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New asymmetric quantum codes over F_q

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Abstract Two families of new asymmetric quantum codes are constructed in this paper. The first family is the asymmetric quantum codes with length $n = q^m - 1$ over F_q, where $q \geq 5$ is a prime power. The second one is the asymmetric quantum codes with length $n = 3^m - 1$. These asymmetric quantum codes are derived from the CSS construction and pairs of nested BCH codes. Moreover, let the defining set $T_1 = T_2^{-q}$, then the real Z-distance of our asymmetric quantum codes are much larger than $\delta_{\max} + 1$, where $\delta_{\max}$ is the maximal designed distance of dual-containing narrow-sense BCH code, and the parameters presented here have better than the ones available in the literature.

Keywords Asymmetric quantum code · BCH code · CSS construction

1 Introduction

Quantum codes are powerful tool for fighting against noise in quantum communication and quantum computation. In general, symmetric quantum codes are adapted to deal with the qubit-flip errors σ_x, phase-flip errors σ_z and the combined qubit-phase-flip errors σ_y, which are all equally likely [1–3]. However, in most cases, qubit-flip and phase-flip errors have different probabilities. In fact, the noise in physical qubits is fundamentally asymmetric. That is to say, the phase-flip errors occur more frequently

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than qubit-flip errors. This concept was first noted by Steane in [4]. In this error model, asymmetric quantum codes are more efficient to protect quantum information than the symmetric quantum codes [5,6]. Therefore, in recent papers, quantum error-correcting codes theory has been extended to the asymmetric case, and some good parameters of asymmetric quantum codes were presented. Aly et al. [7–9] derived some families of asymmetric quantum codes from classical BCH and RS codes over finite fields, these asymmetric quantum codes based on imprimitive narrow-sense BCH codes for certain values of code lengths, dimensions, and various minimum distance were presented. Sarvepalli et al. [10] exploited the construction of some new families of asymmetric quantum stabilizer codes from pairs of nested classical BCH codes and finite geometry LDPC codes. Wang et al. [11] extend the characterization of nonadditive symmetric quantum codes to the asymmetric case, established a relationship of asymmetric quantum codes with classical error-correcting codes and obtained an asymptotic bound on asymmetric quantum codes from algebraic geometry codes. Ezerman [12,13] proposed CSS-like constructions and utilized pairs of nested subfield linear codes under one of the Euclidean, trace Euclidean, Hermitian, and trace Hermitian inner products constructed many best-performing CSS-like asymmetric quantum codes. La Guardia [14–16] presented the parameters of new asymmetric quantum codes which derived from the CSS construction as well as the Hermitian construction applied, respectively, to two classical nested BCH codes where one of them was additionally Euclidean (Hermitian) dual-containing.

In this paper, we study the construction of q-ary asymmetric quantum codes from pairs of nested primitive narrow-sense BCH codes. Here we present the parameters of asymmetric quantum codes of length $n = q^m - 1$ where $q \geq 5$ and $q = 3$. We give our discussion in two cases, one case is $m = 2l(l \geq 2)$ and the other case is $m = 2l + 1(l \geq 2)$. Get rid of the restriction on $\delta_z \leq \delta_{\max}$, our Z-distance of asymmetric quantum codes can be much larger than $\delta_{\max} + 1$ given in [16], where $\delta_{\max} = q^{\frac{m}{2}} - 1, m = 2l$ and $\delta_{\max} = q^{\lceil \frac{m}{2} \rceil} - q + 1, m = 2l + 1$ are the maximal designed distances of dual containing narrow-sense BCH code of length $n = q^m - 1$, see [17].

This paper is organized as follows: In Sect. 2, we recall the concept of cyclotomic cosets, BCH codes and asymmetric quantum codes. In Sect. 3, we construct new families of asymmetric quantum codes derived from Euclidean dual-containing BCH codes. In Sect. 4, we compare the parameters of the new codes with the ones available in the literature. Finally, the paper is summarized with a discussion in Sect. 5.

2 Preliminary

It is well known that there is a close relationship between cyclotomic cosets and cyclic codes. This suggests us to use q-cyclotomic cosets of modulo n to characterize BCH codes over $\mathbf{F}_q$. Let $n = q^m - 1$ denote the code length where $\gcd(n, q) = 1$ and $m = \mathrm{ord}_n(q)$, $\mathcal{B}^{\perp}$ denotes the Euclidean dual of BCH code $\mathcal{B}$, and an asymmetric quantum BCH code Q is denoted by $[[n, k, d_z/d_x]]_q$. For more details, we refer the reader to [18–21].

Definition 2.1 [20] If $\gcd(q, n) = 1$, the q-cyclotomic coset of modulo n containing x is defined by

$$C_x = \{x, xq, xq^2, \ldots, xq^{k-1}\}(\bmod\, n),$$

where k is the smallest positive integer such that $q^k x \equiv x(\bmod\, n)$.

A cyclic code of length $n = q^m - 1$ over $\mathbf{F}_q$ is called a BCH code with designed distance δ if its generator polynomial

$$g(x) = \prod_{z \in T}(x - \xi^z), T = C_b \cup C_{b+1} \cup \cdots \cup C_{b+\delta-2},$$

where C_x denotes the q-cyclotomic coset of modulo n containing x, ξ is a primitive element of $\mathbf{F}_{q^m}$ and $m = \mathrm{ord}_n(q)$ is the multiplicative order of q modulo n given by [21–23]. According to the concept of defining set, such a BCH code can also be defined, see following Definition 2.2.

Definition 2.2 [20,21] Let $\gcd(q, n) = 1$. If ξ is a primitive n-th root of unity in some field containing $\mathbf{F}_q$, $T = C_b \cup C_{b+1} \cup \cdots \cup C_{b+\delta-2} = T_{[b,b+\delta-2]}$, the cyclic code of length n with defining set T is called a BCH code of designed distance δ. If $b = 1$, then $\mathcal{C}$ is called a narrow-sense BCH code, if $n = q^m - 1$, then $\mathcal{C}$ is called primitive.

Lemma 2.1 [20] *If $\gcd(q, n) = 1$, $\mathcal{C}$ is a cyclic code over $\mathbf{F}_q$ with defining set T, $\mathcal{C}^{\perp} \subseteq \mathcal{C}$ if and only if $T \cap T^{-1} = \emptyset$, where $T^{-1} = \{n - t(\bmod\, n) \mid t \in T\}$.*

Let $\mathcal{B}_1$ and $\mathcal{B}_2$ be q-ary BCH codes of length n, and with defining set T_1 and T_2, respectively. From above lemma 2.1, we know $\mathcal{B}_1^{\perp} \subseteq \mathcal{B}_2$ if and only if $T_1^{-1} \cap T_2 = \emptyset$. Thus, we have

Lemma 2.2 [20] *Let $\mathcal{B}_1$ and $\mathcal{B}_2$ be q-ary BCH code with defining set T_1 and T_2, then $\mathcal{B}_1^{\perp} \subseteq \mathcal{B}_2$ if and only if $T_1^{\perp} \supseteq T_2$.*

According to [3,7,8,10], if a q-ary asymmetric quantum code be denoted by $[[n, k, d_z/d_x]]_q$, which can control all $\lfloor \frac{d_x-1}{2} \rfloor$ qubit-flip errors and all $\lfloor \frac{d_z-1}{2} \rfloor$ phase-flip errors. At the same time, which can detect $d_x - 1$ qubit-flip errors and $d_z - 1$ phase-flip errors. However, the standard CSS construction for symmetric quantum code can be extended to the constructions of asymmetric quantum code, see [8,10]. The following Theorem 2.3 is CSS construction for asymmetric quantum codes.

Theorem 2.3 [8,10] *For $i = 1, 2$, let $\mathcal{C}_i$ be a classical linear code with parameters $[n, k_i, d_i]_q$. If $\mathcal{C}_1^{\perp} \subseteq \mathcal{C}_2$, then there exists an asymmetric quantum code with parameters $[[n, k(\delta_1) + k(\delta_2) - n, d_z/d_x]]_q$, where $\{d_x, d_z\} = \{d_1, d_2\}$.*

3 Construction of asymmetric quantum codes

There are many previous works which discuss the construction of asymmetric quantum codes from two nested codes as well as restrict themselves to binary or quaternary codes. To such a problem, the main obstacle is the construction of dual distances, a knowledge of which is required to determine the error-correcting capability of the quantum code. In [14–16], La Guardia generalize their previous work and construct asymmetric stabilizer codes over $\mathbf{F}_q$, where $q \geq 3$ is an arbitrary prime power. However, the values of Z-distance of the asymmetric quantum codes provided in the literature were not enough greater. Thus, in this section, we focus on construction of $[[n, k, d_z/d_x]]_q$ asymmetric quantum codes with greater values of Z-distance over $\mathbf{F}_q$ for $q \geq 5$ and $q = 3$, respectively. To construct good parameters asymmetric quantum codes, one need to determine the maximal designed distance of nested primitive narrow-sense BCH codes. In the following, we will discuss such a problem in two cases for each q, one case is $m = 2l + 1$ and the other case is $m = 2l$.

3.1 Construction of asymmetric quantum codes $[[n, k, d_z/d_x]]_{q\geq 5}$

In this subsection, let $q \geq 5$, we apply the CSS construction to construct asymmetric quantum codes of length $n = q^m - 1$ from two nested BCH codes where one of them is Euclidean dual-containing. we first make some notations. Similar to [20], for fixed n, denote $T = \bigcup_{i=1}^{r} C_i$, define $u = \min\{x|x \in T^{-1}\}$ and $v = \max\{y|y \in T^{-1}\}$, and we have following Lemma 3.1.

Lemma 3.1 *If $\mathcal{B}$ is a q-ary narrow-sense BCH code of length n with defining set $T = \cup_{i=1}^{r} C_i$ where $r < \delta_{\max}$, and $T^{\perp} = Z_n \backslash T^{-1}$. Then $\mathcal{B}$ and $\mathcal{B}^{\perp}$ have designed distances $\delta(\mathcal{B}) = r + 1$ and $\delta(\mathcal{B}^{\perp}) \leq \max\{u, n - v - 1\}$, respectively.*

Proof The defining set of BCH code $\mathcal{B}$ is $T = \cup_{i=1}^{r} C_i = T_{[1,r]}$, so we have the maximal designed distance of narrow-sense BCH code $\mathcal{B}$ is $r + 1$, and then we have $\delta(\mathcal{B}) = r + 1$.

Since $T^{\perp} = Z_n \backslash (T^{-1}) = \{0, 1, 2, \ldots, n-1\} - \{n-x|x \in T\} = \{0, 1, 2, \ldots, n-1\} - \{u, u+s, \ldots, v-t, v\} \supseteq \{0, 1, 2, \ldots, u-1, v+1, \ldots, n-1\}$, $T^{\perp}$ contains u or $n - v - 1$ integer. From the Definition 2.2, thus, we have $\delta(\mathcal{B}^{\perp}) \leq \max\{u, n - v - 1\}$.

Case 1. $n = q^m - 1$ **where** $m = 2l + 1$

According to Theorem 2.3, we know that if q-ary BCH codes satisfying $\mathcal{B}_1^{\perp} \subseteq \mathcal{B}_2$ take advantage of CSS construction, one can construct q-ary asymmetric quantum codes from these BCH codes. Hence, we first discuss the conditions regarding Euclidean dual-containing BCH codes in the following Theorem 3.2.

Theorem 3.2 *Let $n = q^m - 1$, $q \geq 5$, and $m = 2l + 1$.*

(I) If $1 \leq i \leq \lceil \frac{q-1}{2} \rceil - 1$. For $\delta_1 = \lceil \frac{q}{2} \rceil + i$, $\delta_1 < \delta_2 \leq \lfloor \frac{q}{2} \rfloor \cdot q^{2l} - (i - 1) \cdot q^{2l} - 1$, then there exist narrow-sense BCH codes satisfying $\mathcal{B}_1^{\perp}(n, \delta_1) \subseteq \mathcal{B}_2(n, \delta_2)$.

(II) If $1 \leq i \leq q - 1$, $1 \leq j \leq l$. For $\delta_1 = i \cdot q^j + 1$, $\delta_1 < \delta_2 \leq q^{2l-j+1} - i$, then there exist narrow-sense BCH codes satisfying $\mathcal{B}_1^{\perp}(n, \delta_1) \subseteq \mathcal{B}_2(n, \delta_2)$.

(III) If $1 \leq i \leq q-2$, $1 \leq j \leq l-1$. For $\delta_1 = q^{j+1} - i$, $\delta_1 < \delta_2 \leq (i+1) \cdot q^{2l-j} - 1$, then there exist narrow-sense BCH codes satisfying $\mathcal{B}_1^{\perp}(n, \delta_1) \subseteq \mathcal{B}_2(n, \delta_2)$.

Proof We only show (II) since the other cases are similar.

Let $n = q^m - 1, m = 2l + 1$. Since $\delta_1 = i \cdot q^j + 1$ where $1 \le i \le q-1, 1 \le j \le l$, then narrow-sense BCH code $\mathcal{B}_1(n, \delta_1)$ with defining set $T_1 = \cup_{t=1}^{\delta_1 - 1} C_t = \cup_{t=1}^{i \cdot q^j} C_t$. If $\delta_1 < \delta_2 \le min\{n - x_i | x_i \in T_1\} = q^{2l-j+1} - i$, then $\mathcal{B}_2(n, \delta_2)$ with defining set $T_2 = \cup_{t=1}^{\delta_2 - 1} C_t$. Especially, if $\delta_i = q \cdot t + 1$ where $t \ge 1, i = 1, 2$, then we can assume $T_i = \cup_{t=1}^{\delta_i - 2} C_t$.

Let $T_1^{-1} = \{n - x_i | x_i \in T_1\}$, since $\mathcal{B}_1^{\perp}$ with defining set $T_1^{\perp} = Zn \backslash T_1^{-1} = \{0, 1, 2, \ldots, n-1\} - \{n - x_i | x_i \in T_1\}$, we can assume $T_2 = C_1 \cup C_2 \cup \cdots C_{\delta_2 - 1}$. If for any $j \in T_2$, from Lemma 3.1, we can deduce that $j \notin T_1^{-1}$, that is to say $T_1^{-1} \cap T_2 = \emptyset$, then $j \in Zn \backslash T_1^{-1}$, and thus one can deduce that $T_2 \subseteq T_1^{\perp}$. From Lemma 2.2, we can conclude $\mathcal{B}_1^{\perp}(n, \delta_1) \subseteq \mathcal{B}_2(n, \delta_2)$ holds.

We know that if two BCH codes satisfy the dual-containing conditions, then an asymmetric quantum codes can be constructed from these two BCH codes. Fortunately, the above Theorem 3.2 presented the exact designed distances for the BCH codes under the dual-containing conditions. Therefore, applying CSS construction, we can derive the following asymmetric quantum codes. In the following Theorem 3.3, our main construction results can be provided. Here, we always use T_δ to denote $T_\delta = \bigcup_{i=1}^{\delta - 1} C_i$ and denote the cardinality of T_δ as $|T_\delta|$, see the following Theorem 3.3.

Theorem 3.3 *Let $n = q^m - 1$, $q \ge 5$, and $m = 2l + 1$.*

(I) For $\delta_1 = \lceil \frac{q}{2} \rceil + i$, $\delta_1 < \delta_2 \le \lfloor \frac{q}{2} \rfloor \cdot q^{2l} - (i-1) \cdot q^{2l} - 1$, then there exist asymmetric quantum codes $[[n, n - |T_{\delta_1}| - |T_{\delta_2}|, d_z \ge \delta_2 / d_x \ge \delta_1]]_q$, where $1 \le i \le \lceil \frac{q-1}{2} \rceil - 1$.

(II) For $\delta_1 = i \cdot q^j + 1$, $\delta_1 < \delta_2 \le q^{2l-j+1} - i$, then there exist asymmetric quantum codes $[[n, n - |T_{\delta_1}| - |T_{\delta_2}|, d_z \ge \delta_2 / d_x \ge \delta_1]]_q$, where $1 \le i \le q - 1$, $1 \le j \le l$.

(III) For $\delta_1 = q^{j+1} - i$, $\delta_1 < \delta_2 \le (i+1) \cdot q^{2l-j} - 1$, then there exist asymmetric quantum codes $[[n, n - |T_{\delta_1}| - |T_{\delta_2}|, d_z \ge \delta_2 / d_x \ge \delta_1]]_q$, where $1 \le i \le q - 2$, $1 \le j \le l - 1$.

Proof We only prove item (II) since the other constructions are similar.

Let $\mathcal{B}_1$ be the narrow-sense BCH code over $\mathbf{F}_q$ of length $n = q^m - 1$. Use T_{δ_1} to denote $T_{\delta_1} = \bigcup_{i=1}^{\delta_1 - 1} C_i$, and the cardinality of T_{δ_1} as $|T_{\delta_1}|$. For $\delta_1 = i \cdot q^j + 1$ where $1 \le i \le q - 1$, $1 \le j \le l$, then there exists narrow-sense BCH code with parameters $[n, n - |T_{\delta_1}|, i \cdot q^j + 1]$. Next, consider another BCH code $\mathcal{B}_2$ with parameters $[n, n - |T_{\delta_2}|, \delta_2]$.

According to Theorem 3.2, we know that if $\delta_1 < \delta_2 \le q^{2l-j+1} - i$, then there exist narrow-sense BCH codes satisfying $\mathcal{B}_1^{\perp}(n, \delta_1) \subseteq \mathcal{B}_2(n, \delta_2)$. Hence, applying the CSS construction in Theorem 2.3, and using the parameters of $B_1(n, \delta_1)$ and $B_2(n, \delta_2)$, q-ary asymmetric quantum codes $[[n, n - |T_{\delta_1}| - |T_{\delta_2}|, d_z \ge \delta_2 / d_x \ge \delta_1]]_{q \ge 5}$ can be constructed.

Summarizing the above discussions, we can conclude (II) holds.

Case 2. $n = q^m - 1$ **where** $m = 2l$

Similar to the discussions of Theorem 3.2, we can also provide the dual containing conditions in the other case $m = 2l$. Then one can easily determine the maximal designed distance of dual-containing narrow-sense BCH code in the following Theorem 3.4.

Theorem 3.4 *Let $n = q^m - 1$, $q \geq 5$ and $m = 2l$.*

(I) If $1 \leq i \leq \lceil \frac{q-1}{2} \rceil - 1$. For $\delta_1 = \lceil \frac{q}{2} \rceil + i$, $\delta_1 < \delta_2 \leq \lfloor \frac{q}{2} \rfloor \cdot q^{2l-1} - (i-1) \cdot q^{2l-1} - 1$, then there exist narrow-sense BCH codes satisfying $\mathcal{B}_1^{\perp}(n, \delta_1) \subseteq \mathcal{B}_2(n, \delta_2)$.

(II) If $1 \leq i \leq q-1$, $1 \leq j \leq l-1$. For $\delta_1 = i \cdot q^j + 1$, $\delta_1 < \delta_2 \leq q^{2l-j} - i$, then there exist narrow-sense BCH codes satisfying $\mathcal{B}_1^{\perp}(n, \delta_1) \subseteq \mathcal{B}_2(n, \delta_2)$.

(III) If $1 \leq i \leq q-2$, $1 \leq j \leq l-1$. For $\delta_1 = q^{j+1} - i$, $\delta_1 < \delta_2 \leq (i+1) \cdot q^{2l-j-1} - 1$, then there exist narrow-sense BCH codes satisfying $\mathcal{B}_1^{\perp}(n, \delta_1) \subseteq \mathcal{B}_2(n, \delta_2)$.

By means of CSS construction and the results of Theorem 3.4, one can determine the parameters of asymmetric quantum codes in the other case $m = 2l$ we have

Theorem 3.5 *Let $n = q^m - 1$, $q \geq 5$ and $m = 2l$.*

(I) For $\delta_1 = \lceil \frac{q}{2} \rceil + i$, $\delta_1 < \delta_2 \leq \lfloor \frac{q}{2} \rfloor \cdot q^{2l-1} - (i-1) \cdot q^{2l-1} - 1$, then there exist asymmetric quantum codes $[[n, n - |T_{\delta_1}| - |T_{\delta_2}|, d_z \geq \delta_2/d_x \geq \delta_1]]_q$, where $1 \leq i \leq \lceil \frac{q-1}{2} \rceil - 1$.

(II) For $\delta_1 = i \cdot q^j + 1$, $\delta_1 < \delta_2 \leq q^{2l-j} - i$, then there exist asymmetric quantum codes $[[n, n - |T_{\delta_1}| - |T_{\delta_2}|, d_z \geq \delta_2/d_x \geq \delta_1]]_q$, where $1 \leq i \leq q-1$, $1 \leq j \leq l-1$.

(III) For $\delta_1 = q^{j+1} - i$, $\delta_1 < \delta_2 \leq (i+1) \cdot q^{2l-j-1} - 1$, then there exist asymmetric quantum codes $[[n, n - |T_{\delta_1}| - |T_{\delta_2}|, d_z \geq \delta_2/d_x \geq \delta_1]]_q$, where $1 \leq i \leq q-2$, $1 \leq j \leq l-1$.

Remark 1 We presented the parameters of asymmetric quantum codes of length $n = q^m - 1$ where $q \geq 5$. Theorems 3.3 and 3.5 give the evidences that the Z-distance of our asymmetric quantum codes is much larger than $\delta_{\max} + 1$ in [16]. However, there are little complex to calculate the exact dimensions for all δ of these asymmetric quantum codes, so we denote the cardinality of T_δ as $|T_\delta|$, and the dimension as $n - |T_\delta|$. But, for fixed the values of length n, and for fixed the special values of d_z and d_x, we will calculate the exact dimensions of the new asymmetric quantum codes in the following Sect. 4.

3.2 Construction of asymmetric quantum codes $[[n, k, d_z/d_x]]_3$

In this subsection, we apply the same technique to construct two families of asymmetric quantum codes $[[n, k, d_z/d_x]]_3$. Similar to the Sect. 3.1, we first present the dual-containing conditions of nested BCH codes in two cases $m = 2l + 1$ and $m = 2l$, respectively. Then, applying the same technique utilized in the previous subsection, one obtains Theorems 3.6 and 3.7:

Theorem 3.6 *Let $n = 3^m - 1$, $m = 2l + 1$.*

(I) If $1 \leq j \leq l$, $1 \leq i \leq 2$. For $\delta_1 = i \cdot q^j + 1$, $\delta_1 < \delta_2 \leq q^{2l-j+1} - i$, then there exist narrow-sense BCH codes satisfying $\mathcal{B}_1^{\perp}(n, \delta_1) \subseteq \mathcal{B}_2(n, \delta_2)$.

(II) If $1 \leq j \leq l-1$. For $\delta_1 = q^{j+1} - 1$, $\delta_1 < \delta_2 \leq 2q^{2l-j} - 1$, then there exist narrow-sense BCH codes satisfying $\mathcal{B}_1^{\perp}(n, \delta_1) \subseteq \mathcal{B}_2(n, \delta_2)$.

Theorem 3.7 *Let $n = 3^m - 1$, $m = 2l$.*

(I) If $1 \leq j \leq l-1$, $1 \leq i \leq 2$. For $\delta_1 = i \cdot q^j + 1$, $\delta_1 < \delta_2 \leq q^{2l-j} - i$, then there exist narrow-sense BCH codes satisfying $\mathcal{B}_1^{\perp}(n, \delta_1) \subseteq \mathcal{B}_2(n, \delta_2)$.

(II) If $1 \le j \le l-1$. *For* $\delta_1 = q^{j+1}-1$, $\delta_1 < \delta_2 \le 2q^{2l-j-1}-1$, *then there exist narrow-sense BCH codes satisfying* $\mathcal{B}_1^{\perp}(n,\delta_1) \subseteq \mathcal{B}_2(n,\delta_2)$.

Ref. [16, Theorem 4.10] gave the parameters of several families of nonbinary asymmetric quantum codes of length $n = q^m - 1$. However, the theorem only provided the code length n of $q \ge 4$ and $m = \mathrm{ord}_n(q) \ge 3$. Obviously, it did not include the case of $q = 3$. On the other hand, if $q = 3$, then the parameters of asymmetric quantum code achieve smaller values of Z-distance and X-distance in [14]. Therefore, we will construct 3-ary asymmetric quantum codes $[[n, k, d_z/d_x]]_3$ with much larger Z-distance. From Theorems 3.6 and 3.7, we can easily obtain the parameters of nested dual-containing BCH codes in $m = 2l+1$ and $m = 2l$ two cases. Then, combining the results of these BCH codes and CSS construction, we can derive Theorems 3.8 and 3.9.

Theorem 3.8 *Let* $n = 3^m - 1$, $m = 2l+1$.

(I) For $\delta_1 = i \cdot q^j + 1$, $\delta_1 < \delta_2 \le q^{2l-j+1} - i$, *then there exist asymmetric quantum codes* $[[n, n-|T_{\delta_1}|-|T_{\delta_2}|, d_z \ge \delta_2/d_x \ge \delta_1]]_3$, *where* $1 \le i \le 2$, $1 \le j \le l$.

(II) For $\delta_1 = q^{j+1}-1$, $\delta_1 < \delta_2 \le 2q^{2l-j}-1$, *then there exist asymmetric quantum codes* $[[n, n-|T_{\delta_1}|-|T_{\delta_2}|, d_z \ge \delta_2/d_x \ge \delta_1]]_3$, *where* $1 \le j \le l-1$.

Theorem 3.9 *Let* $n = 3^m - 1$, $m = 2l$.

(I) For $\delta_1 = i \cdot q^j + 1$, $\delta_1 < \delta_2 \le q^{2l-j} - i$, *then there exist asymmetric quantum codes* $[[n, n-|T_{\delta_1}|-|T_{\delta_2}|, d_z \ge \delta_2/d_x \ge \delta_1]]_3$, *where* $1 \le i \le 2$, $1 \le j \le l-1$.

(II) For $\delta_1 = q^{j+1}-1$, $\delta_1 < \delta_2 \le 2q^{2l-j-1}-1$, *then there exist asymmetric quantum codes* $[[n, n-|T_{\delta_1}|-|T_{\delta_2}|, d_z \ge \delta_2/d_x \ge \delta_1]]_3$, *where* $1 \le j \le l-1$.

4 Code table

In this section, we compare the parameters of the new asymmetric quantum codes and the ones available in the literature. For fixed values of the length n, d_z and d_x, we have computed the dimensions for asymmetric quantum codes derived from BCH codes over $\mathbf{F}_q$. In the following Table, the parameters of the asymmetric quantum codes shown in [16] are denoted by $[[n, k', d_{z'}/d_{x'}]]_q$, and the new code parameters are denoted by $[[n, k, d_z/d_x]]_q$.

Remark 2 Table 1 lists some new asymmetric quantum codes given in Theorems 3.3 and 3.5. For $m = 3, 4$ and $q = 5, 7, 8, 9, 11$, some of the parameters of our asymmetric quantum codes are better than those available in [16]. What is more, some of the asymmetric quantum codes are new ones and are not included in the literature. However, for fixed values of the length n, we only give part results of Theorems 3.4 and 3.5, the discussions of asymmetric quantum codes constructed from pairs of nested BCH codes for all δ may be a little complex. For example, for $q = 9, m = 3, n = 728$, if $d_x \ge 6, 7, 8, 10, 19, 28, 37, 46, 55, 64$, our Z-distance can reach $d_z \ge 323/d_x \ge 6, d_z \ge 242/d_x \ge 7, d_z \ge 161/d_x \ge 8, d_z \ge 80/d_x \ge 10, d_z \ge 79/d_x \ge 19, d_z \ge 78/d_x \ge 28, d_z \ge 77/d_x \ge 37, d_z \ge 76/d_x \ge 46, d_z \ge 75/d_x \ge 55, d_z \ge 74/d_x \ge 64$, but here we only give $d_x \ge 6, 7, 8, 10$ four cases in Table 1. On the other hand, in order to calculate the dimensions, we restrict

Table 1 Sample parameters of asymmetric quantum codes $[[n, k, d_z/d_x]]_{q\geq 5}$

q	m	n	$[[n, k, d_z/d_x]]_q$	$[[n, k', d_{z'}/d_{x'}]]_q$ in [16]
5	3	124	$[[124, 97, d_z \geq 7/d_x \geq 6]]_5$	$[[124, 95, d_{z'} \geq 7/d_{x'} \geq 6]]_5$
			$[[124, 55, d_z \geq 24/d_x \geq 6]]_5$	–
			$[[124, 21, d_z \geq 49/d_x \geq 4]]_5$	–
			...	...
	4	624	$[[624, 588, d_z \geq 7/d_x \geq 6]]_5$	$[[624, 586, d_{z'} \geq 7/d_{x'} \geq 6]]_5$
			$[[624, 244, d_z \geq 124/d_x \geq 6]]_5$	–
			$[[624, 232, d_z \geq 123/d_x \geq 11]]_5$	–
			$[[624, 208, d_z \geq 121/d_x \geq 21]]_5$	–
			...	...
7	3	342	$[[342, 297, d_z \geq 11/d_x \geq 8]]_7$	$[[342, 295, d_{z'} \geq 11/d_{x'} \geq 8]]_7$
			$[[342, 201, d_z \geq 48/d_x \geq 8]]_7$	–
			$[[342, 186, d_z \geq 47/d_x \geq 15]]_7$	–
			$[[342, 141, d_z \geq 44/d_x \geq 36]]_7$	–
			...	...
	4	2400	$[[2400, 2348, d_z \geq 9/d_x \geq 8]]_7$	$[[2400, 2346, d_{z'} \geq 9/d_{x'} \geq 8]]_7$
			$[[2400, 2344, d_z \geq 10/d_x \geq 8]]_7$	$[[2400, 2342, d_{z'} \geq 10/d_{x'} \geq 8]]_7$
			$[[2400, 2340, d_z \geq 11/d_x \geq 8]]_7$	$[[2400, 2338, d_{z'} \geq 11/d_{x'} \geq 8]]_7$
			$[[2400, 2336, d_z \geq 12/d_x \geq 8]]_7$	$[[2400, 2334, d_{z'} \geq 12/d_{x'} \geq 8]]_7$
			$[[2400, 1276, d_z \geq 342/d_x \geq 8]]_7$	–
			$[[2400, 1256, d_z \geq 341/d_x \geq 15]]_7$	–
			$[[2400, 1236, d_z \geq 340/d_x \geq 22]]_7$	–
			...	...
8	3	511	$[[511, 466, d_z \geq 10/d_x \geq 9]]_8$	$[[511, 464, d_{z'} \geq 10/d_{x'} \geq 9]]_8$
			$[[511, 463, d_z \geq 11/d_x \geq 9]]_8$	$[[511, 461, d_{z'} \geq 11/d_{x'} \geq 9]]_8$
			$[[511, 460, d_z \geq 12/d_x \geq 9]]_8$	$[[511, 458, d_{z'} \geq 12/d_{x'} \geq 9]]_8$
			$[[511, 457, d_z \geq 13/d_x \geq 9]]_8$	$[[511, 455, d_{z'} \geq 13/d_{x'} \geq 9]]_8$
			$[[511, 454, d_z \geq 14/d_x \geq 9]]_8$	$[[511, 452, d_{z'} \geq 14/d_{x'} \geq 9]]_8$
			$[[511, 325, d_z \geq 63/d_x \geq 9]]_8$	–
			$[[511, 307, d_z \geq 62/d_x \geq 17]]_8$	–
			$[[511, 289, d_z \geq 61/d_x \geq 25]]_8$	–
			...	...
	4	4095	$[[4095, 4031, d_z \geq 11/d_x \geq 9]]_8$	$[[4095, 4029, d_{z'} \geq 11/d_{x'} \geq 9]]_8$
			$[[4095, 4027, d_z \geq 12/d_x \geq 9]]_8$	$[[4095, 4025, d_{z'} \geq 12/d_{x'} \geq 9]]_8$
			$[[4095, 4023, d_z \geq 13/d_x \geq 9]]_8$	$[[4095, 4021, d_{z'} \geq 13/d_{x'} \geq 9]]_8$
			$[[4095, 4019, d_z \geq 14/d_x \geq 9]]_8$	$[[4095, 4017, d_{z'} \geq 14/d_{x'} \geq 9]]_8$
			$[[4095, 2377, d_z \geq 511/d_x \geq 9]]_8$	–
			$[[4095, 2353, d_z \geq 510/d_x \geq 17]]_8$	–
			$[[4095, 2329, d_z \geq 509/d_x \geq 25]]_8$	–
			$[[4095, 2305, d_z \geq 508/d_x \geq 33]]_8$	–
			...	...

Table 1 continued

q	m	n	$[[n, k, d_z/d_x]]_q$	$[[n, k', d_{z'}/d_{x'}]]_q$ in [16]
9	3	728	$[[728, 677, d_z \geq 11/d_x \geq 10]]_9$	$[[728, 675, d_{z'} \geq 11/d_{x'} \geq 10]]_9$
			$[[728, 674, d_z \geq 12/d_x \geq 10]]_9$	$[[728, 672, d_{z'} \geq 12/d_{x'} \geq 10]]_9$
			$[[728, 671, d_z \geq 13/d_x \geq 10]]_9$	$[[728, 669, d_{z'} \geq 13/d_{x'} \geq 10]]_9$
			$[[728, 668, d_z \geq 14/d_x \geq 10]]_9$	$[[728, 666, d_{z'} \geq 14/d_{x'} \geq 10]]_9$
			$[[728, 665, d_z \geq 15/d_x \geq 10]]_9$	$[[728, 663, d_{z'} \geq 15/d_{x'} \geq 10]]_9$
			$[[728, 662, d_z \geq 16/d_x \geq 10]]_9$	$[[728, 660, d_{z'} \geq 16/d_{x'} \geq 10]]_9$
			$[[728, 331, d_z \geq 161/d_x \geq 6]]_9$	–
			$[[728, 328, d_z \geq 161/d_x \geq 7]]_9$	–
			$[[728, 325, d_z \geq 161/d_x \geq 8]]_9$	–
			...	...
	4	6560	$[[6560, 6492, d_z \geq 11/d_x \geq 10]]_9$	$[[6560, 6490, d_{z'} \geq 11/d_{x'} \geq 10]]_9$
			$[[6560, 6488, d_z \geq 12/d_x \geq 10]]_9$	$[[6560, 6486, d_{z'} \geq 12/d_{x'} \geq 10]]_9$
			$[[6560, 6484, d_z \geq 13/d_x \geq 10]]_9$	$[[6560, 6482, d_{z'} \geq 13/d_{x'} \geq 10]]_9$
			$[[6560, 6480, d_z \geq 14/d_x \geq 10]]_9$	$[[6560, 6478, d_{z'} \geq 14/d_{x'} \geq 10]]_9$
			$[[6560, 6476, d_z \geq 15/d_x \geq 10]]_9$	$[[6560, 6474, d_{z'} \geq 15/d_{x'} \geq 10]]_9$
			$[[6560, 6472, d_z \geq 16/d_x \geq 10]]_9$	$[[6560, 6470, d_{z'} \geq 16/d_{x'} \geq 10]]_9$
			$[[6560, 5952, d_z \geq 165/d_x \geq 10]]_9$	–
			$[[6560, 5888, d_z \geq 165/d_x \geq 28]]_9$	–
			$[[6560, 5704, d_z \geq 165/d_x \geq 79]]_9$	–
			...	...
11	3	1330	$[[1330, 1267, d_z \geq 13/d_x \geq 12]]_{11}$	$[[1330, 1265, d_{z'} \geq 13/d_{x'} \geq 12]]_{11}$
			$[[1330, 1264, d_z \geq 14/d_x \geq 12]]_{11}$	$[[1330, 1262, d_{z'} \geq 14/d_{x'} \geq 12]]_{11}$
			$[[1330, 1261, d_z \geq 15/d_x \geq 12]]_{11}$	$[[1330, 1259, d_{z'} \geq 15/d_{x'} \geq 12]]_{11}$
			$[[1330, 1258, d_z \geq 16/d_x \geq 12]]_{11}$	$[[1330, 1256, d_{z'} \geq 16/d_{x'} \geq 12]]_{11}$
			$[[1330, 1255, d_z \geq 17/d_x \geq 12]]_{11}$	$[[1330, 1253, d_{z'} \geq 17/d_{x'} \geq 12]]_{11}$
			$[[1330, 1252, d_z \geq 18/d_x \geq 12]]_{11}$	$[[1330, 1250, d_{z'} \geq 18/d_{x'} \geq 12]]_{11}$
			$[[1330, 1249, d_z \geq 19/d_x \geq 12]]_{11}$	$[[1330, 1247, d_{z'} \geq 19/d_{x'} \geq 12]]_{11}$
			$[[1330, 1246, d_z \geq 20/d_x \geq 12]]_{11}$	$[[1330, 1244, d_{z'} \geq 20/d_{x'} \geq 12]]_{11}$
			$[[1330, 973, d_z \geq 120/d_x \geq 12]]_{11}$	–
			$[[1330, 946, d_z \geq 119/d_x \geq 23]]_{11}$	–
			$[[1330, 919, d_z \geq 118/d_x \geq 34]]_{11}$	–

$d_z \geq 161/d_x \geq 6$ and $d_z \geq 161/d_x \geq 7$, then one can easily construct two asymmetric quantum codes $[[728, 331, d_z \geq 161/d_x \geq 6]]_9$ and $[[728, 328, d_z \geq 161/d_x \geq 7]]_9$. In fact, if $d_x \geq 6, 7$, our Z-distance can reach 323,242, respectively, it is obviously larger than 161. In a word, we use Table 1 to present evidences of the real Z-distance of our asymmetric quantum codes, which are much larger than $\delta_{\max} + 1$, and some of our asymmetric quantum codes are new ones.

Remark 3 Table 2 shows some new asymmetric quantum codes given in Theorems 3.8 and 3.9. For $q = 3$, $m = 4, 5, 6, 7$, our asymmetric quantum codes constructed from

Table 2 Sample parameters of asymmetric quantum codes $[[n, k, d_z/d_x]]_3$

m	n	$[[n, k, d_z/d_x]]_3$	m	n	$[[n, k, d_z/d_x]]_3$
5	242	$[[242, 22, d_z \geq 80/d_x \geq 4]]_3$	4	80	$[[80, 12, d_z \geq 26/d_x \geq 4]]_3$
		$[[242, 17, d_z \geq 79/d_x \geq 7]]_3$			$[[80, 8, d_z \geq 25/d_x \geq 7]]_3$
		$[[242, 57, d_z \geq 53/d_x \geq 8]]_3$			$[[80, 18, d_z \geq 17/d_x \geq 8]]_3$
		$[[242, 127, d_z \geq 26/d_x \geq 10]]_3$			...
		$[[242, 102, d_z \geq 25/d_x \geq 19]]_3$	6	728	$[[728, 43, d_z \geq 241/d_x \geq 7]]_3$
		...			$[[728, 171, d_z \geq 161/d_x \geq 8]]_3$
7	2186	$[[2186, 1038, d_z \geq 241/d_x \geq 19]]_3$			$[[728, 386, d_z \geq 80/d_x \geq 10]]_3$
		$[[2186, 1339, d_z \geq 161/d_x \geq 26]]_3$			$[[728, 356, d_z \geq 79/d_x \geq 19]]_3$
		$[[2186, 1570, d_z \geq 79/d_x \geq 55]]_3$			$[[728, 419, d_z \geq 53/d_x \geq 26]]_3$
		...			...

pairs of nested BCH codes are all new and are not included in [14,16]. However, similar to Table 1, we still give part results of Theorems 3.8 and 3.9. For example, for $q = 3, m = 7, n = 2186$, if $d_x \geq 4, 7, 8, 10, 19, 26, 28, 55$, our Z-distance can reach $d_z \geq 728/d_x \geq 4, d_z \geq 727/d_x \geq 7, d_z \geq 485/d_x \geq 8, d_z \geq 242/d_x \geq 10, d_z \geq 241/d_x \geq 19, d_z \geq 161/d_x \geq 26, d_z \geq 80/d_x \geq 28, d_z \geq 79/d_x \geq 55$, respectively, but here we only give $d_x \geq 19, 26, 55$ three cases, and hence three asymmetric quantum codes $[[2186, 1038, d_z \geq 241/d_x \geq 19]]_3$, $[[2186, 1339, d_z \geq 161/d_x \geq 26]]_3$ and $[[2186, 1570, d_z \geq 79/d_x \geq 55]]_3$ can be obtained. Additionally, in the particular cases of $d_z \geq 7/d_x \geq 6-l$ and $d_z \geq 8/d_x \geq 6-l$ where $0 \leq l \leq q-2$, our symmetric quantum codes and the ones presented in [14] have the same parameters.

5 Summary

In this paper, we have constructed two families of nonbinary asymmetric quantum codes derived from pairs of nested classical BCH codes by the CSS construction. We generalize our previous work [23] as specified in the following: The asymmetric quantum codes shown in [23] are constructed over the field F_q where $q = 4$, whereas in this paper we have constructed asymmetric quantum codes over F_q, where $q \geq 5$ (q is an arbitrary prime power) or $q = 3$. Furthermore, most of the code parameters shown above are better than the ones available in the literature. Additionally, the quantum codes constructed in this paper can be utilized in quantum channels having great asymmetry, i.e., quantum channels in which the probability of occurrence of phase-shift errors is large when compared to the probability of occurrence of qubit-flip errors.

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