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Variance-based uncertainty relations for incompatible observables

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Abstract We formulate uncertainty relations for arbitrary finite number of incompatible observables. Based on the sum of variances of the observables, both Heisenberg-type and Schrödinger-type uncertainty relations are provided. These new lower bounds are stronger in most of the cases than the ones derived from some existing inequalities. Detailed examples are presented.

1 Introduction

Uncertainty principle is one of the most remarkable features of quantum mechanics. In 1927, Heisenberg [1] introduced the first uncertainty inequality for a pair of canonical

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observables—position x and momentum p. This inequality can be expressed in terms of standard deviation-based product uncertainty relation,

$$\Delta x \Delta p \geq \frac{\hbar}{2},$$

where the standard deviation of an operator Ω is defined by $\Delta \Omega = \sqrt{\langle \Omega^2 \rangle - \langle \Omega \rangle^2}$, $\hbar$ is the Planck constant, $\langle \Omega \rangle$ is the mean value the operator Ω. After that, Robertson [2] generalized the above inequality to any pair of observables A and B, and provided the following uncertainty relation:

$$\Delta A \Delta B \geq \frac{1}{2} |\langle [A, B] \rangle|, \tag{1}$$

where $[A, B] = AB - BA$ is the commutator of A and B. The lower bound of Robertson uncertainty relation (RUR) (1) has an explicit physical meaning. It can be used to capture the noncommutativity of the two observables. A strengthened form of RUR is due to Schrödinger [3], who derived the following Schrödinger uncertainty relation (SUR):

$$(\Delta A)^2 (\Delta B)^2 \geq \left| \frac{1}{2i} \langle [A, B] \rangle \right|^2 + \left| \frac{1}{2} \langle \{A, B\} \rangle - \langle A \rangle \langle B \rangle \right|^2. \tag{2}$$

Uncertainty inequalities of the types of (1) and (2) are often referred to as Heisenberg-type and Schrödinger-type uncertainty relations, respectively.

Recently, Maccone and Pati [4] provided two stronger uncertainty relations based on the sum of variances. These uncertainty inequalities are nontrivial whenever the measured state is not a common eigenstate of the two observables. Thus, these new lower bounds of uncertainty relations can capture better the incompatibility of the two observables. After that, Chen and Fei [5] generalized one of the uncertainty relations in Ref. [4] to arbitrary N incompatible observables.

Variance-based uncertainty relations have many useful applications in quantum information theory, such as entanglement detection [6,7] and quantum spin squeezing (see Ref. [8] and references therein). It is worth noting that there are many other ways to formulate uncertainty relations, such as in terms of entropies [9–14], by use of majorization approach [15–20], based on the skew information [21–23] and the notion of fine-grained uncertainty relation [24–26]. Throughout the paper, we only focus on the variance-based sum uncertainty relations.

In this paper, we provide several new uncertainty relations for arbitrary finite number of incompatible observables. These new lower bounds are shown to be tighter than the ones in the previous works via some examples of spin-$\frac{1}{2}$ particle system. The paper is organized as follows. In Sect. 2, we present a sum uncertainty relation for n incompatible observables. When $n = 2, 3$, the inequality is shown to be stronger than the ones in Ref. [4] and [5] in some cases. In Sect. 3, we derived a product uncertainty relation, each of the factor in which is the sum of uncertainties in the individual incompatible observables. We conclude the paper in Sect. 4.

2 Stronger sum uncertainty relation for n incompatible observables

In this section, we first provide a stronger sum uncertainty relation for n incompatible observables. We have the following Theorem.

Theorem 1 *Let $A_1, A_2, \ldots, A_n$ be n incompatible observables and ρ a quantum state. For each A_i, if $\Delta A_i \neq 0$, then we define $P_i = \widetilde{A_i} \sqrt{\rho} / \Delta A_i$, where $\widetilde{A_i} = A_i - \langle A_i \rangle I$, I is the identity operator. If $\Delta A_i = 0$, we set P_i to be a zero operator. Let M be an $n \times n$ matrix with entries $M_{ij} = \mathrm{Tr}(P_i^\dagger P_j)$. Then, we have*

$$\sum_{i=1}^{n} (\Delta A_i)^2 \geq \frac{1}{\lambda_{\max}(M)} \left[\Delta \left(\sum_{i=1}^{n} A_i \right) \right]^2, \tag{3}$$

where $\lambda_{\max}(M)$ denotes the maximal eigenvalue of M.

Proof We first note that M is a positive semi-definite matrix, since $\sum_{i,j} x_i^* M_{ij} x_j = \| \sum_i x_i P_i \|^2 \geq 0, \forall (x_1, \ldots x_n) \in \mathbb{C}^n$. Then we have $\lambda_{\max}(M) \geq 0$. Taking into account that $\widetilde{\sum_i A_i} = \sum_i \widetilde{A_i}$, we have

$$\left[\Delta \left(\sum_i A_i \right) \right]^2 = \left\langle \left(\widetilde{\sum_i A_i} \right)^2 \right\rangle$$

$$= \left\langle \widetilde{\sum_i A_i} \widetilde{\sum_j A_j} \right\rangle$$

$$= \sum_{i,j} \langle \widetilde{A_i} \widetilde{A_j} \rangle$$

$$= \sum_{i,j} \Delta A_i M_{ij} \Delta A_j$$

$$\leq \lambda_{\max}(M) \sum_i (\Delta A_i)^2.$$

If $\lambda_{\max}(M) = 0$, then M is a zero matrix, and all $\Delta A_i = 0$, which is contrary to the incompatibility of $A_i s$. Hence, $\lambda_{\max}(M) > 0$, and $\sum_i (\Delta A_i)^2 \geq \frac{1}{\lambda_{\max}(M)} \left[\Delta \left(\sum_i A_i \right) \right]^2$ holds from the above inequality. $\square$

In Ref. [4], the authors derived an uncertainty relation for two observables A_1 and A_2:

$$(\Delta A_1)^2 + (\Delta A_2)^2 \geq \frac{1}{2} [\Delta(A_1 + A_2)]^2. \tag{4}$$

If A_1 and A_2 are incompatible, then the new bound in (3) is shown to be better than the one in (4), which is given in the following corollary.
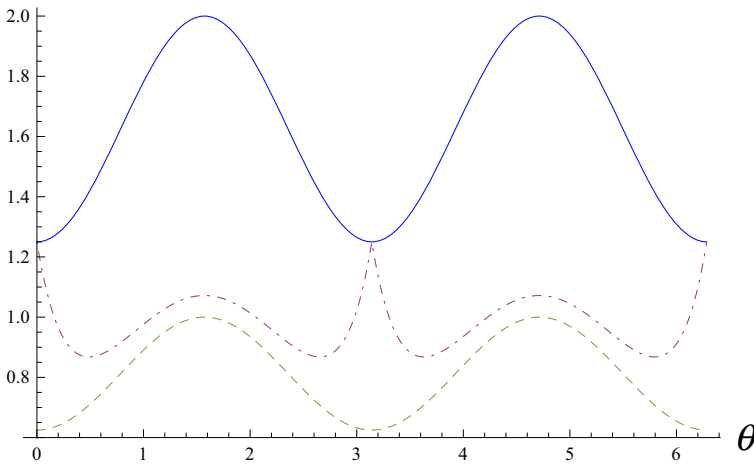
Fig. 1 The *blue solid line* is the sum of the variances $(\Delta\sigma_1)^2 + (\Delta\sigma_3)^2$. The *dot-dashed line* is the bound (3). The *dashed line* is the bound (4) (Color figure online)

Corollary 1 *For two incompatible observables A_1 and A_2, we have*

$$(\Delta A_1)^2 + (\Delta A_2)^2 \geq \frac{1}{\lambda_{\max}(M)}[\Delta(A_1 + A_2)]^2 \geq \frac{1}{2}[\Delta(A_1 + A_2)]^2, \qquad (5)$$

where M is defined in Theorem 1.

Proof We only need to prove that $\lambda_{\max}(M) \leq 2$. If $\Delta A_1 = 0$ or $\Delta A_2 = 0$, then $\lambda_{\max}(M) = 1$. Suppose that $\Delta A_1 \neq 0$ and $\Delta A_2 \neq 0$. Then, $\lambda_{\max}(M) = 1 + |\mathrm{Tr}(P_1^\dagger P_2)|$. Note that $\|P_1\| = \|P_2\| = 1$, we have $|\mathrm{Tr}(P_1^\dagger P_2)| \leq 1$ by the Cauchy-Schwarz inequality. Hence, $\lambda_{\max}(M) \leq 2$ and the last inequality in (5) holds. □

To see that our new lower bound (3) is strictly greater than the one in (4), consider the standard Pauli matrices σ_1 and σ_3. Let the qubit state to be measured and the measured state given by the Bloch vector $\overrightarrow{r} = \left(\frac{\sqrt{3}}{2}\cos\theta, \frac{\sqrt{3}}{2}\sin\theta, 0\right)$. Then we have $\Delta\sigma_1 = \sqrt{1 - \frac{3}{4}\cos^2\theta}$, $\Delta\sigma_3 = 1$, $\Delta(\sigma_1 + \sigma_3) = \sqrt{2 - \frac{3}{4}\cos^2\theta}$, $\lambda_{\max}(M) = 1 + \frac{\sqrt{3}|\sin\theta|}{\sqrt{1+3\sin^2\theta}}$. The comparison between the lower bounds (3) and (4) is shown in Fig. 1. Apparently, our new bound is strictly greater than (4).

Recently, Chen and Fei [5] generalize uncertainty relation (4) to the case of arbitrary n incompatible observables, and present the following variance-based sum uncertainty relation:

$$\sum_{i=1}^{n}(\Delta A_i)^2 \geq \frac{1}{n-2}\left\{\sum_{1\leq i<j\leq n}[\Delta(A_i+A_j)]^2 - \frac{1}{(n-1)^2}\left[\sum_{1\leq i<j\leq n}\Delta(A_i+A_j)\right]^2\right\}.$$

$$(6)$$
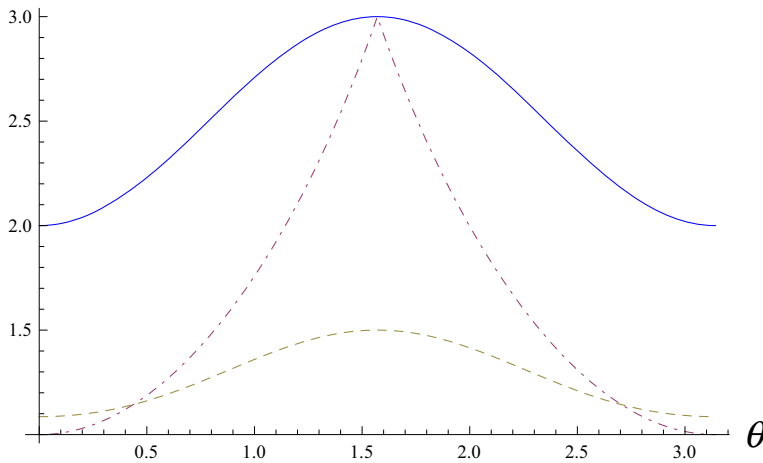
Fig. 2 The *blue solid line* is the sum of the variances $(\Delta\sigma_1)^2 + (\Delta\sigma_2)^2 + (\Delta\sigma_3)^2$. The *dot-dashed line* is the bound (3). The *dashed line* is the bound (6) (Color figure online)

To compare the uncertainty relation (3) with (6), let us consider again the Pauli matrices σ_1, σ_2, σ_3, and the measured state given by the Bloch vector $\vec{r} = (\cos\theta, 0, 0)$, $\theta \in (0, \pi)$. Then, we have $(\Delta\sigma_1)^2 + (\Delta\sigma_2)^2 + (\Delta\sigma_3)^2 = 3 - \cos^2\theta$, $\Delta(\sigma_1 + \sigma_2) = \Delta(\sigma_1 + \sigma_3) = \sqrt{2 - \cos^2\theta}$, $\Delta(\sigma_2 + \sigma_3) = \sqrt{2}$, $\Delta(\sigma_1 + \sigma_2 + \sigma_3) = \sqrt{3 - \cos^2\theta}$, $\lambda_{\max}(M) = 1 + |\cos\theta|$. It is shown in Fig. 2 that for a wide range of θ, our new uncertainty relation (3) is stronger than (6).

3 Uncertainty relation based on the product of sum of variances in the individual observables

In Ref. [27], the author studied the product of sum of uncertainties in the individual observables. They investigated a special case for a set of noncommuting observables. Using the RUR (1) and the convexity of quantum uncertainty, they formulated the product of sum of uncertainties in $\{A_i\}_{i=1}^n$ and $\{B_i\}_{i=1}^n$, which satisfy $[A_i, B_j] = i\delta_{ij}C$, and presented the following uncertainty relation:

$$\left(\sum_i^n \Delta A_i\right)\left(\sum_i^n \Delta B_i\right) \geq \frac{n}{2}|\langle C\rangle|.$$

Here we generalize the idea to any two sets of incompatible observables. We have the following Schrödinger-type uncertainty relation.

Theorem 2 *Let $\{A_i\}_{i=1}^n$ and $\{B_j\}_{j=1}^m$ be two sets of incompatible observables and ρ a quantum state. If $\Delta A_i \neq 0$, then we define $X_i = \widetilde{A_i}\sqrt{\rho}/\Delta A_i$; otherwise, define $X_i = O$; similarly, if $\Delta B_j \neq 0$, we define $Y_j = \widetilde{B_j}\sqrt{\rho}/\Delta B_j$, otherwise $Y_j = O$, $\forall i, j$. Let G be an $n \times m$ matrix with entries $G_{ij} = |\text{Tr}(X_i^\dagger Y_j)|$. If there exist A_i, B_j such that $\langle\widetilde{A_i}\widetilde{B_j}\rangle \neq 0$, then we have*

$$\left[\sum_{i=1}^{n}(\Delta A_i)^2\right]^{\frac{1}{2}}\left[\sum_{j=1}^{m}(\Delta B_j)^2\right]^{\frac{1}{2}}$$

$$\geq \frac{1}{\sigma_{\max}(G)}\sum_{i,j}\left(\left|\frac{1}{2i}\langle[A_i,B_j]\rangle\right|^2+\left|\frac{1}{2}\langle\{A_i,B_j\}\rangle-\langle A_i\rangle\langle B_j\rangle\right|^2\right)^{\frac{1}{2}}, \quad (7)$$

where $\sigma_{\max}(G)$ is the maximal singular value of G.

Proof We first have

$$\sum_{i,j}|\langle\widetilde{A_i}\widetilde{B_j}\rangle| = \sum_{i,j}\Delta A_i G_{ij}\Delta B_j$$

$$\leq \sigma_{\max}(G)\left[\sum_{i=1}^{n}(\Delta A_i)^2\right]^{\frac{1}{2}}\left[\sum_{j=1}^{m}(\Delta B_j)^2\right]^{\frac{1}{2}}.$$

On the other hand,

$$|\langle\widetilde{A_i}\widetilde{B_j}\rangle| = |\langle A_i B_j\rangle - \langle A_i\rangle\langle B_j\rangle|$$

$$= \left(\left|\frac{1}{2i}\langle[A_i,B_j]\rangle\right|^2+\left|\frac{1}{2}\langle\{A_i,B_j\}\rangle-\langle A_i\rangle\langle B_j\rangle\right|^2\right)^{\frac{1}{2}}.$$

If $\sigma_{\max}(G) = 0$, then all $\langle\widetilde{A_i}\widetilde{B_j}\rangle = 0$, which is contrary to our assumption. Taking into account the above formulas, we get (7) directly. $\qquad\square$

Remark It is reasonable to assume that not all $\langle\widetilde{A_i}\widetilde{B_j}\rangle = 0$, since otherwise we have $\sum_i(\Delta A_i)^2\sum_j(\Delta B_j)^2 \geq \sum_{i,j}|\langle\widetilde{A_i}\widetilde{B_j}\rangle|^2 = 0$ by SUR (2), which gives rise to a trivial uncertainty relation.
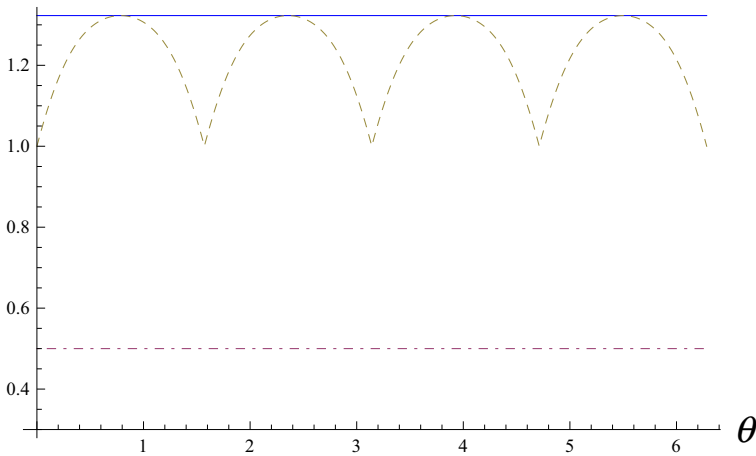
We can also obtain the following Heisenberg-type uncertainty relation from (7).

Corollary 2 *Under the conditions of the Theorem 2, we have*

$$\left[\sum_{i=1}^{n}(\Delta A_i)^2\right]^{\frac{1}{2}}\left[\sum_{j=1}^{m}(\Delta B_j)^2\right]^{\frac{1}{2}} \geq \frac{1}{2\sigma_{\max}(G)}\sum_{i,j}|\langle[A_i,B_j]\rangle|. \quad (8)$$

Noting that the following uncertainty relation holds by RUR (1),

$$\left[\sum_{i=1}^{n}(\Delta A_i)^2\right]^{\frac{1}{2}}\left[\sum_{j=1}^{m}(\Delta B_j)^2\right]^{\frac{1}{2}} \geq \frac{1}{2}\left(\sum_{i,j}|\langle[A_i,B_j]\rangle|^2\right)^{\frac{1}{2}}, \quad (9)$$

Fig. 3 The *blue solid line* is the uncertainties $\Delta\sigma_3\sqrt{(\Delta\sigma_2)^2 + (\Delta\sigma_1)^2}$. The *dot-dashed line* is the bound (9). The *dashed line* is the bound (8) (Color figure online)

we need to compare the lower bounds (8) and (9). Let $A_1 = \sigma_3$, $B_1 = \sigma_1$, $B_2 = \sigma_2$, and the Bloch vector of the measured state $\overrightarrow{r} = (\frac{1}{2}\cos\theta, \frac{1}{2}\sin\theta, 0)$. It is shown in Fig. 3 that the uncertainty relation (8) we derived in this paper is stronger in this case than the one trivially obtained by summing over RURs for all pairs of A_i and B_j.

Let us consider a special case for $n = m$ and $A_i = B_i$. We have the following uncertainty relation.

Corollary 3 *Let $A_1, A_2, \ldots, A_n$ be n incompatible observables. We have*

$$\sum_{i=1}^{n}(\Delta A_i)^2 \geq \frac{1}{\min\{\sigma_{\max}(G), n-1\}} \sum_{1 \leq i < j \leq n} |\langle[A_i, A_j]\rangle|, \tag{10}$$

where G is defined in Theorem 2.

Proof By using RUR (1) and the mean inequality for pairs of observables $A_1, \ldots, A_n$, we get

$$(\Delta A_i)^2 + (\Delta A_j)^2 \geq 2\Delta A_i \Delta A_j \geq |\langle[A_i, A_j]\rangle|, \quad \forall i \neq j.$$

Summing over the above inequalities, we have

$$\sum_{i=1}^{n}(\Delta A_i)^2 \geq \frac{1}{n-1} \sum_{1 \leq i < j \leq n} |\langle[A_i, A_j]\rangle|. \tag{11}$$

We now only need to show that $\sigma_{\max}(G) < n - 1$ for some n, when suitable observables and measured state considered. Let $n = 3$ and $A_i = \sigma_i$, $i = 1, 2, 3$, ρ is given by the Bloch vector $\overrightarrow{r} = (\cos\theta, 0, 0)$, $\theta \in (0, \pi)$. Then, we have $\sigma_{\max}(G) = 1 + |\cos\theta| < 2$. This completes the proof. $\qquad\square$

4 Conclusion

We have formulated uncertainty relations for arbitrary incompatible observables. These new uncertainty inequalities are based on the sum of variances of the observables. The corresponding lower bounds we derived in this paper are shown to be tighter than the previous ones. Thus our bounds capture better the incompatibility of the observables. We have also studied the general form of product of sum of variances in the individual observables and obtained a stronger uncertainty relation. We emphasize that these new bounds are physically realizable in general. These bounds are tight and can be reached by detailed quantum systems, as shown in examples. Moreover, they are also physically measurable. For the uncertainty relation (3), we only need to measure the variance of sum of the observables, and the value of $\lambda_{\max}(M)$ can be estimated beforehand. Similarly, for the uncertainty relation (7), we need to measure the mean value of each observable and the mean value of commutators and anti-commutators of pairs of the observables. The value of $\sigma_{\max}(G)$ can be computed readily. More interesting results could be obtained if multipartite systems are taken into account. As uncertainty relations are tightly related to many quantum tasks, our results might shed new lights on investigating quantum information processing like entanglement detection, quantum spin squeezing, quantum separability criteria, and security analysis of quantum key distribution in quantum cryptography and nonlocality.

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