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Concurrence in the framework of coherent states

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Abstract The concurrence of a two-qubit nonorthogonal pure state is determined through the construction of this state in the language of spin coherent states. The generalization of this method to the case of a class of mixed states is given. The concurrence in this case is nothing but a function of the amplitude of the spin coherent states, it is shown also that probability present an interesting behavior.

Keywords Quantum information · Entanglement · Concurrence · Coherent states

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1 Introduction

The entanglement phenomenon has been studied in quantum mechanics as a specific quantum mechanical nonlocal correlation [1,2]. Recently it has been regarded as a resource for quantum information processing and transmission, we notice the case

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of the quantum cryptography [3–5], the quantum teleportation [6] and the quantum computation [7]. Generally, there are two important problems for quantum entanglement, the first is to find a method to determine whether a given state of a composite quantum system, consisting of two or more subsystem is entangled or not, the second is to define the best measure quantifying the amount of entanglement of a given state. Thus, the characterization and quantification of the entanglement have attracted much attention [8–10] and became a fundamental problem in the field. To quantify the entanglement, various entanglement measures are proposed, such as concurrence [9–11], entanglement of formation [12,13], tangle [14,15] and negativity [16–19].

For bipartite pure states, the concurrence and entanglement of formation which was proposed by Bennett et al. [12] are the best measures of entanglement and are widely accepted. However, for quantifying the entanglement of mixed states there is no such general methods, so the problem in these measures depend on the pure state decompositions, in this way the main difficulty is to find the minimization over all decompositions of mixed state into pure states. For particular cases many minimization can be done analytically [9,20–22], but this problem is not solved in the general case. For the special case of two-qubits, Wootters and Hill [23] have proposed as a measure of entanglement the famous formula for the bipartite concurrence and the associated entanglement of formation.

In quantum optics and quantum information theory, the theory of coherent states (quasi-classical states) is one of the most important and widely used concepts. These states are very useful tool for the study of various problems in physics [24,25]. The concept of coherent sates introduced by Schrödinger [26] in the context of the harmonic oscillator, has been extensively studied in physics [27,28]. Later the term coherent state has become very important in quantum optics due to Glauber [29], who he demonstrated that these states have the interesting property of minimizing the Heisenberg uncertainty relation. The next most important coherent states are spin coherent states or SU(2) coherent states [30], they describe several systems and also have some applications in quantum optics, statistical mechanics and condensed matter physics [27,28]. The general treatment of coherent sates can be found in references [27,30,31]. Spin coherent states are also widely used in quantum information theory [32,33]. In fact, spin coherent states are viewed as the closet quantum states to Glauber's coherent states. These are, inturn, viewed as the closet to classical states. They have raised the idea of studying the entanglement of spin coherent states as a measure of their classicality.

The aim of this paper is to propose a useful method for measuring the entanglement of two-qubit nonorthogonal states which play an important role in the quantum cryptography and quantum information processing. This method is based on spin coherent states. We use the concurrence as a measure of entanglement. We formulate it in terms of the amplitudes of spin coherent states and we study its behavior in detail.

This paper is organized as follows: In Sect. 2 we define the concurrence of a two-qubit state for the pure and mixed cases. In Sect. 3 we give the concurrence in terms of the spin coherent states and we study its behavior in the case of two-qubit nonorthogonal states for both cases. The conclusion is given in Sect. 4.

2 Concurrence of an arbitrary state of two qubits

The most general state of two-qubits can be expressed by

$$|\psi\rangle = a|00\rangle + b|01\rangle + c|10\rangle + d|11\rangle \tag{1}$$

with the normalization condition

$$|a|^2 + |b|^2 + |c|^2 + |d|^2 = 1.$$

The state $|\psi\rangle$ is factorizable or (separable) if $ad = bc$, otherwise it is entangled or (inseparable). For such a state the concurrence $C(\psi)$ is defined by [34]

$$\begin{aligned} C(\psi) &= |\langle\psi|\tilde{\psi}\rangle| \\ &= 2|ad - bc| \end{aligned} \tag{2}$$

where

$$|\tilde{\psi}\rangle = (\sigma_y \otimes \sigma_y)|\psi^*\rangle$$

is the spin-flip operation applied to pure state of two-qubits, and $|\psi^*\rangle$ is the complex conjugate of $|\psi\rangle$.
The concurrence is related to the entanglement of formation by the equation

$$E_f(\psi) = \xi\,(C(\psi)) \tag{3}$$

where

$$\xi(C) = H\left(\frac{1 + \sqrt{1 - C^2}}{2}\right). \tag{4}$$

Here, we have introduced the binary entropy function

$$H(x) = -x \ln x - (1 - x) \ln(1 - x) \tag{5}$$

The function $\xi(C)$ is monotonically increasing and ranges from 0 to 1 as C goes from 0 to 1.

In the case of mixed states it is more convenient to describe the quantum state by the density operator ρ defined by

$$\rho = \sum_i p_i |\psi_i\rangle\langle\psi_i|, \tag{6}$$

where $\{|\psi_i\rangle\}$ is a set (but not necessarily orthogonal) of normalized pure states of two-qubit system, and p_i's are positive real numbers that add up to one.

The state ρ is said to be separable if it can be written as a convex combination of product states, i.e., $\rho = \sum_i p_i \rho_i^R \otimes \rho_i^{R'}$ where $\rho_i^{R,R'}$ is the reduced density matrix given by $\rho_i^{(R,R')} = Tr_{(R',R)}|\psi_i\rangle\langle\psi_i|$.

The concurrence of the mixed state ρ can be defined as the average concurrence of the pure states minimized over all decompositions of ρ [34]

$$C(\rho) = \inf \sum_i p_i C(\psi_i) \tag{7}$$

where $C(\psi_i)$ is the concurrence of the pure state $|\psi_i\rangle$. The creative contribution of Wootters and Hill [23] consists to fining an explicit formula for $C(\rho)$. It is

$$C(\rho) = \max\{\lambda_1 - \lambda_2 - \lambda_3 - \lambda_4, 0\} \tag{8}$$

where λ_i are the square roots of the eigenvalues of the non-Hermitian matrix $\rho\tilde{\rho}$ in descending order. Note that each λ_i is a positive real number ($\tilde{\rho}$ being the spin-flip operation of ρ given by

$$\tilde{\rho} = (\sigma_y \otimes \sigma_y)\rho^*(\sigma_y \otimes \sigma_y)$$

and ρ^* denotes the complex conjugate of ρ).

We can also express the entanglement of formation in terms of the concurrence as

$$E_f(\rho) = \xi\,(C(\rho)) \tag{9}$$

where $C(\rho)$ is given by Eq. 8 and the function ξ is given by Eq. 4.

For bipartite systems with no more than two non vanishing eigenvalues there is an explicit formula for the square of the concurrence [35]. In the case of two-qubits, an obvious expression of concurrence of the mixed state by two orthogonal pure states [36,37].

Indeed, for simplicity we take the density operator given by Eq. 6 as

$$\rho = \mu_1|\mu_1\rangle\langle\mu_1| + \mu_2|\mu_2\rangle\langle\mu_2| \tag{10}$$

i.e., a mixed state of two-qubits with no more than two non vanishing eigenvalues denoted as μ_1 and μ_2. Without loss of generality, we take the pure states $|\mu_1\rangle$ and $|\mu_2\rangle$ as follows:

$$\begin{aligned}|\mu_1\rangle &= a_1|00\rangle + b_1|01\rangle + c_1|10\rangle + d_1|11\rangle\\ |\mu_2\rangle &= a_2|00\rangle + b_2|01\rangle + c_2|10\rangle + d_2|11\rangle.\end{aligned} \tag{11}$$

The concurrence of two-qubit mixed state ρ is

$$C^2(\rho) = \left(\mu_1^2 C_1^2 + \mu_2^2 C_2^2\right) + \frac{1}{2}\mu_1\mu_2|\mathcal{C}_+ - \mathcal{C}_-|^2 - \frac{1}{2}\mu_1\mu_2\left|(\mathcal{C}_+ - \mathcal{C}_-)^2 - 4\mathcal{C}_1\mathcal{C}_2\right| \tag{12}$$

where

$$C_i = |\mathcal{C}_i| = 2|a_i d_i - b_i c_i| \quad (i = 1, 2)$$

is the concurrence of the pure state $|\mu_i\rangle$, and

$$C_\pm = |\mathcal{C}_\pm| = |(a_1 \pm a_2)(d_1 \pm d_2) - (b_1 \pm b_2)(c_1 \pm c_2)|$$

is the concurrence of the pure state $|\mu_\pm\rangle = \frac{1}{\sqrt{2}}(|\mu_1\rangle \pm |\mu_2\rangle)$. $\mathcal{C}_i$ and $\mathcal{C}_\pm$ are the corresponding complex concurrences.

3 Concurrence in the framework of spin coherent states

3.1 The case of pure states

The coherent states have been established to be an important type of robust states which are extensively applied for various quantum information processing and transmission tasks. Those coherent states are chooser since they are easy to generate experimentally and convenient to use.

Generally a qubit can be represented through a phase factor as [33]

$$\begin{aligned} |\theta, \varphi\rangle &= \exp\left[-\frac{\theta}{2}\left(\sigma_+ \mathrm{e}^{-i\varphi} - \sigma_- \mathrm{e}^{i\varphi}\right)\right]|1\rangle \\ &= \cos\frac{\theta}{2}|0\rangle + \mathrm{e}^{i\varphi}\sin\frac{\theta}{2}|1\rangle \end{aligned} \tag{13}$$

where $\sigma\pm = \sigma_x \pm i\sigma_y$, (σ_x, σ_y) are the Pauli matrices and (θ, φ) are real parameters.

One can demonstrate that the above equation represents a spin-coherent state of the Klauder-Perelomov [30]

$$\begin{aligned} |\xi, j\rangle &= R(\xi)|0, j\rangle \\ &= \exp\left(\eta(\xi)J_+ - \eta^*(\xi)J_-\right)|0, j\rangle \\ &= \frac{1}{\left(1 + |\xi|^2\right)^j}\sum_{n=0}^{2j}\binom{2j}{n}^{\frac{1}{2}}\xi^n|n, j\rangle \end{aligned} \tag{14}$$

where $R(\xi)$ is the rotation operator. J_+ and J_- represent the raising and lowering operators of SU(2) Lie algebra satisfying the commutation relation

$$[J_+, J_-] = 2J_z; \quad [J_z, J_\pm] = \pm J_\pm \tag{15}$$

acting on an irreductible unitary representation as follows

$$J_{\pm}|j,m\rangle = \sqrt{(j \mp m)(j \pm m + 1)}|j, m \pm 1\rangle; \quad J_z|j,m\rangle = m|j,m\rangle. \tag{16}$$

The spin coherent states are obtained by successively applying the raising operator on the state $|j,-j\rangle$.

$$|\xi, j\rangle = \frac{1}{\left(1+|\xi|^2\right)^j} \sum_{m=-j}^{j} \left[\frac{(2j)!}{(j+m)!(j-m)!}\right]^{\frac{1}{2}} \xi^{j+m}|j,m\rangle. \tag{17}$$

A change of variable $n = j + m$ will give the formula in (14).

For a particle with spin$\frac{1}{2}$ it gives

$$\begin{aligned} \left|\xi, \frac{1}{2}\right\rangle &= \frac{1}{\left(1+|\xi|^2\right)^{\frac{1}{2}}} \sum_{n=0}^{1} \binom{1}{n}^{\frac{1}{2}} \xi^n \left|n, \frac{1}{2}\right\rangle \\ &= \frac{1}{\left(1+|\xi|^2\right)^{\frac{1}{2}}} \left(\left|0, \frac{1}{2}\right\rangle + \xi \left|1, \frac{1}{2}\right\rangle\right) \end{aligned} \tag{18}$$

and for $\xi = \tan\left(\frac{\theta}{2}\right) \mathrm{e}^{i\varphi}$ we find

$$|\xi\rangle = \cos\frac{\theta}{2}|0\rangle + \mathrm{e}^{i\varphi} \sin\frac{\theta}{2}|1\rangle \tag{19}$$

where $|0, \frac{1}{2}\rangle \equiv |0\rangle$ and $|1, \frac{1}{2}\rangle \equiv |1\rangle$ are the basis states.

A separable pure state of two-qubits can be expressed as $|\theta_1, \varphi_1\rangle \otimes |\theta_2, \varphi_2\rangle$. Consequently the pure state that represents the simplest extension of the separable pure state to entangled pure state of two-qubits can be given by the unnormalized state [33]

$$||\psi\rangle = \cos\theta|\theta_1, \varphi_1\rangle \otimes |\theta_2, \varphi_2\rangle + \mathrm{e}^{i\varphi} \sin\theta|\theta_1', \varphi_1'\rangle \otimes |\theta_2', \varphi_2'\rangle \tag{20}$$

$|\theta_1, \varphi_1\rangle$ and $|\theta_1', \varphi_1'\rangle$ are normalized states of the qubit 1 and $|\theta_2, \varphi_2\rangle$ and $|\theta_2', \varphi_2'\rangle$ are states of the qubit 2, such as

$$\langle\theta_1, \varphi_1|\theta_1', \varphi_1'\rangle \neq 0; \quad \langle\theta_2, \varphi_2|\theta_2', \varphi_2'\rangle \neq 0.$$

From Eq. 18, we obtain

$$||\psi\rangle = \cos\theta|\alpha\rangle \otimes |\beta\rangle + \mathrm{e}^{i\varphi} \sin\theta|\alpha'\rangle \otimes |\beta'\rangle \tag{21}$$

where

$$|\alpha\rangle = \frac{1}{\sqrt{\left(1+|\alpha|^2\right)}}\left(|0\rangle + \alpha|1\rangle\right)$$
$$|\beta\rangle = \frac{1}{\sqrt{\left(1+|\beta|^2\right)}}\left(|0\rangle + \beta|1\rangle\right)$$
$$|\alpha'\rangle = \frac{1}{\sqrt{\left(1+|\alpha'|^2\right)}}\left(|0\rangle + \alpha'|1\rangle\right)$$
$$|\beta'\rangle = \frac{1}{\sqrt{\left(1+|\beta'|^2\right)}}\left(|0\rangle + \beta'|1\rangle\right)$$

are respectively the states for each qubit.

In this case Eq. 21 becomes

$$\begin{aligned} ||\psi\rangle &= \frac{\cos\theta}{\sqrt{(1+|\alpha|^2)(1+|\beta|^2)}}\left(|00\rangle + \beta|01\rangle + \alpha|10\rangle + \alpha\beta|11\rangle\right) \\ &\quad + \frac{e^{i\varphi}\sin\theta}{\sqrt{(1+|\alpha'|^2)(1+|\beta'|^2)}}\left(|00\rangle + \beta'|01\rangle + \alpha'|10\rangle + \alpha'\beta'|11\rangle\right) \\ &= a|00\rangle + b|01\rangle + c|10\rangle + d|11\rangle \end{aligned} \quad (22)$$

where

$$\begin{aligned} a &= \lambda + \gamma \\ b &= \beta\lambda + \beta'\gamma \\ c &= \alpha\lambda + \alpha'\gamma \\ d &= \alpha\beta\lambda + \alpha'\beta'\gamma \end{aligned}$$

with

$$\lambda = \frac{\cos\theta}{\sqrt{(1+|\alpha|^2)(1+|\beta|^2)}}; \quad \gamma = \frac{e^{i\varphi}\sin\theta}{\sqrt{(1+|\alpha'|^2)(1+|\beta'|^2)}}.$$

Finally the normalized pure state of two-qubits may be written as

$$|\psi\rangle = \frac{1}{\sqrt{\mathcal{N}}}\left(a|00\rangle + b|01\rangle + c|10\rangle + d|11\rangle\right) \quad (23)$$

where

$$\mathcal{N} = \langle\psi||\psi\rangle = |a|^2 + |b|^2 + |c|^2 + |d|^2.$$

The concurrence of this state is given by

$$\begin{aligned} C &= |\langle\psi|\tilde{\psi}\rangle| \\ &= 2\left|\frac{\lambda\gamma}{\mathcal{N}}(\alpha-\alpha')(\beta-\beta')\right| \end{aligned} \tag{24}$$

- $C=0$ for $\lambda=0,\ \ \gamma=0,\ \ \alpha=\alpha'$ or $\beta=\beta'$, which corresponds to a factorizable state $|\psi\rangle$.
- $2\left|\frac{\lambda\gamma}{\mathcal{N}}(\alpha-\alpha')(\beta-\beta')\right|=1$, for a Bell state.

3.2 The case of mixed states

In this section, we present the advantages of the expression given by Eq. 12; it depends on concurrence of pure states and their simple combinations. Furthermore, it can be solved easily analytically and describes an important result in quantum information area.

Let us consider a class of mixed states given by a statistical mixture of two pure states of two-qubit system

$$\rho=\sum_i p_i|\psi_i\rangle\langle\psi_i| \quad (i=1,2) \tag{25}$$

where

$$|\psi_i\rangle=\frac{1}{\sqrt{\mathcal{N}_i}}\left(a_i|00\rangle+b_i|01\rangle+c_i|10\rangle+d_i|11\rangle\right)$$

are the pure states of two-qubits, with

$$\begin{aligned} a_i &= \lambda_i+\gamma_i \\ b_i &= \beta_i\lambda_i+\beta_i'\gamma_i \\ c_i &= \alpha_i\lambda_i+\alpha_i'\gamma_i \\ d_i &= \alpha_i\beta_i\lambda_i+\alpha_i'\beta_i'\gamma_i \\ \mathcal{N}_i &= \langle\psi_i||\psi_i\rangle=|a_i|^2+|b_i|^2+|c_i|^2+|d_i|^2. \end{aligned}$$

The expression of the concurrence of the mixed state of two orthogonal states given by Eq. 12 can be directly generalized to the case of mixed state of two nonorthogonal states [38]. So, in our case we have

$$C^2(\rho)=(p_1^2C_1^2+p_2^2C_2^2)+\frac{1}{2}p_1p_2|\mathcal{C}_+-\mathcal{C}_-|^2-\frac{1}{2}p_1p_2|(\mathcal{C}_+-\mathcal{C}_-)^2-4\mathcal{C}_1\mathcal{C}_2| \tag{26}$$

where

$$C_i = |\mathcal{C}_i| = 2\left|\frac{\lambda_i\gamma_i}{\mathcal{N}_i}(\alpha_i - \alpha_i')(\beta_i - \beta_i')\right|$$

is concurrence of the pure state $|\psi_i\rangle$, and

$$\begin{aligned} C_\pm &= |\mathcal{C}_\pm| \\ &= \left|\frac{\lambda_1\gamma_1}{\mathcal{N}_1}(\alpha_1 - \alpha_1')(\beta_1 - \beta_1') + \frac{\lambda_2\gamma_2}{\mathcal{N}_2}(\alpha_2 - \alpha_2')(\beta_2 - \beta_2')\right. \\ &\quad \pm \frac{1}{\sqrt{\mathcal{N}_1\mathcal{N}_2}}(\lambda_1\lambda_2(\alpha_1 - \alpha_2)(\beta_1 - \beta_2) + \lambda_1\gamma_2(\alpha_1 - \alpha_2')(\beta_1 - \beta_2') \\ &\quad \left. + \lambda_2\gamma_1(\alpha_1' - \alpha_2)(\beta_1' - \beta_2) + \gamma_1\gamma_2(\alpha_1' - \alpha_2')(\beta_1' - \beta_2'))\right| \end{aligned} \tag{27}$$

is the concurrence of the pure state $|\psi_\pm\rangle = \frac{1}{\sqrt{2}}(|\psi_1\rangle \pm |\psi_2\rangle)$.

The simplified expression of the concurrence in the Wootters measure of entanglement present some important features

- The concurrence is bounded

$$(p_1C_1 - p_2C_2)^2 \leq C^2(\rho) \leq (p_1C_1 + p_2C_2)^2 \tag{28}$$

where

$$(p_1C_1 - p_2C_2)^2 = 4\left(p_1\left|\frac{\lambda_1\gamma_1}{\mathcal{N}_1}(\alpha_1-\alpha_1')(\beta_1-\beta_1')\right| - p_2\left|\frac{\lambda_2\gamma_2}{\mathcal{N}_2}(\alpha_2-\alpha_2')(\beta_2-\beta_2')\right|\right)^2$$

and

$$(p_1C_1 + p_2C_2)^2 = 4\left(p_1\left|\frac{\lambda_1\gamma_1}{\mathcal{N}_1}(\alpha_1-\alpha_1')(\beta_1-\beta_1')\right| + p_2\left|\frac{\lambda_2\gamma_2}{\mathcal{N}_2}(\alpha_2-\alpha_2')(\beta_2-\beta_2')\right|\right)^2$$

are respectively the lower and upper bounds of concurrence.

- If the coefficients a_i, b_i, c_i and d_i are real numbers, then we have
For $0 \geq (\mathcal{C}_+ - \mathcal{C}_-)^2 \geq 4\mathcal{C}_1\mathcal{C}_2$, the concurrence is equal to the upper bound

$$C^2(\rho) = 4\left(p_1\left|\frac{\lambda_1\gamma_1}{\mathcal{N}_1}(\alpha_1 - \alpha_1')(\beta_1 - \beta_1')\right| + p_2\left|\frac{\lambda_2\gamma_2}{\mathcal{N}_2}(\alpha_2 - \alpha_2')(\beta_2 - \beta_2')\right|\right)^2 \tag{29}$$

For $0 \leq (\mathcal{C}_+ - \mathcal{C}_-)^2 \leq 4\mathcal{C}_1\mathcal{C}_2$, it is

$$C^2(\rho) = 4\left(p_1\left|\frac{\lambda_1\gamma_1}{\mathcal{N}_1}(\alpha_1-\alpha_1')(\beta_1-\beta_1')\right| - p_2\left|\frac{\lambda_2\gamma_2}{\mathcal{N}_2}(\alpha_2-\alpha_2')(\beta_2-\beta_2')\right|\right)^2 + \frac{4p_1p_2}{\mathcal{N}_1\mathcal{N}_2}\big(\lambda_1\lambda_2(\alpha_1-\alpha_2)(\beta_1-\beta_2)+\lambda_1\gamma_2(\alpha_1-\alpha_2')(\beta_1-\beta_2') + \lambda_2\gamma_1(\alpha_1'-\alpha_2)(\beta_1'-\beta_2)+\gamma_1\gamma_2(\alpha_1'-\alpha_2')(\beta_1'-\beta_2')\big)^2 \tag{30}$$

For $\mathcal{C}_1\mathcal{C}_2 \leq 0$, the concurrence equal to the lower bound

$$C^2(\rho) = 4\left(p_1\left|\frac{\lambda_1\gamma_1}{\mathcal{N}_1}(\alpha_1-\alpha_1')(\beta_1-\beta_1')\right| - p_2\left|\frac{\lambda_2\gamma_2}{\mathcal{N}_2}(\alpha_2-\alpha_2')(\beta_2-\beta_2')\right|\right)^2. \tag{31}$$

- When $\mathcal{C}_+ = \mathcal{C}_-$, i.e.,

$$\lambda_1\lambda_2(\alpha_1-\alpha_2)(\beta_1-\beta_2)+\gamma_1\gamma_2(\alpha_1'-\alpha_2')(\beta_1'-\beta_2') = \lambda_1\gamma_2(\alpha_1-\alpha_2')(\beta_2'-\beta_1) + \lambda_2\gamma_1(\alpha_1'-\alpha_2)(\beta_2-\beta_1')$$

  the concurrence reaches the lower bound

$$C^2(\rho) = 4\left(p_1\left|\frac{\lambda_1\gamma_1}{\mathcal{N}_1}(\alpha_1-\alpha_1')(\beta_1-\beta_1')\right| - p_2\left|\frac{\lambda_2\gamma_2}{\mathcal{N}_2}(\alpha_2-\alpha_2')(\beta_2-\beta_2')\right|\right)^2. \tag{32}$$

- In the case where $|\psi_1\rangle$ or $|\psi_2\rangle$ is factorizable, (i.e., $C_1 = 0$ or $C_2 = 0$), the concurrence of the mixed state becomes

$$C^2(\rho) = 4p_2^2\left|\frac{\lambda_2\gamma_2}{\mathcal{N}_2}(\alpha_2-\alpha_2')(\beta_2-\beta_2')\right|^2 \tag{33}$$

  or

$$C^2(\rho) = 4p_1^2\left|\frac{\lambda_1\gamma_1}{\mathcal{N}_1}(\alpha_1-\alpha_1')(\beta_1-\beta_1')\right|^2 \tag{34}$$

  This expression of concurrence shows that the probability p_1 (or p_2) and the pure state $|\psi_1\rangle$ (or $|\psi_2\rangle$) contain the information about the entanglement of the mixed state ρ.
  Without loss of generality, we consider simple case where $\alpha_i = \beta_i$ and $\alpha_i' = \beta_i'$. The concurrence has the simplified form

$$C^2(\rho) = 4p_2^2\left|\frac{\lambda_2\gamma_2}{\mathcal{N}_2}(\alpha_2-\alpha_2')^2\right|^2 \tag{35}$$

or

$$C^2(\rho) = 4p_1^2 \left| \frac{\lambda_1\gamma_1}{\mathcal{N}_1} (\alpha_1 - \alpha_1')^2 \right|^2 . \tag{36}$$

For simplicity we take α and α' to be real and set θ and φ to their values at the extremum of C (i.e., $\partial^2 C/\partial\theta\partial\varphi = 0$) [39]. This corresponds to $\theta = \pi/4$ and $\varphi = 0$, $C^2(\rho)$ is then given by

$$C^2(\rho) = p_i^2 \frac{(\alpha_i - \alpha_i')^4}{(2\,\alpha_i^2\,\alpha_i'^2 + \alpha_i'^2 + 2\alpha_i\alpha_i' + \alpha_i^2 + 2)^2}$$
$$(i = 1 \text{ for } C_2 = 0; \quad i = 2 \text{ for } C_1 = 0) \tag{37}$$

$C^2(\rho)$ can also be written as

$$C^2(\rho) = \left(\frac{p_i}{1 + 2X_i} \right)^2 \tag{38}$$

where

$$X_i = \left(\frac{\alpha_i\alpha_i' + 1}{\alpha_i - \alpha_i'} \right)^2 . \tag{39}$$

Here $X_i \in [0, \infty[$, where we can distinguish two interesting cases:

- $X_i = 0$ ($\alpha_i = -1/\alpha_i'$) corresponds to $C^2(\rho) = p_i^2$ which reduces in the case of $p_i = 1$ to the maximally entangled pure states ($C^2(\rho) = C^2(\psi) = 1$). Thus, in this case we have a mixed state ρ as a mixture of a maximally entangled pure state and a separable pure state. These mixed states present an important class of mixed states [40,41] which are widely studied and applied in many quantum information processing and transmission tasks.
- $X_i \to \infty$ ($\alpha_i = \alpha_i'$) which corresponds to the factorizable states ($C^2(\rho) = 0$).

At this point an interesting case is worth notifying: the completely mixed state for which the density operator is simply $\rho = \frac{1}{d} I$ where d is the dimension of the Hilbert space (in our case $\rho = \frac{1}{4} I$), this corresponds to $p_1 = p_2 = \frac{1}{2}$. The plot of $C^2(\rho)$ as a function of α and α' (Figs. 1, 2) shows that the maximal value of the concurrence $C(\rho)$ is $\frac{1}{2}$ rather than 1 as for the pure state. This can be interpreted as reflecting the fact that in a completely mixed state the concurrence is equally shared by the two subsystems.

4 Conclusion

In this paper, we have showed how the concurrence of two-qubit nonorthogonal pure states can be expressed in terms of the spin coherent states. We have then analyzed these states and calculated the concurrence that allowed us to determine the conditions for minimal and maximal entanglement.

Fig. 1 $C^2(X, p)$

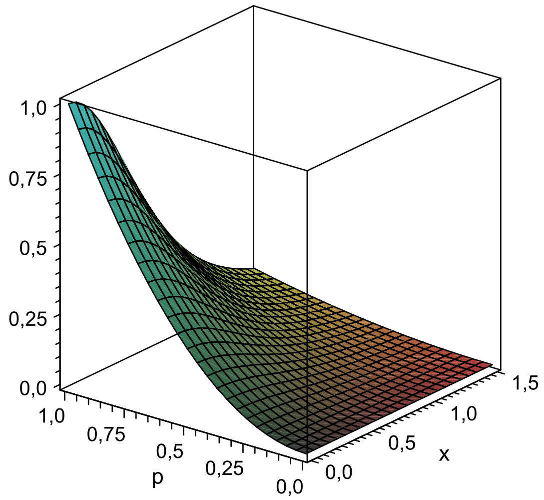
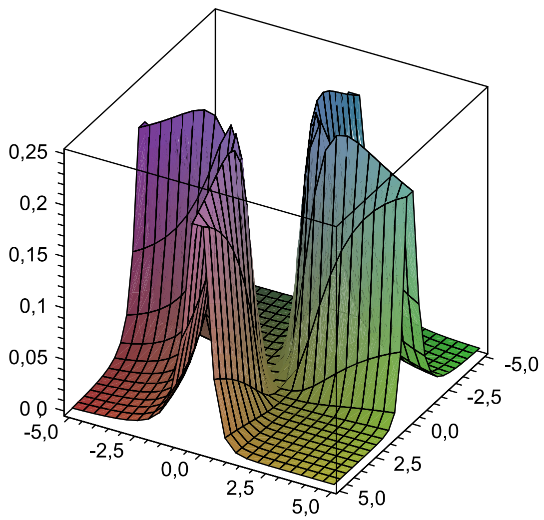


Fig. 2 $C^2(\alpha, \alpha')$. For $p_1 = p_2 = \frac{1}{2}$



We have generalized this formalism of concurrence to the case of a class of mixed states. Using a simplified and evident expression of the concurrence in terms of the concurrences of the pure states and their simple combinations we have calculated the square of the concurrence of the mixed state and studied its behavior as a function of the amplitudes of spin coherent states and the probabilities.

By studying a simple case we found that the concurrence of the mixed state cannot be higher than the probability of one of the subsystems. Furthermore, in the case of completely mixed state it cannot exceed one half.

In this way, it is shown that the coherent states are useful elements to determine and measure the entanglement of two-qubit states, and their use is not only of theoretical but also of some practical importance having in mind their experimental accessibility [42].

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