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Improving the payoffs of cooperators in three-player cooperative game using weak measurements

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Abstract In this paper, an efficient method is proposed to improve the payoffs of cooperators in cooperative three-player quantum game under the action of amplitude damping, bit flip and depolarizing channels using weak measurements. It is shown that the payoffs of cooperators can be enhanced to a great extent in the case of amplitude damping channel, and the payoff sudden death can be avoided in the case of bit flip and depolarizing channels. Moreover, the payoffs of cooperators tend to a constant by changing weak measurement strength in spite of sufficiently strong decoherence.

Keywords Three-player cooperative game · Decoherence · The payoff · Weak measurement

1 Introduction

The game theory was initially developed for in economics by von Neumann and Morgenstern [1]. After the important contributions given by John Nash [2], the game theory has been a branch of applied mathematics with many applications, e.g., modeling the behavior of biological, economical and computer systems. It aims to capture mathematical behavior in strategic situations, in which an individuals success in making a choice depends on the choice of the other players. Papers by Meyer [3] and Eisert

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et al. [4] paved the way for the creation of the new field of quantum game theory. Starting from the works of Meyer and Eisert, a number of classical games have been converted into the realm of quantum mechanics [5–18], such as quantum matching pennies game [8], Prisoner's dilemma quantum game [14], Quantum Parrondo's games [15] and quantum cooperative games [16]. Miszczak et al. [19] studied a qubit flip game on a Heisenberg spin chain. It has been shown that being well aware of the dimensionality of the system, a player can achieve a mean payoff equal to almost 1. Sharif et al. [20] proposed the quantum solution to a three-player Kolkata restaurant problem. The Kolkata paise restaurant (KPR) [21] is a repeated game similar to the minority games, played between a large number of agents among whom there are no interactions. In recent years, the quantum minority games [22–25] have attracted much attention. They have been analyzed under the influence of decoherence by Flitney and Hollenberg [26].

Quantum games in the presence of decoherence have been studied by a number of authors [27–34]. In Ref. [35], effect of quantum decoherence in a three-player quantum Kolkata restaurant problem was investigated by using tripartite entangled qutrit states. In Ref. [36], Khan et al. studied the behavior of cooperative multiplayer quantum games using different quantum channels. The result shows that in the case of depolarizing channel, the game is a no-payoff game irrespective of the degree of entanglement in the initial state for the larger values of decoherence parameter. The decoherence makes the cooperators worse off. Thus, the main issue is how to battle against the decoherence and improve the payoff.

Quantum strategies and quantum entanglement lead quantum players to harness the outcome of the game in their favor. In Ref. [37], Gawron et al. investigated the quantum version of the coin flip game under decoherence. The goal is to optimize the players control pulses and thus allow them to achieve higher probability of winning compared with the Pauli strategy. It turns out that it is possible to optimize the strategy of both players in the low noise scenario. However, if the rate of decoherence is high enough, the players strategies have little impact on the game outcome.

In 1988, Weak measurement was introduced by Aharonov, Albert and Vaidman (AAV) [38]. Weak measurement is very useful and can help understand many counterintuitive quantum phenomena, for example, Hardy's paradoxes [39–41]. Recently, the weak measurement has been applied as a practically implementable method for protecting entanglement and quantum fidelity of quantum states undergoing decoherence through the amplitude damping channel [42–46]. In Ref. [45], by using weak measurement and quantum measurement reversal, Kim et al. experimentally demonstrated a scheme for protecting entanglement from amplitude damping decoherence. However, the study on the improvement of the payoffs in the quantum games in the presence of decoherence via using weak measurements is not involved so far.

In this paper, we propose a method to improve the payoffs of cooperators in cooperative three-player quantum game under the action of amplitude damping, bit flip and depolarizing channels by using weak measurements. It is shown that the payoffs of cooperators can be enhanced greatly.

2 Three-player cooperative game

2.1 Classical form

A classical three-person normal form game [47] is given by three nonempty sets Σ_A, Σ_B and Σ_C, the strategy sets of the players A, B and C and three real valued functions P_A, P_B and P_C defined on $\Sigma_A \times \Sigma_B \times \Sigma_C$. The product space $\Sigma_A \times \Sigma_B \times \Sigma_C$ is the set of all tuples (σ_A,σ_B,σ_C) with $\sigma_A \epsilon \Sigma_A$, $\sigma_B \epsilon \Sigma_B$ and $\sigma_C \epsilon \Sigma_C$. A strategy is understood as such a tuple (σ_A,σ_B,σ_C) and P_A, P_B, P_C are payoff functions of the three players. The game is usually denoted as $\Gamma = \{\Sigma_A, \Sigma_B, \Sigma_C; P_A, P_B, P_C\}$.

Each of three players A, B and C chooses one of the two strategies 0, 1. Players A and B are said to be cooperators if they choose the same strategy. If the three players choose the same strategy, there is no payoff; otherwise, the two players who have chosen the same strategy receive a equal fixed amount each from the loser. The loser is one who chooses strategy different from the other two players. Hence, the classical three persons symmetric cooperative game is a zero-sum game. And, the mixed strategies $\frac{1}{2}[00] + \frac{1}{2}[11]$, $\frac{1}{2}[0] + \frac{1}{2}[1]$ are optimal for cooperators and the loser, respectively. In this case, the game is also symmetric with zero sum.

2.2 Quantum form

In quantum form of the three-player game, the players implement their strategies by applying the unitary operators in their possession on the initial quantum state. The strategy set of each player consists of two strategies I and σ_x, where I is the single qubit identity operator and σ_x is the Pauli spin flip operator. The initial three qubits entangled state is prepared by an arbiter, and then, the qubits are sent to all players at random, one qubit for each player. The players execute their strategies on their own qubit, and the final state is returned to the arbiter. Upon receiving all the qubits, the arbiter will perform measurement and the corresponding payoffs of the players are declared.

There are two different ways for each player to perform his/her strategies. The classical probability operator (CPO), then each player is allowed to perform I and σ_x with probability x and $1 - x$, respectively. In CPO, the players work exactly the way as in a classical game. The only difference is that they lay their bet on a quantum state. The quantum superposed operator (QSO), then each player is allowed to perform an unitary operation that is a linear combination of I and σ_x: $\sqrt{x}I + \sqrt{1-x}\sigma_x$. This corresponds to a physical implementation that each player is allowed to use linear superposition of the identity operator I and the σ_x. For the three cooperative quantum game, because A and B are cooperators, in term of the rules, they take the same strategies. Which implies A and B do the same operations on their respective qubits: $U_{AB} = \sqrt{q}I \bigotimes I + \sqrt{1-q}\sigma_x \bigotimes \sigma_x$, while C takes operation $U_C = \sqrt{r}I + \sqrt{1-r}\sigma_x$.

3 Theoretical framework

In this paper, we propose a method to improve the payoffs of cooperators in cooperative three-player quantum game under the action of decoherence channels by using weak

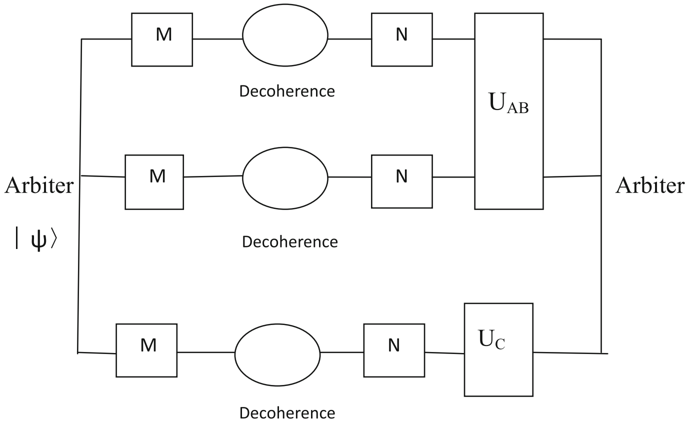

Fig. 1 Schematic diagram of our approach to improve the payoffs of cooperators in three-players cooperative game using weak measurements. We perform two weak measurements M and N, before and after the decoherence channels. The operator U_{AB} represents the strategy of the two cooperators and U_C is the strategy of the third player

measurements. Specifically, we consider a simple schemes as shown in Fig. 1. The scheme is weak measurement M + decoherence + weak measurement N +strategy operator.

For the three-player game, we consider the initial state as $|\psi\rangle = \cos(Q/2)|000\rangle + \sin(Q/2)|111\rangle$, where $Q\epsilon[0, \pi/2]$ is a measure of entanglement.

A quantum channel transfers information from one place to another. During the course of transformation, the source of information may interact with the channel with the many degrees of freedom and thus lead to the information damage. The effect of quantum channels on the state of a system is a completely positive and trace-preserving map that is described in terms of Kraus operators.

$$\rho_{\text{in}} = |\psi\rangle\langle\psi| \mapsto \varepsilon_{\text{channel}} = \sum_l K_l|\psi\rangle\langle\psi|K_l^\dagger \tag{1}$$

The operator K_l satisfies the CPTP relation $\sum_l K_l^\dagger K_l = I$. K_l can be expressed in terms of the linear operator $E_l^{A,B,C}$ as $K_l = E_l^A \otimes E_l^B \otimes E_l^C$.

In order to improve the payoff, we should perform two weak measurements M and N, before and after the decoherence channels, respectively. The two weak measurements can be written, respectively, as a nonunitary quantum operation [48]

$$M = \begin{bmatrix} 1 & 0 \\ 0 & m \end{bmatrix} \qquad N = \begin{bmatrix} n & 0 \\ 0 & 1 \end{bmatrix} \tag{2}$$

where m and n are the measurement strengths.

After these weak measurements being implemented, the state is,

$$\rho_G = \frac{N\varepsilon_{\text{channel}}(M\rho_{in}M^\dagger)N^\dagger}{Tr(N\varepsilon_{\text{channel}}(M\rho_{in}M^\dagger)N^\dagger)} \tag{3}$$

where $\varepsilon_{\text{channel}}$ is defined by Eq. (1).

The final density matrix of the game after the players execute their moves is given by

$$\rho_l = \frac{(U_{AB} \bigotimes U_C)\rho_G (U_{AB} \bigotimes U_C)^\dagger}{Tr((U_{AB} \bigotimes U_C)\rho_G (U_{AB} \bigotimes U_C)^\dagger)} \tag{4}$$

The operator $U_{AB} = \sqrt{q} I \bigotimes I + \sqrt{1-q}\sigma_x \bigotimes \sigma_x$, represents the strategy of the two cooperators and $U_C = \sqrt{r} I + \sqrt{1-r}\sigma_x$, is the strategy of the third player. The payoff function of the player is given by [16]

$$P_{A,B,C} = Tr(P'_{A,B,C}\rho_l) \tag{5}$$

where $P'_{A,B,C}$ is the payoff operator for players A, B or C, which is given by

$$P'_{A,B,C} = \Sigma_{i=1}^{8}(\alpha_i, \beta_i, \gamma_i) \times \rho_{ii}^{l} \tag{6}$$

with ρ_{ii}^{l} are the diagonal elements of the final density matrix ρ_l of the game. $\alpha_i's$, $\beta_i's$, and $\gamma_i's$ are the elements of the payoff matrix of the three-player game. According to the rules of the game, the values of the matrix elements are

$$\begin{aligned}
&\alpha_1 = \alpha_8 = 0, \alpha_2 = \alpha_3 = \alpha_6 = \alpha_7 = 1, \alpha_4 = \alpha_5 = -2\\
&\beta_1 = \beta_8 = 0, \beta_2 = \beta_4 = \beta_5 = \beta_7 = 1, \beta_3 = \beta_6 = -2\\
&\gamma_1 = \gamma_8 = 0, \gamma_3 = \gamma_4 = \gamma_5 = \gamma_6 = 1, \gamma_2 = \gamma_7 = -2
\end{aligned} \tag{7}$$

4 Improving payoff for the amplitude damping channel using weak measurements

A single qubit Kraus operators for amplitude damping channel is

$$E_0 = \begin{bmatrix} 1 & 0 \\ 0 & \sqrt{1-p} \end{bmatrix} \qquad E_1 = \begin{bmatrix} 0 & \sqrt{p} \\ 0 & 0 \end{bmatrix} \tag{8}$$

where p is the decoherence parameter.

If there is no weak measurement, according to Eq. (5), then the payoff function of cooperators A and B is given by

$$\begin{aligned}
P_{A,B}^{AD1} = [&q + r - 2\,q\,r + 4\,p^2\,q\,\sin\left(\frac{Q}{2}\right)^2 + 4\,p^2\,r\,\sin\left(\frac{Q}{2}\right)^2\\
&- 4\,p\,q\,\sin\left(\frac{Q}{2}\right)^2 - 4\,p\,r\,\sin\left(\frac{Q}{2}\right)^2\\
&+ 8\,p\,q\,r\,\sin\left(\frac{Q}{2}\right)^2 - 8\,p^2\,q\,r\,\sin\left(\frac{Q}{2}\right)^2\\
&- \frac{1}{2}]/[4\,\sqrt{q}\,\sqrt{r}\,\sin(Q)\,(1-p)^{\frac{3}{2}}\,\sqrt{1-q}\,\sqrt{1-r} + 1] + \frac{1}{2}
\end{aligned} \tag{9}$$

By using weak measurement, when the initial state is maximally entangled ($Q = \pi/2$), the payoff function of cooperators A and B is given by (see "Appendix 1").

$$P_{A,B}^{AD3} = \frac{(q+r-2qr)[m^6(pn^2+p-1)^2(pn^2-p+1)+n^6]+4m^3n^3\sqrt{qr(1-p)^3(1-q)(1-r)}}{m^6(pn^2-p+1)^3+n^6+8m^3n^3\sqrt{qr(1-p)^3(1-q)(1-r)}} \tag{10}$$

To gain the optimal payoff, we let

$$\frac{\partial P_{A,B}^{AD3}}{\partial q} = 0$$
$$\frac{\partial P_{A,B}^{AD3}}{\partial r} = 0 \tag{11}$$

From above equations, we get $q = r = 1/2$. Substituting them into Eq. (10), the maximum payoff of cooperators becomes

$$P_{A,B,\max}^{AD} = \frac{2m^6 n^2 p\,(p-1)\left(pn^2 - p + 1\right)}{m^6\left(pn^2 - p + 1\right)^3 + n^6 + 2m^3 n^3 (1-p)^{\frac{3}{2}}} + \frac{1}{2} \tag{12}$$

In the classical form of the game, the maximum values of payoffs that define the Nash equilibrium of the game is a fixed point. Whereas by using weak measurements, in the presence of decoherence, the Nash equilibrium under the action of amplitude damping channels is a function of decoherence parameter p, and measurement strengths m and n.

The payoff of player C is negative and twice the payoff function of a cooperator, that is,

$$P_C^{AD3} = -2P_{A,B}^{AD3}$$
$$P_{C,\max}^{AD} = -2P_{A,B,\max}^{AD} \tag{13}$$

Certainly, $P_A^{AD} + P_B^{AD} + P_C^{AD} = 0$, that is, our scheme is still a zero-sum game by using weak measurements.

In order to compare with the results in Ref. [36], we choose that the initial state is maximally entangled ($Q = \pi/2$) and the probability parameters $q = r = 0.2$. These parameters are the same as those in Ref. [36]. Thus, according to Eq. (10), the payoff of player $A(B)$ for the amplitude damping channel against the decoherence parameter p with $q = r = 0.2$, $n = 0.9$ for different m is shown in Fig. 2. From this figure, we can see that under the action of amplitude damping channel, the payoff of cooperators reaches to a minimum. By using numerical method to let $\frac{\partial P_{A,B}^{AD1}}{\partial p}|_{q=r=0.2,Q=\pi/2} = \frac{800p-400-4\sqrt{1-p}(32p^2-96p+91)}{[16(1-p)^{(3/2)}+25]^2} = 0$, we can get the minimum at $p = 0.6311$. Instead, the payoff can be improved greatly by using weak measurement. Moreover, when $n = 0.9$, with the decrease in m, the payoff becomes a fixed value gradually. When $m = 0.1$, the fixed value is 0.32 regardless of the decoherence parameter (see Fig. 2d). It seems that decoherence has no effect on the payoff of cooperators.

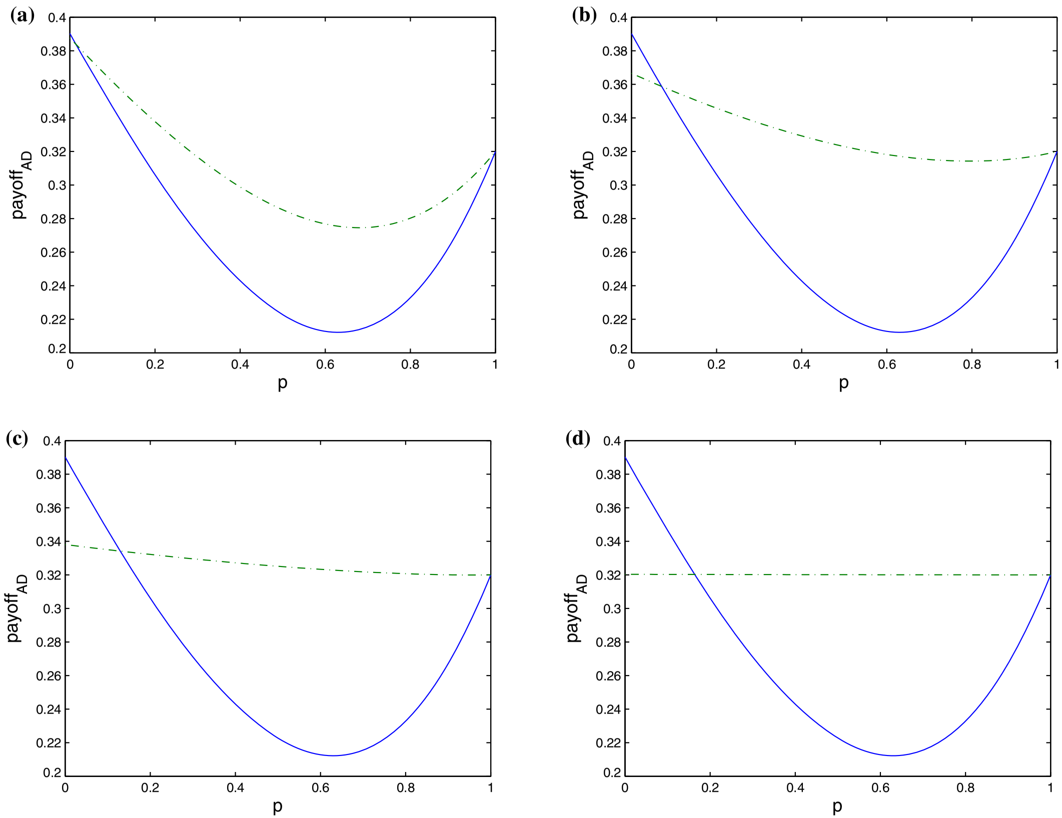

Fig. 2 The payoff of player $A(B)$ for the amplitude damping channel against the decoherence parameter p with $q = r = 0.2$, $Q = \pi/2$, $n = 0.9$. **a** $m = 0.8$; **b** $m = 0.6$; **c** $m = 0.4$; **d** $m = 0.1$. The *dashed-dotted line* and the *solid line* stand for using weak measurement and no weak measurement, respectively

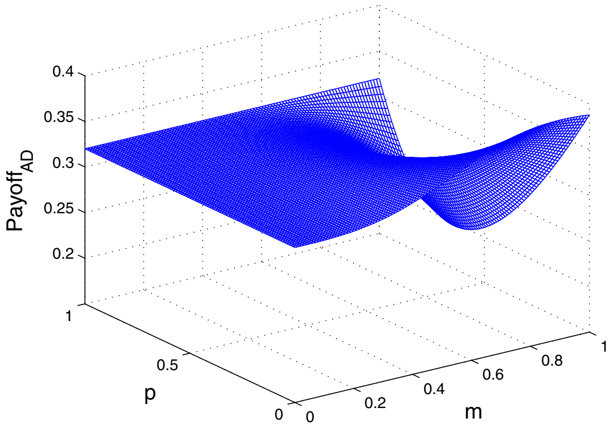

Fig. 3 The payoff of player $A(B)$ for the amplitude damping channel against the decoherence parameter p and m with $q = r = 0.2$, $Q = \pi/2$, $n = 0.9$

Figure 3 displays the dependence of payoff $P_{A(B)}$ on the decoherence parameter p and the measurement strength m with $q = r = 0.2$, $Q = \pi/2$, and $n = 0.9$. It shows that the cooperators' payoff tends to a constant value when m becomes small in spite of sufficiently strong decoherence. Moreover, it is interesting to observe that, $\lim_{m\to 0} P_{A,B}^{AD} = 0.32$. This result can be explained from Eq. (10). From the equation, we get $\lim_{m\to 0} P_{A,B}^{AD} = q + r - 2qr$. When $q = r = 0.2$, the limit is 0.32. If choose $q = 0.001$, $r = 0.999$, then $\lim_{m\to 0} P_{A,B}^{AD} = 0.998$, which means that by choosing the appropriate parameters, the payoff of a cooperator can be approximated to 1 under the action of weak measurement.

5 Improving payoff for the bit flip channel using weak measurements

A single qubit Kraus operators for the bit flip channel is

$$E_0 = \sqrt{1-p}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad E_1 = \sqrt{p}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \tag{14}$$

Without weak measurement, according to Eq. (5), the payoff function of cooperators A and B is given by

$$P_{A,B}^{Bf1} = \frac{(2\,p-1)^2}{2} + \frac{(2\,p-1)^2\;(q+r-2\,q\,r) - \frac{(2\,p-1)^2}{2}}{4\,\sqrt{q}\,\sqrt{r}\;\sin(Q)\;\sqrt{1-q}\,\sqrt{1-r}+1} \tag{15}$$

By using weak measurement, when the initial state is maximally entangled ($Q = \pi/2$), the payoff function of cooperators A and B $P_{A,B}^{Bf3}$ is given by (see "Appendix 2").

$$P_{A,B}^{Bf3} = \{(q+r-2qr)[m^6(pn^2+p-1)^2(pn^2-p+1)+(p-n^2p+n^2)(p+n^2p-n^2)^2] + 4m^3n^3(2p-1)^2\sqrt{qr(1-q)(1-r)}\}/\{m^6(pn^2-p+1)^3+(p-n^2p+n^2)^3 + 8m^3n^3\sqrt{qr(1-q)(1-r)}\} \tag{16}$$

The maximization of $P_{A,B}^{Bf3}$ with respect to q and r, leads to $q = r = 1/2$, and the maximum payoff of cooperators becomes

$$\begin{aligned} P_{A,B,\max}^{Bf} = & \left[2n^2 p (p-1) \left(p + 2m^3 n - m^6 p - n^2 p + m^6 + n^2 + m^6 n^2 p\right)\right] \\ & / \Big[n^2 \left(3m^6 p^3 - 6m^6 p^2 + 3m^6 p - 3p^3 + 3p^2\right) - 3m^6 p \\ & + n^4 \left(-3m^6 p^3 + 3m^6 p^2 + 3p^3 - 6p^2 + 3p\right) \\ & + n^6 \left(m^6 p^3 - p^3 + 3p^2 - 3p + 1\right) + m^6 + p^3 \\ & + 2m^3 n^3 + 3m^6 p^2 - m^6 p^3\Big] + \frac{1}{2} \end{aligned} \tag{17}$$

Unlike the equilibrium payoff in the classical form of the three players game, this payoff depends on decoherence parameter p, and measurement strengths m and n under the action of bit flip channels by using weak measurements.

As in the amplitude damping channel, the calculation results show that the payoff of player C is negative and twice the payoff function of a cooperator in the case of bit flip channel. So, our scheme is still a zero-sum game using weak measurements under the action of bit flip channel.

According to Eq. (16), the payoff of player $A(B)$ for the bit flip channel against the decoherence parameter p with $q = r = 0.2$, $m = 0.9$ for different n is shown in Fig. 4. From this figure, we can see that the game is no-payoff around the decoherence parameter $p = 0.5$ under the influence of bit flip channel. This result can be obtained directly from Eq. (15). Obviously, when $p = 0.5$, $P_{A,B}^{Bf1} = 0$. However, we are able to enhance payoff greatly, and to avoid the payoff sudden death by using weak measurement. Moreover, when $m = 0.9$, with the decrease in n, the payoff is improved to a great extent.

In Fig. 5, we plot the dependence of payoff of player $A(B)$ for the bit flip channel on the decoherence parameter p and weak measurement strength n with $q = r = 0.2$, $Q = \pi/2$, $m = 0.9$. We can observe that except $n = 1$, the payoff sudden death can be avoided for all strengths of decoherence by using weak measurement. The payoff of player $A(B)$ tends to a constant value when n becomes small in spite of sufficiently strong decoherence. Moreover, it is interesting to observe that, $\lim_{n\to 0} P_{A,B}^{Bf} = 0.32$. This result can be explained from Eq. (16). From the equation, we get $\lim_{n\to 0} P_{A,B}^{Bf} = q + r - 2qr$. When $q = r = 0.2$, we obtain $\lim_{n\to 0} P_{A,B}^{Bf} = 0.32$. This result is the same as that discussed in the case of amplitude damping channel.

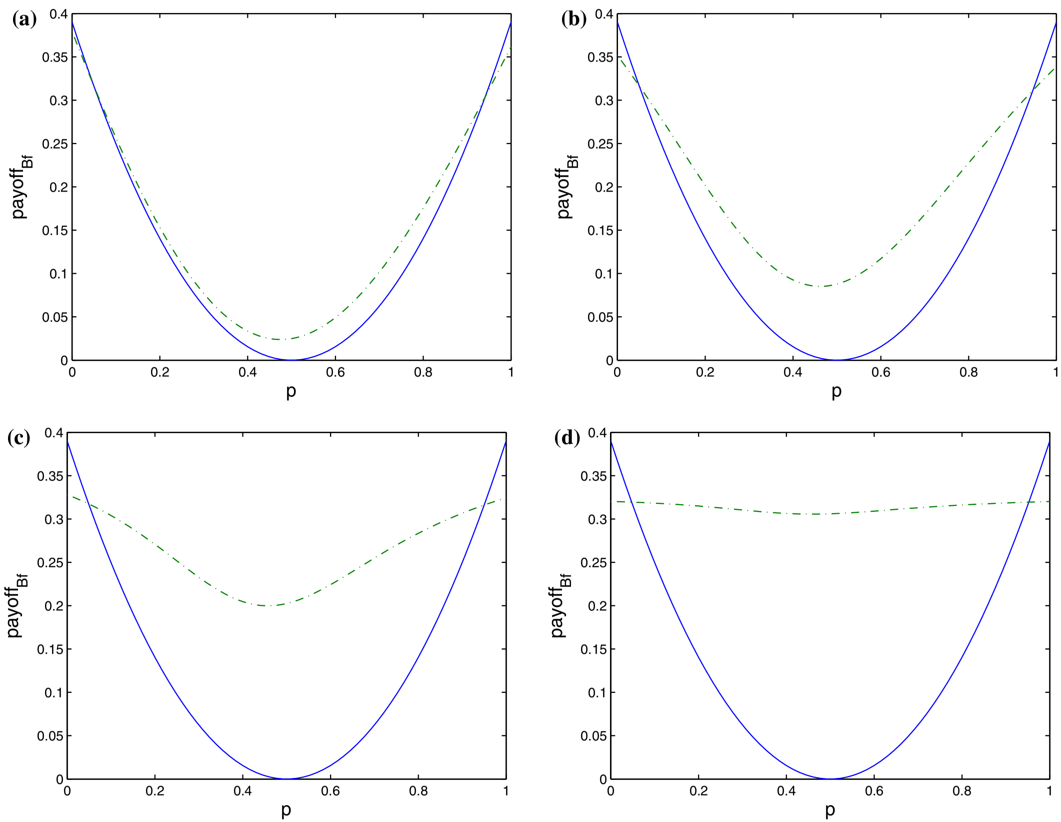

Fig. 4 The payoff of player $A(B)$ for the bit flip channel against the decoherence parameter p with $q = r = 0.2$, $Q = \pi/2$, $m = 0.9$. **a** $n = 0.7$; **b** $n = 0.5$; **c** $n = 0.3$; **d** $n = 0.1$. The *dashed-dotted line* and the *solid line* stand for using weak measurement and no weak measurement respectively

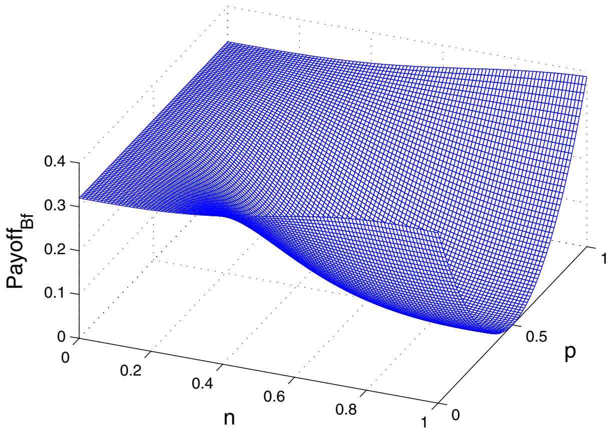

Fig. 5 The payoff of player $A(B)$ for the bit flip channel against the decoherence parameter p and n with $q = r = 0.2$, $Q = \pi/2$, $m = 0.9$

6 Improving payoff for the depolarizing channel using weak measurements

A single qubit Kraus operators for depolarizing channel is

$$E_0=\sqrt{1-p}\begin{bmatrix}1&0\\0&1\end{bmatrix}\quad E_1=\sqrt{\frac{p}{3}}\begin{bmatrix}0&1\\1&0\end{bmatrix}\quad E_2=\sqrt{\frac{p}{3}}\begin{bmatrix}0&-i\\i&0\end{bmatrix}\quad E_3=\sqrt{\frac{p}{3}}\begin{bmatrix}1&0\\0&-1\end{bmatrix} \tag{18}$$

Although the analytical solutions for the cooperators' payoffs are possible, the formulas are rather lengthy and we will not give them in this paper. Instead, in Fig. 6, we plot the payoff of player $A(B)$ for the depolarizing channel against the decoherence parameter p with $q = r = 0.2$, $Q = \pi/2$, $m = 0.9$ for different n. From this figure, we can see that the game becomes a no-payoff game around a decoherence of 75 % under the influence of depolarizing channel. But, we are able to enhance payoff to a great extent, and to circumvent payoff sudden death by using weak measurement. Moreover, when $m = 0.9$, as the decrease in n, the payoff is improved clearly.

In Fig.7, we plot the dependence of payoff of player $A(B)$ for the depolarizing channel on the decoherence parameter p and the measurement strength n with $q = r = 0.2$, $Q = \pi/2$, $m = 0.9$. We conclude that except $n = 1$, the payoff sudden death can be avoided for all values of the decoherence parameter by using weak measurement. Moreover, it is interesting to observe that $\lim_{n\to 0} P_{A,B}^{Dp} = 0.32$ in spite of sufficiently strong decoherence.

The results of calculation for the depolarizing channel show that the Nash equilibrium is a function of decoherence parameter p, and measurement strengths m and n. And, the payoff of player C is negative and twice the payoff function of a cooperator. This means that our scheme is still a zero-sum game using weak measurements in

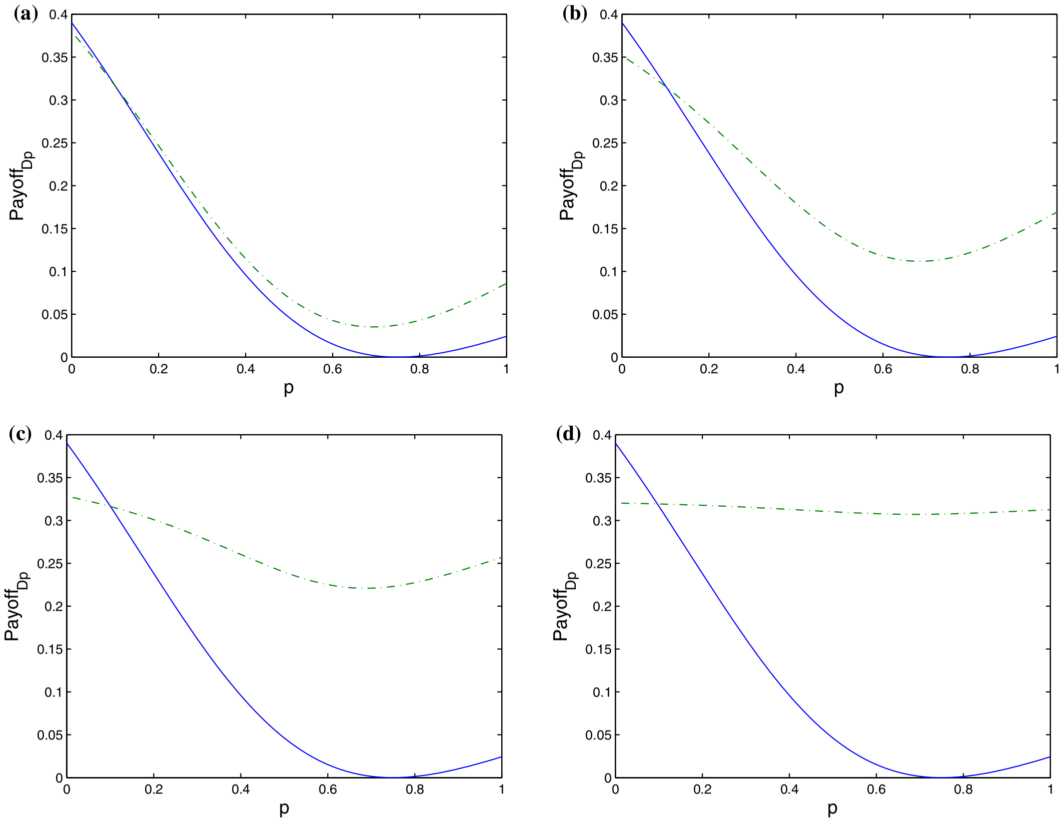

Fig. 6 The payoff of player $A(B)$ for the depolarizing channel against the decoherence parameter p with $q = r = 0.2$, $Q = \pi/2$, $m = 0.9$. **a** $n = 0.7$; **b** $n = 0.5$; **c** $n = 0.3$; **d** $n = 0.1$. The *dashed-dotted line* and the *solid line* stand for using weak measurement and no weak measurement respectively

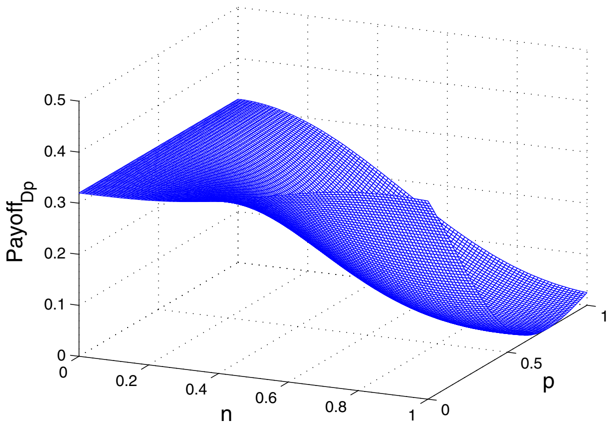

Fig. 7 The payoff of player $A(B)$ for the depolarizing channel against the decoherence parameter p and n with $q = r = 0.2$, $Q = \pi/2$, $m = 0.9$

the case of depolarizing channel. These results are the same as those in the amplitude damping and bit flip channel.

7 Conclusion and outlook

In this paper, we have proposed a method to improve the payoffs of cooperators in cooperative three-player quantum game under the action of amplitude damping, bit flip and depolarizing channels by using weak measurements. It is shown that the payoffs of cooperators can be enhanced greatly in the case of amplitude damping channel, and the payoff sudden death can be avoided in the case of bit flip and depolarizing channels. Moreover, we have found that the payoffs of cooperators tend to a constant by changing weak measurement strength irrespective of sufficiently strong decoherence.

Some experimental realizations of quantum games are described in Refs. [49–51]. Moreover, weak measurements have been experimentally realized in various physical systems, e.g., solid systems [52], superconducting phase qubits [53] and linear optic devices [54]. We hope that our proposal can be realized by using such or other systems. Although we have illustrated our protocol for quantum game with tripartite entangled qubits, generalization to quantum game with entangled qutrits and quantum game with more than three players seems straightforward. And, we believe that weak measurement scheme can be used to improve the payoff in some more general channels.

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Appendix 1

In this Appendix, we present the payoff function of cooperators A and B by using Eq. (5) for the amplitude damping channel using weak measurements as follows:

$$
\begin{aligned}
P_{A,B}^{AD2} = \Bigg[& n^6\, q \cos\left(\frac{Q}{2}\right)^2 + n^6\, r \cos\left(\frac{Q}{2}\right)^2 + m^6\, q \sin\left(\frac{Q}{2}\right)^2 + m^6\, r \sin\left(\frac{Q}{2}\right)^2 \\
& - 2\, n^6\, q\, r \cos\left(\frac{Q}{2}\right)^2 - 3\, m^6\, p\, q \sin\left(\frac{Q}{2}\right)^2 - 3\, m^6\, p\, r \sin\left(\frac{Q}{2}\right)^2 \\
& - 2\, m^6\, q\, r \sin\left(\frac{Q}{2}\right)^2 + 3\, m^6\, p^2\, q \sin\left(\frac{Q}{2}\right)^2 - m^6\, p^3\, q \sin\left(\frac{Q}{2}\right)^2 \\
& + 3\, m^6\, p^2\, r \sin\left(\frac{Q}{2}\right)^2 - m^6\, p^3\, r \sin\left(\frac{Q}{2}\right)^2 + 6\, m^6\, p\, q\, r \sin\left(\frac{Q}{2}\right)^2 \\
& - m^6\, n^2\, p\, q \sin\left(\frac{Q}{2}\right)^2 - m^6\, n^2\, p\, r \sin\left(\frac{Q}{2}\right)^2 - 6\, m^6\, p^2\, q\, r \sin\left(\frac{Q}{2}\right)^2 \\
& + 2\, m^6\, p^3\, q\, r \sin\left(\frac{Q}{2}\right)^2 + 2\, m^6\, n^2\, p^2\, q \sin\left(\frac{Q}{2}\right)^2 - m^6\, n^2\, p^3\, q \sin\left(\frac{Q}{2}\right)^2 \\
& - m^6\, n^4\, p^2\, q \sin\left(\frac{Q}{2}\right)^2 + m^6\, n^4\, p^3\, q \sin\left(\frac{Q}{2}\right)^2 + m^6\, n^6\, p^3\, q \sin\left(\frac{Q}{2}\right)^2 \\
& + 2\, m^6\, n^2\, p^2\, r \sin\left(\frac{Q}{2}\right)^2 - m^6\, n^2\, p^3\, r \sin\left(\frac{Q}{2}\right)^2 - m^6\, n^4\, p^2\, r \sin\left(\frac{Q}{2}\right)^2 \\
& + m^6\, n^4\, p^3\, r \sin\left(\frac{Q}{2}\right)^2 + m^6\, n^6\, p^3\, r \sin\left(\frac{Q}{2}\right)^2 - 4\, m^6\, n^2\, p^2\, q\, r \sin\left(\frac{Q}{2}\right)^2 \\
& + 2\, m^6\, n^2\, p^3\, q\, r \sin\left(\frac{Q}{2}\right)^2 + 2\, m^6\, n^4\, p^2\, q\, r \sin\left(\frac{Q}{2}\right)^2 \\
& - 2\, m^6\, n^4\, p^3\, q\, r \sin\left(\frac{Q}{2}\right)^2 - 2\, m^6\, n^6\, p^3\, q\, r \sin\left(\frac{Q}{2}\right)^2 \\
& + 2\, m^6\, n^2\, p\, q\, r \sin\left(\frac{Q}{2}\right)^2 + 4\, m^3\, n^3\, \sqrt{q}\, \sqrt{r}\, \cos\left(\frac{Q}{2}\right) \sin\left(\frac{Q}{2}\right) (1-p)^{\frac{3}{2}} \\
& \sqrt{1-q}\, \sqrt{1-r} [/] n^6 \cos\left(\frac{Q}{2}\right)^2 + m^6 \sin\left(\frac{Q}{2}\right)^2 - 3\, m^6\, p \sin\left(\frac{Q}{2}\right)^2 \\
& + 3\, m^6\, p^2 \sin\left(\frac{Q}{2}\right)^2 - m^6\, p^3 \sin\left(\frac{Q}{2}\right)^2 - 6\, m^6\, n^2\, p^2 \sin\left(\frac{Q}{2}\right)^2 \\
& + 3\, m^6\, n^2\, p^3 \sin\left(\frac{Q}{2}\right)^2 + 3\, m^6\, n^4\, p^2 \sin\left(\frac{Q}{2}\right)^2 - 3\, m^6\, n^4\, p^3 \sin\left(\frac{Q}{2}\right)^2 \\
& + m^6\, n^6\, p^3 \sin\left(\frac{Q}{2}\right)^2 + 3\, m^6\, n^2\, p \sin\left(\frac{Q}{2}\right)^2 + 8\, m^3\, n^3\, \sqrt{q}\, \sqrt{r}\, \cos\left(\frac{Q}{2}\right) \\
& \sin\left(\frac{Q}{2}\right) (1-p)^{\frac{3}{2}} \sqrt{1-q}\, \sqrt{1-r} \Bigg]
\end{aligned}
$$

When the initial state is maximally entangled, i.e., $Q = \pi/2$, we obtain the payoff function Eq. (10).

Appendix 2

In this Appendix, we present the payoff function of cooperators A and B by using Eq. (5) for the bit flip channel using weak measurements as follows:

$$
\begin{aligned}
P_{A,B}^{Bf2} = \Bigg[& n^6 q \cos\left(\frac{Q}{2}\right)^2 + n^6 r \cos\left(\frac{Q}{2}\right)^2 + p^3 q \cos\left(\frac{Q}{2}\right)^2 + p^3 r \cos\left(\frac{Q}{2}\right)^2 \\
& + m^6 q \sin\left(\frac{Q}{2}\right)^2 + 7m^6 r \sin\left(\frac{Q}{2}\right)^2 - n^4\, p\, q \cos\left(\frac{Q}{2}\right)^2 \\
& - 3\, n^6\, p\, q \cos\left(\frac{Q}{2}\right)^2 - n^4\, p\, r \cos\left(\frac{Q}{2}\right)^2 - 3\, n^6\, p\, r \cos\left(\frac{Q}{2}\right)^2 \\
& - 2\, n^6\, q\, r \cos\left(\frac{Q}{2}\right)^2 - 2\, p^3\, q\, r \cos\left(\frac{Q}{2}\right)^2 - 3\, m^6\, p\, q \sin\left(\frac{Q}{2}\right)^2 \\
& - 3\, m^6\, p\, r \sin\left(\frac{Q}{2}\right)^2 - 2\, m^6\, q\, r \sin\left(\frac{Q}{2}\right)^2 - n^2\, p^2\, q \cos\left(\frac{Q}{2}\right)^2 \\
& + n^2\, p^3\, q \cos\left(\frac{Q}{2}\right)^2 + 2\, n^4\, p^2\, q \cos\left(\frac{Q}{2}\right)^2 - n^4\, p^3\, q \cos\left(\frac{Q}{2}\right)^2 \\
& + 3\, n^6\, p^2\, q \cos\left(\frac{Q}{2}\right)^2 - n^6\, p^3\, q \cos\left(\frac{Q}{2}\right)^2 - n^2\, p^2\, r \cos\left(\frac{Q}{2}\right)^2 \\
& + n^2\, p^3\, r \cos\left(\frac{Q}{2}\right)^2 + 2\, n^4\, p^2\, r \cos\left(\frac{Q}{2}\right)^2 - n^4\, p^3\, r \cos\left(\frac{Q}{2}\right)^2 \\
& + 3\, n^6\, p^2\, r \cos\left(\frac{Q}{2}\right)^2 - n^6\, p^3\, r \cos\left(\frac{Q}{2}\right)^2 + 3\, m^6\, p^2\, q \sin\left(\frac{Q}{2}\right)^2 \\
& - m^6\, p^3\, q \sin\left(\frac{Q}{2}\right)^2 + 3\, m^6\, p^2\, r \sin\left(\frac{Q}{2}\right)^2 - m^6\, p^3\, r \sin\left(\frac{Q}{2}\right)^2 \\
& + 6\, m^6\, p\, q\, r \sin\left(\frac{Q}{2}\right)^2 + 2\, n^2\, p^2\, q\, r \cos\left(\frac{Q}{2}\right)^2 - 2\, n^2\, p^3\, q\, r \cos\left(\frac{Q}{2}\right)^2 \\
& - 4\, n^4\, p^2\, q\, r \cos\left(\frac{Q}{2}\right)^2 + 2\, n^4\, p^3\, q\, r \cos\left(\frac{Q}{2}\right)^2 - 6\, n^6\, p^2\, q\, r \cos\left(\frac{Q}{2}\right)^2 \\
& + 2\, n^6\, p^3\, q\, r \cos\left(\frac{Q}{2}\right)^2 - m^6\, n^2\, p\, q \sin\left(\frac{Q}{2}\right)^2 - m^6\, n^2\, p\, r \sin\left(\frac{Q}{2}\right)^2 \\
& - 6\, m^6\, p^2\, q\, r \sin\left(\frac{Q}{2}\right)^2 + 2\, m^6\, p^3\, q\, r \sin\left(\frac{Q}{2}\right)^2 + 2\, m^6\, n^2\, p^2\, q \sin\left(\frac{Q}{2}\right)^2 \\
& - m^6\, n^2\, p^3\, q \sin\left(\frac{Q}{2}\right)^2 - m^6\, n^4\, p^2\, q \sin\left(\frac{Q}{2}\right)^2 + m^6\, n^4\, p^3\, q \sin\left(\frac{Q}{2}\right)^2 \\
& + m^6\, n^6\, p^3\, q \sin\left(\frac{Q}{2}\right)^2 + 2\, m^6\, n^2\, p^2\, r \sin\left(\frac{Q}{2}\right)^2 - m^6\, n^2\, p^3\, r \sin\left(\frac{Q}{2}\right)^2
\end{aligned}
$$

$$
\begin{aligned}
&- m^6 n^4 p^2 r \sin\left(\frac{Q}{2}\right)^2 + m^6 n^4 p^3 r \sin\left(\frac{Q}{2}\right)^2 + m^6 n^6 p^3 r \sin\left(\frac{Q}{2}\right)^2 \\
&+ 2 n^4 p q r \cos\left(\frac{Q}{2}\right)^2 + 6 n^6 p q r \cos\left(\frac{Q}{2}\right)^2 - 4 m^6 n^2 p^2 q r \sin\left(\frac{Q}{2}\right)^2 \\
&+ 2 m^6 n^2 p^3 q r \sin\left(\frac{Q}{2}\right)^2 + 2 m^6 n^4 p^2 q r \sin\left(\frac{Q}{2}\right)^2 \\
&- 2 m^6 n^4 p^3 q r \sin\left(\frac{Q}{2}\right)^2 - 2 m^6 n^6 p^3 q r \sin\left(\frac{Q}{2}\right)^2 \\
&+ 2 m^6 n^2 p q r \sin\left(\frac{Q}{2}\right)^2 + 4 m^3 n^3 \sqrt{q}\sqrt{r} \cos\left(\frac{Q}{2}\right) \\
&\sin\left(\frac{Q}{2}\right) \sqrt{1-q}\sqrt{1-r} - 16 m^3 n^3 p \sqrt{q}\sqrt{r} \cos\left(\frac{Q}{2}\right) \\
&\sin\left(\frac{Q}{2}\right) \sqrt{1-q}\sqrt{1-r} + 16 m^3 n^3 p^2 \sqrt{q}\sqrt{r} \\
&\cos\left(\frac{Q}{2}\right) \sin\left(\frac{Q}{2}\right) \sqrt{1-q}\sqrt{1-r}[/] n^6 \cos\left(\frac{Q}{2}\right)^2 + p^3 \cos\left(\frac{Q}{2}\right)^2 \\
&+ m^6 \sin\left(\frac{Q}{2}\right)^2 + 3 n^4 p \cos\left(\frac{Q}{2}\right)^2 - 3 n^6 p \cos\left(\frac{Q}{2}\right)^2 - 3 m^6 p \sin\left(\frac{Q}{2}\right)^2 \\
&+ 3 n^2 p^2 \cos\left(\frac{Q}{2}\right)^2 - 3 n^2 p^3 \cos\left(\frac{Q}{2}\right)^2 - 6 n^4 p^2 \cos\left(\frac{Q}{2}\right)^2 + 3 n^4 p^3 \cos\left(\frac{Q}{2}\right)^2 \\
&+ 3 n^6 p^2 \cos\left(\frac{Q}{2}\right)^2 - n^6 p^3 \cos\left(\frac{Q}{2}\right)^2 + 3 m^6 p^2 \sin\left(\frac{Q}{2}\right)^2 - m^6 p^3 \sin\left(\frac{Q}{2}\right)^2 \\
&- 6 m^6 n^2 p^2 \sin\left(\frac{Q}{2}\right)^2 + 3 m^6 n^2 p^3 \sin\left(\frac{Q}{2}\right)^2 + 3 m^6 n^4 p^2 \sin\left(\frac{Q}{2}\right)^2 \\
&- 3 m^6 n^4 p^3 \sin\left(\frac{Q}{2}\right)^2 + m^6 n^6 p^3 \sin\left(\frac{Q}{2}\right)^2 + 3 m^6 n^2 p \sin\left(\frac{Q}{2}\right)^2 \\
&+ 8 m^3 n^3 \sqrt{q}\sqrt{r} \cos\left(\frac{Q}{2}\right) \sin\left(\frac{Q}{2}\right) \sqrt{1-q}\sqrt{1-r}\Big]
\end{aligned}
$$

When the initial state is maximally entangled, i.e., $Q = \pi/2$, we obtain the payoff function Eq. (16).

References

1. von-Neumann, J., Morgenstein, O.: The Theory of Games and Economic Behaviour. Princeton University Press, Princeton, NJ (1944)
2. Nash, J.: Equilibrium points in n-person games. Proc. Nat. Acad. Sci. **36**, 48 (1950)
3. Meyer, D.A.: Quantum strategies. Phys. Rev. Lett. **82**, 1052 (1999)
4. Eisert, J., Wilkens, M., Lewenstein, M.: Quantum games and quantum strategies. Phy. Rev. Lett. **83**, 3077 (1999)
5. Marinatto, L., Weber, T.: A quantum approach to static games of complete information. Phys. Lett. A **272**, 291 (2000)
6. Flitney, A.P., Abbott, D.: Quantum games with decoherence. J. Phys. A **38**, 449 (2005)

7. Cheon, T., Iqbal, A.: Bayesian Nash equilibria and Bell inequalities. J. Phys. Soc. Jpn. **77**, 024801 (2008)
8. Iqbal, A., Abbott, D.: Quantum matching pennies game. J. Phy. Soc. Jpn. **78**, 014803 (2009)
9. Iqbal, A., Cheon, T., Abbott, D.: Probabilistic analysis of three-player symmetric quantum games played using the Einstein–Podolsky–Rosen–Bohm setting. Phys. Lett. A **372**, 6564 (2008)
10. Iqbal, A., Toor, A.H.: Evolutionarily stable strategies in quantum games. Phys. Lett. A **280**, 249 (2001)
11. Flitney, A.P., Abbott, D.: Quantum version of the Monty Hall problem. Phys. Rev. A **65**, 062318 (2002)
12. Iqbal, A., Toor, A.H.: Quantum mechanics gives stability to Nash equilibrium. Phys. Rev. A **65**, 022036 (2002)
13. Eisert, J., Wilkens, M.: Quantum games. J. Mod. Opt. **47**, 2543 (2000)
14. Ramzan, M., Khan, M.K.: Noise effects in a three-player Prisoner's dilemma quantum game. J. Phys. A Math. Theor. **41**, 435302 (2008)
15. Flitney, A.P., Ng, J., Abbott, D.: Quantum Parrondo's games. Phys. A **314**, 35 (2002)
16. Iqbal, A., Toor, A.H.: Quantum cooperative games. Phys. Lett. A **293**, 103 (2002)
17. Johnson, N.F.: Playing a quantum game with a corrupted source. Phys. Rev. A **63**, 020302(R) (2001)
18. DAriano, G.M., Gill, R.D., Keyl, M., Kuemmerer, B., Maassen, H., Werner, R.F.: The quantum Monty Hall problem. Quant. Inf. Comp. **2**, 355 (2002)
19. Miszczak, J.A., Gawron, P., Puchala, Z.: Qubit flip game on a Heisenberg spin chain. arXiv:1108.0642 [quant-ph], (2011)
20. Puya, S., Hoshang, H.: Quantum solution to a three player Kolkata restaurant problem using entangled qutrits. arXiv:1111.1962 [quant-ph], (2011)
21. Chakrabarti, A.S., Chakrabarti, B.K., Chatterjee, A., Mitra, M.: The Kolkata paise restaurant problem and resource utilization. Phys. A **388**, 2420–2426 (2009)
22. Benjamin, S.C., Hayden, P.M.: Multiplayer quantum games. Phys. Rev. A **64**, 030301(R) (2001)
23. Chen, Q., Wang, Y.: N-player quantum minority game. Phys. Lett. A **A 327**, 98,102 (2004)
24. Flitney, A.P., Greentree, A.D.: Coalitions in the quantum Minority game: classical cheats and quantum bullies. Phys. Lett. A **362**, 132 (2007)
25. Schmid, C., Flitney, A.P.: Experimental implementation of a four-player quantum game. arXiv:0901.0063v1 [quant-ph], (2008)
26. Flitney, A.P., Hollenberg, L.C.L.: Multiplayer quantum minority game with decoherence. Quantum Inf. Comput. **7**, 111 (2007)
27. Flitney, A.P., Abbott, D.: Quantum games with decoherence. J. Phys. A Math. Gen. **38**, 449 (2005)
28. Khan, S., Ramzan, M., Khan, M.K.: Quantum Parrondos games under decoherence. Int. J. Theor. Phys **49**, 31 (2010)
29. Khan, S., Ramzan, M., Khan, M.K.: Quantum Monty Hall problem under decoherence. Commun. Theor. Phys. **54**, 47 (2010)
30. Gawron, P., Miszczak, J.A., Sladkowski, J.: Noise effects in quantum magic squares game. Int. J. Quant. Inf. **6**, 667 (2008)
31. Gawron, P.: Noisy quantum Monty Hall game. Fluct. Noise Lett. **9**, 9 (2010)
32. Chen, L.K., Ang, H., Kiang, D., Kwek, L.C., Lo, C.F.: Quantum prisoner dilemma under decoherence. Phys. Lett. A **316**, 317 (2003)
33. Zhu, X., Kuang, L.M.: The influence of entanglement and decoherence on the quantum Stackelberg duopoly game. J. Phys. A Math. Theor. **40**, 7729 (2007)
34. Ramzan, M., Nawaz, A., Toor, A.H., Khan, M.K.: The effect of quantum memory on quantum games. J. Phys. A Math. Theor. **41**, 055307 (2008)
35. Ramzan, M.: Three-player quantum Kolkata restaurant problem under decoherence. Quantum Inf. Process. **12**(1), 577–586 (2013)
36. Khan, S., Ramzan, M., Khan, M.K.: Decoherence effects on multiplayer cooperative quantum games. Commun. Theor. Phys. **56**, 228–234 (2011)
37. Gawron, P., Kurzyk, D., Pawela, L.: Decoherence effects in the quantum qubit flip game using Markovian approximation. Quantum. Inf. Process. **13**, 665–682 (2014)
38. Aharonov, Y., Albert, D.Z., Vaidman, L.: How the result of a measurement of a component of the spin of a spin-1/2 particle can turn out to be 100. Phys. Rev. Lett. **60**, 1351 (1988)
39. Aharonov, Y., Botero, A., Pospescu, S., Reznik, B., Tollaksen, J.: Revisiting Hardy's paradox: counterfactual statements, real measurements, entanglement and weak values. Phys. Lett. A **301**, 130 (2002)
40. Lundeen, J.S., Steinberg, A.M.: Experimental joint weak measurement on a photon pair as a probe of Hardys paradox. Phys. Rev. Lett. **102**, 020404 (2009)

41. Yokota, K., Yamamoto, T., Koashi, M., Imoto, N.: Direct observation of Hardy's paradox by joint weak measurement with an entangled photon pair. N. J. Phys. **11**, 033011 (2009)
42. Korotkov, A.N., Jordan, A.N.: Undoing a weak quantum measurement of a solid-state qubit. Phys. Rev. Lett. **97**, 166805 (2006)
43. Kim, Y.S., Cho, Y.W., Ra, Y.S., Kim, Y.H.: Reversing the weak quantum measurement for a photonic qubit. Opt. Express **17**, 11978 (2009)
44. Lee, J.C., Jeong, Y.C., Kim, Y.S., Kim, Y.H.: Experimental demonstration of decoherence suppression via quantum measurement reversal. Opt. Express **19**, 16309 (2011)
45. Kim, Y.S., Lee, J.C., Kwon, O., Kim, Y.H.: Protecting entanglement from decoherence using weak measurement and quantum measurement reversal. Nat. Phys. **8**, 117 (2012)
46. Man, Z.X., Xia, Y.J., An, N.B.: Enhancing entanglement of two qubits undergoing independent decoherences by local pre- and postmeasurements. Phys. Rev. A **86**, 052322 (2012)
47. Burger, E., Freund, J.E.: Introduction to the Theory of Games. Prentice-Hall, Englewood Cliffs (1963)
48. Wang, S.C., Yu, Z.W., Zou, W.J., Wang, X.B.: Protecting quantum states from decoherence of finite temperature using weak measurement. Phys. Rev. A **89**, 022318 (2014)
49. Du, J., Li, H., Xu, X., Shi, M., Wu, J., Zhou, X., Han, R.: Experimental realization of quantum games on a quantum computer. Phys. Rev. Lett. **88**, 137902 (2002)
50. Prevedel, R., Stefanov, A., Walther, P., Zeilinger, A.: Experimental realization of a quantum game on a one-way quantum computer. N. J. Phys. **9**, 205 (2007)
51. Kolenderski, P., Sinha, U., Youning, L., Zhao, T., Volpini, M., Cabello, A., Laflamme, R., Jennewein, T.: Aharon–Vaidman quantum game with a Young-type photonic qutrit. Phys. Rev. A **86**, 012321 (2012)
52. Korotkov, A.N., Jordan, A.N.: Undoing a weak quantum measurement of a solid-state qubit. Phys. Rev. Lett. **97**, 166805 (2006)
53. Katz, N., Neeley, M., Ansmann, M., Bialczak, R.C., Hofheinz, M., Lucero, E., OConnell, A., Wang, H., Cleland, A.N., Martinis, J.M., Korotkov, A.N.: Reversal of the weak measurement of a quantum state in a superconducting phase qubit. Phys. Rev. Lett. **101**, 200401 (2008)
54. Kim, Y.S., Cho, Y.W., Ra, Y.S., Kim, Y.H.: Reversing the weak quantum measurement for a photonic qubit. Opt. Express **17**, 11978 (2009)