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Hierarchical quantum information splitting with eight-qubit cluster states

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Abstract In the paper, a scheme is proposed for hierarchical quantum information splitting with an unknown eight-qubit cluster state. The Boss Alice wants to distribute a quantum secret to seven distant agents who are divided into two grades. Three agents are in the upper grade and four agents are in the lower grade. Every agent of the upper grade only needs the collaboration of three of the other six agents to get the secret, but all the agents of the lower grade need the collaboration of all the other six agents. In other words, different agents in different grades have different authorities to recover Boss' secret. And the agent in upper grade is more powerful than the one in the lower grades which needs more information to recover the secret.

Keywords Quantum secret · Hierarchical quantum information splitting · Cluster state · Unitary operation

1 Introduction

Entanglement is an unique physical resource of quantum physics with many applications in quantum information processing and quantum computation [1]. Quantum teleportation is an important technique for information transmission between two or more parties, with a distributed entangled state and a classical communication channel. Since Bennett et al. [2] presented the first protocol of quantum teleportation through an entangled channel of EPR pair in 1993, many teleportation protocols have been devised by using multi-partite entangled states, such as the prototype-GHZ states [3], generalized W states [4] and cluster states [5–7]. In 1999, Hillery et al.

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first demonstrated that an entangled three-qubit GHZ state can be used for quantum information splitting(QIS) [8]. The basic idea of QIS is to share quantum information among a group of participants so that the original information cannot be completely reconstructed by any one of the parties individually.In the Ref. [9–12] QIS are also articulated.

Although entanglement is the significant resource of quantum information science, and plays a powerful role in transferring quantum states, actually not all entangled states can be used to implement perfect quantum teleportation. Whether an entangled state can implement teleportation is determined by the property of entanglement [13]. Thus teleportation can show some properties of entangled states, especially multipartite entangled states [14]. QIS can be considered as a generalization of classical secret sharing to quantum scenario. Classical secret sharing is one of the most important information-security cryptographic protocols and is germane to online auctions, electronic voting, shared electronic banking, cooperative activation of bombs, and so on. Also, QIS has extensive applications in quantum information science, such as creating joint checking accounts containing quantum money [15], secure distributed quantum computation [16], and so on.

In the last decade, quantum information splitting has been attracting much attention [7–17], and a scheme has already been realized experimentally [18]. Up to now, in the existing schemes of QIS, quantum states (such as Bell states [25], four-qubit (five-qubit, six-qubit) cluster state [11,14], genuinely entangled five-qubit (six-qubit) state [26,27], GHZ states [28]) are chosen as quantum channel. In essence, QIS equals to controlled teleportation [25]. That is to say, all the agents in a QIS scheme can be considered as the controllers to an unique receiver to recover the quantum information in a quantum teleportation. In the case of teleporting an unknown single-qubit state, Zhan et al. [29] have investigated a deterministic teleportation via high-dimensional entangled state. Yin et al. [30] proposed a scheme for teleportation of simplified four-qubit cluster state. Nie et al. [31] have discussed the quantum teleportation by GHZ state.

It is well known that the n-qubit($n > 3$) cluster state is maximally connected with better persistency than the GHZ state [32]. In other words, cluster state has the properties of both the GHZ-class and the W-class entangled states, and is more difficult to be destroyed by local operations than GHZ-class states. Briegel [33], Raussendorf [34], Schlingemann [35] and Walther [36] investigated that cluster state can be used in one way quantum computation and quantum error correction, and it has been experimentally proved [36]. It is a remarkable fact that all of the aforementioned schemes are focused on the symmetric case which every agent has the same status, i.e., the same authority for getting the sender's secret. In other words, every agent can recover the secret(quantum state) successfully with the help of the other agents. The symmetry of quantum channel leads all the receivers to have the same power to recover the secret. But in the real world, based on a certain purpose, the boss wants to send his/her secret to different agents with different grades to recover corresponding information. Thus, a more general QIS scheme should contain the asymmetric case in which different agents have different authorities to recover the secret information. Wang et al. [19] introduced the concept of hierarchical QIS which is (1, 2)-hierarchy model of quantum information splitting. Then they investigated (2, 3)-hierarchy model [20]. In the two schemes, they adopted four-qubit and six-qubit cluster state as the quantum channel respectively. It is worth noting that the two models are discussed with maximally entangled channel. As mentioned before, eight-qubit cluster state has better persistency than the GHZ state. And it has been experimentally proved to realize [37]. So quantum information splitting of asymmetric case is an important part of quantum information processing.

Based on the above reasons, it is worth discussing quantum information splitting based on asymmetric and non-maximally entangled channel. It gives us motivation to study new application of non-maximally entangled channel for QIS protocol. In this paper, we propose a new scheme for hierarchical quantum information splitting of an unknown single-qubit state.In our scheme a non-maximally entangled eight-qubit cluster state is chosen as the quantum channel shared by the sender and other agents. The eight-qubit cluster state is easy to be adopted to investigate the QIS. The sender performs Bell-state measurement on her qubits pair, then the agents makes the projective measurements on his qubit. Finally the receiver applies some appropriately transformations on his qubit according to the measured results from both the sender and part or all of other agents. Thus the task of QIS of an unknown single-qubit state is completed. In our scheme, the agent ($\text{Bob}_1, \text{Bob}_2, \text{Bob}_3$) in upper grade is more powerful than in the lower grades($\text{Bob}_4, \text{Bob}_5, \text{Bob}_6$ and Bob_7). The agents in upper grade need less information than the one in the lower grade to recover the secret. Nowadays, the hierarchical QIS has been investigated with two grades. But in real world, all the receivers may be divided into more than two grades in order to satisfy some physical facts. Thus it is meaningful to make further discussion of hierarchical QIS.

The paper is organized as follows. In Sect. 2, we describe the hierarchical quantum information splitting with non-maximally eight-qubit cluster states in detail. In Sect. 3, the conclusion is obtained.

2 Hierarchical quantum information splitting with eight-qubit cluster states

Suppose Alice has an unknown single-qubit state, which can be described as follows

$$|\varphi\rangle_x = \alpha|0\rangle_x + \beta|1\rangle_x, \tag{1}$$

where α, β are complex numbers with $|\alpha|^2 + |\beta|^2 = 1$. As the Boss of QIS, Alice wants to share the secret quantum state $|\psi\rangle_x$ with her seven agents, i.e., $\text{Bob}_1, \text{Bob}_2, \ldots, \text{Bob}_7$. She chooses an unknown no-maximally entangled eight-qubit cluster state

$$|\mathcal{C}_8\rangle_{1234578} = (a|0000000\rangle + a|00001111\rangle + b|11110000\rangle - b|11111111\rangle)_{12345678}, \tag{2}$$

where a, b are arbitrary nonzero real numbers and satisfying $|a|^2 + |b|^2 = \frac{1}{2}$ and $|a| \geq |b|$. The particles x and 1 belong to Alice, the particle i belongs to Bob_{i-1}, $i = 2, 3, \ldots, 7$ respectively.

The state of the whole system is

$$|\varphi\rangle_{x12345678} = |\varphi\rangle_x \otimes |\mathcal{C}_8\rangle_{12345678}. \tag{3}$$

In order to express the state of the whole system easily, we denote

$$|\mathcal{G}\rangle_{2345678} = |000\rangle(|0000\rangle + |1111\rangle), |\mathcal{G}'\rangle_{2345678} = |111\rangle(|0000\rangle - |1111\rangle). \tag{4}$$

Then

$$\begin{aligned} |\varphi\rangle_{x12345678} &= a\alpha|00\rangle_{x1}|\mathcal{G}\rangle_{2345678} + b\alpha|01\rangle_{x1}|\mathcal{G}'\rangle_{2345678} \\ &\quad + a\beta|10\rangle_{x1}|\mathcal{G}\rangle_{2345678} + b\beta|11\rangle_{x1}|\mathcal{G}'\rangle_{2345678}. \end{aligned} \tag{5}$$

In the scheme, Alice wants to distribute her secret $|\varphi\rangle_x$ to her seven agents such that any one of them can recover the secret state with the assistance of part or all of the others. Because the Bell-state measurement for particles can be well realized according to recent relational reports [21–23], at first, Alice performs a joint measurement on her particles x and 1 using the Bell basis $|\Psi^{\pm}\rangle_{x1}$ and $|\Phi^{\pm}\rangle_{x1}$. The four Bell states are given by

$$|\Psi^{\pm}\rangle_{x1} = \frac{1}{\sqrt{2}}(|00\rangle \pm |11\rangle)_{x1}, |\Phi^{\pm}\rangle_{x1} = \frac{1}{\sqrt{2}}(|01\rangle \pm |10\rangle)_{x1}. \tag{6}$$

After performing Bell measurement, the state of whole system will be collapsed into one of four possible entangled states $|\psi^{\pm}\rangle_{2345678}$ and $|\phi^{\pm}\rangle_{2345678}$.

$$\begin{aligned} |\psi^{\pm}\rangle_{2345678} &= |\Psi^{\pm}\rangle_{x1} \otimes |\varphi\rangle_{x12345678} = a\alpha|\mathcal{G}\rangle_{2345678} \pm b\beta|\mathcal{G}'\rangle_{2345678}, \\ |\phi^{\pm}\rangle_{2345678} &= |\Phi^{\pm}\rangle_{x1} \otimes |\varphi\rangle_{x12345678} = b\alpha|\mathcal{G}'\rangle_{2345678} \pm a\beta|\mathcal{G}'\rangle_{2345678}. \end{aligned} \tag{7}$$

And then Alice tells the measurement outcomes to all her agents via classical channel. According to the non-cloning theorem [24], only one particle allows to be in the state $|\varphi\rangle$. Hence any one of Alice's seven agents Bob$_i$ $(i = 1, 2, \ldots, 7)$, but not all, can recover the state $|\varphi\rangle$.

Without loss of generality, we assume that all the Bobs agree to let Bob$_1$ possess the secret. Now we rewrite $|\psi^{\pm}\rangle_{2345678}$ and $|\phi^{\pm}\rangle_{2345678}$ as

$$\begin{aligned} |\psi^{\pm}\rangle_{2345678} &= (a\alpha|0\rangle_2 \pm b\beta|1\rangle_2) \cdot |\psi^1\rangle_{345678} + (a\alpha|0\rangle_2 \mp b\beta|1\rangle_2) \cdot |\psi^2\rangle_{345678}, \\ |\phi^{\pm}\rangle_{2345678} &= (a\beta|0\rangle_2 \pm b\alpha|1\rangle_2) \cdot |\phi^1\rangle_{345678} + (a\beta|0\rangle_2 \mp b\alpha|1\rangle_2) \cdot |\phi^2\rangle_{345678}, \end{aligned} \tag{8}$$

where

$$\begin{aligned} |\psi^1\rangle_{345678} = \frac{1}{\sqrt{2}}[&|+\rangle_3|+\rangle_4|0000\rangle_{5678} + |+\rangle_3|-\rangle_4|1111\rangle_{5678} \\ &+|-\rangle_3|+\rangle_4|1111\rangle_{5678} + |-\rangle_3|-\rangle_4|0000\rangle_{5678}], \end{aligned}$$

$$
\begin{aligned}
|\psi^2\rangle_{345678} &= \frac{1}{\sqrt{2}}[|+\rangle_3|+\rangle_4|1111\rangle_{5678} + |+\rangle_3|-\rangle_4|0000\rangle_{5678} \\
&\quad +|-\rangle_3|+\rangle_4|0000\rangle_{5678} + |-\rangle_3|-\rangle_4|1111\rangle_{5678}], \\
|\phi^1\rangle_{345678} &= \frac{1}{\sqrt{2}}[|+\rangle_3|+\rangle_4|0000\rangle_{5678} + |+\rangle_3|-\rangle_4|1111\rangle_{5678} \\
&\quad +|-\rangle_3|+\rangle_4|1111\rangle_{5678} + |-\rangle_3|-\rangle_4|0000\rangle_{5678}], \\
|\phi^2\rangle_{345678} &= \frac{1}{\sqrt{2}}[|+\rangle_3|+\rangle_4|1111\rangle_{5678} + |+\rangle_3|-\rangle_4|0000\rangle_{5678} \\
&\quad +|-\rangle_3|+\rangle_4|0000\rangle_{5678} + |-\rangle_3|-\rangle_4|1111\rangle_{5678}], \\
|+\rangle &= \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle), |-\rangle = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle).
\end{aligned}
$$

In order to assist Bob_1 to reconstruct Alice's secret state, the other agents need to measure their particles using appropriate bases and broadcast the outcomes to other suitable agents. Obviously, if Bob_4, Bob_5, Bob_6 and Bob_7 measure their particles using the basis $\{|0\rangle, |1\rangle\}$, their outcomes are always correlated, i.e., anyone's outcome can deduce the others'. This shows that only one of them, to be referred to as "Bob_*", is needed to inform Bob_1 of his single-particle measurement outcome. Suppose that Bob_2 and Bob_3 measure their particle 3 and 4 using the basis $\{|+\rangle, |-\rangle\}$ respectively. Then Bob_1 can reconstruct the state $|\varphi\rangle$ on particle 2 by appropriate local operations based on the measurement outcomes of Bob_2, Bob_3 and Bob_*. In other words, if Bob_1 wants to reconstruct the original state $|\varphi\rangle$, he only needs the result of three of six other agents, i.e., Bob_2, Bob_3 and Bob_*.

Because the parameters a and b are arbitrary nonzero real numbers in quantum channel, Bob_1 recovers the secret state with a certain probability. Now we describe the process to recover the secret state by Bob_1 in detail.

According to Eq. (8), without loss of generality, suppose Bob_2, Bob_3 and Bob_4's measurement results are $|-\rangle_3, |-\rangle_4$ and $|0\rangle_5$, and Alice's measurement result is $|\Phi^-\rangle_{x1}$, then the joint state of whole system will be collapsed to

$$|\phi\rangle_2 = a\beta|0\rangle_2 - b\alpha|1\rangle_2. \tag{9}$$

Now, in order to recover the original state $|\varphi\rangle$, Bob_1 performs an unitary transformation $U = \sigma_2^z\sigma_2^x$ on her particles 2, which transforms $|\phi\rangle_2$ into

$$|\phi^1\rangle_2 = b\alpha|0\rangle_2 + a\beta|1\rangle_2, \tag{10}$$

where $\sigma^x = |0\rangle\langle 1| + |1\rangle\langle 0|$ and $\sigma^z = |0\rangle\langle 0| - |1\rangle\langle 1|$ are the usual Pauli operators.

As proposed before, the state taken as the quantum channel,which are shared among Alice and $\text{Bob}_i\,(i = 1, 2, ..., 7)$, i.e., Bob_1 has complete knowledge of non-maximally entangled states. Therefore Bob_1 introduces an auxiliary particle m in the initial state $|0\rangle_m$. Now, the compositive state of the particles 2 and m in Bob_1's position is

$$|\phi\rangle_{2m} = |\phi^1\rangle_2|0\rangle_m = b\alpha|00\rangle_{2m} + a\beta|10\rangle_{2m}. \tag{11}$$

This compositive state can be rewritten as

$$|\phi\rangle_{2m} = \frac{1}{2}(|G_1\rangle \otimes |E_1\rangle + |G_2\rangle \otimes |E_2\rangle), \tag{12}$$

where $|G_1\rangle = \alpha|0\rangle_2 + \beta|1\rangle_2$, $|G_2\rangle = \alpha|0\rangle_2 - \beta|1\rangle_2$, $|E_1\rangle = b|0\rangle_m + a|1\rangle_m$, $|E_2\rangle = b|0\rangle_m - a|1\rangle_m$.

From above Eq. (12), one can see that Bob_1 can get the state $|G_i\rangle(i = 1, 2)$ of his particle 2 provided that $|E_i\rangle(i = 1, 2)$ are obtained via appropriate measurements on his auxiliary particle m. Unfortunately, the states $|E_i\rangle(i = 1, 2)$ are not orthogonal in general. As a consequence, they cannot be differentiated deterministically. In order to distinguish the two states with a certain probability, Bob adopts to perform an optimal POVM measurement on the auxiliary particle m. The POVM takes the following form

$$P_1 = \frac{1}{\omega}|M_1\rangle\langle M_1|,\ P_2 = \frac{1}{\omega}|M_2\rangle\langle|M_2|,\ P_3 = I - P_1 - P_2. \tag{13}$$

where

$$|M_1\rangle = \frac{1}{\sqrt{\eta}}\left(\frac{1}{b}|0\rangle + \frac{1}{a}|1\rangle\right)_m,\ |M_2\rangle = \frac{1}{\sqrt{\eta}}\left(\frac{1}{b}|0\rangle - \frac{1}{a}|1\rangle\right)_m,\ \eta = \frac{1}{b^2} + \frac{1}{a^2} = \frac{1}{2a^2b^2}. \tag{14}$$

I is an identity operator, and the coefficient ω, which relates to a and b, should be able to assure P_3 to be a positive operator. In order to determine exactly coefficient ω, we would like to rewrite the three operators $P_i(i = 1, 2, 3)$ in the matrix forms.

$$P_1 = \frac{1}{\omega\eta}\begin{pmatrix} \frac{1}{b^2} & \frac{1}{ab} \\ \frac{1}{ab} & \frac{1}{a^2} \end{pmatrix},\ P_2 = \frac{1}{\omega\eta}\begin{pmatrix} \frac{1}{b^2} & -\frac{1}{ab} \\ -\frac{1}{ab} & \frac{1}{a^2} \end{pmatrix},\ P_3 = \begin{pmatrix} 1 - \frac{2}{\omega\eta b^2} & 0 \\ 0 & 1 - \frac{2}{\omega\eta a^2}, \end{pmatrix}. \tag{15}$$

Evidently, to let P_3 be a positive operator, the parameter ω should be an appropriate value satisfying $\omega \geq 1$, as is strongly relative to a and b.

After performing the above POVM operation on the auxiliary particle m, Bob_1 can definitively get $P_i(i = 1, 2)$ with the probability

$$p(P_i) = {}_{2m}\langle\phi|P_i|\phi\rangle_{2m} = \frac{1}{\omega\eta}, i = 1, 2. \tag{16}$$

Alternatively, in terms of the POVM value, Bob can positively conclude the state $|E_i\rangle(i = 1, 2)$ of the particle m. Once Bob_1 determines the $|E_i\rangle(i = 1, 2)$, this means he also knows the state $|G_i\rangle(i = 1, 2)$ of his particle 2. As a consequence, Bob_1 can reconstruct the original state in his particle 2 by performing an appropriate unitary operation. That is, if Bob knows that the state of his particle 2 is $|G_1\rangle$ or $|G_2\rangle$, he needs only to perform the unitary operation $I_2 = |0\rangle_2\langle 0| + |1\rangle_2\langle 1|$ or $\sigma_2^z = |0\rangle_2\langle 0| - |1\rangle_2\langle 1|$ respectively. According to the above equation, one can see that the QSTS probability depends on the parameter ω and the small coefficients of the states taken as the quantum channel. As mentioned before, ω can be varied from 1 to 2, however, it should still be carefully chosen such that P_3 is a nonnegative operator. If $a = b = \frac{1}{2}$ and one can choose $\omega = 1$, the success probability is 1, that is to say, the quantum channel consists of maximally entangled states, and the present QIS scheme turns out to be deterministic.

Because $|\mathcal{C}_8\rangle$ is unchanged under the permutation of particles 2, 3 and 4, the above method on particle 2 can apply to particles 3 and 4. In other words, Bob_1, Bob_2 and Bob_3 have the same status in our QIS scheme.

Similarly, according to the symmetric structure of $|\mathcal{C}_8\rangle$, Bob_4, Bob_5, Bob_6 and Bob_7 have the same status in this scheme. Without loss of generality, we suppose that Bob_7 can possess Alice's secret, i.e., recover the state $|\varphi\rangle$ with authorization by all the other agents. Based on the assumption, the states $|\psi^{\pm}\rangle_{2345678}$ and $|\phi^{\pm}\rangle_{2345678}$ can be rewritten as

$$\begin{aligned}
|\psi^{\pm}\rangle_{2345678} &= |\psi^{1}\rangle_{234567}(a\alpha|+\rangle_8 \pm b\beta|-\rangle_8) + |\psi^{2}\rangle_{234567}(a\alpha|-\rangle_8 \pm b\beta|+\rangle_8) \\
&\quad + |\psi^{3}\rangle_{234567}(a\alpha|+\rangle_8 \mp b\beta|-\rangle_8) + |\psi^{4}\rangle_{234567}(a\alpha|-\rangle_8 \mp b\beta|+\rangle_8) \\
|\phi^{\pm}\rangle_{2345678} &= |\psi^{1}\rangle_{234567}(b\alpha|-\rangle_8 \pm a\beta|+\rangle_8) + |\psi^{2}\rangle_{234567}(b\alpha|+\rangle_8 \pm a\beta|-\rangle_8) \\
&\quad - |\psi^{3}\rangle_{234567}(b\alpha|-\rangle_8 \mp a\beta|+\rangle_8) - |\psi^{4}\rangle_{234567}(b\alpha|+\rangle_8 \mp a\beta|-\rangle_8),
\end{aligned} \tag{17}$$

where

$$\begin{aligned}
|\psi^{1}\rangle_{234567} = (&|{+}{+}{+}{+}{+}{+}\rangle + |{+}{+}{+}{+}{-}{-}\rangle + |{+}{+}{+}{-}{+}{-}\rangle \\
&+ |{+}{+}{+}{-}{-}{+}\rangle + |{+}{-}{-}{+}{+}{+}\rangle + |{+}{-}{-}{+}{-}{-}\rangle \\
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&+ |{-}{-}{+}{+}{+}{+}\rangle + |{-}{-}{+}{+}{-}{-}\rangle + |{-}{-}{+}{-}{+}{-}\rangle \\
&+ |{-}{-}{+}{-}{-}{+}\rangle)_{234567} \\
|\psi^{2}\rangle_{234567} = (&|{+}{+}{+}{+}{+}{-}\rangle + |{+}{+}{+}{+}{-}{+}\rangle + |{+}{+}{+}{-}{+}{+}\rangle \\
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|\psi^{3}\rangle_{234567} = (&|{+}{+}{-}{+}{+}{+}\rangle + |{+}{+}{-}{+}{-}{-}\rangle + |{+}{+}{-}{-}{+}{-}\rangle \\
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&+ |{-}{-}{-}{+}{+}{+}\rangle + |{-}{-}{-}{+}{-}{-}\rangle + |{-}{-}{-}{-}{+}{-}\rangle \\
&+ |{-}{-}{-}{-}{-}{+}\rangle)_{234567}
\end{aligned}$$

$$
\begin{aligned}
|\psi^4\rangle_{234567} = (&|++-++-\rangle + |++-+-+\rangle + |++--++\rangle \\
&+|++----\rangle + |+-+++-\rangle + |+-++-+\rangle \\
&+|+-+-++\rangle + |+-+---\rangle + |-++++-\rangle \\
&+|-+++-+\rangle + |-++-++\rangle + |-++---\rangle \\
&+|---++-\rangle + |---+-+\rangle + |----++\rangle \\
&+|------\rangle)_{234567}
\end{aligned}
$$

From the Eq. (17), it is clear that Bob_7 can reconstruct the state $|\varphi\rangle$ if and only if all the other six Bobs measure their particles in the basis $\{|+\rangle, |-\rangle\}$ and broadcast their outcomes. That is to say, if Bob_7 wants to recover the original state $|\varphi\rangle$, he must need the aidance of all the other six agents (Bob_i, $i = 1, 2, 3, 4, 5, 6$). Once the measurement basses of Alice and Bob_1,. . ., Bob_6 are chosen, the state of whole system will be collapsed into one of the following eight states

$$
a\alpha|+\rangle_8 \pm b\beta|-\rangle_8, a\alpha|-\rangle_8 \pm b\beta|+\rangle_8, b\alpha|+\rangle_8 \pm a\beta|-\rangle_8, b\alpha|-\rangle_8 \pm a\beta|+\rangle_8. \quad (18)
$$

When Bob_7 receives the state with the form of Eq.(18), he performs a Hardamard transformation and a certain Pauli operation on particle 8, then the state will be transformed into

$$
|\psi^1\rangle_8 = a\alpha|0\rangle_8 + b\beta|1\rangle, |\psi^2\rangle_8 = b\alpha|0\rangle_8 + a\beta|1\rangle, \quad (19)
$$

Bob_7 should make corresponding operations which are listed in Table 1 to transform Eqs. (18, 19) according to Alice's Bell-state measurement outcomes and the other Bobs' single-qubit measurement outcomes, where $\mathcal{H} = |0\rangle\langle 0| + |0\rangle\langle 1| + |1\rangle\langle 0| - |1\rangle\langle 1|$ is the Hardamard transformation. The Set $S_i (i = 1, 2, 3, 4)$ in Table 1 are

$$
\begin{aligned}
S_1 = \{&|++++++\rangle, |++++--\rangle, |+++-+-\rangle, |+++--+\rangle, \\
&|+--+++\rangle, |+--+--\rangle, |+---+-\rangle, |+----+\rangle, \\
&|-+-+++\rangle, |-+-+--\rangle, |-+--+-\rangle, |-+---+\rangle, \\
&|--++++\rangle, |--++--\rangle, |--+-+-\rangle, |--+--+\rangle\}
\end{aligned} \quad (20)
$$

$$
\begin{aligned}
S_2 = \{&|+++++-\rangle, |++++-+\rangle, |+++-++\rangle, |+++---\rangle, \\
&|+--++-\rangle, |+--+-+\rangle, |+---++\rangle, |+-----\rangle, \\
&|-+-++-\rangle, |-+-+-+\rangle, |-+--++\rangle, |-+----\rangle, \\
&|--+++-\rangle, |--++-+\rangle, |--+-++\rangle, |--+---\rangle\}
\end{aligned} \quad (21)
$$

$$
\begin{aligned}
S_3 = \{&|++-+++\rangle, |++-+--\rangle, |++--+-\rangle, |++---+\rangle, \\
&|+-++++\rangle, |+-++--\rangle, |+-+-+-\rangle, |+-+--+\rangle, \\
&|-+++++\rangle, |-+++--\rangle, |-++-+-\rangle, |-++--+\rangle, \\
&|---+++\rangle, |---+--\rangle, |----+-\rangle, |-----+\rangle\}
\end{aligned} \quad (22)
$$

Table 1 The corresponding operation that Bob_7 should perform for transforming the state in Eq. (18) to the state in Eq. (19), according to the measurement outcomes of Alice and the other six agents (Bob_i, $i = 1, 2, 3, 4, 5, 6$)

Alice's outcomes	The outcomes of $\text{Bob}_{1,2,3,4,5,6}$	Bob_7's operation	The state of Bob_7 after performing transfer
$\|\Psi^+\rangle$	$\|\phi_i\rangle \in S_1$	$\mathcal{H}$	
$\|\Psi^-\rangle$	$\|\phi_i\rangle \in S_1$	$\sigma_8^z\mathcal{H}$	
$\|\Psi^+\rangle$	$\|\phi_i\rangle \in S_2$	$\sigma_8^x\mathcal{H}$	
$\|\Psi^-\rangle$	$\|\phi_i\rangle \in S_2$	$\sigma_8^x\sigma_8^z\mathcal{H}$	$\|\psi^1\rangle_8$
$\|\Psi^+\rangle$	$\|\phi_i\rangle \in S_3$	$\sigma_8^z\mathcal{H}$	
$\|\Psi^-\rangle$	$\|\phi_i\rangle \in S_3$	$\mathcal{H}$	
$\|\Psi^+\rangle$	$\|\phi_i\rangle \in S_4$	$\sigma_8^z\sigma_8^x\mathcal{H}$	
$\|\Psi^-\rangle$	$\|\phi_i\rangle \in S_4$	$\sigma_8^x\mathcal{H}$	
$\|\Phi^+\rangle$	$\|\phi_i\rangle \in S_1$	$\sigma_8^x\mathcal{H}$	
$\|\Phi^-\rangle$	$\|\phi_i\rangle \in S_1$	$\sigma_8^x\mathcal{H}$	
$\|\Phi^+\rangle$	$\|\phi_i\rangle \in S_2$	$\mathcal{H}$	
$\|\Phi^-\rangle$	$\|\phi_i\rangle \in S_2$	$\sigma_8^z\mathcal{H}$	$\|\psi^2\rangle_8$
$\|\Phi^+\rangle$	$\|\phi_i\rangle \in S_3$	$\sigma_8^z\sigma_8^x\mathcal{H}$	
$\|\Phi^-\rangle$	$\|\phi_i\rangle \in S_3$	$\sigma_8^x\mathcal{H}$	
$\|\Phi^+\rangle$	$\|\phi_i\rangle \in S_4$	$\sigma_8^z\mathcal{H}$	
$\|\Phi^-\rangle$	$\|\phi_i\rangle \in S_4$	$\mathcal{H}$	

$$\begin{aligned} S_4 = \{ & |++-++-\rangle, |++-+-+\rangle, |++--++\rangle, |++----\rangle, \\ & |+-+++-\rangle, |+-++-+\rangle, |+-+-++\rangle, |+-+---\rangle, \\ & |-++++-\rangle, |-+++-+\rangle, |-++-++\rangle, |-++---\rangle, \\ & |---++-\rangle, |---+-+\rangle, |----++\rangle, |------\rangle\} \end{aligned} \tag{23}$$

Once the state in Bob_7's position is $|\psi^1\rangle_8$ or $|\psi^2\rangle_8$, Bob_7 can apply POVM method to recover the original state with a certain probability as before. From the above scheme, if the agents want to reconstruct the secret state $|\varphi\rangle$, $\text{Bob}_i (i = 1, 2, 3)$ only needs the assistance of two other Bob_j, $j = 1, 2, 3$ and any one of the other four Bobs (i.e., Bob_i, $i = 4, 5, 6, 7$). But one of $\text{Bob}_j (j = 4, 5, 6, 7)$ needs the help of all of the other Bobs in order to achieve the same aim. That is to say, every agent has different authority for getting the secret. Hence all the agents are divided into different grade. Some agents in upper grade only need assistance of part of agents to achieve the secret, but some other agents need all other agents' aidance. It shows that some agents (i.e., Bob_1, Bob_2, Bob_3) are in a higher grade relative to the other agents (i.e., Bob_4, Bob_5, Bob_6 and Bob_7).

From the above scheme, we can know Alice's secret is distributed to $\text{Bob}_{1,2,3}$ and $\text{Bob}_{4,5,6,7}$ asymmetrically. Obviously the more information is known by some agent, the less collaborations are needed. It is consistent with the actual physical fact.

In our scheme, since the parameters a and b in the non-maximally entangled quantum channel $|\mathcal{C}\rangle_8$ are unknown, the agent Bob_i must use the POVM method to

Table 2 The corresponding operation that Bob_1 should perform for recover the state $|\varphi\rangle$, according to the measurement outcomes of Alice, Bob_2, Bob_3 and Bob_*, where a and b are known to Bob_1

Alice's outcomes	Bob_2's outcomes	Bob_3's outcomes	Bob*'s outcomes	Bob_1's operation
$\|\psi^+\rangle_{2345678}$	$\|+\rangle$	$\|+\rangle(\|-\rangle)$	$\|0\rangle(\|1\rangle)$	$H_{a^{-1},b^{-1}}$
$\|\psi^+\rangle_{2345678}$	$\|+\rangle$	$\|+\rangle(\|-\rangle)$	$\|1\rangle(\|0\rangle)$	$H_{a^{-1},-b^{-1}}$
$\|\psi^+\rangle_{2345678}$	$\|-\rangle$	$\|+\rangle(\|-\rangle)$	$\|0\rangle(\|1\rangle)$	$H_{a^{-1},-b^{-1}}$
$\|\psi^+\rangle_{2345678}$	$\|-\rangle$	$\|+\rangle(\|-\rangle)$	$\|1\rangle(\|0\rangle)$	$H_{a^{-1},b^{-1}}$
$\|\psi^-\rangle_{2345678}$	$\|+\rangle$	$\|+\rangle(\|-\rangle)$	$\|0\rangle(\|1\rangle)$	$H_{a^{-1},-b^{-1}}$
$\|\psi^-\rangle_{2345678}$	$\|+\rangle$	$\|+\rangle(\|-\rangle)$	$\|1\rangle(\|0\rangle)$	$H_{a^{-1},b^{-1}}$
$\|\psi^-\rangle_{2345678}$	$\|-\rangle$	$\|+\rangle(\|-\rangle)$	$\|0\rangle(\|1\rangle)$	$H_{a^{-1},b^{-1}}$
$\|\psi^-\rangle_{2345678}$	$\|-\rangle$	$\|+\rangle(\|-\rangle)$	$\|1\rangle(\|0\rangle)$	$H_{a^{-1},-b^{-1}}$
$\|\phi^+\rangle_{2345678}$	$\|+\rangle$	$\|+\rangle(\|-\rangle)$	$\|0\rangle(\|1\rangle)$	$\sigma_{b^{-1},a^{-1}}$
$\|\phi^+\rangle_{2345678}$	$\|+\rangle$	$\|+\rangle(\|-\rangle)$	$\|1\rangle(\|0\rangle)$	$\sigma_{-b^{-1},a^{-1}}$
$\|\phi^+\rangle_{2345678}$	$\|-\rangle$	$\|+\rangle(\|-\rangle)$	$\|0\rangle(\|1\rangle)$	$\sigma_{-b^{-1},a^{-1}}$
$\|\phi^+\rangle_{2345678}$	$\|-\rangle$	$\|+\rangle(\|-\rangle)$	$\|1\rangle(\|0\rangle)$	$\sigma_{b^{-1},a^{-1}}$
$\|\phi^-\rangle_{2345678}$	$\|+\rangle$	$\|+\rangle(\|-\rangle)$	$\|0\rangle(\|1\rangle)$	$\sigma_{-b^{-1},a^{-1}}$
$\|\phi^-\rangle_{2345678}$	$\|+\rangle$	$\|+\rangle(\|-\rangle)$	$\|1\rangle(\|0\rangle)$	$\sigma_{b^{-1},a^{-1}}$
$\|\phi^-\rangle_{2345678}$	$\|-\rangle$	$\|+\rangle(\|-\rangle)$	$\|0\rangle(\|1\rangle)$	$\sigma_{b^{-1},a^{-1}}$
$\|\phi^-\rangle_{2345678}$	$\|-\rangle$	$\|+\rangle(\|-\rangle)$	$\|1\rangle(\|0\rangle)$	$\sigma_{-b^{-1},a^{-1}}$

Table 3 The corresponding local operation that Bob_7 should perform to recover the state $|\varphi\rangle$, according to the measurement outcomes of Alice and the other six agents (Bob_i, $i = 1, 2, 3, 4, 5, 6$), where a and b are known to Bob_7

Alice's outcomes	The outcomes of $Bob_{1,2,3,4,5,6}$	The operations of Bob_7
$\|\Psi^+\rangle(\|\Psi^-\rangle)$	$\|\phi_i\rangle \in S_1$	$H_{a^{-1},b^{-1}}\mathcal{H}(H_{a^{-1},-b^{-1}}\mathcal{H})$
$\|\Psi^+\rangle(\|\Psi^-\rangle)$	$\|\phi_i\rangle \in S_2$	$\sigma_{a^{-1},b^{-1}}\mathcal{H}(\sigma_{a^{-1},-b^{-1}}\mathcal{H})$
$\|\Psi^+\rangle(\|\Psi^-\rangle)$	$\|\phi_i\rangle \in S_3$	$H_{a^{-1},-b^{-1}}\mathcal{H}(H_{a^{-1},b^{-1}}\mathcal{H})$
$\|\Psi^+\rangle(\|\Psi^-\rangle)$	$\|\phi_i\rangle \in S_4$	$\sigma_{a^{-1},-b^{-1}}\mathcal{H}(\sigma_{a^{-1},b^{-1}}\mathcal{H})$
$\|\Phi^+\rangle(\|\Phi^-\rangle)$	$\|\phi_i\rangle \in S_1$	$\sigma_{b^{-1},a^{-1}}\mathcal{H}(\sigma_{b^{-1},-a^{-1}}\mathcal{H})$
$\|\Phi^+\rangle(\|\Phi^-\rangle)$	$\|\phi_i\rangle \in S_2$	$H_{b^{-1},a^{-1}}\mathcal{H}(H_{b^{-1},-a^{-1}}\mathcal{H})$
$\|\Phi^+\rangle(\|\Phi^-\rangle)$	$\|\phi_i\rangle \in S_3$	$\sigma_{b^{-1},-a^{-1}}\mathcal{H}(\sigma_{b^{-1},a^{-1}}\mathcal{H})$
$\|\Phi^+\rangle(\|\Phi^-\rangle)$	$\|\phi_i\rangle \in S_4$	$H_{b^{-1},-a^{-1}}\mathcal{H}(H_{b^{-1},a^{-1}}\mathcal{H})$

reconstruct the secret. If the parameters a and b in the quantum channel $|\mathcal{C}\rangle_8$ are deterministic which are known to every agents, all the agents can recover the secret state with the help of part or all of the other agents using the corresponding operations listed in Tables 2 and 3. Table 2 shows that Bob_1 preforms the corresponding operation with the help of the other agents. And the corresponding operations that Bob_7 should perform to recover the secret state are in Table 3 according to the measurement outcomes of Alice and Bob_i $(i = 1, 2, ..., 6)$ in Table 3. Accroding to symmetry of quantum channel, Bob_2 and Bob_3 can recover the secret state with the operation as same as

Bob_1's operation on particles 3 and 4 respectively. Because $\text{Bob}_i (i = 3, 4, 5, 6)$ has the same status with Bob_7, $\text{Bob}_i (i = 3, 4, 5, 6)$ can performs the same operation in Table 3 on particle $i + 1 (i = 4, 5, 6, 7)$. The local operations $H_{a,b}$, $\sigma_{a,b}$ in Tables 2 and 3 are defined as $H_{a,b} = a|0\rangle\langle 0| + b|1\rangle\langle 1|$, $\sigma_{a,b} = a|0\rangle\langle 1| + b|1\rangle\langle 0|$.

3 Conclusions

In the paper, we propose a scheme for (3, 4)-hierarchy model of quantum information splitting with an unknown eight-qubit cluster state. In the scheme, seven agents are divided into two grades. Three of them are in upper grade and the other four agents are in lower grade. The agents in the upper grade have the larger authority than in the lower grade. To achieve the same aim, the agents in different grades have different authorities. In the paper, we discuss the parameters a and b in quantum channel are unknown and known. The scheme is different with the existed results in which the quantum channel is arbitrary and symmetric. Especially, if the parameters of quantum channel a and b satisfy $a = b = \frac{1}{2}$, the QIS scheme becomes the symmetric case, which all the agents have the authority to recover the secret with the help of all the other agents. In this paper our scheme is easy to help the agent to recover the secret state. And it gives us new motivity to investigate hierarchical QIS protocol which has more than two grades for different agents.

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