Quantum Inf Process (2010) 9:497–508
DOI 10.1007/s11128-009-0154-5

Residual entanglement with negativity for pure four-qubit quantum states

David Sena Oliveira · Rubens Viana Ramos

Received: 1 February 2009 / Accepted: 12 November 2009 / Published online: 1 December 2009

Abstract In this work we study the entanglement of pure four-qubit quantum states. The analysis is realized, firstly, through the comparison between two different entanglement measures: the Groverian entanglement measure and the residual entanglement calculated with negativities. After, we use the last to measure the entanglement of several four-qubit states and the variation of the entanglement when the four-qubit state is processed by a two-qubit gate.

Keywords Four-qubit states · Entanglement measures · Negativity

PACS 03.67.-a · 03.67.Mn

1 Introduction

Quantum entanglement is the key property that allows the design of powerful quantum communication protocols and quantum algorithms. Hence, the analysis of entanglement through its classification and quantification is a crucial task in quantum information. Well established entanglement measures for two qubit states (pure and mixed) [1–4] and pure three-qubit states [5–7] have been reported. Considering multi-qubit states with $n > 3$, some entanglement measures have been considered [8,9] and two interesting are the Groverian entanglement measure [10–12] and the generalization of the residual entanglement based on the negativity proposed in [7]. In this work we use

D. S. Oliveira · R. V. Ramos (✉)
Department of Teleinformatic Engineering, Federal University of Ceara, Campus do Pici, C.P. 6007, 60455-740 Fortaleza, Brazil
e-mail: rubens@deti.ufc.br

D. S. Oliveira
e-mail: shalom.david@gmail.com

the last two measures to study the entanglement of some pure four-qubit states having four-way entanglement, including graph states [13,14].

2 Entanglement measures for pure four-qubit states

In this work we are concerned only with the Groverian entanglement measure and the residual entanglement calculated with negativities. Thus, in this section we give a brief review of the first and we introduce a useful version of the second.

The Groverian entanglement is an operational measure based on the quantum search algorithm proposed by Grover [10–12]. Basically, it relates the entanglement of the quantum state that represents the database with the probability of finding a marked state. The lower the entanglement of the input state the higher is the probability of finding the marked state. In order to maximize the probability of the quantum search to find the marked state, local unitary operation are allowed. In this way, for an input state $|\psi\rangle$, the success probability of the quantum search is

$$P_{\max}(\psi) = \max_{U_1,\ldots,U_n} \frac{1}{N} \sum_{m=0}^{N-1} \left| \langle m| (U_G)^k (U_1 \otimes \cdots \otimes U_n) |\psi\rangle \right|^2, \tag{1}$$

where U_G is the Grover operator that is applied k times, $|m\rangle$ is the marked state, $N = 2^n$ and n is the number of qubits, and $U_1, \ldots, U_n$ are the allowed local unitary operations that can be changed in order to maximize the success probability. Equation (1) can be rewritten as [10]

$$P_{\max}(\psi) = \max_{|e_1,\ldots,e_n\rangle} |\langle e_1, \ldots, e_n|\psi\rangle|^2 . \tag{2}$$

In (2), $|e_i\rangle$ is a pure one qubit state. At last, the Groverian entanglement measure is given by

$$E_G(\psi) = \sqrt{1 - P_{\max}(\psi)}. \tag{3}$$

As can be seen in (2)–(3), the Groverian entanglement measure consists in finding the closest disentangled state (formed by the tensor product of single-qubit states) of the state whose entanglement one wishes to measure. The distance measure used is the fidelity. Considering four-qubit states, one has

$$\begin{aligned} &P_{\max}(\psi) \\ &= \max_{|e_1,\ldots,e_n\rangle} \left| \begin{bmatrix} \big(\langle 0|\cos(\theta_1) + \langle 1| e^{i\varphi_1}\sin(\theta_1)\big) \otimes \big(\langle 0|\cos(\theta_2) + \langle 1| e^{i\varphi_2}\sin(\theta_2)\big) \otimes \\ \big(\langle 0|\cos(\theta_3) + \langle 1| e^{i\varphi_3}\sin(\theta_3)\big) \otimes \big(\langle 0|\cos(\theta_4) + \langle 1| e^{i\varphi_4}\sin(\theta_4)\big) \end{bmatrix} |\psi\rangle \right|^2 \end{aligned} \tag{4}$$

and the maximization is taken over the angles θ_i and φ_i: $\partial P/\partial\theta_i = \partial P/\partial\varphi_j = 0$ for all $i, j = 1, 2, 3, 4$. Here, instead of looking for analytical equations for $P_{\max}$ and E_G, as it was done in [11,12], we developed a genetic algorithm that can find $P_{\max}$ and E_G for any number of qubits. For example, for the W state with n qubits

$$|W_n\rangle = \frac{1}{\sqrt{n}} \left(|00\ldots01\rangle + |00\ldots10\rangle + \cdots + |01\ldots00\rangle + |10\ldots00\rangle\right), \quad (5)$$

the Groverian entanglement can be analytically calculated by the equation [10]

$$E_G\left(W_n\right) = \sqrt{1 - \left(\frac{n-1}{n}\right)^{n-1}}. \quad (6)$$

The comparison between the analytical result given by (6) and the numerical result achieved by our genetic algorithm, can be seen in Fig. 1.

In Fig. 2 one can see the Groverian entanglement of the state $a\,|000\rangle + \sqrt{1-a^2}\,|111\rangle$, obtained numerically. Its maximum occurs when $a = 1/\sqrt{2}$, $E_G = 0.707$.

The second entanglement measure to be considered is the residual entanglement. It was firstly proposed in [6] and modified and generalized in [7], where instead of using the concurrence for measuring the bipartite entanglements, the negativity was employed. Considering pure three-qubit quantum states, the three-way entanglement can be measured by the residual entanglement measure π_3 defined in [7] as:

$$\pi_3 = \frac{1}{3}\left(\pi_A + \pi_B + \pi_C\right) \quad (7)$$

$$\pi_A = N^2_{A\_BC} - N^2_{AB} - N^2_{AC} \quad (8)$$

$$\pi_B = N^2_{B\_AC} - N^2_{AB} - N^2_{BC} \quad (9)$$

$$\pi_C = N^2_{C\_AB} - N^2_{AC} - N^2_{BC}, \quad (10)$$

where, for example, $N^2_{A\_BC} = 2\left(1 - T_r\rho^2_A\right)$ is the negativity of the system composed by the single subsystem A and the bipartite system BC, while $N_{AB} =$

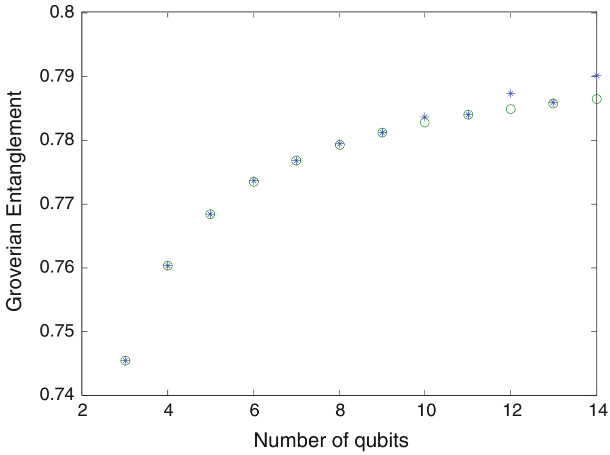

Fig. 1 Groverian entanglement for W states having 3–14 qubits. Analytical (o) and numerical (*) results

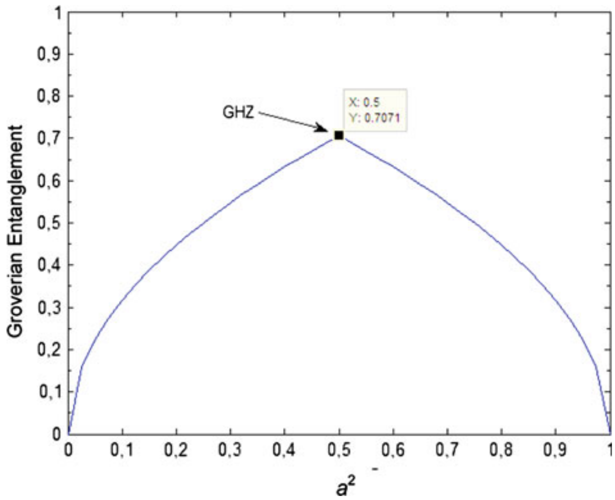

Fig. 2 Groverian entanglement for the state $a\,|000\rangle + \sqrt{1-a^2}\,|111\rangle$ versus a^2

$Tr\sqrt{\rho_{AB}^{T_A}\left(\rho_{AB}^{T_A}\right)^{\dagger}} - 1$ is the (normalized) negativity of the subsystems AB [4]. In general $\pi_A \neq \pi_B \neq \pi_C$ and, for W-class states, π_3 may be larger than zero. The extension for four-qubit states is straightforward

$$\pi_4 = \sqrt[4]{\pi_A \pi_B \pi_C \pi_D} \tag{11}$$
$$\pi_A = N_{A\_BCD}^2 - N_{AB}^2 - N_{AC}^2 - N_{AD}^2 \tag{12}$$
$$\pi_B = N_{B\_ACD}^2 - N_{AB}^2 - N_{BC}^2 - N_{BD}^2 \tag{13}$$
$$\pi_C = N_{C\_ABD}^2 - N_{AC}^2 - N_{BC}^2 - N_{CD}^2 \tag{14}$$
$$\pi_D = N_{D\_ABC}^2 - N_{AD}^2 - N_{BD}^2 - N_{CD}^2. \tag{15}$$

Here, we use the geometric mean instead of arithmetic mean because the latter is not zero for four-qubit states formed by the tensor product of a single-qubit and a three-way entangled tripartite state. Now, we show the properties that make π_4 a natural entanglement measure for pure four-qubit states. Firstly, we show that π_4 is invariant under local unitary (LU) operations. Let the state $|\psi\rangle$ to be a four-qubit state and $|\phi\rangle = U_A \otimes U_B \otimes U_C \otimes U_D|\psi\rangle$. Then,

$$\begin{aligned} N_{A\_BCD}^2(|\phi\rangle) &= 2\left(1 - Tr\left[\rho_A^2(|\phi\rangle)\right]\right) \\ &= 2\left(1 - Tr\left\{\left[U_A\rho_A(|\psi\rangle)U_A^{\dagger}\right]^2\right\}\right) \\ &= 2\left(1 - Tr\left[U_A\rho_A(|\psi\rangle)U_A^{\dagger}U_A\rho_A(|\psi\rangle)U_A^{\dagger}\right]\right) \\ &= 2\left(1 - Tr\left[U_A\rho_A^2(|\psi\rangle)U_A^{\dagger}\right]\right) \end{aligned}$$

$$= 2\left(1 - Tr\left[\rho_A^2\left(|\psi\rangle\right)\right]\right) = N_{A\_BCD}^2\left(|\psi\rangle\right). \tag{16}$$

The result (16) comes from the fact that the trace operation is the sum of the eigenvalues which are invariant under a unitary operation. Similarly, N_{B_ACD}, N_{C_ABD}, N_{D_ABC} are also LU invariant. The bipartite negativity proposed by Vidal and Werner is, itself, LU invariant, hence, $N_{AB}(\rho_{AB}) = N_{AB}[(U_A \otimes U_B)\rho_{AB}(U_A \otimes U_B)^\dagger]$. Regarding $|\phi\rangle = U_A \otimes U_B \otimes U_C \otimes U_D|\psi\rangle$ one has $\rho_{AB}(|\phi\rangle) = Tr_{CD}(|\phi\rangle\langle\phi|) = (U_A \otimes U_B)\rho_{AB}(|\psi\rangle)(U_A \otimes U_B)^\dagger$, hence $N_{AB}[\rho_{AB}(|\phi\rangle)] = N_{AB}[\rho_{AB}(|\psi\rangle)]$. Similarly, N_{AC}, N_{AD}, N_{BC}, N_{BD}, and N_{CD} are also LU invariant. Since all negativities used for definition of π_4 are LU invariant, π_4 is also LU invariant.

The proof that $\pi_4(|\psi\rangle) \geq 0$ comes straightforward from

$$N_{A_1A_2}^2 + N_{A_1A_3}^2 + \cdots + N_{A_1A_k}^2 \leq N_{A_1\_A_2\ldots A_k}^2, \tag{17}$$

proved in [7]. Now, let us assume that $\pi_4(|\psi\rangle) = 0$. This, implies that at least one $\pi_x(|\psi\rangle) = 0$ $(x = A, B, C, D)$. If $\pi_x(|\psi\rangle) = 0$ then one of the following conditions is satisfied: I) $(N_{x\_zyw} = 0, N_{xz} = N_{xy} = N_{xw} = 0)$ or II) $(N_{x\_zyw} = N_{xz}$ and $N_{xy} = N_{xw} = 0)$ or III) $(N_{x\_zyw} = N_{xy}$ and $N_{xz} = N_{xw} = 0)$ or IV) $(N_{x\_zyw} = N_{xw}$ and $N_{xy} = N_{xz} = 0)$. The condition (I) is satisfied if $|\psi\rangle = |\psi_x\rangle \otimes |\psi_{zyw}\rangle$($|\psi_{zyw}\rangle$ can have tripartite, bipartite or no entanglement). The condition (II) is satisfied if $|\psi\rangle = |\psi_{xz}\rangle \otimes |\psi_{yw}\rangle$($|\psi_{yw}\rangle$ can have bipartite or no entanglement), the condition (III) is satisfied if $|\psi\rangle = |\psi_{xy}\rangle \otimes |\psi_{zw}\rangle$ and the condition (IV) is satisfied if $|\psi\rangle = |\psi_{xw}\rangle \otimes |\psi_{zy}\rangle$. Hence, $\pi_4(|\psi\rangle) = 0$ only if $|\psi\rangle$ is a product state.

As an example, let us consider the state [13]

$$|\chi\rangle = \frac{|\xi_0\rangle + |\xi_1\rangle}{\sqrt{2}} \tag{18}$$

$$|\xi_0\rangle = [\cos(\theta)\,|0000\rangle - \sin(\theta)\,|0011\rangle - \sin(\phi)\,|0101\rangle + \cos(\phi)\,|0110\rangle]/\sqrt{2} \tag{19}$$

$$|\xi_1\rangle = [\cos(\phi)\,|1001\rangle + \sin(\phi)\,|1010\rangle + \sin(\theta)\,|1100\rangle + \cos(\theta)\,|1111\rangle]/\sqrt{2}. \tag{20}$$

The four-way entanglement measured by π_4 versus the angles θ and ϕ can be seen in Fig. 3.

In Fig. 3 one can see that $\pi_4 = 1$ when $\phi = \pi/2 - \theta$. In fact, it can be checked that $N(\rho_{A\_BCD}) = N(\rho_{B\_ACD}) = N(\rho_{C\_ABD}) = N(\rho_{D\_ABC}) = 1$ and $N(\rho_{AB}) = N(\rho_{AC}) = N(\rho_{AD}) = N(\rho_{BC}) = N(\rho_{BD}) = N(\rho_{CD}) = 0$ when $\phi = \pi/2 - \theta$. In Table 1 one can see the entanglement of several four-qubit states measured by E_G and π_4.

As it can be seen in Table 1, there are four-qubit states which have no four-way entanglement ($|\psi_1\rangle, \ldots, |\psi_4\rangle, |\xi_0\rangle, |\xi_1\rangle$) but their E_G is larger than zero. This happens because the Groverian entanglement detects the presence of any type of entanglement. For example, the state $|\psi_1\rangle$ has a bipartite entanglement and, hence, $|\psi_1\rangle$ cannot be written as a tensor product of four single-qubits states, thus, $P_{\max}$ will be

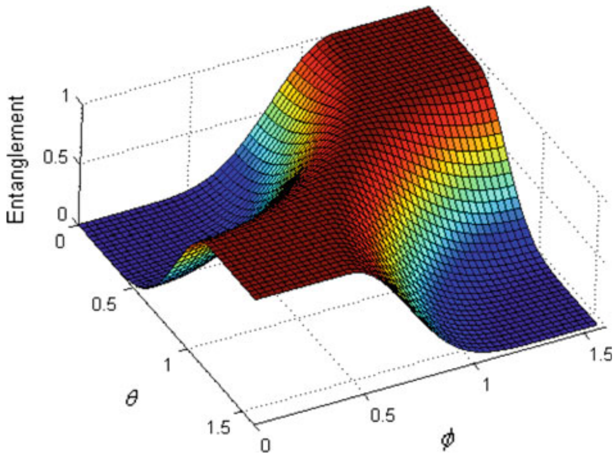

Fig. 3 Entanglement of χ, (18)–(20), versus θ and ϕ, measured by π_4

Table 1 Entanglement of several four-qubit states measured with E_G and π_4

Quantum state	E_G	π_4
$\lvert\psi_1\rangle = (\lvert 01\rangle + \lvert 10\rangle)/2^{1/2} \otimes (\lvert 0\rangle + \lvert 1\rangle)/2^{1/2} \otimes (\lvert 0\rangle + \lvert 1\rangle)/2^{1/2}$	0.707	0
$\lvert\psi_2\rangle = (\lvert 00\rangle + \lvert 11\rangle)/2^{1/2} \otimes (\lvert 00\rangle + \lvert 11\rangle)/2^{1/2}$	0.866	0
$\lvert\psi_3\rangle = (\lvert 000\rangle + \lvert 111\rangle)/2^{1/2} \otimes (\lvert 0\rangle + \lvert 1\rangle)/2^{1/2}$	0.707	0
$\lvert\psi_4\rangle = (\lvert 001\rangle + \lvert 010\rangle + \lvert 100\rangle)/3^{1/2} \otimes (\lvert 0\rangle + \lvert 1\rangle)/2^{1/2}$	0.745	0
$\lvert\xi_0\rangle = (\lvert 0000\rangle - \lvert 0011\rangle - \lvert 0101\rangle + \lvert 0110\rangle)/2$	0.707	0
$\lvert\xi_1\rangle = (\lvert 1001\rangle + \lvert 1010\rangle + \lvert 1100\rangle + \lvert 1111\rangle)/2$	0.707	0
$\lvert\psi_5\rangle \equiv \lvert\chi\rangle = (\lvert\xi_0\rangle + \lvert\xi_1\rangle)/2^{1/2}$	0.866	1
$\lvert\psi_6\rangle = (\lvert 0000\rangle + \lvert 1111\rangle)/2^{1/2}$	0.707	1
$\lvert\psi_7\rangle = (\lvert 0001\rangle + \lvert 0010\rangle + \lvert 0100\rangle + \lvert 1000\rangle)/2$	0.7603	0.6213
$\lvert\psi_8\rangle = (\lvert 0000\rangle + \lvert 0101\rangle + \lvert 1000\rangle + \lvert 1110\rangle)/2$	0.707	0.7140
$\lvert\psi_9\rangle = (\lvert 0000\rangle + \lvert 1011\rangle + \lvert 1101\rangle + \lvert 1110\rangle)/2$	0.81	0.9306
$\lvert\psi_{10}\rangle = (\lvert 0001\rangle + \lvert 0110\rangle + \lvert 1000\rangle)/3^{1/2}$	0.81	0.7995
$\lvert\psi_{11}\rangle = (\lvert 0000\rangle + \lvert 0111\rangle + \lvert 1011\rangle + \lvert 1100\rangle)/2$	0.866	1
$\lvert\psi_{12}\rangle = (\lvert 0000\rangle - \lvert 0101\rangle + \lvert 1010\rangle + \lvert 1111\rangle)/2$	0.866	1

lower than one making E_G larger than zero. It is interesting to observe that the states $\lvert\psi_6\rangle$, $\lvert\psi_7\rangle$, $\lvert\psi_8\rangle$, $\lvert\psi_9\rangle$ and $\lvert\psi_{10}\rangle$ correspond to different classes of entanglement [15]. This can be detected by $\pi_4$but not by E_G. Since the Groverian entanglement measure is not a reliable measure for genuine four-way entanglement, hereafter we are going to use only the residual entanglement π_4.

Now, let us consider the following graph states having four qubits [14]

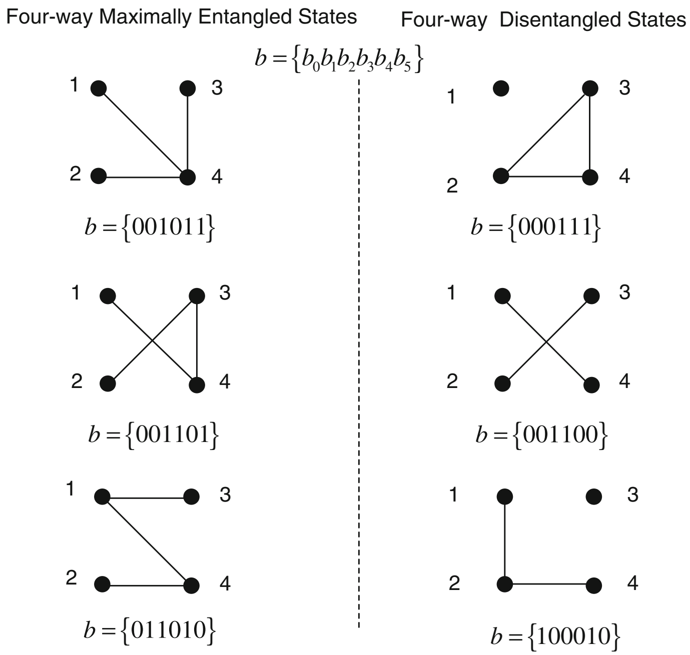

Fig. 4 Maximally entangled and disentanglement graph states according to π_4

$$|G\rangle = U_{12}^{b_0}U_{13}^{b_1}U_{14}^{b_2}U_{23}^{b_3}U_{24}^{b_4}U_{34}^{b_5}\left(\frac{|0\rangle+|1\rangle}{\sqrt{2}}\right)^{\otimes 4}. \tag{21}$$

$$U_{ij} = |0\rangle\langle 0|_i \otimes I_j + |1\rangle\langle 1|_i \otimes Z_j \tag{22}$$

In (21) U_{ij} is a controlled-phase gate applied on the qubits i and j, and $b_k \in \{0, 1\}$ ($k = 0, \ldots, 5$). It can be checked (there are only 64 possibilities) that a state in (21) is maximally entangled, $\pi_4 = 1$, if the graph that represents the state (for example, there is an edge between vertices 1 and 2 if $b_0 = 1$) is completely connected, otherwise the state has not genuine four-way entanglement, $\pi_4 = 0$. This has been pointed out in [16]. Some examples can be seen in Fig. 4.

An interesting search for highly entangled four-qubit states was realized analytically in [17] and numerically in [18]. The states obtained in [17] and [18] were, respectively,

$$|\psi_{HS}\rangle = \frac{1}{\sqrt{6}}\left[|1100\rangle + |0011\rangle + \left(-\frac{1}{2}+\frac{\sqrt{3}}{2}i\right)(|1010\rangle+|0101\rangle) + \left(-\frac{1}{2}+\frac{\sqrt{3}}{2}i\right)^2(|1001\rangle+|0110\rangle)\right] \tag{23}$$

$$|\psi_4\rangle = \frac{1}{2}\left[|0000\rangle + \left(\frac{|0\rangle+|1\rangle}{\sqrt{2}}\right)|011\rangle + |1101\rangle + \left(\frac{|0\rangle-|1\rangle}{\sqrt{2}}\right)|110\rangle\right] \tag{24}$$

For these states the residual entanglement with negativity is $\pi_4(|\psi_{HS}\rangle) = \pi_4(|\psi_4\rangle) = 1$. This result is different from the conclusion expressed in [18], which is based on a numerical optimization using the hill climbing algorithm. In [18] the authors found $E_{NPT}(|\psi_{HS}\rangle) \approx 6.0981$ while $E_{NPT}(|\psi_4\rangle) \approx 5.9142$, where E_{NPT} is an entanglement measure also based on the negative partial transpose criterion. The conclusion in [18] is that $|\psi_4\rangle$ is a highly entangled state but not a maximally entangled one. From our results and from the fact that the difference between $E_{NPT}(|\psi_{HS}\rangle)$ and $E_{NPT}(|\psi_4\rangle)$ may be a numerical error (common in optimization algorithms), we believe the authors in [18] have succeeded in finding a new maximally entangled four-qubit state: $|\psi_4\rangle$. Finally, the quantum state $|\psi_m\rangle = [|0000\rangle + (|0111\rangle + |1011\rangle + |1101\rangle + |1110\rangle)/2^{1/2}]/3^{1/2}$, proposed in [9], is also a maximally entangled four-qubit state $\pi_4(|\psi_m\rangle) = 1$.

3 Analysis of four-way entanglement variation

Initially, we are interested in observing the four-way entanglement variation after the propagation of a two-qubit state through two quantum noisy channels. Let us assume that each qubit of the quantum state $(|00\rangle_{AB} + |11\rangle_{AB})/2^{1/2}$ is sent through a noisy channel, modeled by the interaction with environment through the unitary operations U_A for qubit A and U_B for qubit B. This scheme is shown in Fig. 5.

Assuming $U_A = \exp(i\theta X \otimes X)$ and $U_B = \exp(i\phi X \otimes X)$, and that the initial state of the environment is $|0\rangle$ for both noisy channels, the entanglement of the total four-qubit state $|\Psi\rangle_{e_1 ABe_2}$ and the entanglement of the bipartite state ρ_{AB} at the channel's output can be calculated. The quantum states are given by

$$|\Psi\rangle_{e_1ABe_2} = (U_A \otimes U_B)\,|0\rangle_{e_1}\left(\frac{|00\rangle + |11\rangle}{\sqrt{2}}\right)_{AB}|0\rangle_{e_2} \tag{25}$$

$$\rho_{AB} = Tr_{e_1e_2}\left(|\Psi\rangle_{e_1ABe_2}\right) \tag{26}$$

In Fig. 6 one can see the entanglement $\pi_4\left(|\Psi\rangle_{e_1ABe_2}\right)$ and the Vidal–Werner negativity of ρ_{AB}, $N(\rho_{AB})$, versus the angles θ and ϕ. As expected, the larger π_4 the lower is N and vice-versa.

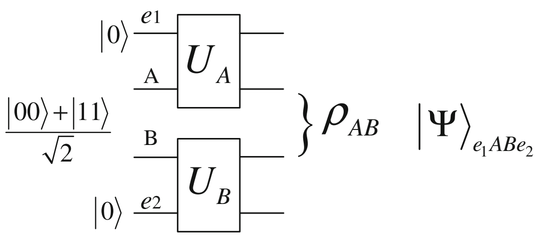

Fig. 5 Two entangled qubits sent through noisy channels modeled by unitary operations U_A and U_B

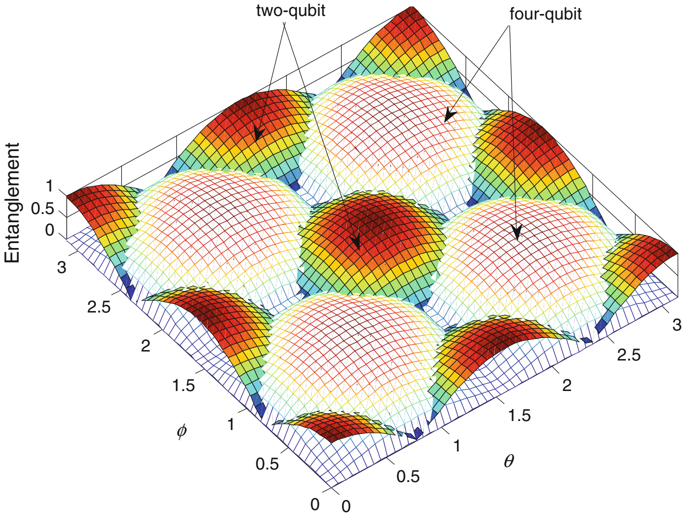

Fig. 6 Entanglements of the four-qubit and bipartite states shown in Fig. 5

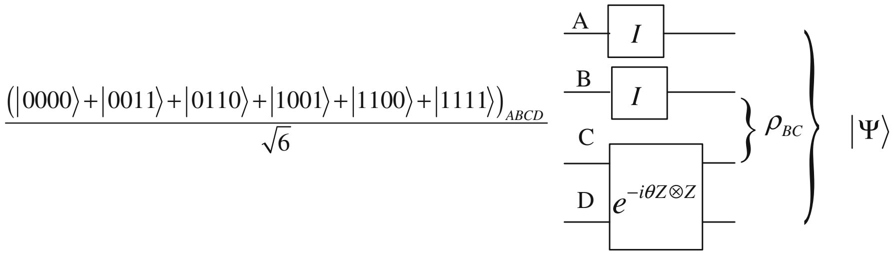

Fig. 7 Quantum circuit for observation of entanglement death and survival

Now, let us consider the scheme shown in Fig. 7, where I is the identity gate. The output state $|\psi\rangle$ is

$$\begin{aligned} |\psi\rangle &= \left(I \otimes I \otimes \mathrm{e}^{-i\theta Z\otimes Z}\right) \frac{(|0000\rangle + |0011\rangle + |0110\rangle + |1001\rangle + |1100\rangle + |1111\rangle)_{ABCD}}{\sqrt{6}} \\ &= \frac{\left(\mathrm{e}^{-i\theta}\,|0000\rangle + \mathrm{e}^{-i\theta}\,|0011\rangle + \mathrm{e}^{i\theta}\,|0110\rangle + \mathrm{e}^{i\theta}\,|1001\rangle + \mathrm{e}^{-i\theta}\,|1100\rangle + \mathrm{e}^{-i\theta}\,|1111\rangle\right)_{ABCD}}{\sqrt{6}}. \end{aligned} \tag{27}$$

For such state, one has

$$\rho_{BC} = Tr_{AD}\left(|\psi\rangle\right) = \begin{bmatrix} 1/3 & 0 & 0 & 1/3\cos(2\theta) \\ 0 & 1/6 & 0 & 0 \\ 0 & 0 & 1/6 & 0 \\ 1/3\cos(2\theta) & 0 & 0 & 1/3 \end{bmatrix} \tag{28}$$

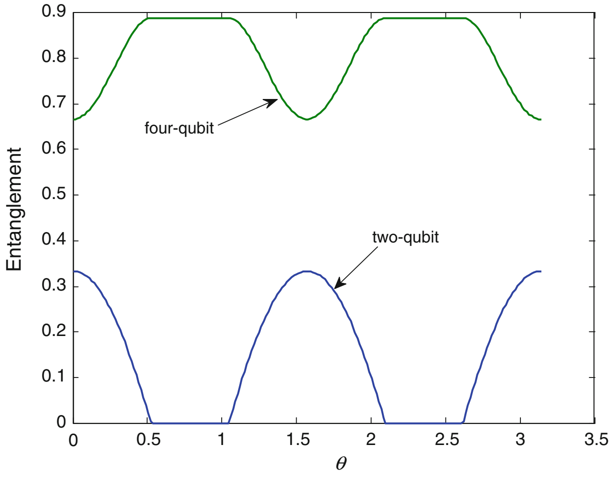

Fig. 8 Entanglements of the four-qubit and bipartite states given in (27) and (28), respectively

whose concurrence, calculated using Wootters' equation [1], is given by

$$C\left(\rho_{BC}\right)=2\max\left\{0,\frac{1}{9}\cos^2\left(2\theta\right)-\frac{1}{36}\right\}. \tag{29}$$

Hence, ρ_{BC} is disentangled for θ belonging to the intervals $[\pi/6,\pi/3]$ and $[2\pi/3,5\pi/6]$. Figure 8 shows $\pi_4(|\psi\rangle)$ and $N(\rho_{BC})$, versus θ. As can be seen in Fig. 8, the bipartite entanglement death and survival is followed, respectively, by an increase and decrease of the four-way entanglement.

Finally, let us consider the scheme shown in Fig. 9 where $|\Phi^{\pm}\rangle=(|00\rangle\pm|11\rangle)/2^{1/2}$ and $|\Psi^{\pm}\rangle=(|01\rangle\pm|10\rangle)/2^{1/2}$.
The output state $|\psi\rangle$ and its individual bipartite states are

$$\begin{aligned}|\psi\rangle&=\left(I\otimes \mathrm{e}^{i\theta X\otimes X}\otimes I\right)\left(\left|\Phi^{+}\right\rangle\left|\Phi^{+}\right\rangle+\left|\Phi^{-}\right\rangle\left|\Phi^{-}\right\rangle+\left|\Psi^{+}\right\rangle\left|\Psi^{+}\right\rangle+\left|\Psi^{-}\right\rangle\left|\Psi^{-}\right\rangle\right)_{ABCD}/2\\&=\frac{1}{2}\left[\begin{array}{l}\cos\left(\theta\right)|0000\rangle+i\sin\left(\theta\right)|0011\rangle+\cos\left(\theta\right)|0101\rangle+i\sin\left(\theta\right)|0110\rangle+\\ i\sin\left(\theta\right)|1001\rangle+\cos\left(\theta\right)|1010\rangle+i\sin\left(\theta\right)|1100\rangle+\cos\left(\theta\right)|1111\rangle\end{array}\right]\end{aligned} \tag{30}$$

$$\rho_{AB}=\rho_{AD}=\rho_{BC}=\rho_{CD}=I^{(4)}/4 \tag{31}$$

$$\rho_{AC}=\rho_{BD}=\begin{bmatrix}\cos^2\left(\theta\right) & 0 & 0 & \cos^2\left(\theta\right)\\ 0 & \sin^2\left(\theta\right) & \sin^2\left(\theta\right) & 0\\ 0 & \sin^2\left(\theta\right) & \sin^2\left(\theta\right) & 0\\ \cos^2\left(\theta\right) & 0 & 0 & \cos^2\left(\theta\right)\end{bmatrix}. \tag{32}$$

The eigenvalues of $\rho_{AC}^{T_A}\left(=\rho_{BD}^{T_B}\right)$ are $[1/2-\cos^2(\theta),-1/2+\cos^2(\theta),1/2,1/2]$, hence, ρ_{AC} and ρ_{BD} are disentangled only when $\theta=\pi/4$ or $\theta=3\pi/4$. Figure 10 shows $\pi_4(|\psi\rangle)$ and $N(\rho_{AC})(=N(\rho_{BD}))$, versus θ.

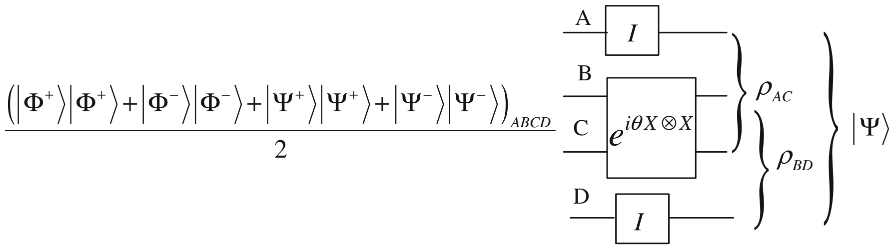

Fig. 9 Quantum circuit for observation of four-way (π_4) and bipartite entanglement variations

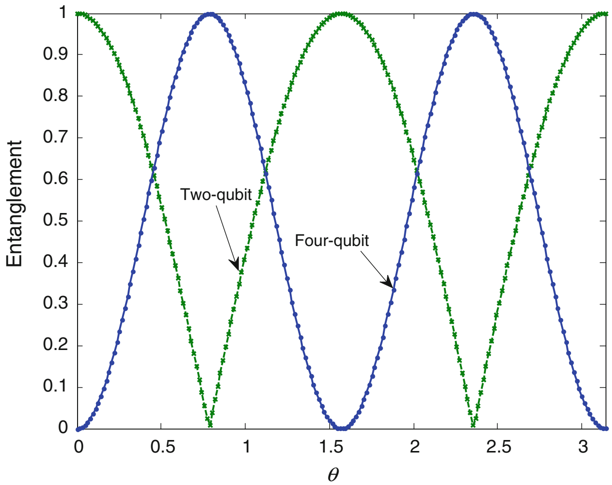

Fig. 10 Four-way ($|\psi\rangle$ given in (30)) and bipartite entanglements ($N(\rho_{AC}) = N(\rho_{BD})$) versus θ

Although the two-qubit gate is applied at qubits B and C, ρ_{BC} is always separable. Furthermore, the four-way entanglement varies in a smooth way, while the bipartite entanglement has a discontinuity at the points where it goes to zero. Furthermore, from Fig. 10 we see that the state obtained from (30) with $\theta = \pi/4$, is also a maximally entangled four-qubit state.

4 Conclusions

We have introduced a four-way entanglement measure: the residual entanglement based on negativity, π_4. However, differently from initially proposed in the literature, we use the geometric mean instead of arithmetic mean because the latter is not zero for four-qubit states formed by the tensor product of a single-qubit and a three-way entangled tripartite state. The residual entanglement π_4 was compared with the Groverian entanglement. The entanglements of several four-qubit states were calculated and the residual entanglement is considered more appropriate, for the four-way entanglement calculation, than the Groverian entanglement. This happen because the Groverian entanglement will be larger than zero for any quantum state that is not fully decomposable in the tensor product of single-qubit states, indicating a false four-way entanglement in states of the type $|\psi_x\rangle \otimes |\psi_{zyw}\rangle(|\psi_{zyw}\rangle$ has three-way entanglement), $|\psi_{xz}\rangle \otimes |\psi_{yw}\rangle(|\psi_{xz}\rangle$ and $|\psi_{yw}\rangle$ have bipartite entanglement), $|\psi_x\rangle \otimes |\psi_z\rangle \otimes |\psi_{yw}\rangle(|\psi_{yw}\rangle$ has bipartite entanglement). On the other hand, as shown earlier, $\pi_4$is zero for any quantum state that has not four-way entanglement.

After that, we analyzed the variation of global (four-way) and sub-global (bipartite) entanglements of four-qubit states processed by two-qubit gates. As expected, when the bipartite entanglement (of two-qubits) increases (decreases) the four-way entanglement decreases (increases). However, this entanglement exchange is not a trivial question and it depends on the particular four-qubit state considered. For example, the last case considered, (30), four bipartite states (ρ_{AB}, ρ_{AD}, ρ_{BC}, and ρ_{BD}) have always a null entanglement while two bipartite states (ρ_{AC} and ρ_{BD},) have equal entanglement behavior, which increases (decreases) when the four-way entanglement decreases (increases). On the other hand, the four-way entanglement of the quantum state $\cos(\theta)|0000\rangle + \sin(\theta)|1111\rangle$ depends on θ while all bipartite states have zero entanglement for any value of θ.

Acknowledgments This work was supported by the Brazilian agencies FUNCAP/CAPES and CNPq (process no. 303514/2008-6).

References

1. Wootters, W.K.: Entanglement of formation of an arbitrary state of two qubits. Phys. Rev. Lett. **80**, 2245 (1996)
2. Vedral, V., Plenio, M.B., Ripping, M.A., Knight, P.L.: Quantifying entanglement. Phys. Rev. Lett. **78**, 2275 (1997)
3. Bennett, C.H., Popescu, S., Rohrlich, D., Smolin, J., Thapliyal, A.: Exact and asymptotic measures of multipartite pure-state entanglement. Phys. Rev. A **63**, 012307 (2001)
4. Vidal, G., Werner, R.F.: A computable measure of entanglement. quant-ph/0102117 (2001)
5. Pan, F., Liu, D., Lu, G., Draayer, J.P.: Extremal entanglement for triqubit pure states. Phys. Lett. A **336**, 384 (2005)
6. Coffman, V., Kundu, J., Wootters, W.K.: Distributed entanglement. Phys. Rev. A **61**, 052306 (2000)
7. Ou, Y.-C., Fan, H.: Monogamy inequality in terms of negativity for three-qubti states. Phys. Rev. A **75**, 062308/1–062308/5 (2007)
8. Yu, C.S., Song, H.: Free entanglement measure of multiparticle quantum states. Phys. Lett. A **330**, 377 (2004)
9. Love, P.J., van den Brink, A.M., Smirnov, A.Y., Amin, M.H.S., Grajcat, M., Il'ichev, E., Izmalkov, A., Zagoskin, A.M.: A characterization of global entanglement. Quant. Inf. Proc. **6**(3), 187–195 (2007)
10. Shimoni, Y., Shapira, D., Biham, O.: Characterization of pure quantum state of multiple qubits using the Groverian entanglement measure. Phys. Rev. A **69**, 062303/1–062303/4 (2004)
11. Chamoli, A., Bhandari, C.M.: Success rate and entanglement measure in Grover's search algorithm for certain kinds of four qubit states. Phys. Lett. A **346**, 17–26 (2005)
12. Chamoli, A., Bhandari, C.M.: Groverian entanglement measure and evolution of entanglement in search algorithm for n(=3,5)-qubit systems with real coefficients. Quant. Inf. Process. **6**(4), 255–271 (2007)
13. Yeo, Y., Chua, W.K.: Teleportation and dense coding with genuine multipartite entanglement. Phys. Rev. Lett. **96**, 060502/1–060502/4 (2006)
14. Ye, M.-Y., Lin, X.-M.: A genuine four-partite entangled state, xxx.lanl.gov quant-ph 0801.0908 (2008)
15. Verstraete, F., Dehaene, J., De Moor, B., Verschelde, H.: Four qubits can be entangled in nine different ways. Phys. Rev. A **65**(5), 052112/1–052112/5 (2002)
16. Cai, J.-M., Zhou, Z.-W., Guo, G.-C.: Fully multi-qubit entangled states. xxx.lanl.gov quant-ph 0609186 (2007)
17. Higuchi, A., Sudbery, A.: How entangled can two couples get? Phys. Lett. A **273**, 213–217 (2000)
18. Brown, I.D.K., Stepney, S., Sudbery, A., Braunstein, S.L.: Searching for highly entangled multi-qubit states. J. Phys. A Math. Gen. **38**, 1119–1131 (2005)