steady state by Theorem 1. To find this steady state, it suffices to solve for ρ the system of linear equations of $\mathcal{L}(\rho) = 0$ and $\sum_{j=1}^3 \rho_{jj} = 1$. Here, \mathcal{L} is given by Eq. (8) and $\rho = (\rho_{jk})_{3 \times 3}$. After massive calculation, we obtain a unique solution to the system of the linear equations, which is given by

$$\begin{bmatrix} \frac{2}{7} & -\frac{1}{28} + \frac{1}{28}i & \frac{1}{14} \\ -\frac{1}{28} - \frac{1}{28}i & \frac{3}{7} & -\frac{1}{28} - \frac{1}{28}i \\ \frac{1}{14} & -\frac{1}{28} + \frac{1}{28}i & \frac{2}{7} \end{bmatrix} \tag{19}$$

It can be verified that the above matrix is a positive definite density matrix. To distinguish this steady state from other density states, we denote it by ρ_∞ . According to Theorem 1, for CTOQW on this path, we have $\lim_{t \rightarrow \infty} e^{t\mathcal{L}}\rho = \rho_\infty$ for any initial state ρ . This implies that the limiting probability distribution over the three sites, namely v_1, v_2 and v_3 , is $(\frac{2}{7}, \frac{3}{7}, \frac{2}{7})$ regardless of the initial state of CTOQW.

We now turn to the two aforesaid extreme cases of continuous-time processes on this graph. For CTRW, it is interesting to note that the long-time limit of all transition probabilities is $\lim_{t \rightarrow \infty} p(t) = 1/3$. In fact, every CTRW whose transfer matrix follows directly from the Laplacian matrix will eventually decay at long times to the equipartition value $1/N$ and is independent of the connectivity of the graph [6].

Fig. 2 Star with three edges



For CTQW, the eigenvalues and the corresponding unit eigenvectors of Laplacian matrix L are as follows:

$$\lambda_1 = 0, \psi_1 = \left(\frac{\sqrt{3}}{3}, \frac{\sqrt{3}}{3}, \frac{\sqrt{3}}{3} \right)^T \tag{20}$$

$$\lambda_2 = 1, \psi_2 = \left(-\frac{\sqrt{2}}{2}, 0, \frac{\sqrt{2}}{2} \right)^T \tag{21}$$

$$\lambda_3 = 3, \psi_3 = \left(\frac{\sqrt{6}}{6}, -\frac{\sqrt{6}}{3}, \frac{\sqrt{6}}{6} \right)^T \tag{22}$$

By standard computation rule, the limiting average probability of finding the quantum walker at a site v initially launching at a site u is given by $P_\infty(v|u) = \sum_{j=1}^3 |\langle v|\psi_j\rangle|^2 |\langle u|\psi_j\rangle|^2$.

We now show the limiting average probability distributions of the CTQW launching from two specific states:

Case 1 When the walker is launched at v_1 (or v_3), then the limiting average probability distribution is $(\frac{7}{18}, \frac{4}{18}, \frac{7}{18})$.

Case 2 When the walker is launched at v_2 , then the limiting average probability distribution is $(\frac{2}{9}, \frac{5}{9}, \frac{2}{9})$.

As shown above, the asymptotic behaviors of these three continuous-time processes on the path are quite distinct. Unlike CTQW, both CTOQW and CTRW have unique limiting distributions which are independent of initial states/probability distributions.

Finally, we consider the CTOQW on a star with three edges (this graph is often called a claw and is shown in Fig. 2).

The Laplacian matrix and the matrix of transition probabilities associated with this graph are as follows:

$$L = \begin{bmatrix} 3 & -1 & -1 & -1 \\ -1 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ -1 & 0 & 0 & 1 \end{bmatrix}, M = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1/3 & 0 & 0 & 0 \\ 1/3 & 0 & 0 & 0 \\ 1/3 & 0 & 0 & 0 \end{bmatrix} \tag{23}$$

Since the star is connected, there is a unique steady state denoted by ρ_∞ such that $\lim_{t \rightarrow \infty} e^{t\mathcal{L}}\rho = \rho_\infty$ for any initial state ρ according to Theorem 1. By a similar approach employed in the second example, we obtain the steady state given by:

$$\rho_\infty = \begin{bmatrix} \frac{11}{26} & -\frac{2}{39} - \frac{1}{39}i & -\frac{2}{39} - \frac{1}{39}i & -\frac{2}{39} - \frac{1}{39}i \\ -\frac{2}{39} + \frac{1}{39}i & \frac{5}{26} & \frac{2}{39} & \frac{2}{39} \\ -\frac{2}{39} + \frac{1}{39}i & \frac{2}{39} & \frac{5}{26} & \frac{2}{39} \\ -\frac{2}{39} + \frac{1}{39}i & \frac{2}{39} & \frac{2}{39} & \frac{5}{26} \end{bmatrix} \tag{24}$$

It should be noted that the limiting density matrices of CTOQW such as the ones given by Eqs. (19) and (24) can be very complicated and unusual even though the underlying graphs are simple. It also should be noted that the nonzeroness of the off-diagonal elements of the limiting density matrices indicates the existence of quantum coherence throughout the evolution of the CTOQW on the graphs. This finding suggests that quantum coherence persists throughout the evolution of the CTOQW when the underlying topology is certain irregular graphs (such as a path or a star as shown in the examples). In contrast, quantum coherence will eventually vanish from the open quantum system when the underlying topology is a regular graph (such as a cycle) by Theorem 1.

4 Concluding remarks and related questions

In this article, we examine the evolution of CTOQW on finite graphs. In general, for dynamical processes on graphs, one is interested not only in determining the shape of the limiting state (also known as steady state) of the processes and the condition under which the limiting state exists, but also in determining the relation between steady states and the structures of the underlying graphs. We have shown that a CTOQW always converges to a steady state regardless of the initial state when a graph is connected. When the graph is both connected and regular, it is shown that the steady state is the maximally mixed state. As shown by the examples in this article, the steady states of CTOQW can be very unusual and complicated even though the underlying graphs are simple. Our examples with the steady states of CTOQW, to our best knowledge, may be the first examples that were reported in literature for semigroups with explicit steady states determined by graphs.

CTOQW on graphs can be extended to lattices. For instance, consider CTOQW on one-dimensional lattice \mathbb{Z} . One may choose H in Eq. (3) to be the *Laplacian*

matrix of the lattice \mathbb{Z} and choose the swap operator $B_{x,x-1} = \frac{\sqrt{2}}{2}|x\rangle\langle x-1|$ and $B_{x,x+1} = \frac{\sqrt{2}}{2}|x\rangle\langle x+1|$ as Lindblad operators. It would be interesting to investigate the asymptotic behavior of ρ_t/\sqrt{t} .

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