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Deterministic remote preparation of an arbitrary qubit state using a partially entangled state and finite classical communication

Congyi Hua[1] · **Yi-Xin Chen**[1]

Abstract We propose a deterministic remote state preparation (RSP) scheme for preparing an arbitrary (including pure and mixed) qubit, where a partially entangled state and finite classical communication are used. To our knowledge, our scheme is the first RSP scheme that fits into this category. One other RSP scheme proposed by Berry shares close features, but can only be used to prepare an arbitrary pure qubit. Even so, our scheme saves classical communication by approximate 1 bit per prepared qubit under equal conditions. When using a maximally entangled state, the classical communication for our scheme is 2 bits, which agrees with Lo's conjecture on the resource cost. Furthermore Alice can switch between our RSP scheme and a standard teleportation scheme without letting Bob know, which makes the quantum channel multipurpose.

1 Introduction

Relying on quantum entanglement [1], quantum communication protocols can present abilities that are unachievable by their classical counterparts. Notable examples include quantum key distribution [2–4], quantum teleportation [5,6] and quantum secure direct communications [7–12]. By using two entangled qubits and 2 bits of classical communication as resources, quantum teleportation enables Alice to transmit an *unknown* qubit state to Bob without physically transporting it. A variation of teleportation is remote state preparation (RSP) [13–17], wherein Alice produces a *known* qubit state at

✉ Yi-Xin Chen
yixinchenzimp@zju.edu.cn

1 Zhejiang Institute of Modern Physics, Zhejiang University, Hangzhou 310027, China

Bob's location by using entanglement and classical communication. Depending on the resource requirement, preparable state ensemble, and even successful rate, different kind of RSP scheme has been devised for carrying out the specific task.

In most RSP schemes, maximal entanglement, oftenest a Bell state, is assumed to be accessible. For example, the first RSP scheme proposed by Pati [14] uses a Bell state, realizing deterministic preparation of a qubit from a fixed great circle on the Bloch sphere. Other later proposed schemes extended the preparable ensembles to varying degrees for preparing more general qubits, like deterministic preparations of arbitrary pure qubits [18–20], probabilistic preparations of arbitrary qubits [20,21], and more resently, schemes for deterministic preparations of arbitrary qubits were also implemented [22,23]. Unfortunately, these mentioned schemes are unadaptable to partially entangled resource states, which may occur in the real world. To devise RSP schemes that work with partial entanglement, new ways must be found.

Although it is more difficult to devise RSP schemes employing partial entanglement, Ye et al. [24] proved that, at the expense of increased classical communication, it is possible to use a partially entangled state to perform deterministic RSP of an arbitrary pure state. Based on the proof, Berry [25] proposed an explicit scheme for performing such an RSP. Thus, the two-dimensional case of Berry's scheme provides a complete technique for preparing an arbitrary pure qubit. In Ref. [26], we also proposed an optimization procedure that can be incorporated into Berry's scheme to reduce unnecessary classical communication. Despite all these efforts, to our best knowledge, as yet there is no deterministic RSP scheme that use partial entanglement to prepare arbitrary (include pure and mixed) qubit states, which we reported here.

In Sect. 2, by utilizing the ensembles that can be prepared using partial entanglement with only two bits of classical communication, we give our RSP scheme which trades off classical communication against entanglement to implement the preparation of an arbitrary qubit. Our results agree with Lo's conjecture on the resource cost for deterministic RSP. At the end of Sect. 2, we briefly summarize our scheme, and show Alice can switch between this RSP and a standard teleportation without letting Bob know. In Sect. 3, we introduce a geometrical tool called Voronoi diagram for the calculation of the classical communication requirement for our scheme. The calculation shows our scheme saves approximate 1 bit of classical communication when compared with Berry's scheme for the preparation of an arbitrary pure qubit. The underlying cause of the resource saving will be explained. In Sect. 4, we draw our conclusions.

2 Deterministic remote preparation of an arbitrary qubit

In this section, we present a remote state preparation (RSP) scheme for Alice to prepare an arbitrary qubit state at Bob's location deterministically. The resources for performing this task are a partially entangled two-qubit state and finite bits of classical communication from Alice to Bob. The qubit to be prepared is known to Alice but unknown to Bob. Furthermore, we restrict the scheme to be oblivious, where the state to be prepared is known to Alice but unknown to Bob.

The initial setup of our RSP scheme is as follows. Assume Alice and Bob have shared an entangled resource state in the form

$$|r\rangle = r_0 \, |00\rangle_{AB} + r_1 \, |11\rangle_{AB} \,,$$

or, equivalently,

$$|r\rangle = \cos \frac{\theta_r}{2} \, |00\rangle_{AB} + \sin \frac{\theta_r}{2} \, |11\rangle_{AB} \,, \tag{1}$$

where $0 \leq r_1 \leq r_0$ and $0 \leq \theta_r \leq \pi$. Any entangled pure two-qubit state can be brought to this form via local unitary operations. The entanglement of $|r\rangle$ is quantified by the Von Neumann entropy of either of $|r\rangle$'s reduced states, namely,

$$E(|r\rangle) = -r_0^2 \log r_0^2 - r_1^2 \log r_1^2. \tag{2}$$

The scheme starts by a positive operator valued measurement (POVM) [27,28] on Alice's system A. POVM is the most general kind of quantum measurements, it has some features that cannot be achieved within the context of projective measurement. The POVM operators Alice performs in our RSP scheme are given by

$$\left\{ M_m = p_m \left(\frac{1}{r_0} |0\rangle \langle 0| + \frac{1}{r_1} |1\rangle \langle 1| \right) \rho_m^T \cdot \left(\frac{1}{r_0} |0\rangle \langle 0| + \frac{1}{r_1} |1\rangle \langle 1| \right) \right\}_{m=0}^3, \tag{3}$$

where the superscript T denotes the transposition, and the values of p_m and ρ_m are to be determined later. By implementing this POVM, Alice obtains a measurement outcome m with probability calculated as $\langle r| \, M_m \, |r\rangle = p_m$, and the corresponding state of Bob's post-measurement system will be $\mathrm{tr}_A \left(M_m |r\rangle \langle r| \right) / p_m = \rho_m$. Based on the nature of measurement on quantum system, Bob's collapsed system is always interrelated with Alice's measurement result, but intrinsically random. In order to transform Bob's system to the state as Alice desired every time, Alice needs to instruct Bob to perform a unitary operation U_m chosen from $\{I, \sigma_3, \sigma_1, -\sigma_3 \sigma_1\}$ according her measurement outcome m. This can be done by sending to Bob 2 bits of classical information from Alice. The unitary operation set $\{I, \sigma_3, \sigma_1, -\sigma_3 \sigma_1\}$ is purposely duplicated from a standard teleportation scheme, as we will discuss at the end of the section. After the unitary operation, no matter which U_m Bob performs, all $U_m \rho_m U_m^\dagger$ should lead to the same state.

Now, based on the above setup, we give the preparable ensemble of pure qubits in Theorem 1. This ensemble will be generalized to include mixed qubits by Corollary 1.

Theorem 1 *Using a preshared resource state $|r\rangle$ given by Eq. (1) and 2 bits of classical communication from Alice to Bob, any pure qubit from the ensemble*

$$\left\{ |i(\theta_i, \phi_i)\rangle = \cos \frac{\theta_i}{2} |0\rangle + e^{i\phi_i} \sin \frac{\theta_i}{2} |1\rangle \,\middle|\, \phi_i \in [0, 2\pi) \right.$$

$$\left. and \ \theta_i \in [0, \theta_r] \cup [\pi - \theta_r, \pi] \right\}$$

can be remotely prepared. Particularly if $|r\rangle$ is a maximal entangled state, the above ensemble is represented by the entire Bloch sphere consisting of every pure qubit.

Proof Assume the pure qubit that Alice wants to prepare is expressed as

$$|i\rangle = \cos\frac{\theta_i}{2}|0\rangle + e^{i\phi_i}\sin\frac{\theta_i}{2}|1\rangle \equiv i_0|0\rangle + e^{i\phi_i}i_1|1\rangle, 0 \le \theta_i \le \pi, 0 \le \phi_i < 2\pi.$$

Here θ_i and ϕ_i are the polar and azimuthal angles of $|i\rangle$ in the Bloch sphere representation. For Bob, before he receives the 2 classical bits from Alice, his system is in the state

$$\sum_{m=0}^{3} p_m \rho_m = r_0^2|0\rangle\langle 0| + r_1^2|1\rangle\langle 1|. \tag{4}$$

Substitute $\rho_m \equiv U_m^\dagger |i\rangle\langle i|U_m$ with $U_m \in \{I, \sigma_3, \sigma_1, -\sigma_3\sigma_1\}$ into Eq. (4), we obtain

$$\left(i_0^2 P_1 + i_1^2 P_2\right)|0\rangle\langle 0| + \left(i_0^2 P_2 + i_1^2 P_1\right)|1\rangle\langle 1| = r_0^2|0\rangle\langle 0| + r_1^2|1\rangle\langle 1|, \tag{5}$$

where $P_1 \equiv 2p_0 = 2p_1$ and $P_2 \equiv 2p_2 = 2p_3$. Moreover, for a legitimate POVM in Eq. (3), $P_1, P_2 \ge 0$ and $P_1 + P_2 = 1$ are implied. Equation (5) can be used as the necessary and sufficient condition for checking the preparablity of a pure qubit $|i\rangle$ (also see Eq. (3) in Ref. [24]). It is easy to see, as long as $r_0 \le \max\{i_0, i_1\}$, Eq. (5) is soluble for non-negative P_1, P_2 with $P_1 + P_2 = 1$. Except for $r_0 = i_0 = \frac{1}{\sqrt{2}}$, where P_1 can be any value in [0, 1], the universal solution to Eq. (5) is

$$P_1 = \left(r_0^2 - i_0^2\right)/2\left(\frac{1}{2} - i_0^2\right). \tag{6}$$

For consistency, we only use the universal solution in the follow discussion.

On the Bloch sphere, the ensemble of states that satisfies $r_0 \le \max\{i_0, i_1\}$ is represented by an antipodal pair of spherical caps with $\theta_i \in [0, \theta_r] \cup [\pi - \theta_r, \pi]$. Particularly when $|r\rangle$ is maximally entangled, we have $\theta_i \in [0, \pi/2] \cup [\pi/2, \pi]$, which means the ensemble is represented by the entire Bloch sphere consisting of every pure qubit. $\qquad\square$

A mixed qubit describes a two-dimensional quantum system whose state is not completely known. One can suppose such a system is in a pure qubit state $|i\rangle$ with probability p and in the maximally mixed qubit state with probability $1 - p$. The density matrix for such a mixed qubit can be expressed as

$$\rho_i(p, \theta_i, \phi_i) = p|i(\theta_i, \phi_i)\rangle\langle i(\theta_i, \phi_i)| + (1 - p)\frac{I}{2},$$

where

$$|i(\theta_i, \phi_i)\rangle = \cos\frac{\theta_i}{2}|0\rangle + e^{i\phi_i}\sin\frac{\theta_i}{2}|1\rangle.$$

Here p, θ_i and ϕ_i uniquely identify the Bloch vector $\boldsymbol{r}_i = (p \sin \theta_i \cos \phi_i,$ $p \sin \theta_i \sin \phi_i,\ p \cos \theta_i)$, which is related to the position of ρ_i in the Bloch ball. When $p = 1$, ρ_i degenerates into the pure qubit $|i\rangle$, while from the pure $|i\rangle$ to the mixed ρ_i, the Bloch vector shrinks by a factor p. One can verify, if Alice sends Bob totally random classical bits (00, 01, 10 or 11 each with probability 1/4), Bob's system will end up with the maximally mixed qubit represented by the density matrix $I/2$. Therefore, if Alice wants to remotely prepare ρ_i, she only needs to replace the bits used for preparing $|i\rangle$ by totally random bits with probability $1 - p$. Obviously we have the following corollary.

Corollary 1 *Using a preshared resource state ketr given by Eq.* (1) *and 2 bits of classical communication from Alice to Bob, any qubit state from the ensemble*

$$\left\{ \rho_i(p, \theta_i, \phi_i) = p\, |i(\theta_i, \phi_i)\rangle \langle i(\theta_i, \phi_i)| + (1 - p) \frac{I}{2} \,\middle|\, p \in [0, 1],\ \phi_i \in [0, 2\pi) \right.$$

$$\left. and\ \theta_i \in [0, \theta_r] \cup [\pi - \theta_r, \pi] \right\}$$

can be remotely prepared. Particularly if $|r\rangle$ is maximally entangled, the above ensemble is represented by the entire Bloch ball consisting of every pure and mixed qubit.

Lo [13] conjectured that, with unlimited entanglement resource, deterministic preparation of an arbitrary pure qubit at Bob's location requires 2 bits of classical communication. Bennet et al. [15] have proved in a more restricted condition, where Bob is restricted to perform a unitary operation and is oblivious to the prepared state, the RSP must use at least 2 bits of classical communication. Our setup meets this condition, because when $|r\rangle$ is maximally entangled we have $P_1 = 1/2$ from Eq. (6), which means Bob cannot extract from the classical communication any information about the prepared state (thus is oblivious). Now, by treating Theorem 1 and Corollary 1 as complements to Bennet et al.'s proof, we know 2 bits of classical communication are both necessary and sufficient (even for preparing an arbitrary qubit).

Due to the strictly increasing relation between θ_r and $E(|r\rangle)$, a less entangled $|r\rangle$ will lead to smaller preparable ensembles in Theorem 1 and Corollary 1. In the following discussion, we will show how to trade off classical communication for reduced entanglement to enable preparation of an arbitrary target qubit ρ_t.

We use C_1 to denote the spherical cap lying at the north pole mentioned in Theorem 1, and C_{-1} the antipodal one. $C_{1,-1}$ is used to denote the pair of C_1 and C_{-1}. The size of $C_{1,-1}$ can be directly measured by the parameter θ_r. Obviously, if $|r\rangle$ is partially entangled, $C_{1,-1}$ will not cover the entire Bloch sphere. In order to prepare a target state ρ_t with Bloch vector $\boldsymbol{r}_t$ outside the convex hull of $C_{1,-1}$, Alice and Bob need to proceed as follows. First, before the preparation begins, they need to determine a K-element rotation operation set $\{R_j\}_{j=1}^{K}$ which is $|r\rangle$-dependent. After $\{R_j\}_{j=1}^{K}$ is determined, Alice deliberately prepares the intermediate state ρ_i that is exactly $R_j^{-1} \rho_t R_j$ and has a Bloch vector inside the convex hull of $C_{1,-1}$. Lastly, by

sending Bob $\log K$ bits, Alice instructs him to choose R_j from $\{R_j\}_{j=1}^{K}$ to transform his system into the target state ρ_t.

Visually, the effect of a rotation operation on a state is rotating the state's Bloch vector by a fixed angle about some axis of the Bloch ball. Let's suppose each R_j maps the spherical cap C_1 to C_j (and thus C_{-1} to C_{-j}). Although the position of $C_{j,-j}$ may vary, all of them are the same size as $C_{1,-1}$ and the union of $\{C_{j,-j}\}_{j=1}^{K}$ must cover the entire Bloch sphere. As pointed out earlier, if $E(|r\rangle)$ decreases, the size of $C_{j,-j}$ will decrease simultaneously. This generally results in an increased classical communication cost as K tends to become larger to ensure total coverage. The resource trade-off is inevitable for an RSP scheme of this type, but making $\{C_{j,-j}\}_{j=1}^{K}$ uniformly distributed can avoid overcommunication. The problem of how to uniformly distribute $\{C_{j,-j}\}_{j=1}^{K}$ can be rephrased as how to construct uniformly distributed $2K$ points with antipodal symmetry on the Bloch sphere, since we can use these points as the spherical caps' centers. The points construction method we use has been put into Appendix.

Now we summarize our scheme for remote preparation of an arbitrary qubit ρ_t as the two-stage procedure below.

Stage 1. Using the POVM (3) and the unitary operations $I, \sigma_3, \sigma_1, -\sigma_3\sigma_1$ to prepare an intermediate state $\rho_i = R_j^{-1}\rho_t R_j$ that belongs to the ensemble given in Corollary 1. The classical communication cost here is 2 bits. If $|r\rangle$ is maximally entangled, by setting $\{R_j\}_{j=1}^{K} \equiv \{I\}$, ρ_t can be prepared within this stage.

Stage 2. If $|r\rangle$ is non-maximally entangled, Bob performs R_j from the predefined set $\{R_j\}_{j=1}^{K}$ to transform ρ_i to ρ_t. The classical communication cost in this stage is $\log K$ bits, which is traded off against $E(|r\rangle)$.

Different from RSP, teleportation can only be carried out when a maximally entangled resource state is available [5]. In a standard teleportation scheme, where a Bell state $|\Psi^+\rangle \equiv \frac{1}{\sqrt{2}}|00\rangle + \frac{1}{\sqrt{2}}|11\rangle$ is used, Bob needs to apply a unitary operation chosen from $\{I,\sigma_3,\sigma_1, -\sigma_3\sigma_1\}$ after he receives the outcome of Alice's Bell basis measurement. The same unitary operation set is used in our RSP scheme. As we have said, this choice is made on purpose, because one can see when $|r\rangle$ is maximally entangled, i.e., $|r\rangle = |\Psi^+\rangle$, regardless of the target state the probability that Bob uses any one of these unitary operations in both schemes is always 1/4. There is no chance for Bob to tell which scheme is being performed, and Alice can switch between teleportation and RSP unilaterally.

3 Classical communication cost

The total classical communication cost for our scheme is $2 + \log K$ bits. For a given K, the entanglement of the resource state $|r\rangle$ cannot below a certain lower bound, otherwise there always exist some unpreparable qubit states for our scheme. To calculate the lower bound of $E(|r\rangle)$, we need to make use of a geometry tool called *Voronoi diagram*. A Voronoi diagram is a partition of a space into regions based on distance to some specific points called sites. For each site, the corresponding region,
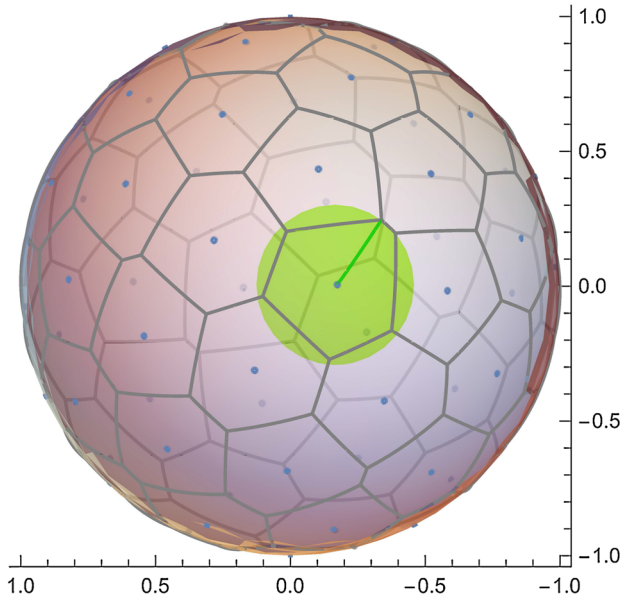
Fig. 1 Voronoi diagram generated from uniformly distributed 64 points with antipodal symmetry on the Bloch sphere. The mesh on the sphere shows the Voronoi cells corresponding to these points. The *green segment* is the longest site-vertex *geodesic line*. To cover the entire Bloch sphere, every C_j must be no smaller than the *green area* (Color figure online)

called Voronoi cell, consists all points closer to this site than to any other. All Voronoi cells are polygon-shaped with edges equidistant from two sites and vertices equidistant from three or more sites. Figure 1 gives an illustration of the Voronoi diagram generated from uniformly distributed 64 ($K = 32$) points with antipodal symmetry.

If we treat the centers of C_j, $j = \pm 1, \ldots, \pm K$, as sites denoted by s_j, a Voronoi diagram can be generated. The necessary and sufficient condition for the union of all C_j's to cover the entire Bloch sphere is that every C_j covers the Voronoi cell corresponding to s_j. We denote by $v_{j,k}$ the kth vertex of the Voronoi cell based on s_j. After obtaining the coordinates of all $v_{i,j}$ numerically by computer [29], the lower bound of $E(|r\rangle)$ can be calculated from Eq. (2) and

$$\max\big\{ \arccos(v_{j,k} \cdot s_j)\big| \text{ for all valid } (j, k)\big\} \le \theta_r.$$

In the extreme situation where the resource state $|r\rangle$ is unentangled, i.e., $\theta_r = 0$, C_j is a point. In this situation, K needs to be infinite large if we want the union of all C_j's to cover the entire Bloch sphere. For a finite K, using the point sets from Appendix, the results for $K = 2^n$, $n = 1, 2, \ldots, 10$ are presented both in Table 1 and in Fig. 2.

An earlier RSP scheme proposed by Berry [25] can be used for preparing arbitrary pure qubits. We find it also fits into our two-stage procedure summarized in Sect. 2, just by replacing mixed ρ_i and ρ_t with pure $|i\rangle$ and $|t\rangle$, respectively. The main difference is the intermediate state ensemble in Berry's scheme is represented by the spherical cap on Bloch sphere given by

Table 1 For K with a value no more than 128, we use the optimal point sets as input in calculation, while when $K = 256, 512, 1024$ we use Koay's point sets (see "Appendix")

K	2	4	8	16	32
Total bits cost	3	4	5	6	7
Lower bound of $E(|r\rangle)$	1	0.744008	0.502988	0.236295	0.155618

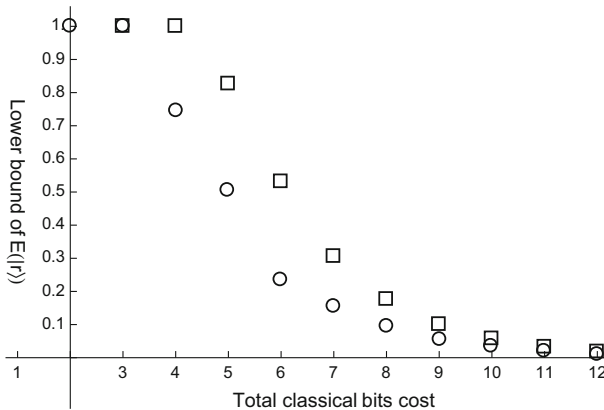| K	64	128	256	512	1024
Total bits cost	8	9	10	11	12
Lower bound of $E(|r\rangle)$	0.094967	0.056478	0.033252	0.018274	0.010069

Fig. 2 The total classical bits cost versus the lower bound of $E(|r\rangle)$ for RSP of general qubits. The *circles* represent the required resource in the RSP scheme proposed in this paper for preparing an arbitrary (including pure and mixed) qubit. The *squares* represent the result of Berry's scheme (after optimization) for preparing an arbitrary pure qubit based on the data from Ref. [26]

$$\left\{ |i\rangle = \cos\frac{\theta_i}{2} |0\rangle + e^{i\phi_i} \sin\frac{\theta_i}{2} |1\rangle \,\middle|\, \theta_i \in [0, \theta_r] \right\},$$

which is only half-size of the intermediate state ensemble in the Theorem 1. So two classical bits of communication will not be sufficient for Berry's scheme for preparing an arbitrary pure qubit, even when the resource state is maximally entangled. The double-sized intermediate state ensemble in our scheme reduces the number of elements in devising the unitary operation set $\{R_j\}_{j=1}^K$ to nearly half and eventually cut down the total classical communication cost by approximate 1 bit per pure qubit. The approximation is caused by the different symmetries employed in constructing uniformly distributed spherical caps in these two schemes. In Fig. 2, we include the result from [26] for comparison.

4 Conclusions

We have proposed an RSP scheme for remotely preparing a general qubit by using any pure entangled state and finite classical bits. Our scheme can be treated as a two-stage

procedure. If a maximally entangled resource state is available, the target qubit can be directly prepared in the first stage with 2 bits of classical communication, which agrees with Lo's conjecture on the resource cost for deterministic RSP. If the resource state is only partially entangled, an additional rotation operation will be performed in the second stage to transform the intermediate state prepared in the first stage to the final target state. The total classical communication cost is shown to be traded off against the resource state's entanglement. To the best of our knowledge, our scheme is the first deterministic RSP scheme for preparing an arbitrary qubit using a partially entangled state and finite classical communication. Theoretically, Our technique can be generalized to a higher dimension, but the geometry of qudits ($d > 2$) may be hard to deal with.

Our scheme also shares the same unitary operation set with the standard teleportation scheme. The benefit is when the resource state $|r\rangle$ is maximally entangled, Alice can switch between teleportation and RSP without letting Bob know, because no matter in which scheme Bob always performs a unitary operation $U_m \in \{I, \sigma_3, \sigma_1, -\sigma_3\sigma_1\}$ with probability 1/4. This feature can make an entangled channel more versatile without sacrificing flexibility.

Appendix: Uniform distribution of antipodally symmetric points on the unit sphere

On a sphere, point sets with antipodal symmetry have special importance in both scientific and engineering fields, many works has been published for generating such sets. The methods for generating a $2K$-element set with antipodal symmetry usually contain a minimization procedure of electrostatic potential energy. For the number of elements within a few hundreds, the point sets are tabulated online [30]. However, for the number of points in these point sets beyond a few hundreds, the optimization procedure will become unwieldy. To solve this problem, we can use instead some constructive methods to generate nearly uniform point sets with antipodal symmetry, which give very close results especially when K is large. In this work for generating antipodally symmetric point sets with $K \geq 256$, we use a simple deterministic points construction scheme proposed by Koay [31]. For the unit sphere, the spherical coordinates $(1, \theta_i, \phi_{i,j})$ of the points on the upper hemisphere are given by:

$$\theta_i = \left(i - \frac{1}{2}\right)\frac{\pi}{[N]}, \qquad\qquad i = 1, 2, \ldots, [N],$$
$$\phi_{i,j} = \left(j - \frac{1}{2}\right)\frac{2\pi}{K_i}, \qquad\qquad j = 1, 2, \ldots, K_i.$$

where N is the solution to $N = \frac{K}{2}\sin\frac{\pi}{4N}$, $[\cdot]$ is the function which gives the integer closest to the input, and

$$K_i = \begin{cases} \left\lceil \frac{2\pi \sin \theta_i}{\pi \csc \frac{\pi}{4[N]}} K \right\rceil, & i = 1, 2, \ldots, [N] - 1, \\ K - \sum_{i=1}^{[N]-1} K_i, & i = [N]. \end{cases}$$

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