Quantum Inf Process (2017) 16:77
DOI 10.1007/s11128-017-1520-3

Tighter entanglement monogamy relations of qubit systems

Zhi-Xiang Jin[1] · **Shao-Ming Fei**[1,2]

Received: 15 October 2016 / Accepted: 4 January 2017 / Published online: 3 February 2017

Abstract Monogamy relations characterize the distributions of entanglement in multipartite systems. We investigate monogamy relations related to the concurrence C and the entanglement of formation E. We present new entanglement monogamy relations satisfied by the α-th power of concurrence for all $\alpha \geq 2$, and the α-th power of the entanglement of formation for all $\alpha \geq \sqrt{2}$. These monogamy relations are shown to be tighter than the existing ones.

Keywords Entanglement · Monogamy · Concurrence · Entanglement of formation

1 Introduction

Quantum entanglement [1–8] is an essential feature of quantum mechanics. As one of the fundamental differences between quantum entanglement and classical correlations, a key property of entanglement is that a quantum system entangled with one of other subsystems limits its entanglement with the remaining ones. The monogamy relations give rise to the distribution of entanglement in the multipartite setting. Monogamy is also an essential feature allowing for security in quantum key distribution [9].

For a tripartite system A, B and C, the usual monogamy of an entanglement measure $\mathcal{E}$ implies that [10] the entanglement between A and BC satisfies $\mathcal{E}_{A|BC} \geq \mathcal{E}_{AB} + \mathcal{E}_{AC}$. Such monogamy relations are not always satisfied by all entanglement measures for all quantum states. It has been shown that the squared concurrence C^2 [11,12] and the squared entanglement of formation E^2 [13] satisfy the monogamy relations for

✉ Zhi-Xiang Jin
jzxjinzhixiang@126.com

1 School of Mathematical Sciences, Capital Normal University, Beijing 100048, China

2 Max-Planck-Institute for Mathematics in the Sciences, 04103 Leipzig, Germany

multi-qubit states. It is further proved that [14] C^α and E^α satisfy the monogamy inequalities for $\alpha \geq 2$ and $\alpha \geq \sqrt{2}$, respectively.

In this paper, we show that the monogamy inequalities obtained so far can be made tighter. We establish entanglement monogamy relations for the α-th power of the concurrence C and the entanglement of formation E which are tighter than those in [14], which give rise to finer characterizations of the entanglement distributions among the multipartite qubit states.

2 Tighter monogamy relation of concurrence

We first consider the monogamy inequalities related to concurrence. Let H_X denote a discrete finite dimensional complex vector space associated with a quantum subsystem X. For a bipartite pure state $|\psi\rangle_{AB}$ in vector space $H_A \otimes H_B$, the concurrence is given by [15–17]

$$C(|\psi\rangle_{AB}) = \sqrt{2\left[1 - \mathrm{Tr}(\rho_A^2)\right]}, \tag{1}$$

where ρ_A is the reduced density matrix by tracing over the subsystem B, $\rho_A = \mathrm{Tr}_B(|\psi\rangle_{AB}\langle\psi|)$. The concurrence for a bipartite mixed state ρ_{AB} is defined by the convex roof extension

$$C(\rho_{AB}) = \min_{\{p_i, |\psi_i\rangle\}} \sum_i p_i C(|\psi_i\rangle),$$

where the minimum is taken over all possible decompositions of $\rho_{AB} = \sum_i p_i |\psi_i\rangle\langle\psi_i|$, with $p_i \geq 0$ and $\sum_i p_i = 1$ and $|\psi_i\rangle \in H_A \otimes H_B$.

For an N-qubit pure state $|\psi\rangle_{AB_1\ldots B_{N-1}} \in H_A \otimes H_{B_1} \otimes \cdots \otimes H_{B_{N-1}}$, the concurrence $C(|\psi\rangle_{A|B_1\ldots B_{N-1}})$ of the state $|\psi\rangle_{A|B_1\ldots B_{N-1}}$, viewed as a bipartite state under the partitions A and $B_1, B_2, \ldots, B_{N-1}$, satisfies the Coffman–Kundu–Wootters (CKW) inequality [11,12],

$$C^2_{A|B_1,B_2\ldots,B_{N-1}} \geq C^2_{A|B_1} + C^2_{A|B_2} + \cdots + C^2_{A|B_{N-1}}, \tag{2}$$

where $C_{AB_i} = C(\rho_{AB_i})$ is the concurrence of $\rho_{AB_i} = \mathrm{Tr}_{B_1\ldots B_{i-1}B_{i+1}\ldots B_{N-1}}(|\psi\rangle_{AB_1\ldots B_{N-1}}\langle\psi|)$, $C_{A|B_1,B_2\ldots,B_{N-1}} = C(|\psi\rangle_{A|B_1\ldots B_{N-1}})$. It is further proved that for $\alpha \geq 2$, one has [14],

$$C^\alpha_{A|B_1,B_2\ldots,B_{N-1}} \geq C^\alpha_{A|B_1} + C^\alpha_{A|B_2} + \cdots + C^\alpha_{A|B_{N-1}}. \tag{3}$$

In fact, as the characterization of the entanglement distribution among the subsystems, the monogamy inequalities satisfied by the concurrence can be refined and becomes tighter. Before finding tighter monogamy relations of concurrence, we first introduce a Lemma.

Lemma 1 *For any $2 \otimes 2 \otimes 2^{n-2}$ mixed state $\rho \in H_A \otimes H_B \otimes H_C$, if $C_{AB} \geq C_{AC}$, we have*

$$C^\alpha_{A|BC} \geq C^\alpha_{AB} + \frac{\alpha}{2} C^\alpha_{AC}, \tag{4}$$

for all $\alpha \geq 2$.

Proof For arbitrary $2 \otimes 2 \otimes 2^{n-2}$ tripartite state ρ_{ABC}, one has [11,18], $C^2_{A|BC} \geq C^2_{AB} + C^2_{AC}$. If $C_{AB} \geq C_{AC}$, we have

$$\begin{aligned} C^\alpha_{A|BC} &\geq (C^2_{AB} + C^2_{AC})^{\frac{\alpha}{2}} = C^\alpha_{AB} \left(1 + \frac{C^2_{AC}}{C^2_{AB}}\right)^{\frac{\alpha}{2}} \\ &\geq C^\alpha_{AB} \left[1 + \frac{\alpha}{2} \left(\frac{C^2_{AC}}{C^2_{AB}}\right)^{\frac{\alpha}{2}}\right] = C^\alpha_{AB} + \frac{\alpha}{2} C^\alpha_{AC}, \end{aligned}$$

where the second inequality is due to the inequality $(1+t)^x \geq 1 + xt \geq 1 + xt^x$ for $x \geq 1,\ 0 \leq t \leq 1$. □

In the Lemma, without loss of generality, we have assumed that $C_{AB} \geq C_{AC}$, since the subsystems A and B are equivalent. Moreover, in the proof of the Lemma we have assumed $C_{AB} > 0$. If $C_{AB} = 0$ and $C_{AB} \geq C_{AC}$, then $C_{AB} = C_{AC} = 0$. The lower bound is trivially zero. For multipartite qubit systems, we have the following Theorem.

Theorem 1 *For any* $2 \otimes 2 \otimes \cdots \otimes 2$ *mixed state* $\rho \in H_A \otimes H_{B_1} \otimes \cdots \otimes H_{B_{N-1}}$, *if* $C_{AB_i} \geq C_{A|B_{i+1}\ldots B_{N-1}}$ *for* $i = 1, 2, \ldots, m$, *and* $C_{AB_j} \leq C_{A|B_{j+1}\ldots B_{N-1}}$ *for* $j = m+1, \ldots, N-2$, $\forall\ 1 \leq m \leq N-3$, $N \geq 4$, *we have*

$$\begin{aligned} C^\alpha_{A|B_1B_2\ldots B_{N-1}} &\geq C^\alpha_{A|B_1} \\ &+ \frac{\alpha}{2} C^\alpha_{A|B_2} + \cdots + \left(\frac{\alpha}{2}\right)^{m-1} C^\alpha_{A|B_m} \\ &+ \left(\frac{\alpha}{2}\right)^{m+1} (C^\alpha_{A|B_{m+1}} + \cdots + C^\alpha_{A|B_{N-2}}) \\ &+ \left(\frac{\alpha}{2}\right)^m C^\alpha_{A|B_{N-1}} \end{aligned} \tag{5}$$

for all $\alpha \geq 2$.

Proof By using the inequality (4) repeatedly, one gets

$$\begin{aligned} C^\alpha_{A|B_1B_2\ldots B_{N-1}} &\geq C^\alpha_{A|B_1} + \frac{\alpha}{2} C^\alpha_{A|B_2\ldots B_{N-1}} \\ &\geq C^\alpha_{A|B_1} + \frac{\alpha}{2} C^\alpha_{A|B_2} + \left(\frac{\alpha}{2}\right)^2 C^\alpha_{A|B_3\ldots B_{N-1}} \\ &\geq \cdots \geq C^\alpha_{A|B_1} + \frac{\alpha}{2} C^\alpha_{A|B_2} + \cdots + \left(\frac{\alpha}{2}\right)^{m-1} C^\alpha_{A|B_m} \\ &+ \left(\frac{\alpha}{2}\right)^m C^\alpha_{A|B_{m+1}\ldots B_{N-1}}. \end{aligned} \tag{6}$$

As $C_{AB_j} \leq C_{A|B_{j+1}\dots B_{N-1}}$ for $j = m+1, \dots, N-2$, by (4) we get

$$
\begin{aligned}
C^{\alpha}_{A|B_{m+1}\dots B_{N-1}} &\geq \frac{\alpha}{2} C^{\alpha}_{A|B_{m+1}} + C^{\alpha}_{A|B_{m+2}\dots B_{N-1}} \\
&\geq \frac{\alpha}{2} (C^{\alpha}_{A|B_{m+1}} + \cdots + C^{\alpha}_{A|B_{N-2})} + C^{\alpha}_{A|B_{N-1}}.
\end{aligned} \tag{7}
$$

Combining (6) and (7), we have Theorem 1. □

As for $\alpha \geq 2$, $(\alpha/2)^m \geq 1$ for all $1 \leq m \leq N-3$, comparing with the monogamy relation (3), our formula (5) in Theorem 1 gives a tighter monogamy relation with larger lower bounds. In Theorem 1 we have assumed that some $C_{AB_i} \geq C_{A|B_{i+1}\dots B_{N-1}}$ and some $C_{AB_j} \leq C_{A|B_{j+1}\dots B_{N-1}}$ for the $2 \otimes 2 \otimes \cdots \otimes 2$ mixed state $\rho \in H_A \otimes H_{B_1} \otimes \cdots \otimes H_{B_{N-1}}$. If all $C_{AB_i} \geq C_{A|B_{i+1}\dots B_{N-1}}$ for $i = 1, 2, \dots, N-2$, then we have the following conclusion:

Theorem 2 *If $C_{AB_i} \geq C_{A|B_{i+1}\dots B_{N-1}}$ for all $i = 1, 2, \dots, N-2$, then we have*

$$
C^{\alpha}_{A|B_1\dots B_{N-1}} \geq C^{\alpha}_{A|B_1} + \frac{\alpha}{2} C^{\alpha}_{A|B_2} + \cdots + \left(\frac{\alpha}{2}\right)^{N-2} C^{\alpha}_{A|B_{N-1}}. \tag{8}
$$

Example 1 Let us consider the three-qubit state $|\psi\rangle$ which can be written in the generalized Schmidt decomposition form [19,20],

$$
|\psi\rangle = \lambda_0|000\rangle + \lambda_1 e^{i\varphi}|100\rangle + \lambda_2|101\rangle + \lambda_3|110\rangle + \lambda_4|111\rangle, \tag{9}
$$

where $\lambda_i \geq 0,\ i = 0, \dots, 4$ and $\sum_{i=0}^{4} \lambda_i^2 = 1$. From the definition of concurrence, we have $C_{A|BC} = 2\lambda_0\sqrt{\lambda_2^2 + \lambda_3^2 + \lambda_4^2}$, $C_{A|B} = 2\lambda_0\lambda_2$, and $C_{A|C} = 2\lambda_0\lambda_3$. Set $\lambda_0 = \lambda_1 = \lambda_2 = \lambda_3 = \lambda_4 = \frac{\sqrt{5}}{5}$. One gets $C^{\alpha}_{A|BC} = (\frac{2\sqrt{3}}{5})^{\alpha}$, $C^{\alpha}_{A|B} + C^{\alpha}_{A|C} = 2(\frac{2}{5})^{\alpha}$, $C^{\alpha}_{A|B} + \frac{\alpha}{2} C^{\alpha}_{A|C} = \left(1 + \frac{\alpha}{2}\right)(\frac{2}{5})^{\alpha}$. The "residual" entanglement from our result is given by $y_1 = C^{\alpha}_{A|BC} - C^{\alpha}_{A|B} - \frac{\alpha}{2} C^{\alpha}_{A|C} = (\frac{2\sqrt{3}}{5})^{\alpha} - \left(1 + \frac{\alpha}{2}\right)(\frac{2}{5})^{\alpha}$ and the "residual" entanglement from (3) is given by $y_2 = C^{\alpha}_{A|BC} - C^{\alpha}_{A|B} - C^{\alpha}_{A|C} = (\frac{2\sqrt{3}}{5})^{\alpha} - 2(\frac{2}{5})^{\alpha}$. One can see that our result is better than that in [14] for $\alpha \geq 2$, see Fig. 1.

We can also derive a tighter upper bound of $C^{\alpha}_{A|B_1 B_2\dots B_{N-1}}$ for $\alpha < 0$.

Theorem 3 *For any $2 \otimes 2 \otimes \cdots \otimes 2$ mixed state $\rho \in H_A \otimes H_{B_1} \otimes \cdots \otimes H_{B_{N-1}}$ with $C_{AB_i} \neq 0$, $i = 1, 2, \dots, N-1$, we have*

$$
C^{\alpha}_{A|B_1 B_2\dots B_{N-1}} < \tilde{M} \left(C^{\alpha}_{A|B_1} + C^{\alpha}_{A|B_2} + \cdots + C^{\alpha}_{A|B_{N-1}} \right) \tag{10}
$$

for all $\alpha < 0$, where $\tilde{M} = \frac{1}{N-1}$.

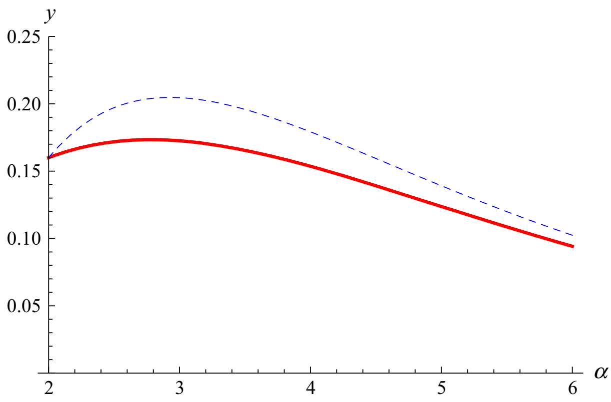

Fig. 1 y is the "residual" entanglement as a function of α: *solid* (*red*) line y_1 from our result, *dashed* (*blue*) line y_2 from the result in [14] (Color figure online)

Proof Similar to the proof of Theorem 1, for arbitrary tripartite state we have

$$\begin{aligned} C^{\alpha}_{A|B_1B_2} &\leq (C^2_{AB_1}+C^2_{AB_2})^{\frac{\alpha}{2}} \\ &= C^{\alpha}_{AB_1}\left(1+\frac{C^2_{AB_2}}{C^2_{AB_1}}\right)^{\frac{\alpha}{2}} < C^{\alpha}_{AB_1}, \end{aligned} \tag{11}$$

where the first inequality is due to $\alpha < 0$ and the second inequality is due to $(1+\frac{C^2_{AB_2}}{C^2_{AB_1}})^{\frac{\alpha}{2}} < 1$. On the other hand, we have

$$\begin{aligned} C^{\alpha}_{A|B_1B_2} &\leq (C^2_{AB_1}+C^2_{AB_2})^{\frac{\alpha}{2}} \\ &= C^{\alpha}_{AB_2}\left(1+\frac{C^2_{AB_1}}{C^2_{AB_2}}\right)^{\frac{\alpha}{2}} < C^{\alpha}_{AB_2}. \end{aligned} \tag{12}$$

From (11) and (12) we obtain

$$C^{\alpha}_{A|B_1B_2} < \frac{1}{2}(C^{\alpha}_{AB_1}+C^{\alpha}_{AB_2}). \tag{13}$$

By using the inequality (13) repeatedly, one gets

$$\begin{aligned} C^{\alpha}_{A|B_1B_2\ldots B_{N-1}} &< \frac{1}{2}\left(C^{\alpha}_{A|B_1}+C^{\alpha}_{A|B_2\ldots B_{N-1}}\right) \\ &< \frac{1}{2}C^{\alpha}_{A|B_1}+\left(\frac{1}{2}\right)^2 C^{\alpha}_{A|B_2}+\left(\frac{1}{2}\right)^2 C^{\alpha}_{A|B_3\ldots B_{N-1}} \end{aligned} \tag{14}$$

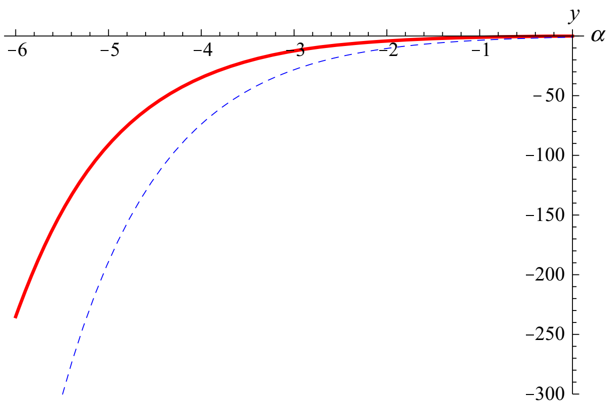

Fig. 2 y is the "residual" entanglement as a function of α: *red line* (*solid line*) from our Theorem 2; *blue line* (*dashed line*) from the result in [14] (Color figure online)

$$< \cdots < \frac{1}{2} C^{\alpha}_{A|B_1} + \left(\frac{1}{2}\right)^2 C^{\alpha}_{A|B_2} + \cdots + \left(\frac{1}{2}\right)^{N-2} C^{\alpha}_{A|B_{N-2}} + \left(\frac{1}{2}\right)^{N-2} C^{\alpha}_{A|B_{N-1}}.$$

By cyclically permuting the sub-indices $B_1, B_2, \ldots, B_{N-1}$ in (14) we can get a set of inequalities. Summing up these inequalities we have (10). □

As the factor $\tilde{M} = \frac{1}{N-1}$ is less than one, the inequality (10) is tighter than the one in [14]. This factor $\tilde{M}$ depends on the number of partite N. Namely, for larger multipartite systems, the inequality (10) gets even tighter than the one in [14].

Example 2 Let us consider again the three-qubit state (9). In this case, we have $N = 3$ and $\tilde{M} = 1/2$. Taking the same parameters used in Example 1, we have $C^{\alpha}_{A|BC} = (\frac{2\sqrt{3}}{5})^{\alpha}$, $C^{\alpha}_{A|B} + C^{\alpha}_{A|C} = 2(\frac{2}{5})^{\alpha}$, $\tilde{M}(C^{\alpha}_{A|B} + C^{\alpha}_{A|C}) = (\frac{2}{5})^{\alpha}$. Comparing the function of $y_1 = C^{\alpha}_{A|BC} - \tilde{M}C^{\alpha}_{A|B} - \tilde{M}C^{\alpha}_{A|C} = (\frac{2\sqrt{3}}{5})^{\alpha} - (\frac{2}{5})^{\alpha}$ with $y_2 = C^{\alpha}_{A|BC} - C^{\alpha}_{A|B} - C^{\alpha}_{A|C} = (\frac{2\sqrt{3}}{5})^{\alpha} - 2(\frac{2}{5})^{\alpha}$, one can see that our result is better than the one from [14], see Fig. 2.

Remark In (10) we have assumed that all C_{AB_i}, $i = 1, 2, \ldots, N-1$, are nonzero. In fact, if one of them is zero, the inequality still holds if one removes this term from the inequality. Namely, if $C_{AB_i} = 0$, then one has $C^{\alpha}_{A|B_1B_2\ldots B_{N-1}} < \frac{1}{2}C^{\alpha}_{A|B_1} + \cdots + \left(\frac{1}{2}\right)^{i-1} C^{\alpha}_{A|B_{i-1}} + \left(\frac{1}{2}\right)^{i} C^{\alpha}_{A|B_{i+1}} + \cdots + \left(\frac{1}{2}\right)^{N-3} C^{\alpha}_{A|B_{N-2}} + \left(\frac{1}{2}\right)^{N-3} C^{\alpha}_{A|B_{N-1}}$. Similar to the analysis in proving Theorem 2, one gets $C^{\alpha}_{A|B_1B_2\ldots B_{N-1}} < \frac{1}{N-1}(C^{\alpha}_{A|B_1} + \cdots + C^{\alpha}_{A|B_{i-1}} + C^{\alpha}_{A|B_{i+1}} + \cdots + C^{\alpha}_{A|B_{N-1}})$, for $\alpha < 0$.

3 Tighter monogamy inequality for EoF

The entanglement of formation (EoF) [21,22] is a well defined important measure of entanglement for bipartite systems. Let H_A and H_B be m- and n-dimensional ($m \leq n$) vector spaces, respectively. The EoF of a pure state $|\psi\rangle \in H_A \otimes H_B$ is defined by

$$E(|\psi\rangle) = S(\rho_A), \tag{15}$$

where $\rho_A = \mathrm{Tr}_B(|\psi\rangle\langle\psi|)$ and $S(\rho) = -\mathrm{Tr}(\rho \log_2 \rho)$. For a bipartite mixed state $\rho_{AB} \in H_A \otimes H_B$, the entanglement of formation is given by

$$E(\rho_{AB}) = \min_{\{p_i, |\psi_i\rangle\}} \sum_i p_i E(|\psi_i\rangle) \tag{16}$$

with the minimum taking over all possible decompositions of ρ_{AB} in a mixture of pure states $\rho_{AB} = \sum_i p_i |\psi_i\rangle\langle\psi_i|$, where $p_i \geq 0$ and $\sum_i p_i = 1$.

Denote $f(x) = H\left(\frac{1+\sqrt{1-x}}{2}\right)$, where $H(x) = -x \log_2(x) - (1-x)\log_2(1-x)$. From (15) and (16), one has $E(|\psi\rangle) = f\left(C^2(|\psi\rangle)\right)$ for $2 \otimes m$ ($m \geq 2$) pure state $|\psi\rangle$, and $E(\rho) = f\left(C^2(\rho)\right)$ for two-qubit mixed state ρ [23]. It is obvious that $f(x)$ is a monotonically increasing function for $0 \leq x \leq 1$. $f(x)$ satisfies the following relations:

$$f^{\sqrt{2}}(x^2 + y^2) \geq f^{\sqrt{2}}(x^2) + f^{\sqrt{2}}(y^2), \tag{17}$$

where $f^{\sqrt{2}}(x^2 + y^2) = [f(x^2 + y^2)]^{\sqrt{2}}$.

It has been show that the entanglement of formation does not satisfy the inequality $E_{AB} + E_{AC} \leq E_{A|BC}$ [24]. In [25], the authors showed that EoF is a monotonic function $E^2(C^2_{A|B_1B_2\ldots B_{N-1}}) \geq E^2(\sum_{i=1}^{N-1} C^2_{AB_i})$. It is further proved that for $N-$qubit systems, one has [14]

$$E^{\alpha}_{A|B_1B_2\ldots B_{N-1}} \geq E^{\alpha}_{A|B_1} + E^{\alpha}_{A|B_2} + \cdots + E^{\alpha}_{A|B_{N-1}} \tag{18}$$

for $\alpha \geq \sqrt{2}$, where $E_{A|B_1B_2\ldots B_{N-1}}$ is the entanglement of formation of ρ in bipartite partition $A|B_1B_2\ldots B_{N-1}$, and E_{AB_i}, $i = 1, 2, \ldots, N-1$, is the entanglement of formation of the mixed states $\rho_{AB_i} = \mathrm{Tr}_{B_1B_2\ldots B_{i-1}, B_{i+1}\ldots B_{N-1}}(\rho)$. In fact, generally we can prove the following results.

Theorem 4 *For any N-qubit mixed state $\rho \in H_A \otimes H_{B_1} \otimes \cdots \otimes H_{B_{N-1}}$, if $C_{AB_i} \geq C_{A|B_{i+1}\ldots B_{N-1}}$ for $i = 1, 2, \ldots, m$, and $C_{AB_j} \leq C_{A|B_{j+1}\ldots B_{N-1}}$ for $j = m+1, \ldots, N-2$, $\forall\ 1 \leq m \leq N-3$, $N \geq 4$, the entanglement of formation $E(\rho)$ satisfies*

$$\begin{aligned} E^{\alpha}_{A|B_1B_2\ldots B_{N-1}} \geq\ & E^{\alpha}_{A|B_1} + tE^{\alpha}_{A|B_2} \cdots + t^{m-1}E^{\alpha}_{A|B_m} \\ & + t^{m+1}(E^{\alpha}_{A|B_{m+1}} + \cdots + E^{\alpha}_{A|B_{N-2}}) \\ & + t^m E^{\alpha}_{A|B_{N-1}}, \end{aligned} \tag{19}$$

for $\alpha \geq \sqrt{2}$, where $t = \alpha/\sqrt{2}$.

Proof For $\alpha \geq \sqrt{2}$, we have

$$\begin{aligned} f^{\alpha}(x^2+y^2) &= \left(f^{\sqrt{2}}(x^2+y^2)\right)^t \\ &\geq \left(f^{\sqrt{2}}(x^2)+f^{\sqrt{2}}(y^2)\right)^t \\ &\geq \left(f^{\sqrt{2}}(x^2)\right)^t + t\left(f^{\sqrt{2}}(y^2)\right)^t \\ &= f^{\alpha}(x^2)+tf^{\alpha}(y^2), \end{aligned} \tag{20}$$

where the first inequality is due to the inequality (17), and the second inequality is obtained from a similar consideration in the proof of the second inequality in (4).

Let $\rho = \sum_i p_i|\psi_i\rangle\langle\psi_i| \in H_A \otimes H_{B_1} \otimes \cdots \otimes H_{B_N-1}$ be the optimal decomposition of $E_{A|B_1B_2\ldots B_{N-1}}(\rho)$ for the N-qubit mixed state ρ, we have

$$\begin{aligned} &E_{A|B_1B_2\ldots B_{N-1}}(\rho) \\ &= \sum_i p_i E_{A|B_1B_2\ldots B_{N-1}}(|\psi_i\rangle) \\ &= \sum_i p_i f\left(C^2_{A|B_1B_2\ldots B_{N-1}}(|\psi_i\rangle)\right) \\ &\geq f\left(\sum_i p_i C^2_{A|B_1B_2\ldots B_{N-1}}(|\psi_i\rangle)\right) \\ &\geq f\left(\left[\sum_i p_i C_{A|B_1B_2\ldots B_{N-1}}(|\psi_i\rangle)\right]^2\right) \\ &\geq f\left(C^2_{A|B_1B_2\ldots B_{N-1}}(\rho)\right), \end{aligned}$$

where the first inequality is due to that $f(x)$ is a convex function. The second inequality is due to the Cauchy–Schwarz inequality: $(\sum_i x_i^2)^{\frac{1}{2}}(\sum_i y_i^2)^{\frac{1}{2}} \geq \sum_i x_i y_i$, with $x_i = \sqrt{p_i}$ and $y_i = \sqrt{p_i}C_{A|B_1B_2\ldots B_{N-1}}(|\psi_i\rangle)$. Due to the definition of concurrence and that $f(x)$ is a monotonically increasing function, we obtain the third inequality. Therefore, we have

$$\begin{aligned} &E^{\alpha}_{A|B_1B_2\ldots B_{N-1}}(\rho) \\ &\geq f^{\alpha}(C^2_{AB_1}+C^2_{AB_2}+\cdots+C^2_{AB_{m-1}}) \\ &\geq f^{\alpha}(C^2_{A|B_1})+tf^{\alpha}(C^2_{A|B_2})\cdots+t^{m-1}f^{\alpha}(C^2_{A|B_m}) \\ &\quad +t^{m+1}\left(f^{\alpha}(C^2_{A|B_{m+1}})+\cdots+f^{\alpha}(C^2_{A|B_{N-2}})\right) \\ &\quad +t^m f^{\alpha}(C^2_{A|B_{N-1}}) \\ &= E^{\alpha}_{A|B_1}+tE^{\alpha}_{A|B_2}\cdots+t^{m-1}E^{\alpha}_{A|B_m} \\ &\quad +t^{m+1}(E^{\alpha}_{A|B_{m+1}}+\cdots+E^{\alpha}_{A|B_{N-2}})+t^m E^{\alpha}_{A|B_{N-1}}, \end{aligned}$$

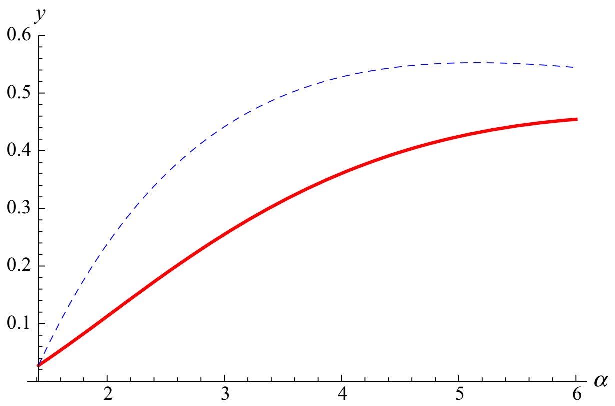

Fig. 3 y is the residual entanglement as a function of α: *red (solid) line* from our results; *blue (dashed) line* from the result in [14] (Color figure online)

where we have used the monogamy inequality in (2) for $N-$qubit states ρ to obtain the first inequality. By using (20) and the similar consideration in the proof of Theorem 1, we get the second inequality. Since for any $2 \otimes 2$ quantum state ρ_{AB_i}, $E(\rho_{AB_i}) = f\left[C^2(\rho_{AB_i})\right]$, one gets the last equality. □

As the factor $t = \alpha/\sqrt{2}$ is greater or equal to one for $\alpha \geq \sqrt{2}$, (19) is obviously tighter than (18). Moreover, similar to the concurrence, for the case that $C_{AB_i} \geq C_{A|B_{i+1}\dots B_{N-1}}$ for all $i = 1, 2, \dots, N-2$, we have a simple tighter monogamy relation for entanglement of formation:

Theorem 5 *If* $C_{AB_i} \geq C_{A|B_{i+1}\dots B_{N-1}}$ *for all* $i = 1, 2, \dots, N-2$, *we have*

$$E^{\alpha}_{A|B_1B_2\dots B_{N-1}} \geq E^{\alpha}_{A|B_1} + \frac{\alpha}{\sqrt{2}}E^{\alpha}_{A|B_2} + \cdots + \left(\frac{\alpha}{\sqrt{2}}\right)^{N-2} E^{\alpha}_{A|B_{N-1}} \tag{21}$$

for $\alpha \geq \sqrt{2}$.

Example 3 Let us consider the W state, $|W\rangle = \frac{1}{\sqrt{3}}(|100\rangle + |010\rangle + |001\rangle)$. We have $E_{AB} = E_{AC} = 0.55$, $E_{A|BC} = 0.92$. Let $y_1 = E^{\alpha}_{A|BC} - E^{\alpha}_{A|B} - \frac{\alpha}{\sqrt{2}}E^{\alpha}_{A|C}$ denote the residual entanglement from our formula (21), and $y_2 = E^{\alpha}_{A|BC} - E^{\alpha}_{A|B} - E^{\alpha}_{A|C}$ the residual entanglement from formula (18). It is easily verified that our results are better than the one in [14] for $\alpha \geq \sqrt{2}$, see Fig. 3.

4 Conclusion

Entanglement monogamy is a fundamental property of multipartite entangled states. We have investigated the monogamy relations related to the concurrence and EoF, and presented tighter entanglement monogamy relations of C^α and E^α for $\alpha \geq 2$ and $\alpha \geq \sqrt{2}$, respectively. Monogamy relations characterize the distributions of entanglement in multipartite systems. Tighter monogamy relations imply finer characterizations of the entanglement distribution. Our approach may be also used to study further the monogamy properties related to other quantum entanglement measures such as negativity and quantum correlations such as quantum discord.

Acknowledgements We thank Bao-Zhi Sun, Xue-Na Zhu and Xian Shi for very useful discussions. This work is supported by NSFC under Number 11275131.

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