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Asymptotic velocity of a position-dependent quantum walk

Akito Suzuki[1]

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Abstract We consider a position-dependent coined quantum walk on $\mathbb{Z}$ and assume that the coin operator $C(x)$ satisfies

$$\|C(x) - C_0\| \leq c_1 |x|^{-1-\epsilon}, \quad x \in \mathbb{Z} \setminus \{0\}$$

with positive c_1 and ϵ and $C_0 \in U(2)$. We show that the Heisenberg operator $\hat{x}(t)$ of the position operator converges to the asymptotic velocity operator $\hat{v}_+$ so that

$$\text{s-}\lim_{t\to\infty} \exp\left(i\xi \frac{\hat{x}(t)}{t}\right) = \Pi_{\mathrm{p}}(U) + \exp(i\xi\hat{v}_+)\Pi_{\mathrm{ac}}(U)$$

provided that U has no singular continuous spectrum. Here $\Pi_{\mathrm{p}}(U)$ (resp., $\Pi_{\mathrm{ac}}(U)$) is the orthogonal projection onto the direct sum of all eigenspaces (resp., the subspace of absolute continuity) of U. We also prove that for the random variable X_t denoting the position of a quantum walker at time $t \in \mathbb{N}$, X_t/t converges in law to a random variable V with the probability distribution

$$\mu_V = \|\Pi_{\mathrm{p}}(U)\Psi_0\|^2 \delta_0 + \|E_{\hat{v}_+}(\cdot)\Pi_{\mathrm{ac}}(U)\Psi_0\|^2,$$

where Ψ_0 is the initial state, δ_0 the Dirac measure at zero, and $E_{\hat{v}_+}$ the spectral measure of $\hat{v}_+$.

Keywords Quantum walk · Asymptotic velocity · Weak limit

✉ Akito Suzuki
akito@shinshu-u.ac.jp

[1] Division of Mathematics and Physics, Faculty of Engineering, Shinshu University, Wakasato, Nagano 380-8553, Japan

1 Introduction

The weak limit theorems for discrete time quantum walks have been studied in various models (for reviews, see [7,12]). In his papers [5,6], Konno first proved the weak limit theorem for a position-independent quantum walk on $\mathbb{Z}$. Grimmett et al. [4] simplified the proof and extended the result to higher dimensions. For position-dependent qunatum walks on $\mathbb{Z}$, the weak limit theorems were obtained by Konno et al. [9], Endo and Konno [2], and Endo et al. [3].

We consider a position-dependent quantum walk on $\mathbb{Z}$ given by a unitary evolution operator U:

$$(U\Psi)(x) = P(x+1)\Psi(x+1) + Q(x-1)\Psi(x-1), \quad x \in \mathbb{Z},$$

where Ψ is a state vector in the Hilbert space $\mathcal{H} = \ell^2(\mathbb{Z}; \mathbb{C}^2)$ of states and

$$P(x) = \begin{pmatrix} a(x) & b(x) \\ 0 & 0 \end{pmatrix}, \quad Q(x) = \begin{pmatrix} 0 & 0 \\ c(x) & d(x) \end{pmatrix}.$$

Let $C(x) = P(x) + Q(x) \in U(2)$ and S be a shift operator such that $U = SC$. Suppose that there exists a unitary matrix $C_0 = P_0 + Q_0 \in U(2)$ such that

$$\|C(x) - C_0\| \le c_1 |x|^{-1-\epsilon}, \quad x \in \mathbb{Z} \setminus \{0\} \tag{1.1}$$

with positive c_1 and ϵ independent of x. Here $\|M\|$ stands for the operator norm of a matrix $M \in M_2(\mathbb{C})$. A typical example is the quantum walks with one defect [1,8,9,13], which clearly satisfies (1.1). We note that the condition (1.1) allows not only finite but also infinite defects, whereas the models introduced in [2,3] do not satisfy (1.1). The unitary operator $U_0 = SC_0$ also defines an evolution of a position-independent quantum walk on $\mathbb{Z}$ and satisfies

$$(U_0\Psi)(x) = P_0\Psi(x+1) + Q_0\Psi(x-1), \quad x \in \mathbb{Z}$$

with $C_0 = P_0 + Q_0$. Let $\hat{x}$ be the position operator defined by $(\hat{x}\Psi)(x) = x\Psi(x), \quad x \in \mathbb{Z}$. and $\hat{x}_0(t) = U_0^{-t}\hat{x}U_0^t$ the Heisenberg operator of $\hat{x}$ at time $t \in \mathbb{N}$ with the evolution U_0. In Grimmett et al. [4] essentially proved that the operator $\hat{x}_0(t)/t$ weakly converges to the asymptotic velocity operator $\hat{v}_0$ so that

$$\text{w-}\lim_{t\to\infty} \exp\left(i\xi \frac{\hat{x}_0(t)}{t}\right) = \exp\left(i\xi\hat{v}_0\right), \quad \xi \in \mathbb{R}. \tag{1.2}$$

Let $X_t^{(0)}$ be the random variable denoting the position of a quantum walker at time $t \in \mathbb{N}$ with the evolution operator U_0. Then, the characteristic function of $X_t^{(0)}/t$ is given by

$$\mathbb{E}\left(e^{i\xi X_t^{(0)}/t}\right) = \left\langle \Psi_0, e^{i\xi\hat{x}_0(t)/t}\Psi_0 \right\rangle, \quad \xi \in \mathbb{R},$$

where Ψ_0 is the initial state of the quantum walker. Hence, (1.2) means that the random variable $X_t^{(0)}/t$ converges in law to a random variable V_0, which represents the linear spreading of the quantum walk: $X_t^{(0)} \sim tV_0$.

In this paper, we derive the asymptotic velocity $\hat{v}_+$ for the Heisenberg operator $\hat{x}(t) = U^{-t}\hat{x}U^t$ with the evolution U of the position-dependent quantum walk. The decaying condition (1.1) implies that $U - U_0$ is a trace class operator and allows us to prove the existence and completeness of the wave operator

$$W_+ = \text{s-}\lim_{t\to\infty} U^{-t}U_0^t\Pi_{\text{ac}}(U_0)$$

using a discrete analogue of the Kato–Rosenblum Theorem (see [11] for details), where $\Pi_{\text{ac}}(U_0)$ is the orthogonal projection onto the subspace of absolute continuity of U_0. We also prove that

$$\text{s-}\lim_{t\to\infty} \exp\left(i\xi\frac{\hat{x}_0(t)}{t}\right) = \exp\left(i\xi\hat{v}_0\right), \quad \xi \in \mathbb{R}$$

under a reasonable condition, which is essentially the same as that of [4]. Furthermore, we assume that U has no singular continuous spectrum. Then, we prove that

$$\text{s-}\lim_{t\to\infty} \exp\left(i\xi\frac{\hat{x}(t)}{t}\right) = \Pi_{\text{p}}(U) + \exp(i\xi\hat{v}_+)\Pi_{\text{ac}}(U), \tag{1.3}$$

where $\Pi_{\text{p}}(U)$ is the orthogonal projection onto the direct sum of all eigenspaces of U and $\hat{v}_+ = W_+\hat{v}_0W_+^*$. We believe that the absence of a singular continuous spectrum can be checked with a concrete example such as the one-defect model. As a consequence of (1.3), we have the following weak limit theorem. Let X_t be the random variable denoting the position of a quantum walker at time $t \in \mathbb{N}$ with the evolution operator U and the initial state Ψ_0. We prove that X_t/t converges in law to a random variable V with a probability distribution

$$\mu_V = \|\Pi_{\text{p}}(U)\Psi_0\|^2\delta_0 + \|E_{\hat{v}_+}(\cdot)\Pi_{\text{ac}}(U)\Psi_0\|^2,$$

where δ_0 is the Dirac measure at zero and $E_{\hat{v}_+}$ the spectral measure of $\hat{v}_+$.

The remainder of this paper is organized as follows. In Sect. 2, we present the precise definition of the model and our results. Section 3 is devoted to the proof of the existence and completeness of the wave operator. In Sect. 4, we construct the asymptotic velocity.

2 Definition of the model

Let $\mathcal{H} = \ell^2(\mathbb{Z};\mathbb{C}^2)$ be the Hilbert space of the square summable functions $\Psi : \mathbb{Z} \to \mathbb{C}^2$. We define a shift operator S and a coin operator C on $\mathcal{H}$ as follows. For a vector

$\Psi = \begin{pmatrix} \Psi^{(0)} \\ \Psi^{(1)} \end{pmatrix} \in \mathcal{H}$, $S\Psi$ is given by

$$(S\Psi)(x) = \begin{pmatrix} \Psi^{(0)}(x+1) \\ \Psi^{(1)}(x-1) \end{pmatrix}, \quad x \in \mathbb{Z}.$$

Let $\{C(x)\}_{x \in \mathbb{Z}} \subset U(2)$ be a family of unitary matrices with

$$C(x) = \begin{pmatrix} a(x) & b(x) \\ c(x) & d(x) \end{pmatrix}.$$

$C\Psi$ is given by

$$(C\Psi)(x) = C(x)\Psi(x), \quad x \in \mathbb{Z}.$$

We define an evolution operator as $U = SC$. U satisfies

$$(U\Psi)(x) = P(x+1)\Psi(x+1) + Q(x-1)\Psi(x-1), \quad x \in \mathbb{Z}$$

with

$$P(x) = \begin{pmatrix} a(x) & b(x) \\ 0 & 0 \end{pmatrix}, \quad Q(x) = \begin{pmatrix} 0 & 0 \\ c(x) & d(x) \end{pmatrix}.$$

For a matrix $M \in M(2, \mathbb{C})$, we use $\|M\|$ to denote the operator norm in $\mathbb{C}^2$: $\|M\| = \sup_{\|\boldsymbol{x}\|_{\mathbb{C}^2}=1} \|M\boldsymbol{x}\|_{\mathbb{C}^2}$. We suppose that:

(A.1) There exists a unitary matrix $C_0 = \begin{pmatrix} a_0 & b_0 \\ c_0 & d_0 \end{pmatrix} \in U(2)$ such that

$$\|C(x) - C_0\| \le c_1 |x|^{-1-\epsilon}, \quad x \in \mathbb{Z} \setminus \{0\}$$

with some positive c_1 and ϵ independent of x.

We denote by $\mathscr{T}_1$ the set of trace class operators.

Lemma 2.1 *Let U satisfy (A.1) and set $U_0 = SC_0$. Then, $U - U_0 \in \mathscr{T}_1$.*

Proof Let $T = U - U_0$ and $T(x) = C(x) - C_0$. Then,

$$T^*T = (C - C_0)^*(C - C_0) \tag{2.1}$$

is the multiplication operator by the matrix-valued function $T(x)^*T(x)$. Let $t_i(x)$ ($i = 1, 2$) be the eigenvalues of the Hermitian matrix $T(x)^*T(x) \in M(2, \mathbb{C})$ and take an orthonormal basis (ONB) $\{\tau_i(x)\}_{i=1,2}$ of corresponding eigenvectors for all $x \in \mathbb{Z}$.

We use $|\xi\rangle\langle\eta|$ to denote the operator on $\mathcal{H}$ defined by $|\xi\rangle\langle\eta|\Psi = \langle\eta, \Psi\rangle\xi$. Then, we have

$$T^*T = \sum_{i=1,2}\sum_{x\in\mathbb{Z}} t_i(x)|\tau_{i,x}\rangle\langle\tau_{i,x}|, \tag{2.2}$$

where $\{\tau_{i,x}\}$ is the ONB given by

$$\tau_{i,x}(y) = \delta_{xy}\tau_i(x), \quad y \in \mathbb{Z}.$$

Since $T^*(x)T(x) \geq 0$, we have $t_i(x) \geq 0$. By (A.1), we know that

$$\max_{i=1,2} t_i(x) \leq c_1^2|x|^{-2-2\epsilon}.$$

Hence, we have

$$\mathrm{Tr}|T| = \sum_{x\in\mathbb{Z}}\sum_{i=1,2} t_i(x)^{1/2} \leq 2c_1\sum_{x\in\mathbb{Z}}|x|^{-1-\epsilon} < \infty,$$

which means that $T \in \mathscr{T}_1$. Since $\mathscr{T}_1$ is an ideal, $U - U_0 = ST \in \mathscr{T}_1$. □

Example 2.1 (one-defect model) Let $C_0, C_0' \in U(2)$ be unitary matrices with $C_0 \neq C_0'$ and set

$$C(x) = \begin{cases} C_0', & x = 0 \\ C_0, & x \neq 0. \end{cases}$$

$U = SC$ satisfies (A.1), because $C(x) - C_0 = 0$ if $x \neq 0$.

Example 2.2 Let $C_0 \in U(2)$ be a unitary matrix and $\{C(x)\} \subset U(2)$ a family of unitary matrices. Assume that

$$\max_{i,j}|(C(x) - C_0)_{ij}| \leq c_1|x|^{-1-\epsilon}, \quad x \in \mathbb{Z}\setminus\{0\},$$

where M_{ij} denotes the ij-component of a matrix M. Then, $U = SC$ satisfies (A.1), because all norms on a finite-dimensional vector space are equivalent.

We prove the following theorem in Sect. 3 using a discrete analogue of the Kato–Rosenblum theorem.

Theorem 2.1 *Let U and U_0 be as above and assume that (A.1) holds. Then,*

$$W_+ = s\text{-}\lim_{t\to\infty} U^{-t}U_0^t\Pi_{\mathrm{ac}}(U_0)$$

exists and is complete.

In what follows, we introduce the asymptotic velocity $\hat{v}_0$, obtained first in [4], of the quantum walk with the evolution U_0 as follows. Let

$$\hat{U}_0(k) = \begin{pmatrix} e^{ik} & 0 \\ 0 & e^{-ik} \end{pmatrix} C_0, \quad k \in [0, 2\pi).$$

Since $\hat{U}_0(k) \in U(2)$, $\hat{U}_0(k)$ is represented as

$$\hat{U}_0(k) = \sum_{i=1,2} \lambda_i(k) |u_j(k)\rangle\langle u_j(k)|,$$

where $\lambda_j(k)$ is an eigenvalue of $\hat{U}_0(k)$ and $u_j(k)$ is the corresponding eigenvector with $\|u_j(k)\| = 1$. The function $k \mapsto e^{ik}$ is analytic, and so is $\lambda_j(k)$. We need the following assumption on $u_j(k)$:

(A.2) The functions $k \mapsto u_j(k)$ are continuously differentiable in k with

$$\sup_{k\in[0,2\pi)} \left\| \frac{\mathrm{d}}{\mathrm{d}k} u_j(k) \right\|_{\mathbb{C}^2} < \infty.$$

Let $\mathcal{K}$ be the Hilbert space of square integrable functions $f:[0, 2\pi) \to \mathbb{C}^2$ with norm

$$\|f\|_{\mathcal{K}} = \left(\int_0^{2\pi} \frac{\mathrm{d}k}{2\pi} \|f(k)\|^2_{\mathbb{C}^2} \right)^{1/2}.$$

Let $\mathcal{F}_0:\mathcal{H} \to \mathcal{K}$ be the discrete Fourier transform given by

$$(\mathscr{F}\Psi)(k) = \sum_{x\in\mathbb{Z}} e^{-ik\cdot x} \Psi(x), \quad \Psi \in \mathcal{H}.$$

We also use $\hat{\Psi}(k) = \begin{pmatrix} \hat{\Psi}^{(0)}(k) \\ \hat{\Psi}^{(1)}(k) \end{pmatrix}$ to denote the Fourier transform of Ψ. The asymptotic velocity $\hat{v}_0$ is the self-adjoint operator defined by

$$\hat{v}_0 = \mathscr{F}^{-1} \left(\int_{[0,2\pi)}^{\oplus} \frac{\mathrm{d}k}{2\pi} \sum_{j=1,2} \left(\frac{i\lambda'_j(k)}{\lambda_j(k)} \right) |u_j(k)\rangle\langle u_j(k)| \right) \mathscr{F}$$

The position operator $\hat{x}$ is a self-adjoint operator defined by

$$(\hat{x}\Psi)(x) = x\Psi(x), \quad x \in \mathbb{Z}$$

with domain

$$D(\hat{x}) = \left\{ \Psi \in \mathcal{H} \;\middle|\; \sum_{x \in \mathbb{Z}} |x|^2 \|\Psi(x)\|_{\mathbb{C}^2}^2 < \infty \right\}.$$

Let $\hat{x}_0(t) = U_0^{-t} \hat{x} U_0^t$ be the Heisenberg operator of $\hat{x}$ for the evolution U_0.

Theorem 2.2 *Let $\hat{v}_0$ and $\hat{x}_0$ be as above. Suppose that (A.2) holds. Then,*

$$s\text{-}\lim_{t \to \infty} \exp\left(i\xi \frac{\hat{x}_0(t)}{t} \right) = \exp(i\xi \hat{v}_0), \quad \xi \in \mathbb{R}. \tag{2.3}$$

Proof By [10, TheoremVIII.21], (2.3) holds if and only if

$$\text{s-}\lim_{t \to \infty} \left(\frac{\hat{x}_0(t)}{t} - z \right)^{-1} = (\hat{v}_0 - z)^{-1}, \quad z \in \mathbb{C} \setminus \mathbb{R},$$

which is proved in Sect. 4.1. □

Example 2.3 (i) Let $C_0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. Then, $\hat{U}_0(k)$ has eigenvalues 1 and -1, which are independent of k. By definition, $\hat{v}_0 = 0$. Hence, the random variable $X_t^{(0)}/t$ converges in law to a random variable V_0 with a probability distribution δ_0.

(ii) Let $C_0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. $\hat{U}_0(k)$ has eigenvalues e^{ik} and $-e^{-ik}$. Hence, $\hat{v}_0$ has eigenvalues -1 and 1. The random variable $X_t^{(0)}/t$ converges in law to a random variable V_0 with a probability distribution $\|\Psi^{(0)}\|^2 \delta_{-1} + \|\Psi^{(1)}\|^2 \delta_1$.

(iii) Let C_0 be the Hadamard matrix. The eigenvalues of $\hat{U}_0(k)$ are given by $\lambda_j(k) = ((-1)^j w(k) + i \sin k)/\sqrt{2}$ ($j = 1, 2$), where $w(k) = \sqrt{1 + \cos^2 k}$. Hence, $\hat{v}_0$ has no eigenvalue. The corresponding eigenvectors

$$u_j(k) = \sqrt{\frac{w(k) + (-1)^j \cos k}{2w(k)}} \begin{pmatrix} e^{ik} \\ (-1)^j w(k) - \cos k \end{pmatrix}$$

form an ONB of $\mathbb{C}^2$ and satisfy (A.2). The random variable $X_t^{(0)}/t$ converges in law to a random variable V_0 with a probability distribution $\|E_{\hat{v}_0}(\cdot)\Psi_0\|^2$, where $E_{\hat{v}_0}$ is the spectral measure of $\hat{v}_0$. Let us consider the Hadmard walk starting from the origin. Let the initial state Ψ_0 satisfy $\Psi_0(0) = \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$ ($|\alpha|^2 + |\beta|^2 = 1$) and $\Psi(x) = 0$ if $x \neq 0$. Then,

$$d\|E_{\hat{v}_0}(v)\Psi_0\|^2 = (1 - c_{\alpha,\beta} v) f_K\left(v; \frac{1}{\sqrt{2}}\right) \mathrm{d}v,$$

where $c_{\alpha,\beta} = |\alpha|^2 - |\beta|^2 + \alpha\bar{\beta} + \bar{\alpha}\beta$,

$$f_K(v;r) = \frac{\sqrt{1-r^2}}{\pi(1-v^2)\sqrt{r^2-v^2}} I_{(-r,r)}(v)$$

is the Konno function, and I_A is the indicator function of a set A. For more details, the reader can consult [4,7].

Let $\hat{x}(t) = U^{-t}\hat{x}U$ be the Heisenberg operator of $\hat{x}$ and define the asymptotic velocity $\hat{v}_+$ for the evolution U by

$$\hat{v}_+ = W_+ \hat{v}_0 W_+^*.$$

We need the following assumption:

(A.3) The singular continuous spectrum of U is empty.

We are now in a psition to state our main result, which is proved in Sect. 4.2.

Theorem 2.3 *Let $\hat{x}(t)$ and $\hat{v}_+$ be as above. Suppose that (A.1)–(A.3) hold. Then,*

$$s\text{-}\lim_{t\to\infty} \exp\left(i\xi\frac{\hat{x}(t)}{t}\right) = \Pi_{\rm p}(U) + \exp\left(i\xi\hat{v}_+\right)\Pi_{\rm ac}(U), \quad \xi \in \mathbb{R}.$$

Let X_t be the random variable denoting the position of the walker at time $t \in \mathbb{N}$ with the initial state Ψ_0. We use $\Pi_{\rm p}(U)$ to denote the orthogonal projection onto the direct sum of all eigenspaces of U and E_A to denote the spectral projection of a self-adjoint operator A.

Corollary 2.4 *Let X_t be as above. Suppose that (A.1)–(A.3) hold. Then, X_t/t converges in law to a random variable V with a probability distribution*

$$\mu_V = \|\Pi_{\rm p}(U)\Psi_0\|^2\delta_0 + \|E_{\hat{v}_+}(\cdot)\Pi_{\rm ac}(U)\Psi_0\|^2,$$

where δ_0 is the Dirac measure at zero.

Proof From Theorem 2.1, s-$\lim_{t\to\infty} U_0^{-t}U^t\,\Pi_{\rm ac}(U)$ exists and is equal to W_+^*. Then, W_+ is unitary from $\mathrm{Ran}\,W_+^* = \mathrm{Ran}\,\Pi_{\rm ac}(U_0)$ to $\mathrm{Ran}\,W_+ = \mathrm{Ran}\,\Pi_{\rm ac}(U)$. Since, by Lemma 4.1, U_0 is strongly commuting with $\hat{v}_0$, we know, from the intertwining property $UW_+ = W_+U_0$, that U is also strongly commuting with $\hat{v}_+$. Hence, $\hat{v}_+$ is strongly commuting with $\Pi_{\rm ac}(U)$ and $e^{i\xi\hat{v}_+}\Pi_{\rm ac}(U) = \Pi_{\rm ac}(U)e^{i\xi\hat{v}_+}$. Hence, by Theorem 2.3, $\exp(i\xi\hat{x}(t)/t)\Psi_0$ converges strongly to $\Pi_{\rm p}(U)\Psi_0 + e^{i\xi\hat{v}_+}\Pi_{\rm ac}(U)\Psi_0$ and

$$\lim_{t\to\infty}\mathbb{E}(e^{i\xi X_t/t}) = \langle\Psi_0, \Pi_{\rm p}(U)\Psi_0 + e^{i\xi\hat{v}_+}\Pi_{\rm ac}(U)\Psi_0\rangle = \|\Pi_{\rm p}(U)\Psi_0\|^2 + \int_{-\infty}^{\infty} e^{i\xi v} d\|E_{\hat{v}_+}(v)\Pi_{\rm ac}(U)\Psi_0\|^2 = \int_{-\infty}^{\infty} e^{i\xi v} d\mu_V(v),$$

which proves the corollary. □

Example 2.4 Let C_0 be the Hadmard matrix and $C(x)$ satisfy (A.1). As seen in Example 2.3 (iii), (A.2) is satisfied and the spectrum of U_0 is purely absolutely continuous. Let $\Psi_+ \in \mathcal{H}$ satisfy $\Psi_+(0) = \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$ ($|\alpha|^2 + |\beta|^2 = 1$) and $\Psi_+(x) = 0$ if $x \neq 0$. By Example 2.3,

$$d\|E_{\hat{v}_+}(v)\Pi_{\rm ac}(U)W_+\Psi_+\|^2 = d\|E_{\hat{v}_0}(v)\Psi_+\|^2 = (1 - c_{\alpha,\beta}v) f_K\left(v; \frac{1}{\sqrt{2}}\right) {\rm d}v.$$

Let $\Psi_{\rm p} \in {\rm Ran}\Pi_{\rm p}(U_0)$ be a unit vector and take the initial state Ψ_0 as $\Psi_0 = C_1\Psi_{\rm p} + C_2 W_+\Psi_+$ ($|C_1|^2 + |C_2|^2 = 1$). Suppose that $U = SC$ satisfies (A.3). By Corollary 2.4, X_t/t converges in law to V with a probability distribution μ_V and

$$\mu_V({\rm d}v) = |C_1|^2\delta_0({\rm d}v) + |C_2|^2(1 - c_{\alpha,\beta}v) f_K\left(v; \frac{1}{\sqrt{2}}\right) {\rm d}v.$$

3 Wave operator

To prove Theorem 2.1, we use the following general proposition:

Proposition 3.1 *Let U and U_0 be unitary operators on a Hilbert space $\mathcal{H}$ and suppose that $U - U_0 \in \mathscr{T}_1$. The following limit exists:*

$$W_+ = s\text{-}\lim_{t\to\infty} U^{-t}U_0^t\Pi_{\rm ac}(U_0)$$

Proof of Theorem 2.1 Since, by Lemma 2.1, $U - U_0 \in \mathscr{T}_1$, the wave operator W_+ exists. If we interchange the roles of U and U_0, then the proposition says that the limit s-$\lim_{t\to\infty} U_0^{-t}U^t\Pi_{\rm ac}(U)$ also exists, which implies that W_+ is complete. This completes the proof. □

In the remainder of this section, we suppose that $U - U_0 \in \mathscr{T}_1$ and prove Proposition 3.1. This is done by a discrete analogue of [11, Theorem 6.2]. We use $\mathcal{H}_{\rm ac}$ and $\mathcal{H}_{\rm p}$ to denote the subspaces of absolute continuity and the direct sum of all eigenspaces of U_0. Let E_0 be the spectral measure of U_0 with $E_0([0, 2\pi)) = I$. Let

$$\mathcal{H}_{\rm ac,0} = \{\psi \in \mathcal{H}_{\rm ac} \mid d\|E_0(\lambda)\psi\|^2 = G_\psi(\lambda)^2 d\lambda \text{ and } G_\psi \in L^2 \cap L^\infty\},$$

where $L^2 = L^2([0, 2\pi))$ and $L^\infty = L^\infty([0, 2\pi))$. Although the following lemma may be well known, we give proofs for completeness.

Lemma 3.1 *$\mathcal{H}_{\rm ac,0}$ is dense in $\mathcal{H}_{\rm ac}$.*

Proof For all $\psi \in \mathcal{H}_{\rm ac}$, there exists a positive function $F \in L^1$ such that $d\|E_0(\lambda)\psi\|^2 = F(\lambda)d\lambda$. Let $B_n = F^{-1}([0, n])$, and let χ_{B_n} be the characteristic function of B_n. We set $G_n = \sqrt{F}\chi_{B_n}$ and $\psi_n = E_0(B_n)\Psi$. Then, $G_n \in L^2 \cap L^\infty$ and $\|E_0(B)\psi_n\|^2 = \int_B G_n(\lambda)^2 d\lambda$. Hence, $\psi_n \in \mathscr{H}_{\rm ac,0}$ and $\psi = \lim_n \psi_n$. This completes the proof. □

Lemma 3.2 *Let $\phi \in \mathcal{H}$ and $\psi \in \mathcal{H}_{\mathrm{ac},0}$. Then,*

$$\sum_{t\in\mathbb{Z}} \left|\langle \phi, U_0^t \psi\rangle\right|^2 \leq 2\pi \|\phi\|^2 \sup_\lambda G_\psi(\lambda)^2.$$

Proof Let $\psi \in \mathcal{H}_{\mathrm{ac},0}$ and $\mathcal{L} = L^2([0,2\pi), G_\psi^2(\lambda)d\lambda)$. Let H_0 be the self-adjoint operator defined by $\langle \xi, H_0\eta\rangle = \int_0^{2\pi} \lambda d\langle \xi, E_0(\lambda)\eta\rangle$ $(\xi,\eta \in \mathcal{H})$. Let $\mathscr{U}:\mathcal{L} \to \mathcal{H}$ be an injection defined by $\mathscr{U} f = f(H_0)\psi$ $(f \in \mathcal{L})$. Then $\mathscr{U}1 = \psi$ and $\mathscr{U}e^{it\lambda} = U_0^t\psi$ $(t \in \mathbb{N})$. We use Π to denote the orthogonal projection onto $U\mathcal{L}$. Let $\phi \in \mathcal{H}$ and $F = \mathscr{U}^{-1}\Pi\phi \in \mathcal{L}$. Then, we have

$$\langle \phi, U_0^t\psi\rangle = \int_0^{2\pi} e^{it\lambda}\bar{F}(\lambda)G_\psi(\lambda)^2 d\lambda = 2\pi\,\widehat{\bar{F}G_\psi^2}(t).$$

Hence, by Parseval's identity, we obtain

$$\begin{aligned}\sum_{t\in\mathbb{Z}} |\langle \phi, U_0^t\psi\rangle|^2 &= 2\pi \int_0^{2\pi} |\bar{F}(\lambda)G_\psi(\lambda)^2|^2 d\lambda \\ &\leq 2\pi \sup_\lambda G_\psi(\lambda)^2 \int_0^{2\pi} |\bar{F}(\lambda)|^2 G_\psi(\lambda)^2 d\lambda \\ &\leq 2\pi \sup_\lambda G_\psi(\lambda)^2 \|\Pi\phi\|^2.\end{aligned}$$

This completes the proof. □

Let $W_t = U^{-t}U_0^t$.

Lemma 3.3 *Let $t, s \in \mathbb{N}$ $(s \neq t)$. Then,* $\text{s-}\lim_{r\to\infty}(W_t - W_s)U_0^r\Pi_{\mathrm{ac}}(U_0) = 0$.

Proof For $t, s \in \mathbb{N}$ $(t > s)$, we have $W_t = \sum_{k=s+1}^t (W_k - W_{k-1}) + W_s$ and $W_k - W_{k-1} = U^{-k}(-T)U_0^{k-1}$, where $T = U - U_0 \in \mathscr{T}_1$. Since $\mathscr{T}_1$ is an ideal, we know that

$$W_t - W_s = \sum_{k=s+1}^{t} U^{-k}(-T)U_0^{k-1} \in \mathscr{T}_1.$$

In particular, $W_t - W_s$ is compact. Let H_0 be the self-adjoint operator defined in the proof of Lemma 3.2. Since $\text{w-}\lim_{r\to\infty} e^{irH_0}\Pi_{\mathrm{ac}}(H_0) = 0$, we have

$$\text{s-}\lim_{r\to\infty}(W_t - W_s)U_0^r\Pi_{\mathrm{ac}}(U_0) = \text{s-}\lim_{r\to\infty}(W_t - W_s)e^{irH_0}\Pi_{\mathrm{ac}}(H_0) = 0.$$

This completes the proof. □

Proof of Proposition 3.1 By Lemma 3.1, it suffices to prove that, for $\psi \in \mathcal{H}_{\mathrm{ac},0}$,

$$\|(W_t - W_s)\psi\| \to 0, \quad t, s \to \infty.$$

Because

$$\|(W_t - W_s)\psi\|^2 = \langle \psi, W_t^*(W_t - W_s)\psi\rangle - \langle \psi, W_s^*(W_t - W_s)\psi\rangle,$$

we need only to prove that

$$\langle \psi, W_t^*(W_t - W_s)\psi\rangle \to 0, \quad t, s \to \infty.$$

By direct calculation, we have, for $r > 1$,

$$\begin{aligned} &W_t^*(W_t - W_s) - U_0^{-r} W_t^*(W_t - W_s)U_0^r \\ &= U_0^{-r} W_t^* W_s U_0^r - W_t^* W_s \\ &= \sum_{k=0}^{r-1} \left(U_0^{-k-1} W_t^* W_s U_0^{k+1} - U_0^{-k} W_t^* W_s U_0^k \right). \end{aligned}$$

Since

$$U_0^{-k-1} W_t^* W_s U_0^{k+1} - U_0^{-k} W_t^* W_s U_0^k = U_0^{-k-t-1} \left(T U^{t-s} - U^{t-s} T \right) U_0^{s+k},$$

we obtain

$$\begin{aligned} &W_t^*(W_t - W_s) - U_0^{-r} W_t^*(W_t - W_s)U_0^r \\ &= \sum_{k=0}^{r-1} U_0^{-k-t-1} \left(T U^{t-s} - U^{t-s} T \right) U_0^{s+k}. \end{aligned}$$

Since, by Lemma 3.3, $\text{s-}\lim_{r\to\infty} U_0^{-r} W_t^*(W_t - W_s)U_0^r \psi = 0$, we have

$$\begin{aligned} W_t^*(W_t - W_s)\psi &= \sum_{k=0}^{\infty} U_0^{-k-t-1} \left(T U^{t-s} - U^{t-s} T \right) U_0^{s+k}\psi \\ &= Z_{t,s}((U_0 T)U^{t-s} - (U_0 U^{t-s})T)\psi, \end{aligned}$$

where

$$Z_{t,s}(A) = \sum_{k=0}^{\infty} U_0^{-k-t} A U_0^{k+s}.$$

By Lemma 3.4 below, we know that

$$|\langle \psi, W_t^*(W_t - W_s)\psi\rangle| \le |\langle \psi, Z_{t,s}((U_0T)U^{t-s})\psi\rangle| + |\langle \psi, Z_{t,s}(U_0U^{t-s})T)\psi\rangle| \to 0, \quad t, s \to \infty.$$

This completes the proof. □

Lemma 3.4 *Let $Y \in \mathscr{T}_1$ and $\{Q(t,s)\}$ be a family of bounded operators with $\sup_{t,s}\|Q(t,s)\| < \infty$. Then, for all $\psi \in \mathcal{H}_{\mathrm{ac},0}$,*

(1) $\lim_{t,s\to\infty}\left\langle \psi, Z_{t,s}(YQ(t,s))\psi\right\rangle = 0$;
(2) $\lim_{t,s\to\infty}\left\langle \psi, Z_{t,s}(Q(t,s)Y)\psi\right\rangle = 0$.

Proof Let $Y = \sum_{n=1}^{\infty}\lambda_n|\psi_n\rangle\langle\phi_n|$ be the canonical expansion of the compact operator Y. Since $Y \in \mathscr{T}_1$, $\sum_n \lambda_n < \infty$. Then, by the Cauchy–Schwartz inequality, we have

$$\begin{aligned}\left|\left\langle \psi, Z_{t,s}(YQ(t,s))\psi\right\rangle\right| &\le \sum_{n=1}^{\infty}\sum_{k=0}^{\infty}\lambda_n\left|\left\langle U_0^{k+t}\psi, \psi_n\right\rangle\left\langle \phi_n, Q(t,s)U_0^{k+s}\psi\right\rangle\right| \\ &\le I_1(t,s)^{1/2}\times I_2(t,s)^{1/2},\end{aligned}$$

where

$$\begin{aligned}I_1(t) &= \sum_{n=1}^{\infty}\sum_{k=0}^{\infty}\lambda_n\left|\left\langle \psi_n, U_0^{k+t}\psi\right\rangle\right|^2, \\ I_2(t,s) &= \sum_{n=1}^{\infty}\sum_{k=0}^{\infty}\lambda_n\left|\left\langle Q(t,s)^*\phi_n, U_0^{k+s}\psi\right\rangle\right|^2.\end{aligned}$$

By Lemma 3.2, we have

$$I_2(t,s) \le 2\pi \sup_{\lambda} G_\psi(\lambda)^2 \sup_{t,s}\|Q(t,s)\|\sum_n \lambda_n < \infty,$$

where we have used the fact that ϕ_n is a normalized vector. Let $u_k = \sum_{n=1}^{\infty}\lambda_n|\left\langle\psi_n, U_0^k\psi\right\rangle|^2$. Then, similar to the above, we observe that $\{u_k\} \in \ell^1(\mathbb{Z})$. Hence, we have

$$\lim_{t\to\infty} I_1(t) = \lim_{t\to\infty}\sum_{k=t}^{\infty} u_k = 0.$$

This proves (i). The same proof works for (ii). □

4 Asymptotic velocity

4.1 Proof of Theorem 2.2

Let

$$\mathcal{H}_0 = \bigcup_{m=0}^{\infty} \{\Psi \in \mathcal{H} \mid \Psi(x) = 0,\ |x| \geq m\}.$$

We use $\mathcal{D}$ to denote a subspace of vectors $\Psi \in \mathcal{H}$ whose Fourier transform $\hat{\Psi}$ is differentiable in k with

$$\sup_{k\in[0,2\pi)} \left\| \frac{\mathrm{d}}{\mathrm{d}k}\hat{\Psi}(k) \right\| < \infty.$$

Note that $\mathcal{H}_0$ is a core for $\hat{x}$, and so is $\mathcal{D}$. Let $D = \mathscr{F}\hat{x}\mathscr{F}^{-1}$. Then, by direct calculation, we know that $(D\hat{\Psi})(k) = i\frac{\mathrm{d}}{\mathrm{d}k}\hat{\Psi}(k)$ for $\Psi \in \mathcal{D}$. We prove the following theorem:

Theorem 4.1 *Suppose that (A.2) holds. Then,*

$$s\text{-}\lim_{t\to\infty} \left(\frac{\hat{x}_0(t)}{t} - z \right)^{-1} = (\hat{v}_0 - z)^{-1}, \quad z \in \mathbb{C} \setminus \mathbb{R}. \tag{4.1}$$

Proof For all $\Psi \in \mathcal{H}$ and $\epsilon > 0$, there exists a vector $\Psi_\epsilon \in \mathcal{D}$ such that $\|\Psi - \Psi_\epsilon\| \leq \epsilon$. Because, by the second resolvent identity,

$$\begin{aligned}
&\left\| \left(\frac{\hat{x}_0(t)}{t} - z \right)^{-1} \Psi - (\hat{v}_0 - z)^{-1}\Psi \right\| \\
&\leq \frac{2\epsilon}{|\mathrm{Im} z|} + \left\| \left(\frac{\hat{x}_0(t)}{t} - z \right)^{-1} \Psi_\epsilon - (\hat{v}_0 - z)^{-1}\Psi_\epsilon \right\| \\
&\leq \frac{2\epsilon}{|\mathrm{Im} z|} + \frac{1}{|\mathrm{Im} z|} \left\| \left(\hat{v}_0 - \frac{\hat{x}_0(t)}{t} \right) (\hat{v}_0 - z)^{-1}\Psi_\epsilon \right\|,
\end{aligned}$$

it suffices to prove that

$$\lim_{t\to\infty} \left\| \left(\hat{v}_0 - \frac{\hat{x}_0(t)}{t} \right) (\hat{v}_0 - z)^{-1}\Psi \right\| = 0, \quad \Psi \in \mathcal{D}.$$

Note that

$$(\hat{v}_0 - z)^{-1} = \mathscr{F}^{-1} \left(\int_{[0,2\pi)}^{\oplus} \mathrm{d}k \sum_{j=1,2} \left(\frac{i\lambda_j'(k)}{\lambda_j(k)} - z \right)^{-1} |u_j(k)\rangle\langle u_j(k)| \right) \mathscr{F}.$$

Since $\lambda_j(k)$ is analytic and $|\lambda_j(k)| = 1$, we observe from (A.2) that $(\hat{v}_0 - z)^{-1}$ leaves $\mathcal{D}$ invariant. Hence, we only need to prove that

$$\lim_{t\to\infty} \left\| \left(\hat{v}_0 - \frac{\hat{x}_0(t)}{t} \right) \Psi \right\| = 0, \quad \Psi \in \mathcal{D}.$$

By direct calculation, we have

$$\begin{aligned} &\left\| \left(\hat{v}_0 - \frac{\hat{x}_0(t)}{t} \right) \Psi \right\|^2 \\ &= \int_0^{2\pi} \mathrm{d}k \left\| \sum_{j=1,2} \left(\frac{i\lambda_j'(k)}{\lambda_j(k)} \right) \langle u_j(k), \hat{\Psi}(k) \rangle u_j(k) - \hat{U}(k)^{-t} \frac{D}{t} \hat{U}(k)^t \hat{\Psi}(k) \right\|^2 \\ &= \int_0^{2\pi} \frac{\mathrm{d}k}{t^2} \left\| \sum_{j=1,2} \lambda_j(k)^t \hat{U}(k)^{-t} \left(i \frac{\mathrm{d}}{\mathrm{d}k} \langle u_j(k), \hat{\Psi}(k) \rangle u_j(k) \right) \right\|^2 . \end{aligned}$$

By the definition of $\mathcal{D}$ and (A.2), we know that

$$\sup_{k\in[0,2\pi)} \left\| \left(i \frac{\mathrm{d}}{\mathrm{d}k} \langle u_j(k), \hat{\Psi}(k) \rangle u_j(k) \right) \right\| < \infty.$$

Hence, we have

$$\left\| \left(\hat{v}_0 - \frac{\hat{x}_0(t)}{t} \right) \Psi \right\| = O(t^{-1}),$$

which completes the proof. □

4.2 Proof of Theorem 2.3

The proof falls naturally into two parts:

Theorem 4.2 *Let U be a unitary operator on $\mathcal{H}$. $\hat{x}(t) = U^{-t} \hat{x} U^t$ satisfies*

$$s\text{-}\lim_{t\to\infty} \exp\left(i\xi \frac{\hat{x}(t)}{t} \right) \Pi_{\mathrm{p}}(U) = \Pi_{\mathrm{p}}(U), \quad \xi \in \mathbb{R}.$$

Theorem 4.3 *Let $U = SC$ and $U_0 = SC_0$ satisfy (A.1) and (A.2). Then,*

$$s\text{-}\lim_{t\to\infty} \exp\left(i\xi \frac{\hat{x}(t)}{t} \right) \Pi_{\mathrm{ac}}(U) = \exp(i\xi \hat{v}_+) \Pi_{\mathrm{ac}}(U), \quad \xi \in \mathbb{R}.$$

Proof of Theorem 2.3 By (A.3), we have

$$\begin{aligned}\text{s-}\lim_{t\to\infty}\exp\left(i\xi\frac{\hat{x}(t)}{t}\right) &= \text{s-}\lim_{t\to\infty}\exp\left(i\xi\frac{\hat{x}(t)}{t}\right)(\Pi_{\mathrm{p}}(U)+\Pi_{\mathrm{p}}(U)) \\ &= \Pi_{\mathrm{p}}(U)+\exp(i\xi\hat{v}_+)\Pi_{\mathrm{ac}}(U).\end{aligned}$$

This prove the theorem. □

It remains to prove Theorems 4.2 and 4.3.

Proof of Theorem 4.2 Let $\mathcal{H}_{\mathrm{p}}(U)$ be the direct sum of all eigenspaces of U. It suffices to prove that, for $\Psi \in \mathcal{H}_{\mathrm{p}}(U)$,

$$\text{s-}\lim_{t\to\infty}\exp\left(i\xi\frac{\hat{x}(t)}{t}\right)\Psi = \Psi.$$

Let λ_n be the eigenvalues of U and take an ONB $\{\eta_n\}_{n=1}^{\infty}$ of $\mathcal{H}_{\mathrm{p}}$ such that $U\eta_n = \lambda_n\eta_n$. We have $\Pi_{\mathrm{p}}(U) = \sum_n |\eta_n\rangle\langle\eta_n|$. Let $\epsilon > 0$. For sufficiently large N, $\Psi_N = \sum_{n=1}^{N}\langle\eta_n, \Psi\rangle\eta_n$ satisfies $\|\Psi - \Psi_N\| \le \epsilon$. Then,

$$\left\|\exp\left(i\xi\frac{\hat{x}(t)}{t}\right)\Psi - \Psi\right\| \le 2\epsilon + \left\|\exp\left(i\xi\frac{\hat{x}(t)}{t}\right)\Psi_N - \Psi_N\right\|.$$

By direct calculation, we have

$$\begin{aligned}\left\|\exp\left(i\xi\frac{\hat{x}(t)}{t}\right)\Psi_N - \Psi_N\right\| &= \left\|\left(\exp\left(i\xi\frac{\hat{x}}{t}\right)-1\right)U^t\Psi_N\right\| \\ &= \left\|\sum_{n=1}^{N}\lambda_n^t\langle\eta_n,\Psi\rangle\left(\exp\left(i\xi\frac{\hat{x}}{t}\right)-1\right)\eta_n\right\| \\ &\le \sum_{n=1}^{N}|\langle\eta_n,\Psi\rangle|\left\|\left(\exp\left(i\xi\frac{\hat{x}}{t}\right)-1\right)\eta_n\right\|. \qquad (4.2)\end{aligned}$$

Since $\lim_{t\to\infty}|1-e^{i\xi x/t}| = 0$, $|1-e^{i\xi x/t}| \le 2$ and $\sum_x \|\eta_n(x)\|_{\mathbb{C}^2}^2 = \|\eta_n\|^2 < \infty$, we have

$$\lim_{t\to\infty}\left\|\left(\exp\left(i\xi\frac{\hat{x}}{t}\right)-1\right)\eta_n\right\|^2 = \lim_{t\to\infty}\sum_{x\in\mathbb{Z}}|e^{i\xi x/t}-1|^2\|\eta_n(x)\|_{\mathbb{C}^2}^2 = 0,$$

which, combined with (4.2), completes the proof. □

Lemma 4.1 $[U_0, \exp(i\xi\hat{v}_0)] = 0$.

Proof By direct calculation, we have

$$[U_0, \exp(i\xi\hat{v}_0)] = \text{s-}\lim_{t\to\infty}\left[U_0, \exp\left(i\xi\frac{\hat{x}_0(t)}{t}\right)\right]$$
$$= \text{s-}\lim_{t\to\infty} U_0\left\{\exp\left(i\xi\frac{\hat{x}_0(t)}{t}\right) - \exp\left(i\xi\frac{\hat{x}_0(t+1)}{t}\right)\right\} = 0.$$

Proof of Theorem 4.3 By (A.1) and (A.2), Theorems 2.1 and 2.2 hold. Then, W_+ is a unitary operator from $\mathcal{H}_{\text{ac}}(U_0)$ to $\mathcal{H}_{\text{ac}}(U)$. Hence, we have

$$\exp(i\xi\hat{v}_+)\Pi_{\text{ac}}(U) = W_+\exp(i\xi\hat{v}_0)W_+^*\Pi_{\text{ac}}(U).$$

By direct calculation, we observe that

$$I(t) := \exp\left(i\xi\frac{\hat{x}(t)}{t}\right)\Pi_{\text{ac}}(U) - \exp(i\xi\hat{v}_+)\Pi_{\text{ac}}(U)$$
$$= W_t\exp\left(i\xi\frac{\hat{x}_0(t)}{t}\right)W_t^*\Pi_{\text{ac}}(U) - W_+\exp(i\xi\hat{v}_0)W_+^*\Pi_{\text{ac}}(U)$$
$$=: \sum_{j=1}^{3} I_j(t),$$

where

$$I_1(t) = W_t\exp\left(i\xi\frac{\hat{x}_0(t)}{t}\right)\left(W_t^* - W_+^*\right)\Pi_{\text{ac}}(U),$$
$$I_2(t) = W_t\left(\exp\left(i\xi\frac{\hat{x}_0(t)}{t}\right) - \exp(i\xi\hat{v}_0)\right)W_+^*\Pi_{\text{ac}}(U),$$
$$I_3(t) = (W_t - W_+)\exp(i\xi\hat{v}_0)W_+^*\Pi_{\text{ac}}(U).$$

Because W_t and $\exp(i\xi\hat{x}_0(t)/t)$ are uniformly bounded, we know from Theorems 2.1 and 2.2 that $\text{s-}\lim_{t\to\infty} I_1(t) = \text{s-}\lim_{t\to\infty} I_2(t) = 0$. Hence, we have

$$I(t) = (W_t - W_+)\exp(i\xi\hat{v}_0)W_+^*\Pi_{\text{ac}}(U) + o(1)$$
$$= (W_t - W_+)\,\Pi_{\text{ac}}(U_0)\exp(i\xi\hat{v}_0)W_+^*\Pi_{\text{ac}}(U)$$
$$+ (W_t - W_+)\,[\exp(i\xi\hat{v}_0), \Pi_{\text{ac}}(U_0)]W_+^*\Pi_{\text{ac}}(U) + o(1),$$

where we have used the fact that $\text{Ran}\,W_+^* = \mathcal{H}_{\text{ac}}(U_0)$. Since, by Lemma 4.1, $[\exp(i\xi\hat{v}_0), \Pi_{\text{ac}}(U_0)] = 0$, we obtain from Theorem 2.1, that $\text{s-}\lim_{t\to\infty} I(t) = 0$. This completes the proof. □

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References

1. Cantero, M.J., Grünbaum, F.A., Moral, L., Velázquez, L.: One-dimensional quantum walks with one defect. Rev. Math. Phys **24**, 52 (2012)
2. Endo, T., Konno, N.: Weak Convergence of the Wojcik Model. arXiv:1412.7874v3
3. Endo, S., Endo, T., Konno, N., Segawa, E., Takei, M.: Weak Limit Theorem of a Two-phase Quantum Walk with One Defect. arXiv:1412.4309v2
4. Grimmett, G., Janson, S., Scudo, P.: Weak limits for quantum random walks. Phys. Rev. E **69**, 026119 (2004)
5. Konno, N.: Quantum random walks in one dimension. Quantum Inf. Process. **1**, 345–354 (2002)
6. Konno, N.: A new type of limit theorems for the one-dimensional quantum random walk. J. Math. Soc. Jpn. **57**, 1179–1195 (2005)
7. Konno, N.: Quantum Walks. Quantum Potential Theory. Lecture Notes in Mathematics, vol. 1954, pp. 309–452. Springer, Berlin (2008)
8. Konno, N.: Localization of an inhomogeneous discrete-time quantum walk on the line. Quantum Inf. Process. **9**, 405–418 (2010)
9. Konno, N., Łuczak, T., Segawa, E.: Limit measure of inhomogeneous discrete-time quantum walks in one dimension. Quantum Inf. Process. **12**, 33–53 (2013)
10. Reed, M., Simon, B.: Methods of Modern Mathematical Physics, vol. I. Academic Press, New York (1972)
11. Simon, B.: Trace Ideals and Their Applications: Second Edition, Mathematical Surveys and Monographs., vol. 120. American Mathematical Society, Providence (2005)
12. Venegas-Andraca, S.E.: Quantum walks: a comprehensive review. Quantum Inf. Process. **11**, 1015–1106 (2012)
13. Wójcik, A., Łuczak, T., Kurzyński, P., Grudka, A., Gdala, T., Bednarska-Bzdega, M.: Trapping a particle of a quantum walk on the line. Phys. Rev. A **85**, 012329 (2012)