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Maximal entanglement entanglement-assisted quantum codes constructed from linear codes

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Abstract An entanglement-assisted quantum error-correcting code (EAQECC) is a generalization of standard stabilizer quantum code. Maximal entanglement EAQECCs can achieve the EA-hashing bound asymptotically. In this work, the construction of quaternary zero radical codes is discussed, including the construction of low- dimensional quaternary codes for all code lengths and higher- dimensional quaternary codes for short lengths. Using the obtained quaternary codes, we construct many maximal entanglement EAQECCs with very good parameters. Some of these EAQECCs are optimal codes, and some of them are better than previously known ones. Combining these results with known bounds, we formulate a table of upper and lower bounds on the minimum distance of any maximal entanglement EAQECCs with length up to 20 channel qubits.

Keywords EAQECC · Maximal entanglement · EA-quantum Plotkin bound · Optimal code

1 Introduction

The entanglement-assisted (EA) stabilizer formalism was proposed by Brun et al. in [1], and it includes the standard stabilizer formalism [2,3] as a special case. It was shown in [1] that classical quaternary (and binary) linear codes which are not self-orthogonal can be transformed into EAQECCs, if shared entanglement is available between the sender and receiver.

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An $[[n, k, d; c]]$ EAQECC that encodes k information qubits into n channel qubits with the help of c pairs of maximally entangled Bell states (ebits) can correct up to $\lfloor \frac{d-1}{2} \rfloor$ errors, where d is the minimum distance of the code. If there is no $[[n, k, d+1; c]]$ for given n, k and c, then an $[[n, k, d; c]]$ EAQECC is *optimal*. In order to figure out the optimality of EAQECCs, some bounds on EAQECCs are invented, such as the EA-quantum Hamming bound for non-degenerate EAQECCs, the EA-quantum Singleton bound, the EA-quantum Plotkin bound and the EA-quantum linear programming bound [1,4–7]. The EA-quantum Plotkin bound reads: If there exits an $[[n, k, d; c]]$, then

$$d \leq \frac{3n \times 4^k}{4(4^k - 1)}.$$

Entanglement is a useful resource; it has been shown that entanglement can increase the rate and error-correcting ability of quantum codes [1,7], and maximal entanglement EAQECCs can have more information qubits and higher minimum distance at the cost of more ebits [7]. Maximal entanglement EAQECCs exploit the maximum amount of entanglement possible, although that much ebits could be an expensive resource in practice. However, known results found in the literature [8–12] and recent research [12–16] suggest that a higher rate and/or better noise suppression capabilities may be achieved by exploiting maximal entanglement. Refs. [8–11] have shown that maximal entanglement EAQECCs can achieve the EA-quantum capacity of a depolarizing channel, which establishes a limit on the performance of EAQECCs, and [12] has shown maximal entanglement EA turbo codes come close to the EA-hashing bound within a few decibels. Refs. [7,13] have shown that some maximal entanglement codes $[[n, k, d; c]]$ are not equivalent to any standard quantum codes $[[n + c, k, d]]$ and have better performance than all $[[n + c, k, d]]$. Even if a maximal entanglement code $[[n, k, d; c]]$ is equivalent to an $[[n + c, k, d]]$ stabilizer code, it may still have better performance than $[[n+c, k, d]]$ stabilizer codes [13]. Refs. [12–14,16] indicate that ebits may be robust against noise when the ebits are not noiseless; in such case, maximal entanglement EAQECCs can also be used to correct errors efficiently on some channels. Thus, it is worthwhile to study maximal entanglement EAQECCs.

It has been shown that maximal entanglement EAQECCs can achieve the EA-hashing bound asymptotically [4,5]. In [4,5], Lai et al. proved that EA repetition codes with parameters $[[n, 1, n; n-1]]$ are optimal for n odd, and EA repetition codes with parameters $[[n, 1, n-1; n-1]]$ are optimal for n even. They also construct many good maximal entanglement EAQECCs with length $n \leq 15$ in [4,5,7] and establish a table of upper and lower bounds on the highest achievable minimum distance of any maximal entanglement EAQECCs for $n \leq 15$. In [17], some maximal entanglement EAQECCs of short length are constructed from caps in projective space and several of these codes improve parameters of the codes given in [4,5].

In this paper, we study constructions of maximal entanglement EAQECCs from quaternary linear codes and manage to improve parameters of the codes in [4,5,7]. We give our discussion in two aspects. The first one is on the construction of quaternary zero radical codes of dimension $k \leq 4$ and related maximal entanglement EAQECCs, and the second one is on the construction of higher-dimensional quaternary zero radical codes of length $n \leq 20$ and related maximal entanglement EAQECCs.

This paper is organized as follows. In Sect. 2, basic concepts on linear codes over the quaternary field $\mathbf{F}_4$ and EAQECCs are reviewed. In Sect. 3, constructions of quaternary zero radical codes of dimension $k \leq 3$ and related maximal entanglement EAQECCs are presented. Section 4 discuss construction of four-dimensional quaternary zero radical codes and EAQECCs. Explicit constructions of higher-dimensional quaternary codes of short lengths and maximal entanglement EAQECCs of distance $d \geq 5$ are given in Sect. 5. Finally, in Sect. 6, optimality of the obtained EAQECCs is discussed and a table of upper and lower bounds on the minimum distance of any maximal entanglement EAQECCs for $n \leq 20$ is presented.

2 Preliminary

In this section, we recall some basic concepts on linear codes over the quaternary field $\mathbf{F}_4$ and EAQECCs and make some preparation for later use.

Let $\mathbf{F}_4 = \{0, 1, \omega, \varpi\}$ be the Galois field with four elements such that $\varpi = 1+\omega = \omega^2, \omega^3 = 1$, and the conjugation is defined by $\bar{x} = x^2$. Let $\mathbf{F}_4^n$ be the n-dimensional space over $\mathbf{F}_4$, an m-dimensional subspace $\mathcal{C}$ of $\mathbf{F}_4^n$ is called an m-dimensional linear code of length n, and is denoted as $\mathcal{C} = [n, m]_4$. If the Hamming distance of $\mathcal{C}$ is d, then it is denoted as $\mathcal{C} = [n, m, d]_4$.

The Hermitian inner product of $\mathbf{u}, \mathbf{v} \in \mathbf{F}_4^n$ is defined to be

$$(\mathbf{u}, \mathbf{v}) = \mathbf{u}\mathbf{v}^{\dagger} = u_1\bar{v_1} + u_2\bar{v_2} + \cdots + u_n\bar{v_n}.$$

The Hermitian dual code of $\mathcal{C} = [n, m]_4$ is $\mathcal{C}^{\perp h} = \{x \mid (x, y)_h = 0, \forall y \in \mathcal{C}\}$, and $\mathcal{C}^{\perp h} = [n, n-m]_4$. A matrix G whose rows form a basis of $\mathcal{C}$ is called a generator matrix of $\mathcal{C}$, and a generator matrix H of $\mathcal{C}^{\perp h}$ is called a parity check matrix of $\mathcal{C}$.

It was shown that for a linear code $\mathcal{C} = [n, m, d]_4$ with parity check matrix H, then $\mathcal{C}^{\perp h}$ EA stabilizes an $[[n, 2m+c-n, d; c]]$ EAQECC, where $c = \text{rank}(HH^{\dagger})$ and $H^{\dagger}$ is the conjugate transpose of H, see [1,18]. Let $k = n-m$. If $c = k$, then the EAQECC has parameters $[[n, n-k, d; k]]$ and it is a maximal entanglement EAQECC. Thus, we can deduce the following lemma from Corollary 2 of [18].

Lemma 2.1 *If $\mathcal{C} = [n, n-k, d]_4$ is a linear code with parity check matrix $H = H_{k\times n}$, then $\mathcal{C}^{\perp h}$ EA stabilizes an $[[n, n-k, d; k]]$ maximal entanglement EAQECC if and only if $k = \text{rank}(HH^{\dagger})$.*

According to [19], a linear code $\mathcal{C} = [n, m, d]_4$ is a subspace of the unitary space $\mathbf{F}_4^n$. Let G and H be generator and check matrices of $\mathcal{C}$, respectively. From $k = \text{rank}(HH^{\dagger}) = n-m$, one can deduce $m = \text{rank}(GG^{\dagger}) = n-k$. In such case, $\mathcal{C}$ and $\mathcal{C}^{\perp h}$ are called *totally non-isotropic* subspaces of $\mathbf{F}_4^n$ in finite geometry, the radical code $R(\mathcal{C})$ of $\mathcal{C}$ and $\mathcal{C}^{\perp h}$ is $R(\mathcal{C}) = \mathcal{C} \cap \mathcal{C}^{\perp h} = \{0\}$; hence, $\mathcal{C}$ (or $\mathcal{C}^{\perp h}$) is also called *zero radical code*. Thus, Lemma 2.1 can be restated as

Lemma 2.1' *If $\mathcal{C} = [n, m = n-k, d]_4$ is a linear code with generator matrix $G = G_{m\times n}$, then $\mathcal{C}^{\perp h}$ EA stabilizes an $[[n, n-k, d; k]]$ maximal entanglement EAQECC if and only if $\mathcal{C}$ is a zero radical code, i.e., $\text{rank}(GG^{\dagger}) = m$.*

Remark 2.1 According to page 4022 of [4] (in the paragraph after Definition 1) or Theorem 2.3 of [6], a maximal entanglement EAQECC must be non-degenerate. If one gives the EA stabilizer group S of a maximal entanglement EAQECC, then S_I must be a trivial group. Hence, a maximal entanglement EAQECC derived from a zero radical quaternary code has the same minimum distance with the underlying classical code. However, a nonzero radical code usually gives an EAQECC with minimum distance different from that of the underlying classical code [6].

The EA-quantum Plotkin bound has the same form as the Plotkin bound for quaternary linear codes [20]; hence, from Lemma 2.1', we have

Corollary 2.2 *If $\mathcal{C} = [n, m = n-k, d]_4$ is a zero radical code saturating the classical Plotkin bound for quaternary linear code, then $\mathcal{C}^{\perp h}$ EA stabilizes an $[[n, m, d; k]]$ maximal entanglement EAQECC saturating the EA-quantum Plotkin bound.*

In the following sections, we will manage to construct zero radical code $\mathcal{C} = [n, m, d]_4$ with large d. To do this, we make some notations for later use.

Let $\mathbf{1_n} = (1, 1, \ldots, 1)_{1\times n}$ and $\mathbf{0_n} = (0, 0, \ldots, 0)_{1\times n}$ to denote the all-one vector and the all-zero vector of length n, respectively. Construct

$$S_2 = \begin{pmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & \omega & \varpi \end{pmatrix} = (\alpha_1, \ldots, \alpha_5),$$

$$S_3 = \begin{pmatrix} S_2 & \mathbf{0}_{2\times 1} & S_2 & S_2 & S_2 \\ \mathbf{0}_5 & 1 & \mathbf{1}_5 & \omega\mathbf{1}_5 & \varpi\mathbf{1}_5 \end{pmatrix} = (\beta_1, \beta_2, \ldots, \beta_{21}),$$

$$S_4 = \begin{pmatrix} S_3 & \mathbf{0}_{3\times 1} & S_3 & S_3 & S_3 \\ \mathbf{0}_{21} & 1 & \mathbf{1}_{21} & \omega\mathbf{1}_{21} & \varpi\mathbf{1}_{21} \end{pmatrix} = (\gamma_1, \gamma_2, \ldots, \gamma_{85}).$$

It is well known that the matrix S_2 generates the $[5, 2, 4]_4$ Simplex code with weight polynomial $1 + 15y^4$, S_3 generates the $[21, 3, 16]_4$ Simplex code with weight polynomial $1 + 63y^{16}$, S_4 generates the $[85, 4, 64]_4$ Simplex code with weight polynomial $1 + 255y^{16}$, and $S_k S_k^{\dagger} = 0$ for $k = 2, 3, 4$, see [20].

Notation 2.1 In the following sections, in each generator matrix of linear codes, we use *2 and 3 to represent ω and ϖ*, respectively. For a matrix P, the conjugate transpose of P is denoted as $P^{\dagger}$, and the juxtaposition $(P, P, \ldots, P)$ of s-copies of P is denoted as sP. An $[n, m, d]_4$ code is denoted as $[n, m, d]$ for short, and the Hermitian dual code of a linear code is called dual code for short.

Remark 2.2 We use a computer to check the parameters of classical codes and the rank of $GG^{\dagger}$ presented in Sects. 3–5. The zero radical codes presented in Sect. 3 have the largest possible minimum distance for given n and k. However, while some zero radical codes given in Sects. 4 and 5 are also optimal, not all of them attain known upper bounds on the minimum distance of a linear code. Nonetheless, their minimum distances appear very good in general. These codes are the best possible among those obtainable by our approach.

3 Construction of $[[n, k, d; n-k]]$ EAQECCs for $k \leq 3$

There are many works, which discuss the existence, construction and classification of quaternary linear codes, see [20–25]. However, little attention was paid on $R(\mathcal{C}) = \mathcal{C} \cap \mathcal{C}^{\perp h}$ for a given optimal quaternary linear code $\mathcal{C}$. From [22,23], we can deduce many known optimal codes are not zero radical codes, such as Simplex codes and McDonald codes. Thus, to construct maximal entanglement EAQECCs from quaternary linear codes, one needs to construct zero radical codes with good parameters. We will discuss such a problem in the following sections. In this section, we focus on construction of $[[n, k, d; n-k]]$ EAQECCs for $k \leq 3$ from zero radical codes.

It is well known that $G = \mathbf{1_n} = (1, 1, \ldots, 1)$ generates the $[n, 1, n]$ optimal code. If n is odd, then this $[n, 1, n]$ code is a zero radical code and it gives an $[[n, 1, n; n-1]]$ maximal entanglement EAQECC. While n is even, then the $[n, 1, n]$ code is not a zero radical code and it cannot give an $[[n, 1, n; n-1]]$ maximal entanglement EAQECC. When n is even, $G' = (1, 1, \ldots, 1, 0)$ generates an $[n, 1, n-1]$ near optimal code, which is a zero radical code. Using this $[n, 1, n-1]$ near optimal linear code, one can deduce an $[[n, 1, n-1; n-1]]$ maximal entanglement EAQECC. This EAQECC is also optimal, see [5].

In the following, we discuss the construction of maximal entanglement EAQECCs from two- and three-dimensional zero radical codes in two cases.

Case 1. Two-dimensional zero radical codes and EAQECCs

The parameters of optimal linear codes of dimension 2 are known [25]. Table 1 lists the optimal parameters.

The optimal $[5s, 2, 4s]$ and $[5s+4, 2, 4s+3]$ codes are not zero radical codes according to [25]. Let $G = G_{2,n}$ be a generator matrix of an optimal $[n, 2]$ code with $n = 5s$ or $n = 5s+4$, then $\text{rank}(GG^{\dagger}) \leq 1$. Hence, the parameters of zero radical $[n, 2]$ codes for $n = 5s$ and $n = 5s+4$ may be $[5s, 2, 4s-1]$ and $[5s+4, 2, 4s+2]$, respectively. Parameters of good zero radical $[n, 2, d]$ codes are listed as following Table 2.

Lemma 3.1 *If $n = 5s + t \geq 2$, then there is an $[n, 2, d]$ zero radical code with parameters as given in Table 2.*

Proof (1) For $n = 5s, 5s+1$ and $s \geq 1$, construct $G_{2,5} = (\alpha_1, \alpha_2, \alpha_3, 2\alpha_4)$, $G_{2,5s} = (G_{2,5}|(s-1)S_2)$, $G_{2,5s+1} = (2\alpha_1, 2\alpha_2, \alpha_3, \alpha_4|(s-1)S_2)$.

Table 1 Parameters of optimal $[n, 2]$ linear codes

n	5s	5s+1	5s+2	5s+3	5s+4
d	4s	4s	4s+1	4s+2	4s+3

Table 2 Parameters of zero radical $[n, 2]$ codes

n	5s	5s+1	5s+2	5s+3	5s+4
d	4s-1	4s	4s+1	4s+2	4s+2

Table 3 Parameters of zero radical $[n, 3]$ codes for length $3 \leq n \leq 21$

n	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21
d	1	1	2	3	4	5	6	6	7	8	9	9	10	11	12	13	13	14	15

Table 4 Parameters of zero radical $[n, 3]$ codes for length $n \geq 22$

n	$21s+1$	$21s+2$	$21s+3$	$21s+4$	$21s+5$	$21s+6$	$21s+7$	$21s+8$	$21s+9$	$21s+10$
d	$16s-1$	$16s$	$16s+1$	$16s+2$	$16s+2$	$16s+3$	$16s+4$	$16s+5$	$16s+6$	$16s+6$
n	$21s+11$	$21s+12$	$21s+13$	$21s+14$	$21s+15$	$21s+16$	$21s+17$	$21s+18$	$21s+19$	$21s+20$
d	$16s+7$	$16s+8$	$16s+9$	$16s+9$	$16s+10$	$16s+11$	$16s+12$	$16s+13$	$16s+13$	$16s+14$

(2) For $n = 5s+2, 5s+3, 5s+4$ and $s \geq 0$, construct $G_{2,5s+2} = (\alpha_1, \alpha_2|sS_2)$, $G_{2,5s+3} = (\alpha_1, \alpha_2, \alpha_3|sS_2)$, $G_{2,5s+4} = (2\alpha_1, \alpha_2, \alpha_3|sS_2)$.

□

It is easy to check that the codes with generator matrices $G_{2,2} = (\alpha_1, \alpha_2)$, $G_{2,3} = (\alpha_1, \alpha_2, \alpha_3)$, $G_{2,4} = (2\alpha_1, \alpha_2, \alpha_3)$, $G_{2,5} = (\alpha_1, \alpha_2, \alpha_3, 2\alpha_4)$, $G_{2,6} = (2\alpha_1, 2\alpha_2, \alpha_3, \alpha_4$ have parameters [2, 2, 1], [3, 2, 2], [4, 2, 2], [5, 2, 3] and [6, 2, 4], respectively, and they are all zero radical codes. From S_2 generates the [5, 2, 4] Simplex code and $S_2 S_2^\dagger = 0$, one can derive: In the above two cases (1) and (2), the codes with generator matrices $G_{2,n}$ have the desired parameters as Table 2, and $\text{rank}(G_{2,n} G_{2,n}^\dagger) = 2$. Hence, the lemma follows.

Using these zero radical $[n, 2]$ codes given in Lemma 3.1, one can deduce

Corollary 3.2 *There are maximal entanglement EAQECCs with the following parameters:* $[[3, 2, 2; 1]]$, $[[4, 2, 2; 2]]$, $[[5s, 2, 4s-1; 5s-2]]$, $[[5s+1, 2, 4s; 5s-1]]$, $[[5s+2, 2, 4s+1; 5s]]$, $[[5s+3, 2, 4s+2; 5s+1]]$ *and* $[[5s+4, 2, 4s+2; 5s+2]]$ *for* $s \geq 1$. *The* $[[3, 2, 2; 1]]$, $[[5s+1, 2, 4s; 5s-1]]$, $[[5s+2, 2, 4s+1; 5s]]$ *and* $[[5s+3, 2, 4s+2; 5s+1]]$ *codes saturate the EA Plotkin bound,* $[[4, 2, 2; 2]]$, $[[5s, 2, 4s-1; 5s-2]]$ *and* $[[5s+4, 2, 4s+2; 5s+2]]$ *codes have distance one less than the EA Plotkin bound.*

Case 2. Three-dimensional zero radical codes and EAQECCs

In this case, we only discuss construction of three-dimensional zero radical codes and EAQECCs. We use Table 3 and Table 4 to give zero radical $[n, 3]$ codes with good parameters. For $n = 21s$ and $s \geq 1$, there is a zero radical $[n, 3]$ code with parameters $[21s, 3, 16s-1]$.

Lemma 3.3 *(1) If* $3 \leq n \leq 21$, *then there is a zero radical* $[n, 3, d]$ *code as Table 3.*
(2) If $n = 21s + t \geq 22$, *then there is an* $[n, 3, d]$ *zero radical code with parameters as given in Table 4.*

Proof (1) If A is a sub-matrix of B, which is formed by columns of B, delete the columns of A from B, the resulting matrix is denoted as $B \setminus A$. Let

$$G_{3,3}=\begin{pmatrix}100\\010\\001\end{pmatrix},\ G_{3,4}=\begin{pmatrix}1100\\0110\\0001\end{pmatrix},\ G_{3,5}=\begin{pmatrix}01101\\10012\\00011\end{pmatrix},\ G_{3,6}=\begin{pmatrix}111110\\013231\\000113\end{pmatrix},$$

$$B_{3,6}=\begin{pmatrix}111110\\001211\\000000\end{pmatrix},\ G_{3,7}=\begin{pmatrix}1110111\\1231123\\2223333\end{pmatrix},\ G_{3,8}=\begin{pmatrix}01110111\\11231013\\22223333\end{pmatrix},$$

$$G_{3,9}=\begin{pmatrix}110110111\\131231123\\112223333\end{pmatrix},\ G_{3,10}=\begin{pmatrix}1101110111\\1311231123\\1122223333\end{pmatrix},\ G_{3,11}=\begin{pmatrix}11111101111\\33012310123\\01222233333\end{pmatrix},$$

$$G_{3,12}=\begin{pmatrix}111111101111\\123012310123\\111222233333\end{pmatrix},\ A_{3,15}=(\beta_7,\beta_8,\ldots,\beta_{21}),\ A_{3,16}=(\beta_6,\beta_7,\ldots,\beta_{21}).$$

Construct

$$G_{3,13}=(S_3\setminus(\beta_1,\ldots,\beta_7,\beta_8)),\ G_{3,14}=(S_3\setminus G_{3,7}),\ G_{3,15}=(S_3\setminus G_{3,6}),$$
$$G_{3,16}=(S_3\setminus(\beta_1,\beta_2,\beta_3,\beta_6,\beta_{10}),\ G_{3,17}=(S_3\setminus G_{3,4}),\ G_{3,18}=(S_3\setminus G_{3,3}),$$
$$G_{3,19}=(G_{3,3}\mid A_{3,16}),\ G_{3,20}=(\beta_3,\ldots,\beta_6\mid\beta_6,\ldots,\beta_{21}),\ G_{3,21}=(A_{3,15}\mid B_{3,6}).$$

Using a computer, it is not difficult to check that $\mathrm{rank}(G_{3,n}G_{3,n}^{\dagger})=3$ for $3\le n\le 21$, and the codes $\mathcal{C}_n$ with generator matrices $G_{3,n}$ have weight polynomials $W_n(z)$ as follows:

$$W_3(z)=1+9z+27z^2+27z^3,$$
$$W_4(z)=1+3z+9z^2+33z^3+18z^4,$$
$$W_5(z)=1+9z^2+15z^3+18z^4+21z^5,$$
$$W_6(z)=1+9z^3+18z^4+27z^5+9z^6,$$
$$W_7(z)=1+12z^4+27z^5+15z^6+9z^7,$$
$$W_8(z)=1+21z^5+21z^6+15z^7+6z^8,$$
$$W_9(z)=1+27z^6+27z^7+9z^9,$$
$$W_{10}(z)=1+3z^6+33z^7+18z^8+3z^9+6z^{10},$$
$$W_{11}(z)=1+15z^7+18z^8+21z^9+9z^{10},$$
$$W_{12}(z)=1+15z^8+33z^9+9z^{10}+3z^{11}+3z^{12},$$
$$W_{13}(z)=1+24z^9+27z^{10}+9z^{11}+3z^{13},$$
$$W_{14}(z)=1+9z^9+15z^{10}+27z^{11}+12z^{12},$$
$$W_{15}(z)=1+9z^{10}+27z^{11}+18z^{12}+9z^{13},$$
$$W_{16}(z)=1+15z^{11}+24z^{12}+21z^{13}+3z^{14},$$
$$W_{17}(z)=1+18z^{12}+33z^{13}+9z^{14}+3z^{15},$$
$$W_{18}(z)=1+27z^{13}+27z^{14}+9z^{15},$$

$$W_{19}(z) = 1 + 6z^{13} + 27z^{14} + 27z^{15} + 3z^{17},$$
$$W_{20}(z) = 1 + 9z^{14} + 33z^{15} + 18z^{16} + 3z^{17},$$
$$W_{21}(z) = 1 + 21z^{15} + 24z^{16} + 15z^{17} + 3z^{18}.$$

Summarizing the previous discussion, hence (1) holds.
(2) For $n = 21s + t \geq 22$. Construct $G_{3,21s} = (G_{3,21} \mid (s-1)S_3)$, $G_{3,21s+1} = (G_{3,6} \mid A_{3,16} \mid (s-1)S_3)$, $G_{3,21s+2} = (G_{3,7} \mid A_{3,16} \mid (s-1)S_3)$, $G_{3,21s+3} = (G_{3,3} \mid sS_3)$, $G_{3,21s+4} = (G_{3,9} \mid A_{3,16} \mid (s-1)S_3)$ and $G_{3,21s+t} = (G_{3,t} \mid sS_3)$ for $5 \leq t \leq 20$.

From $A_{3,16}A_{3,16}^{\dagger} = 0$, $S_3S_3^{\dagger} = 0$ and the discussion of (1), one can deduce (2) holds. □

Using the zero radical codes given in Lemma 3.3, one can derive

Corollary 3.4 *(1) If $s \geq 0, n = 21s + t \geq 5$, then they are the following maximal entanglement EAQECCs:*

$$[[21s+5, 3, 16s+2; 21s+2]], [[21s+6, 3, 16s+3; 21s+3]],$$
$$[[21s+7, 3, 16s+4; 21s+4]], [[21s+8, 3, 16s+5; 21s+5]],$$
$$[[21s+9, 3, 16s+6; 21s+6]], [[21s+10, 3, 16s+6; 21s+7]],$$
$$[[21s+11, 3, 16s+7; 21s+8]], [[21s+12, 3, 16s+8; 21s+9]],$$
$$[[21s+13, 3, 16s+9; 21s+10]], [[21s+14, 3, 16s+9; 21s+11]],$$
$$[[21s+15, 3, 16s+10; 21s+12]], [[21s+16, 3, 16s+11; 21s+13]],$$
$$[[21s+17, 3, 16s+12; 21s+14]], [[21s+18, 3, 16s+13; 21s+15]],$$
$$[[21s+19, 3, 16s+13; 21s+16]], [[21s+20, 3, 16s+14; 21s+17]].$$

(2) If $s \geq 1$, then they are the following maximal entanglement EAQECCs:

$$[[21s, 3, 16s-1; 21s-3]], [[21s+1, 3, 16s-1; 21s-2]],$$
$$[[21s+2, 3, 16s; 21s-1]], [[21s+3, 3, 16s+1; 21s]],$$
$$[[21s+4, 3, 16s+2; 21s+1]].$$

The $[[21s+9, 3, 16s+6; 21s+6]]$, $[[21s+13, 3, 16s+9; 21s+10]]$, $[[21s+17, 3, 16s+12; 21s+14]]$ *and* $[[21s+18, 3, 16s+13; 21s+15]]$ *codes saturate the EA Plotkin bound, the other EAQECCs have distances one less than the EA Plotkin bound.*

4 Construction of $[[n, 4, d; n-4]]$ EAQECC

In this section, we discuss construction of $[[n, 4, d; n-4]]$ EAQECC from four-dimensional zero radical codes. Parameters of zero radical $[n, 4, d]$ codes for $4 \leq n \leq 88$ are given in the following following Table 5.

Table 5 Parameters of zero radical $[n, 4, d]$ codes for length $4 \le n \le 88$

n	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21
d	1	2	2	3	4	5	6	6	7	8	8	9	10	11	11	12	13	14
n	22	23	24	25	26	27	28	29	30	31	32	33	34	35	36	37	38	39
d	14	15	16	17	18	18	19	20	20	21	22	23	23	24	25	26	26	27
n	40	41	42	43	44	45	46	47	48	49	50	51	52	53	54	55	56	57
d	28	29	29	30	31	32	33	33	34	35	36	36	37	38	39	39	40	41
n	58	59	60	61	62	63	64	65	66	67	68	69	70	71	72	73	74	75
d	42	43	44	44	45	46	46	47	48	49	49	50	51	52	53	53	54	55
n	76	77	78	79	80	81	82	83	84	85	86	87	88					
d	56	56	57	58	59	60	60	61	62	62	63	64	64					

Lemma 4.1 *If* $4 \le n \le 88$*, then there is an* $[n, 4, d]$ *zero radical code as in Table 5.*

Proof Ref. [17] proved the lemma holds for $4 \le n \le 13$. Now we prove it also holds for $14 \le n \le 88$. We will prove the lemma holds in two cases.

Case 1. Construction of zero radical $[n, 4, d]$ code with $14 \le n \le 81$ and $n \ne 46, 48, 63, 73$.

In this case, we give 11 special zero radical codes at first, then puncture these codes and give 53 new zero radical codes. We construct 11 matrices at first. Let

$$G_{4,17} = \begin{pmatrix} 00111011110110101 \\ 11020122201130311 \\ 30112221200001122 \\ 01111112233333333 \end{pmatrix}, \; G_{4,26} = \begin{pmatrix} 11111011011111000011011011 \\ 02122101103313100101121003 \\ 00223220130233310031011101 \\ 00033332032022331011223011 \end{pmatrix},$$

$$G_{4,33} = \begin{pmatrix} 001111111111111111111111111111111 \\ 000000001111111222222233333330203 \\ 100122331003322322110023300110320 \\ 010123123020113201231300323123100 \end{pmatrix},$$

$$G_{4,37} = \begin{pmatrix} 1011110111011011111111111110011110111 \\ 0002301123023130102122020301113121230 \\ 0111123333000112233000112233000112223 \\ 0000000000111111111222222222333333333 \end{pmatrix},$$

$$G_{4,45} = \begin{pmatrix} 111111111111111111111111111111111111111111110 \\ 111111111122222222222233333333333330000000001 \\ 122333000011222333000011122233300001112223330 \\ 323012012312013032012302302103101231231231230 \end{pmatrix},$$

$$G_{4,54} = \begin{pmatrix} 010111101111011110110101111011111110111010111111110111 \\ 001123011230112301230011230112301231123001123012301123 \\ 011111222222333300001111112222233330000111111222233333 \\ 111111111111111122222222222222222223333333333333333333 \end{pmatrix}.$$

Delete the columns with index set $\{1, 2, \ldots, 23, 24, 28\}$ from S_4 and denote the resulting matrix as $G_{4,60}$. Delete the columns with index set $\{1, 2, \ldots, 8, 9, 13, 22, 23, 31, 38, 44, 53, 57, 75, 79\}$ from S_4 and denote the resulting matrix as $G_{4,66}$. Delete the columns with index set $\{1, 2, \ldots, 9, 22, 23, 28, 49, 70\}$ from S_4 and denote the resulting matrix as $G_{4,71}$. Delete the columns with index set $\{1, 2, \ldots, 7, 22, 30\}$ from S_4 and denote the resulting matrix as $G_{4,76}$. Delete the columns with index set $\{1, 2, 6, 22\}$ from S_4 and denote the resulting matrix as $G_{4,81}$.

It is not difficult to check that the above 11 matrices generate zero radical codes by a computer, and these 11 codes have parameters as following: $\mathcal{C}_{17} = [17, 4, 11], \mathcal{C}_{26} = [26, 4, 18], \mathcal{C}_{33} = [33, 4, 23], \mathcal{C}_{37} = [37, 4, 26], \mathcal{C}_{45} = [45, 4, 32], \mathcal{C}_{54} = [54, 4, 39], \mathcal{C}_{60} = [60, 4, 44], \mathcal{C}_{66} = [66, 4, 48], \mathcal{C}_{71} = [71, 4, 52], \mathcal{C}_{76} = [76, 4, 56], \mathcal{C}_{81} = [81, 4, 60]$. Puncturing on these 11 codes, we can construct 53 new zero radical codes with parameters as in Table 5. We just give the results of puncturing, for details of the puncturing process please see the "Appendix".

Puncturing $\mathcal{C}_{17}$ on suitable coordinates, one can obtain desired zero radical codes with length $14 \leq n \leq 16$. Puncturing $\mathcal{C}_{26}$ on suitable coordinates, one can obtain desired zero radical codes with length $18 \leq n \leq 25$. Puncturing $\mathcal{C}_{33}$ on suitable coordinates, one can obtain desired zero radical codes with length $27 \leq n \leq 32$. Puncturing $\mathcal{C}_{37}$ on suitable coordinates, one can obtain desired zero radical codes with length $34 \leq n \leq 36$. Puncturing $\mathcal{C}_{45}$ on suitable coordinates, one can obtain desired zero radical codes with length $38 \leq n \leq 44$. Puncturing $\mathcal{C}_{54}$ on suitable coordinates, one can obtain desired zero radical codes with length $47 \leq n \leq 53$ and $n \neq 48$. Puncturing $\mathcal{C}_{60}$ on suitable coordinates, one can obtain desired zero radical codes with length $55 \leq n \leq 59$. Puncturing $\mathcal{C}_{66}$ on suitable coordinates, one can obtain desired zero radical codes with length $61 \leq n \leq 65$ and $n \neq 63$. Puncturing $\mathcal{C}_{71}$ on suitable coordinates, one can obtain desired zero radical codes with length $67 \leq n \leq 70$. Puncturing $\mathcal{C}_{76}$ on suitable coordinates, one can obtain desired zero radical codes with length $73 \leq n \leq 75$. Puncturing $\mathcal{C}_{81}$ on suitable coordinates, one can obtain desired zero radical codes with length $77 \leq n \leq 80$.

Through the above process, we have constructed 64 zero radical codes, leave 11 zero radical codes undetermined.

Case 2. On 11 zero radical codes with lengths $n \in \{46, 48, 63, 72, 82, \ldots, 88\}$.

In this case, we give 11 zero radical codes that are not covered in Case 1. Let

$$G_{4,46} = \begin{pmatrix} 1111111111111110101101101101101101111101011011 \\ 0123231231231231212311312312311311212313112123 \\ 0000112223330001122233300011122233300011222333 \\ 0000000000001111111111122222222222233333333333 \end{pmatrix},$$

$$G_{4,48} = \begin{pmatrix} 111010111010111011111110101101111111011010111111 \\ 023011120110130123030121203012301230113001121232 \\ 000112223330001111223330011222233330000111112223 \\ 000000000001111111111112222222222223333333333333 \end{pmatrix}.$$

Delete the columns with index set $\{1, 3, \ldots, 23, 28\}$ from S_4 and add the $24th$ column of S_4, denote the resulting matrix as $G_{4,63}$. Delete the columns with index set

$\{1, 2, \ldots, 8, 10, 13, 14, 18, 22\}$ from S_4 and denote the resulting matrix as $G_{4,72}$. Delete the columns with index set $\{1, 2, 3, 6\}$ from S_4 and add the $22th$ column of S_4, denote the resulting matrix as $G_{4,82}$. Delete the columns with index set $\{1, 2, 6\}$ from S_4 and add the $22th$ column of S_4, denote the resulting matrix as $G_{4,83}$. Delete the columns with index set $\{3, 4, 8, 22\}$ from S_4 and add the columns of S_4 with index set $\{6, 9, 10\}$, denote the resulting matrix as $G_{4,84}$.

Let $A_{4,3} = (\gamma_6, \gamma_{13}, \gamma_{35})$, $A_{4,4} = (\gamma_1, \gamma_6, \gamma_{13}, \gamma_{35})$, $B_{4,20} = (\gamma_2, \ldots, \gamma_{21})$, $B_{4,21} = (\gamma_1, \gamma_2, \ldots, \gamma_{21})$, $C_{4,19} = (\gamma_2, \gamma_4, \ldots, \gamma_{21},)$, $C_{4,20} = (\gamma_1, \ldots, \gamma_6, \gamma_8, \ldots, \gamma_{21})$ and $D_{4,63} = (\gamma_{23}, \gamma_{24}, \ldots, \gamma_{85})$. Construct $G_{4,85} = (A_{4,3} \mid C_{4,19} \mid D_{4,63})$, $G_{4,86} = (A_{4,3} \mid B_{4,20} \mid D_{4,63})$, $G_{4,87} = (A_{4,4} \mid C_{4,20} \mid D_{4,63})$ and $G_{4,88} = (A_{4,4} \mid B_{4,21} \mid D_{4,63})$.

It is not difficult to check that the above 11 matrices $G_{4,n}$ generate zero radical codes and these 11 codes have parameters as following: $\mathcal{C}_{46} = [46, 4, 33]$, $\mathcal{C}_{48} = [48, 4, 34]$, $\mathcal{C}_{63} = [63, 4, 46]$, $\mathcal{C}_{72} = [72, 4, 53]$, $\mathcal{C}_{82} = [82, 4, 60]$, $\mathcal{C}_{83} = [83, 4, 61]$, $\mathcal{C}_{84} = [84, 4, 62]$, $\mathcal{C}_{85} = [85, 4, 62]$, $\mathcal{C}_{86} = [86, 4, 63]$, $\mathcal{C}_{87} = [87, 4, 64]$ and $\mathcal{C}_{88} = [88, 4, 64]$. □

Summarizing the above discussion, the lemma follows.

Using the zero radical codes given in Lemma 4.1 and the [85, 4, 64] Simplex code, one can deduce

Corollary 4.2 *(1) If* $s \geq 0, n_1 = 85s + n \geq 4$ *and* $n \leq 84$*, then there is an* $[[n_1, 4, 64s + d; n_1 - 4]]$ *maximal entanglement EAQECC, where* n, d *are given in Table 5*

(2) If $s \geq 1, n = 85s + t$ *and* $t \leq 3$*, then they are the following maximal entanglement EAQECCs:* $[[85s, 4, 64 - 2; 85s - 4]]$, $[[85s + 1, 4, 64s - 1; 85s - 3]]$, $[[85s + 2, 4, 64s; 85s - 2]]$, $[[85s + 3, 4, 64s; 85s - 1]]$.

Remark 4.1 According to [24], for $n = 6, \ldots, 13, 19, 20, 21, 24, 25, 26, 32, 33, 35, 36, 40, 41, 45, 46, 50, 51, \ldots, 54, 56, \ldots, 60, 66, 71, 72, 76, 81$, the $[n, 4, d]$ zero radical codes given in Lemma 4.1 are also optimal linear codes. Except these mentioned above codes, the zero radical codes given in this section do not attain known upper bounds on the minimum distance of a linear code. Nonetheless, their minimum distances appear very good in general. These codes are the best possible among those obtainable by our approach. Furthermore, it is an interesting question that whether zero radical codes which do not attain known upper bounds on the minimum distance of a linear code have better parameters than these ones given in this section.

5 Construction of short length $[[n, k, d; n - k]]$ EAQECC with $k \geq 5$

In this section, we discuss construction of $[n, k]$ zero radical codes for $k \geq 5$ and $n \geq 11$ from known codes in [21,23,26] and construct $[[n, k, d; n - k]]$ EAQECCs for $n \leq 20$. The discussion is presented in four cases.

Case 1. Construction of $[n, 5]$ zero radical codes

Let

$$G_{5,11} = \begin{pmatrix} 13302210000 \\ 01330221000 \\ 00133022100 \\ 00013302210 \\ 00001330221 \end{pmatrix}.$$

According to [21], $G_{5,11}$ generates a $[11, 5, 6]$ cyclic zero radical code and its dual is a $[11, 6, 5]$ cyclic zero radical code. Extending this $[11, 5, 6]$ code by $a = (1, 2, 1, 0, 0)^T$, one obtain a $[12, 5, 6]$ zero radical code. Extending this $[12, 5, 6]$ code by $b = (1, 1, 1, 2, 1)^T$, one obtain a $[13, 5, 7]$ zero radical code.

In [23], an optimal $\mathcal{C} = [24, 5, 16]$ with generator matrix $G_{5,24}$ is given and this code is not zero radical code, where

$$G_{5,24} = \begin{pmatrix} 001111100011110011111011 \\ 010001211100221101133103 \\ 120023302323011213313232 \\ 230322011111333221210103 \\ 330222201331203213321320 \end{pmatrix}.$$

Puncturing $\mathcal{C}$ on coordinate sets $\{1, 2, 3, 4, 8\}$, $\{1, 2, 3, 4, 8, 10\}$, $\{1, 2, 3, 4, 5, 8, 9\}$, $\{1, 2, 3, 4, 5, 8, 16, 22\}$, $\{1, 2, 3, 4, 5, 6, 8, 9, 19\}$ and $\{1, 2, 3, 4, 5, 6, 8, 9, 10, 16\}$, one can obtain $[19, 5, 11]$, $[18, 5, 10]$, $[17, 5, 9]$, $[16, 5, 9]$, $[15, 5, 8]$ and $[14, 5, 7]$ zero radical codes, respectively. In [17], a $[20, 5, 12]$ zero radical code is constructed from projective cap. Thus, for each n with $11 \leq n \leq 20$, we have constructed an $[n, 5]$ zero radical code.

Case 2. Construction of $[n, 6]$ zero radical codes and their dual codes

A constacyclic code $\mathcal{C} = [21, 15, 5]$ with generator polynomial $x^6 + \overline{\omega}x^5 + x^4 + \overline{\omega}x^2 + x + \overline{\omega}$ is given in [21]. The dual code of $\mathcal{C}$ is a code $\mathcal{D} = [21, 6, 12]$, and both of $\mathcal{C}$ and $\mathcal{D}$ are zero radical codes. This zero radical code $\mathcal{D}$ has a generator matrix $G_{6,21}$, where

$$G_{6,21} = \begin{pmatrix} 211210221102122100000 \\ 303201310313021010000 \\ 221130310133220001000 \\ 022113031013322000100 \\ 320131031302101000010 \\ 223203322032332000001 \end{pmatrix}.$$

Puncturing $\mathcal{D}$ on coordinate sets $\{1\}$, $\{1,8\}$, $\{1,2,4\}$, $\{1,2,4,5\}$, $\{1,2,3,11,14\}$, $\{1,2,3,4,6,7\}$, $\{1,2,3,4,5,6,7\}$, $\{1,2,3,4,5,6,7,11\}$ and $\{1,2,3,4,5,6,7,8,11\}$, one can obtain $[20, 6, 11]$, $[19,6,10]$, $[18,6,9]$, $[17,6,8]$, $[16,6,8]$, $[15,6,7]$, $[14,6,6]$, $[13,6,5]$ and $[12,6,5]$ zero radical codes, respectively. The dual codes of these codes have parameters $[n, n-6, 5]$. Thus, for $11 \leq n \leq 20$, we have shown that there are an $[n, 6]$ zero radical code and an $[n, n-6, 5]$ zero radical code.

Case 3. Construction of some zero radical codes with distance six.

In [26], a parity check matrix of a [36, 27, 6] code is given; from this parity check matrix, we can deduce two submatrices $G_{7,20}$ and $G_{8,27}$ as follows:

$$G_{7,20}=\begin{pmatrix} 10000001220130330 1231 \\ 01000001212300313 2021 \\ 00100001123310132 2121 \\ 00010011021201031302 \\ 00001010112200113302 \\ 00000101111100000013 \\ 00000000000011111111 \end{pmatrix}, G_{8,27}=\begin{pmatrix} 1000000122013033012310331202 \\ 0100000121230031320210213110 \\ 0010000112331013221210211202 \\ 000100110212010313020102112 \\ 0000101011220011330200000201 \\ 0000010111110000000130022131 \\ 000000000000111111110001223 \\ 000000000000000000001111111 \end{pmatrix}.$$

Let the codes $\mathcal{C}_1$ and $\mathcal{C}_2$ be generated by $G_{7,20}$ and $G_{8,27}$, respectively. These two codes are not zero radical codes; the dual codes of $\mathcal{C}_1$ and $\mathcal{C}_2$ have parameters [20, 13, 6] and [27, 19, 6], respectively.

Puncturing $\mathcal{C}_1$ on coordinate sets {1,2,3}, {1,2,3,6}, {1,2,3,4,6}, {1,2,3,4,5,6}, {1,2,3,4,5,6,7}, we can obtain $[n, 7]$ zero radical codes for $13 \leq n \leq 17$. From these $[n, 7]$ zero radical codes, one can obtain $[n, n-7, 6]$ zero radical codes for $13 \leq n \leq 17$. Puncturing $\mathcal{C}_2$ on coordinate sets {1,2,3,4,5,6,7}, {1,2,3,4,5,6,7,8}, {1,2,3,4,5,6,7, 8,10}, result in $[n, 8]$ zero radical codes for $18 \leq n \leq 20$. From these $[n, 8]$ zero radical codes, one can obtain $[n, n-8, 6]$ zero radical codes for $18 \leq n \leq 20$. Hence, for each n with $13 \leq n \leq 20$, we have constructed a zero radical code of length n and distance 6.

Case 4. Construction of some zero radical codes with $d \geq 7$ or $k \geq 7$.

A cyclic code [19, 9, 8] with generator polynomial $x^{10}+\varpi x^9+\omega x^8+\omega x^7+x^5+\varpi x^3+\varpi x^2+\omega x+1$ is given in [21], this [19, 9, 8] code is a zero radical code and its dual code is a [19, 10, 7] code. Puncturing this [19, 9, 8] code on coordinates set {1, 2}, one can obtain a [17, 9, 6] code and its dual is a [17, 8, 7] code, these two codes are zero radical codes. Puncturing this [19, 9, 8] code on coordinates set {1, 2, 4}, one can obtain a [16, 9, 5] code and its dual is a [16, 7, 7] code, these two codes are zero radical codes.

Next, consider the [19, 10, 7] cyclic code given in [21], whose generator polynomial is $x^9+\omega x^8+\omega x^6+\omega x^5+\varpi x^4+\varpi x^3+\varpi x+1$. Puncturing this [19, 10, 7] code on the first coordinate, one can get a [18, 10, 6] zero radical code and its dual code is a [18, 8, 8] zero radical code.

A [21, 9, 9] cyclic code given in [21] has generator polynomial $x^{12}+\varpi x^{11}+x^{10}+\omega x^8+x^7+\varpi x^5+x^4+\varpi x^2+\omega$, the dual code of this code is a [21, 12, 7] code, but both of them are not zero radical codes. Puncturing this [21, 9, 9] code on coordinates set {1, 2, 3}, one can obtain a [18, 9, 6] code and its dual code is a [18, 9, 7] code, both of these two codes are zero radical codes.

Construct

$$G_{7,17} = \begin{pmatrix} 10101111110000000 \\ 31013012001000001 \\ 11302012100100000 \\ 12322102200010000 \\ 33311013200001000 \\ 32000121100000100 \\ 13320311000000010 \end{pmatrix}, \quad B = (b_1, b_2, b_3) = \begin{pmatrix} 011 \\ 002 \\ 101 \\ 000 \\ 100 \\ 101 \\ 001 \end{pmatrix}.$$

$G_{7,17}$ generates a zero radical code $\mathcal{C} = [17, 7, 7]$. Extending $\mathcal{C}$ by b_1 gives a $[18, 7, 8]$ zero radical code. Extending $\mathcal{C}$ by b_1 and b_2 gives a $[19, 7, 8]$ zero radical code. Extending $\mathcal{C}$ by b_1, b_2 and b_3, gives a $[20, 7, 9]$ zero radical code.

Hence, we have shown that there are [16,7,7], [17,7,7], [17,8,7], [17,9,6], [18,7,8], [18,8,8], [18,9,7], [18,10,6], [19,7,8], [19,9,8], [19,10,7] and [20,7,9] zero radical codes.

Summarizing the above four cases, we have

Theorem 5.1 *They are the following maximal entanglement EAQECCs:*

(1) [[11,5,6;6]], [[12,5,6;7]], [[13,5,7;8]], [[14,5,7;9]], [[15,5,8;10]], [[16,5,9;11]], [[17,5,9;12]], [[18,5,10;13]], [[19,5,11;14]], [[20,5,12;15]].
(2) [[12,6,5;6]], [[13,6,6;7]], [[14,6,6;8]], [[15,6,7;9]], [[16,6,8;10]], [[17,6,8;11]], [[18,6,9;12]], [[19,6,10;13]], [[20,6,11;14]], [[21,6,12;15]] *and* $[[n, n-6, 5; 6]]$ *for* $12 \leq n \leq 21$.
(3) $[[n, n-7, 6; 7]]$ *for* $13 \leq n \leq 17$, *and* $[[n, n-8, 6; 8]]$ *for* $18 \leq n \leq 20$.
(4) [[16,7,7;9]], [[17,7,7;10]], [[18,7,8;11]], [[19,7,8;12]], [[20,7,9;13]]; [[17,8,7;9]], [[18,8,8;10]], [[19,8,8;11]], [[20,8,8;12]]; [[17,9,6;8]], [[18,9,7;9]], [[19,9,8;10]], [[20,9,8;11]]; [[18,10,6;8]], [[19,10,7;9]].

Remark 5.1 When puncturing a given code, we tried all possible coordinates and chose one case that result in zero radical code with highest minimum distance as output. According to [24], the following zero radical codes we give in this section are also optimal linear codes: $[n, 5, d]$ for $11 \leq n \leq 20$ and $n \neq 14, 17$; $[n, 6, d]$ for $n = 13, 16, 19, 20, 21$; $[n, n-6, 5]$ for $13 \leq n \leq 21$; $[n, n-7, 6]$ for $13 \leq n \leq 20$; [18, 8, 8], [19, 9, 8], [18, 10, 6] and [19, 10, 7]. Except these mentioned above codes, the zero radical codes given in this section do not attain known upper bounds on the minimum distance of a linear code. Nonetheless, their minimum distances appear very good in general. These codes are the best possible among those obtainable by our approach.

6 Discussion and conclusion

In this paper, we studied construction of quaternary zero radical codes and maximal entanglement EAQECCs. For each $k \leq 4$ and $n \geq k$, we constructed an $[n, k]$ zero radical code and we also constructed some $[n, k, d]$ zero radical codes for $n \leq 20$, $k \geq 5$ and $d \geq 5$. Based on these results on classical codes, we constructed many maximal entanglement EAQECCs with good parameters. Some of these EAQECCs are optimal codes and saturate the EA Plotkin bound, and some of them have parameters better than known ones in the literature.

In [17], we have constructed [[12,4,7;8]] and [[13,4,8;9]] codes. Our [[9,3,6;6]], [[12,3,8;9]], [[15,3,10;12]], [[12,4,7;8]], [[13,4,8;9]], [[14,4,8;10]], [[15,4,9;11]], [[13,5,7;8]] and [[14,5,7;9]]; which improve on the codes [[9,3,5;6]], [[12,3,7;9]], [[15,3,9;12]], [[12,4,6;8]], [[13,4,7;9]], [[14,4,7;10]], [[15,4,8;11]], [[13,5,6;8]] and [[14,5,6;9]] given in [4,5], respectively.

In [27], it has been proved that an additive quaternary [15, 5, 9] code does not exist; hence, an [[15, 5, 8; 10]] code is optimal. Thus, optimal maximal entanglement EAQECCs of length $n \leq 15$ can be determined except the six codes [[12, 6; 6]], [[14, 5; 9]], [[14, 6; 8]], [[15, 4; 10]], [[15, 6; 9]] and [[15, 7; 8]]. In [28], a method for constructing maximal entanglement EAQECCs from binary linear codes is presented and many higher-dimensional maximal entanglement EAQECCs are constructed from optimal binary codes. We have checked all maximal entanglement EAQECCs with $n \leq 20$ and $d \geq 3$ that constructed from optimal binary codes and find that the resulting EAQECCs cannot be better than our EAQECCs. Combining the results of [4,5,7,17] and the results in Sects. 3–5 with known bounds, we formulate a table (Table 6) of upper and lower bounds on the minimum distance of any maximal entanglement EAQECCs with length up to 20 channel qubits.

Table 6 Lower and upper bounds on the minimum distance of maximal entanglement EAQECCs

$n \setminus k$	1	2	3	4	5	6	7	8	9
3	3	2							
4	3	2	1						
5	5	3	2	2					
6	5	4	3	2	1				
7	7	5	4	3	2	2			
8	7	6	5	4	3	2	1		
9	9	6	**6**	5	4	3	2	2	
10	9	7	6	6	5	4	3	2	1
11	11	8	7	6	6	5	4	3	2
12	11	9	**8**	**7**	6	5–6	5	4	3
13	13	10	9	**8**	**7**	6	5	4	4
14	13	10	9	**8**	**7–8**	**6–7**	6	5	4
15	15	11	**10**	**9–10**	**8**	**7–8**	**6–7**	6	5
16	15	12	**11**	**10–11**	**9**	**8**	**7–8**	6–7	6
17	17	13	**12**	**11–12**	**9–10**	**8–9**	**7–8**	7–8	6–7
18	17	14	13	**11–12**	**10–11**	**9–10**	**8–9**	8–9	7–8
19	19	14	13–14	**12–13**	**11**	**10–11**	**8–9**	**8–9**	**8**
20	19	15	14–15	**13**	**12**	**11–12**	**9–10**	**8–10**	8–9

Table 6 continued

$n \setminus k$	10	11	12	13	14	15	16	17	18	19
11	2									
12	2	1								
13	3	2	2							
14	4	3	2	1						
15	4	4	3	2	2					
16	5	4	4	3	2	1				
17	6	5	4	3–4	3	2	2			
18	6–7	5–6	5	4	3	3	2	1		
19	7	6–7	5–6	5	4	3	3	2	2	
20	7–8	6–7	6–7	5–6	5	4	3	2	2	1

The bold face entries represent improvements over the prior works

In Table 6, many lower bounds for distance of EAQECCs with length $n \geq 16$ are from known constructed codes. To make the bounds in Table 6 tighter, we need to choose other zero radical codes better than that given in Sect. 5 and consider other code constructions (such as that of [7]) to raise the lower bounds. We also plan to explore the construction of EAQECCs from additive quaternary codes to decrease the upper bound.

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Appendix: Puncturing process of codes given in the proof of Lemma 4.1

(1) Puncturing the $\mathcal{C}_{17} = [17, 4, 11]$ code on coordinates sets $\{7\}$, $\{1, 2\}$, $\{1, 2, 6\}$, one can obtain [16, 4, 10], [15, 4, 9] and [14, 4, 8] zero radical codes, respectively.

(2) Puncturing the $\mathcal{C}_{26} = [26, 4, 18]$ code on coordinates sets {2}, {1,2}, {1,2,3}, {1,2,3, 4}, {1,5,6,8,11}, {1,2,3, 5,9,16}, {1,2,…, 6,13} and {1,2,…, 6,8,10}, one can obtain [25, 4, 17], [24, 4, 16], [23, 4, 15], [22, 4, 14], [21, 4, 14], [20, 4, 13], [19, 4, 12] and [18, 4, 11] zero radical codes, respectively.

(3) Puncturing the $\mathcal{C}_{33} = [33, 4, 23]$ code on coordinates sets {3}, {1,2}, {1,2,3}, {1,2,30,31}, {1,2,3,8,17}, {1,2,3,4,5,20}, one can obtain [32, 4, 22], [31, 4, 21], [30, 4, 20], [29, 4, 20], [28, 4, 19], [27, 4, 18] zero radical codes, respectively.

(4) Puncturing the $\mathcal{C}_{37} = [37, 4, 26]$ code on coordinates sets {2}, {1,2}, {1,2,6}, one can obtain [36, 4, 25], [35, 4, 24], [34, 4, 23] zero radical codes, respectively.

(5) Puncturing the $\mathcal{C}_{45} = [45, 4, 32]$ code on coordinates sets {1}, {1,2}, {1,2,4}, {1,5,9,24}, {1,2,4,7,23}, {1,2,3,4,7,23}, {1,2,3,4,5,7,23}, one can obtain zero radical codes [44, 4, 31], [43, 4, 30], [42, 4, 29], [41, 4, 29], [40, 4, 28], [39, 4, 27] and [38, 4, 26], respectively.

(6) Puncturing the $\mathcal{C}_{54} = [54, 4, 39]$ code on coordinates sets {2}, {1,2}, {1,2,3}, {1,2,35,49}, {1,2,3,35,49}, {1,2,3,4,5,35,49}, one can obtain [53, 4, 38], [52, 4, 37], [51, 4, 36], [50, 4, 36], [49, 4, 35] and [47, 4, 33] zero radical codes, respectively.

(7) Puncturing the $\mathcal{C}_{60} = [60, 4, 44]$ code on coordinates sets {7}, {1,2}, {1,2,7}, {1,2,3,5}, {1,2,3,5,9}, one can obtain [59, 4, 43], [58, 4, 42], [57, 4, 41], [56, 4, 40] and [55, 4, 39] zero radical codes, respectively.

(8) Puncturing the $\mathcal{C}_{66} = [66, 4, 48]$ code on coordinates sets {3}, {1,2}, {1,4,7,9}, {1,2,3,4,7}, one can obtain [65,4,47], [64,4,46], [62,4,45] and [61,4,44] zero radical codes, respectively.

(9) Puncturing the $\mathcal{C}_{71} = [71, 4, 52]$ code on coordinates sets {1}, {1,3}, {1,2,4}, {1,2,4,10}, one can obtain [70,4,51], [69,4,50], [68,4,49] and [67,4,49] zero radical codes, respectively.

(9) Puncturing the $\mathcal{C}_{71} = [71, 4, 52]$ code on coordinates sets {1}, {1,3}, {1,2,4}, {1,2,4,10}, one can obtain [70,4,51], [69,4,50], [68,4,49] and [67,4,49] zero radical codes, respectively.

(10) Puncturing the $\mathcal{C}_{76} = [76, 4, 56]$ code on coordinates sets {1}, {1,2}, {1,2,3}, one can obtain [75,4,55], [74,4,54] and [73,4,53] zero radical codes, respectively.

(11) Puncturing the $\mathcal{C}_{81} = [81, 4, 60]$ code on coordinates sets {1}, {1,7}, {1,2,4}, {1,2,4,7}, one can obtain [80,4,59], [79,4,58], [78,4,57] and [77,4,56] zero radical codes, respectively.

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