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On mathematical theory of the duality computers

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Abstract Recently, Long proposed a new type of quantum computers called duality computers or duality quantum computers. The duality computers based on the general quantum interference principle are much more powerful than an ordinary quantum computer. A mathematical theory for the duality computers has been presented by Gudder. However, he pointed out that a paradoxical situation of the mathematical theory occurs between the mixed state formalism and the pure state formalism. This paper argues for Gudder's mathematical theory of the duality computers for the mixed state formalism. First, we point out two problems existing in the pure state description of the duality computers. Then, we present a new mathematical theory of the duality computers for the pure state formalism according with Gudder's mixed state description, generalize the new mathematical theory of the duality computers in the density matrix formalism, and discuss some basic properties of the divider operators and combiner operators of the duality computers. The new mathematical theory can conquer the two problems mentioned above. Finally, we find that the nonunitary operations can be performed on every path of a quantum wave divider of the duality computers. Especially, we discuss in detail that the subwaves interact with environment by a CNOT gate.

Keywords Quantum computers · Duality computers · Mathematical theory · General quantum interference principle

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1 Introduction

Recently Long suggested a new type of quantum computers called duality computers or duality quantum computers [1]. The latest development of the duality computer theory can be found in Long's review article [2]. The duality computers based on the general quantum interference principle is much more powerful than an ordinary quantum computer. Compared to an ordinary quantum computer, the duality computers are a moving quantum computer passing through a multi-slit. In a duality computer, there exist two new computing gates, the quantum wave divider (QWD) and the quantum wave combiner (QWC), in addition to the usual universal gates for quantum computers. A quantum wave can be divided by QWD and recombined by QWC. The duality computers offer the capability to perform separate operations on the subwaves coming out of the different slits, in the so-called duality parallelism. This enables us to perform computation using not only unitary operations, but also linear combinations of unitary operations.

Because a duality computer is more powerful than an ordinary quantum computer, it is interesting to many scholars. Many researchers have investigated its mathematical theory and discussed its application [2–11]. First, Gudder gave the mathematical theory for the duality computers [3]. He provided two descriptions for the mathematical theory of the duality computers. One is in the pure state formalism and the other is in the mixed state formalism. Also, he pointed out that a paradoxical situation occurs between the mixed state formalism and the pure state formalism. Then, to solve the paradox, Long gave the mathematical theory of the duality computers in the density matrix formalism which is in accordance with Gudder's pure state description [4].

Though the advantage of the duality computers, such as "unsorted database search problem may be solved by using only a single query" and "all NP-complete problems may have polynomial algorithms" in the duality computers, is lost in Gudder's mathematical theory for the mixed state formalism, there exits other merit in the mathematical theory for mixed state formalism pointed out by Gudder [3]. The mathematical theory for the mixed state formalism has been studied by some researchers [5–7].

In this paper, we argue for Gudder's mathematical theory for the mixed state formalism. First, we point out some problems existing in Gudder's pure state description of the mathematical theory of the duality computers. Then, to solve the paradox in Ref. [3], we construct a new mathematical theory of the duality computers in the pure state formalism according with Gudder's mixed state description. The new mathematical theory can conquer the problems mentioned above. Finally, we point out that, based on our mathematical theory, nonunitary operations can be performed in every path of a QWD of the duality computers.

This paper is organized as follows. In Sect. 2, we point out some problems existing in Gudder's pure state description of the duality computers. In Sect. 3, we construct a new mathematical theory of the duality computers in the pure state formalism according with Gudder's mixed state description. In Sect. 4, we discuss other operations in addition to the unitary operations. The last section summarizes our work in this paper.

2 Some problems in the pure state description

Gudder pointed out that a paradoxical situation occurs between the mathematical theories for the mixed state formalism and the pure state formalism [3]. Therefore, only one of these two mathematical theories can be correct. We argue for Gudder's mathematical theory for mixed state formalism by pointing out two problems in his pure state description of the mathematical theory of the duality computers.

For convenience, we give the mathematical theory of the duality computers in the pure state formalism in Refs. [3] and [4], first. Let H be a complex Hilbert space and let $p = (p_1, \ldots, p_n)$ be a probability distribution, i.e., $p_i > 0, i = 1, \ldots, n$, and $\sum p_i = 1$. Similarly to Ref. [3], we write $H^{\oplus^n}$ for $\oplus_{i=1}^n H_i$ where $H_i = H, i = 1, \ldots, n$. The divider operator $D_p : H \to H^{\oplus^n}$ is defined as

$$D_p |\psi\rangle = \frac{1}{\|p\|} \oplus_{i=1}^n (p_i |\psi\rangle), \tag{1}$$

where $\|p\| = \sqrt{\sum p_i^2}$.

The combiner operation $C_p : H^{\oplus^n} \to H$ is defined as

$$C_p(|\psi_1\rangle \oplus \cdots \oplus |\psi_n\rangle) = \|p\| \sum_{i=1}^{n} |\psi_i\rangle. \tag{2}$$

From the definitions of the D_p and C_p, we can find two problems existing in them.

First, the pure state formalism description of the duality computers cannot explain well the results of Long's duality quantum search algorithm [1] when there is not a marked state in the database. When there is not a marked state in the database, the duality quantum search algorithm [1] is described in the following.

(1) We prepare the initial state of the duality computers:

$$|\psi_0\rangle = \frac{1}{\sqrt{N}} \sum_{i=0}^{N-1} |i\rangle = \frac{1}{\sqrt{N}}(|0\rangle + |1\rangle + \cdots + |N-1\rangle). \tag{3}$$

(2) Let the duality computers go through a QWD. Then it divides the wave into two subwaves as follows:

$$|\psi_u\rangle = \frac{1}{2\sqrt{N}} \sum_{i=0}^{N-1} |i\rangle = \frac{1}{2\sqrt{N}}(|0\rangle + |1\rangle + \cdots + |N-1\rangle), \tag{4}$$

$$|\psi_d\rangle = \frac{1}{2\sqrt{N}} \sum_{i=0}^{N-1} |i\rangle = \frac{1}{2\sqrt{N}}(|0\rangle + |1\rangle + \cdots + |N-1\rangle). \tag{5}$$

(3) Apply the query to the lower-path subwave $|\psi_d\rangle$, and then $|\psi_d\rangle$ is changed to

$$|\psi_d\rangle' = -\frac{1}{2\sqrt{N}}\sum_{i=0}^{N-1}|i\rangle = -\frac{1}{2\sqrt{N}}(|0\rangle + |1\rangle + \cdots + |N-1\rangle). \tag{6}$$

(4) Combine the subwaves at the QWC, then the wave becomes

$$|\psi_f\rangle = \frac{1}{\sqrt{N}}\sum_{i=0}^{N-1}|i\rangle - \frac{1}{\sqrt{N}}\sum_{i=0}^{N-1}|i\rangle = 0\sum_{i=0}^{N-1}|i\rangle = 0. \tag{7}$$

Similarly, when $p = \left(\frac{1}{2}, \frac{1}{2}\right)$ and $U_1 = -U_2$, Eq. 8 in Ref. [4] becomes

$$C_p\left(\frac{1}{2}U_1\rho U_1^\dagger \oplus \frac{1}{2}U_2\rho U_2^\dagger\right)C_p^\dagger = \sqrt{2}\left(\frac{1}{2}U_1 + \frac{1}{2}U_2\right)\rho\left(\frac{1}{2}U_1 + \frac{1}{2}U_2\right)^\dagger = 0, \tag{8}$$

where $U^\dagger$ denotes the Hermitian conjugate of U.

Notice that the results $|\psi_f\rangle = 0$ and $C_p\left(\frac{1}{2}U_1\rho U_1^\dagger \oplus \frac{1}{2}U_2\rho U_2^\dagger\right)C_p^\dagger = 0$ do not correspond with the physical fact. As is well known, in quantum mechanics, the coefficients of any quantum state should be normalized, and a quantum state should be located at one of its potential energy level when it is measured.

Statement 1 Equation 7 in Ref. [4] should be $D_p\rho D_p^\dagger = \frac{1}{\|P\|^2} \oplus p_i^2\rho$.

Second, the pure state formalism description of the duality computers can not explain well the difference between the results of reusing a QWD on one of pathes of another QWD and using a single QWD. For example, let the probability distribution $p = (\frac{1}{2}, \frac{1}{4}, \frac{1}{4})$ and $|\psi\rangle$ a pure state. After going through a QWD with the probability distribution p, $|\psi\rangle$ is changed to

$$|\psi_1\rangle = \sqrt{\frac{8}{3}}\left(\frac{1}{2}|\psi\rangle \oplus \frac{1}{4}|\psi\rangle \oplus \frac{1}{4}|\psi\rangle\right) = \sqrt{\frac{2}{3}}|\psi\rangle \oplus \sqrt{\frac{1}{6}}|\psi\rangle \oplus \sqrt{\frac{1}{6}}|\psi\rangle. \tag{9}$$

Intuitively, the result is equal to that the quantum state $|\psi\rangle$ goes through a QWD with the probability distribution $p' = (\frac{1}{2}, \frac{1}{2})$ and then the lower subwave goes through another QWD with the probability distribution p'. It can be described as follows.

After going through a QWD with the probability distribution p', the quantum state $|\psi\rangle$ is changed to

$$|\psi_2\rangle = \sqrt{2}\left(\frac{1}{2}|\psi\rangle \oplus \frac{1}{2}|\psi\rangle\right). \tag{10}$$

Then, the lower subwave going through a QWD with the probability distribution p', the quantum state $|\psi_2\rangle$ is changed to

$$|\psi_3\rangle=\sqrt{2}\left(\frac{1}{2}|\psi\rangle \oplus \frac{1}{2}\left(\sqrt{2}\left(\frac{1}{2}|\psi\rangle \oplus \frac{1}{2}|\psi\rangle\right)\right)\right)=\frac{\sqrt{2}}{2}|\psi\rangle \oplus \frac{1}{2}|\psi\rangle \oplus \frac{1}{2}|\psi\rangle. \tag{11}$$

From the above discussion, these two results $|\psi_1\rangle$ and $|\psi_3\rangle$ are not equal. This contradicts with our intuition.

Accordingly, according to the mathematical theory in the pure state description in Refs. [3] and [4], the combiner operator C_p cannot combine partial subwaves of a QWD.

3 Mathematical theory of the duality computers

In this section, we give a new mathematical theory of the duality computers in the pure state formalism according with Gudder's mixed state description [3]. Then, we generalize the mathematical theory of the duality computers in the density matrix formalism, and discuss some basic properties of the divider operators and combiner operators of the duality computers. By the properties of the divider operators and combiner operators, we obtain that the new mathematical theory can conquer the two problems mentioned in Sect. 2.

If the input state of the duality computers is a mixed state, according to Gudder's theory [3], the divider operator D_p is

$$D_p \rho D_p^\dagger = \oplus p_i \rho, \tag{12}$$

where $p = (p_1, \dots, p_n)$ is a probability distribution and ρ is the density matrix of the input state. The combiner operator C_p is

$$C_p(\oplus p_i \rho_i) C_p^\dagger = \sum p_i \rho_i, \tag{13}$$

where ρ_i is a density matrix, $i = 1, \dots, n$. Clearly, when $\rho_i = \rho$, C_p is the inverse of the divider operator D_p.

Let H be a complex Hilbert space and let $p = (p_1, \dots, p_n)$ be a probability distribution. We redefine the **divider operator**, still denoted by $D_p : H \to H^{\oplus^n}$ as

$$D_p |\psi\rangle = \oplus_{i=1}^n \left(\sqrt{p_i}\, |\psi\rangle\right). \tag{14}$$

It is easy to show that the divider operator D_p defined by us, in the pure state formalism description, accords with Gudder's mixed state description.

In accordance with the divider operator D_p defined by us in the pure state formalism and the combiner operator C_p defined by Gudder in the mixed state formalism, the **combiner operator** C, in the pure state formalism, is defined as: $\forall |\psi\rangle \in H$ and $\|\psi\| = 1, \forall q_i > 0,\ i = 1, \dots, m$, and $\sum q_i^2 \le 1$,

$$C(\oplus q_i |\psi\rangle) = \sqrt{\sum q_i^2} |\psi\rangle. \tag{15}$$

Because the above definition only needs the condition $\sum q_i^2 \le 1$, the combiner operator C can combine partial subwaves of a QWD. However, this definition requires that subwaves in all pathes are attenuated copies of $|\psi\rangle$. In other words, the difference of subwaves in all pathes is only the coefficients and the coefficients must be positive. The combiner operator C in density matrix formalism is

$$C(\oplus q_i \rho_i)C^\dagger = \sum q_i \rho_i, \tag{16}$$

where ρ_i is a density matrix and $q_i > 0, i = 1, \ldots, m$, and $\sum q_i^2 \le 1$.

Clearly, the combiner operator C defined by us in density matrix formalism is a generalization of C_p for the mixed states in Ref. [3]. Also, we use C_p to denote the inverse of D_p, the special case of the combiner operator C when $q_i = \sqrt{p_i}, i = 1, \ldots, n$.

Now, we give some basic properties of the divider operator D_p and the combiner operator C in the pure state formalism description.

Theorem 3.1 *The divider operator D_p in the pure state formalism description is a linear isometry. Therefore, D_p is a unitary operator from H onto its range $\mathscr{R}(D_p)$.*

Proof It is clear that D_p is linear. Let $(\cdot, \cdot)$ denote the inner product on H. Similarly to the proof of Lemma 2.1 in Ref. [3], $\forall |\psi\rangle, |\phi\rangle \in H$,

$$\begin{aligned}\left(D_p|\psi\rangle, D_p|\phi\rangle\right) &= \left(\oplus\left(\sqrt{p_i}|\psi\rangle\right), \oplus\left(\sqrt{p_i}|\phi\rangle\right)\right)\\ &= \sum p_i \left(|\psi\rangle, |\phi\rangle\right) = \left(|\psi\rangle, |\phi\rangle\right).\end{aligned} \tag{17}$$

Therefore, D_p is an isometry. Hence, D_p is a linear isometry. Furthermore, D_p is a unitary operator from H onto its range $\mathscr{R}(D_p)$. □

Theorem 3.2 *The combiner operator C in the pure state formalism description is a linear isometry.*

Proof It is clear that C is linear. Similarly to the proof of Lemma 2.2 in Ref. [3], we have that, $\forall |\psi\rangle, |\phi\rangle \in H$,

$$(q_1|\psi\rangle \oplus \cdots \oplus q_m|\psi\rangle, q_1|\phi\rangle \oplus \cdots \oplus q_m|\phi\rangle) = \sum q_i^2 (|\psi\rangle, |\phi\rangle), \tag{18}$$

and

$$\begin{aligned}&(C(q_1|\psi\rangle \oplus \cdots \oplus q_m|\psi\rangle), C(q_1|\phi\rangle \oplus \cdots \oplus q_m|\phi\rangle))\\ &\quad= \left(\sqrt{\sum q_i^2}|\psi\rangle, \sqrt{\sum q_i^2}|\phi\rangle\right) = \sum q_i^2 (|\psi\rangle, |\phi\rangle).\end{aligned} \tag{19}$$

So, C is an isometry. □

Theorem 3.3 *The combiner operator C_p in the pure state formalism description is a linear isometry, and $C_p = D_p^\dagger$.*

Proof From Theorem 3.2, we have that the combiner operator C_p in the pure state formalism description is a linear isometry.

We can easily obtain

$$C_p D_p |\psi\rangle = C_p \left(\oplus \left(\sqrt{p_i}|\psi\rangle\right)\right) = \sqrt{\sum p_i}|\psi\rangle = |\psi\rangle. \tag{20}$$

Therefore, $C_p D_p = I_H$ where I_H is the identity operator on H. In terms of D_p being a unitary operator, we obtain that $C_p = D_p^\dagger$. □

Statement 2 Theorem 2.3, Corollary 2.4, Theorem 2.5, and Corollary 2.6 in Ref. [3] are also true.

Theorem 3.4 *Let $p = (p_1, \ldots, p_n)$ be a probability distribution and ρ_i a density matrix, $i = 1, \ldots, n$. Then $\sum p_i \rho_i$ is a density matrix.*

Proof Straightforward. □

From Theorem 3.4, we know that a quantum state going through a QWD, doing unitary operation on some subwaves, and going through a QWC (the inverse of the QWD) in turn is also a quantum state. Consequently, the first problem mentioned in Sect. 2 does not exist in the new mathematical theory of the duality computers. Notice that a pure state going through a QWD, doing unitary operations on some subwaves, and going through a QWC in turn may not be a pure state.

Theorem 3.5 *According to the new mathematical theory of the duality computers, the following two cases are equal.*

Case 1: A quantum wave is divided by a QWD D_p with $p = (p_1, \ldots, p_n)$ and do unitary operation U_i on the ith path of D_p. Then, the first subwave is divided by another QWD D_q with $q = (q_1, \ldots, q_m)$ and does unitary operation V_j on the jth sub-path of D_q;

Case 2: A quantum wave is divided by a QWD D_r with $r = (p_1q_1, \ldots, p_1q_m, p_2, \ldots, p_n)$ and does unitary operation W_k on the kth path of D_r, where $W_k = \begin{cases} V_k U_1, & k = 1, \ldots, m, \\ U_{k-m+1}, & k = m+1, \ldots, m+n-1. \end{cases}$

Proof Case 1. Let ρ be the density matrix of the input quantum wave. The operation process of Case 1 is described in the following.

(1) Let the duality computers go through the first QWD D_p. Then it divides the quantum wave ρ into n subwaves:

$$D_p \rho D_p^\dagger = \oplus_{i=1}^n p_i \rho. \tag{21}$$

(2) After U_i being used to the ith subwave of D_p, $i = 1, \ldots, n$, these subwaves is changed to $\oplus_{i=1}^n p_i U_i \rho U_i^\dagger$.

(3) After the first subwave going through the second QWD D_q, it is changed to

$$\left(p_1 D_q U_1 \rho U_1^\dagger D_q^\dagger\right) = \left(p_1 q_1 U_1 \rho U_1^\dagger \oplus \cdots \oplus p_1 q_m U_1 \rho U_1^\dagger\right). \tag{22}$$

(4) After V_j being used to the jth sub-subwave of D_q, $j = 1, \ldots, m$, these m sub-subwaves is changed to

$$\oplus_{j=1}^{m} p_1 q_j V_j U_1 \rho U_1^{\dagger} V_j^{\dagger}.$$

The final global quantum state is

$$\begin{aligned}&\left(\oplus_{j=1}^{m} p_1 q_j V_j U_1 \rho U_1^{\dagger} V_j^{\dagger}\right) \oplus \left(\oplus_{i=2}^{n} \left(p_j U_j \rho U_j^{\dagger}\right)\right)\\ &= \left(p_1 q_1 V_1 U_1 \rho U_1^{\dagger} V_1^{\dagger} \oplus \cdots \oplus p_1 q_m V_m U_1 \rho U_1^{\dagger} V_m^{\dagger}\right)\\ &\oplus \left(p_2 U_2 \rho U_2^{\dagger} \oplus \cdots \oplus p_n U_n \rho U_n^{\dagger}\right).\end{aligned} \tag{23}$$

Case 2. It is easily obtained that r is a probability distribution and W_k is a unitary operator, $k = 1, \ldots, m + n - 1$. The operation process of Case 2 is described in the following.

(1) The duality computers going through the QWD D_r, the quantum wave is divided into $m + n - 1$ subwaves:

$$D_r \rho D_r^{\dagger} = (p_1 q_1 \rho \oplus \cdots \oplus p_1 q_m \rho \oplus p_2 \rho \oplus \cdots \oplus p_n \rho). \tag{24}$$

(2) After W_k being used to the kth subwave of D_r, $k = 1, \ldots, m + n - 1$, these subwaves are changed to

$$\begin{aligned}&\left(p_1 q_1 W_1 \rho W_1^{\dagger} \oplus \cdots \oplus p_1 q_m W_m \rho W_m^{\dagger} \oplus p_2 W_{m+1} \rho W_{m+1}^{\dagger} \oplus \cdots\right.\\ &\left.\oplus p_n W_{m+n-1} \rho W_{m+n-1}^{\dagger}\right) = \left(p_1 q_1 V_1 U_1 \rho U_1^{\dagger} V_1^{\dagger} \oplus \cdots \oplus p_1 q_m V_m U_1 \rho U_1^{\dagger} V_m^{\dagger}\right)\\ &\oplus \left(p_2 U_2 \rho U_2^{\dagger} \oplus \cdots \oplus p_n U_n \rho U_n^{\dagger}\right).\end{aligned} \tag{25}$$

Contrasting Eqs. 23 and 25, we finish the proof. □

By the proof of Theorem 3.5, if the QWD D_q in Case 2 is on the ith path of the QWD D_p, $i = 2, \ldots, n$, the corresponding conclusion is also true.

From the special case of Theorem 3.5 when $U_i = V_j = I_H, i = 1, \ldots, n, j = 1, \ldots, m$, we know that the second problem mentioned in Sect.2 does not exist in the new mathematical theory of the duality computers.

4 Nonunitary operations in the duality computers

Previous works on the duality computers usually consider the unitary operations on subwaves. In this section, we discuss nonunitary operations on some pathes of a QWD. As is well known, there are nonunitary operators in quantum theory, such as taking partial trace after interacting with environment. The basic interaction is the CNOT operation. We discuss that a subwave interacts with other qubits in the following.

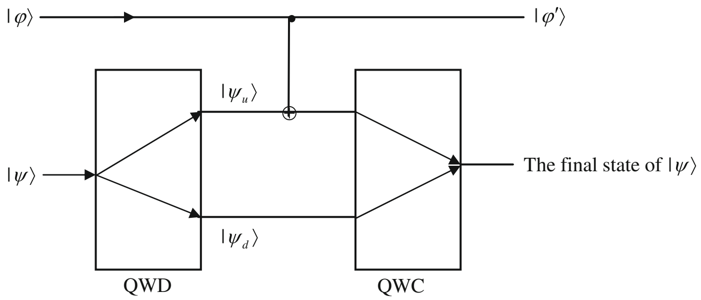

Fig. 1 The subwave is the aim bit

For simplicity, we only discuss the case that after a qubit goes through a QWD D_p, $p = (\frac{1}{2}, \frac{1}{2})$, one subwave interacts with another qubit by a CNOT gate.

First, we consider the case that the first subwave is the aim bit of the CNOT gate. This model is pictured in Fig. 1.

The initial state is

$$|\varphi\rangle \otimes |\psi\rangle = (a|0\rangle + b|1\rangle) \otimes (x|0\rangle + y|1\rangle). \tag{26}$$

After $|\psi\rangle$ goes through the QWD D_p, $|\psi\rangle$ is changed to

$$D_p\,|\psi\rangle = |\psi_u\rangle \oplus |\psi_d\rangle = \left(\frac{x}{\sqrt{2}}|0\rangle + \frac{y}{\sqrt{2}}|1\rangle\right) \oplus \left(\frac{x}{\sqrt{2}}|0\rangle + \frac{y}{\sqrt{2}}|1\rangle\right), \tag{27}$$

and the global state is changed to

$$\begin{aligned}
|\varphi\rangle \otimes (|\psi_u\rangle \oplus |\psi_d\rangle) &= (a|0\rangle + b|1\rangle) \otimes \left[\left(\frac{x}{\sqrt{2}}|0\rangle + \frac{y}{\sqrt{2}}|1\rangle\right) \oplus \left(\frac{x}{\sqrt{2}}|0\rangle + \frac{y}{\sqrt{2}}|1\rangle\right)\right] \\
&= \left[(a|0\rangle + b|1\rangle) \otimes \left(\frac{x}{\sqrt{2}}|0\rangle + \frac{y}{\sqrt{2}}|1\rangle\right)\right] \\
&\quad \oplus \left[(a|0\rangle + b|1\rangle) \otimes \left(\frac{x}{\sqrt{2}}|0\rangle + \frac{y}{\sqrt{2}}|1\rangle\right)\right] \\
&= \left(\frac{ax}{\sqrt{2}}|00\rangle + \frac{ay}{\sqrt{2}}|01\rangle + \frac{bx}{\sqrt{2}}|10\rangle + \frac{by}{\sqrt{2}}|11\rangle\right) \\
&\quad \oplus \left(\frac{ax}{\sqrt{2}}|00\rangle + \frac{ay}{\sqrt{2}}|01\rangle + \frac{bx}{\sqrt{2}}|10\rangle + \frac{by}{\sqrt{2}}|11\rangle\right).
\end{aligned} \tag{28}$$

After $|\varphi\rangle$ and $|\psi_u\rangle$ are interacted by a CONT gate, the global state is changed to

$$\begin{aligned}
|\mu\rangle = &\left(\frac{ax}{\sqrt{2}}|00\rangle + \frac{ay}{\sqrt{2}}|01\rangle + \frac{by}{\sqrt{2}}|10\rangle + \frac{bx}{\sqrt{2}}|11\rangle\right) \\
&\oplus \left(\frac{ax}{\sqrt{2}}|00\rangle + \frac{ay}{\sqrt{2}}|01\rangle + \frac{bx}{\sqrt{2}}|10\rangle + \frac{by}{\sqrt{2}}|11\rangle\right).
\end{aligned} \tag{29}$$

For describing this process clearly, we consider two kinds of special cases: $|\varphi\rangle = |0\rangle$ and $|\varphi\rangle = |1\rangle$.

(1) When $|\varphi\rangle = |0\rangle$, the global state after the CONT gate is

$$|\mu_0\rangle = |0\rangle \left(\frac{x}{\sqrt{2}}|0\rangle + \frac{y}{\sqrt{2}}|1\rangle \right) \oplus |0\rangle \left(\frac{x}{\sqrt{2}}|0\rangle + \frac{y}{\sqrt{2}}|1\rangle \right). \tag{30}$$

Hence, the final global state after the QWC is

$$|\nu_0\rangle = |0\rangle \left[\frac{1}{2}(x|0\rangle + y|1\rangle) \right] + |0\rangle \left[\frac{1}{2}(x|0\rangle + y|1\rangle) \right] = |0\rangle(x|0\rangle + y|1\rangle) = |\varphi\rangle|\psi\rangle. \tag{31}$$

Thus, the density matrix of the final global state after the QWC is

$$\rho_0 = |0\rangle\langle 0| \otimes (x|0\rangle + y|1\rangle)(x|0\rangle + y|1\rangle)^\dagger = |\varphi\rangle\langle\varphi| \otimes |\psi\rangle\langle\psi|. \tag{32}$$

(2) When $|\varphi\rangle = |1\rangle$, the global state after CONT gate is

$$\mu_1 = |1\rangle \left(\frac{y}{\sqrt{2}}|0\rangle + \frac{x}{\sqrt{2}}|1\rangle \right) \oplus |1\rangle \left(\frac{x}{\sqrt{2}}|0\rangle + \frac{y}{\sqrt{2}}|1\rangle \right). \tag{33}$$

Therefore, the density matrix of the final global state after the QWC is

$$\rho_1 = |1\rangle\langle 1| \otimes \left[\frac{1}{2}(y|0\rangle + x|1\rangle)(y|0\rangle + x|1\rangle)^\dagger + \frac{1}{2}(x|0\rangle + y|1\rangle)(x|0\rangle + y|1\rangle)^\dagger \right], \tag{34}$$

i.e.,

$$\rho_1 = |1\rangle\langle 1| \otimes \left(\frac{1}{2}|0\rangle\langle 0| + \frac{y\overline{x} + x\overline{y}}{2}|0\rangle\langle 1| + \frac{y\overline{x} + x\overline{y}}{2}|1\rangle\langle 0| + \frac{1}{2}|1\rangle\langle 1| \right). \tag{35}$$

Unfortunately, the final global state after the QWC is not $|a|^2\rho_0 + |b|^2\rho_1$ in general when $|\varphi\rangle = a|0\rangle + b|1\rangle$. When $|\varphi\rangle = a|0\rangle + b|1\rangle$, the density matrix of the global state after the CONT gate is

$$\begin{aligned} \rho_{CNOT} = & \left(\frac{ax}{\sqrt{2}}|00\rangle + \frac{ay}{\sqrt{2}}|01\rangle + \frac{by}{\sqrt{2}}|10\rangle + \frac{bx}{\sqrt{2}}|11\rangle \right) \\ & \times \left(\frac{ax}{\sqrt{2}}|00\rangle + \frac{ay}{\sqrt{2}}|01\rangle + \frac{by}{\sqrt{2}}|10\rangle + \frac{bx}{\sqrt{2}}|11\rangle \right)^\dagger \\ & \oplus \left(\frac{ax}{\sqrt{2}}|00\rangle + \frac{ay}{\sqrt{2}}|01\rangle + \frac{bx}{\sqrt{2}}|10\rangle + \frac{by}{\sqrt{2}}|11\rangle \right) \\ & \times \left(\frac{ax}{\sqrt{2}}|00\rangle + \frac{ay}{\sqrt{2}}|01\rangle + \frac{bx}{\sqrt{2}}|10\rangle + \frac{by}{\sqrt{2}}|11\rangle \right)^\dagger. \end{aligned} \tag{36}$$

Thus, the final global state after the QWC is

$$\begin{aligned}\rho_{final} = &\left(\frac{ax}{\sqrt{2}}|00\rangle + \frac{ay}{\sqrt{2}}|01\rangle + \frac{by}{\sqrt{2}}|10\rangle + \frac{bx}{\sqrt{2}}|11\rangle\right)\\ &\times\left(\frac{ax}{\sqrt{2}}|00\rangle + \frac{ay}{\sqrt{2}}|01\rangle + \frac{by}{\sqrt{2}}|10\rangle + \frac{bx}{\sqrt{2}}|11\rangle\right)^{\dagger}\\ &+\left(\frac{ax}{\sqrt{2}}|00\rangle + \frac{ay}{\sqrt{2}}|01\rangle + \frac{bx}{\sqrt{2}}|10\rangle + \frac{by}{\sqrt{2}}|11\rangle\right)\\ &\times\left(\frac{ax}{\sqrt{2}}|00\rangle + \frac{ay}{\sqrt{2}}|01\rangle + \frac{bx}{\sqrt{2}}|10\rangle + \frac{by}{\sqrt{2}}|11\rangle\right)^{\dagger},\end{aligned} \tag{37}$$

i.e.,

$$\begin{aligned}&\rho_{final}\\ &= \frac{1}{2}\begin{pmatrix} 2|ax|^2 & 2|a|^2x\overline{y} & a\overline{b}|x|^2 + a\overline{b}x\overline{y} & a\overline{b}|x|^2 + a\overline{b}x\overline{y} \\ 2|a|^2y\overline{x} & 2|ay|^2 & a\overline{b}|y|^2 + a\overline{b}y\overline{x} & a\overline{b}|y|^2 + a\overline{b}y\overline{x} \\ \overline{a}b|x|^2 + \overline{a}by\overline{x} & \overline{a}b|y|^2 + \overline{a}bx\overline{y} & |b|^2 & |b|^2y\overline{x} + |b|^2x\overline{y} \\ \overline{a}b|x|^2 + \overline{a}by\overline{x} & \overline{a}by\overline{x} + \overline{a}b|y|^2 & |b|^2x\overline{y} + |b|^2y\overline{x} & |b|^2 \end{pmatrix}.\end{aligned} \tag{38}$$

In particular, $\rho_{final} = \rho_0$ if $|\varphi\rangle = |0\rangle$; $\rho_{final} = \rho_1$ if $|\varphi\rangle = |1\rangle$.

Now, we consider the case that the second subwave is the control bit of the CNOT gate. This model is pictured in the Fig. 2. Similarly to the discussion of the first case, in the interest of convenience, let the initial state be

$$|\psi\rangle \otimes |\varphi\rangle = (x|0\rangle + y|1\rangle) \otimes (a|0\rangle + b|1\rangle). \tag{39}$$

After $|\psi\rangle$ goes through the QWD D_p, $|\psi\rangle$ is changed to

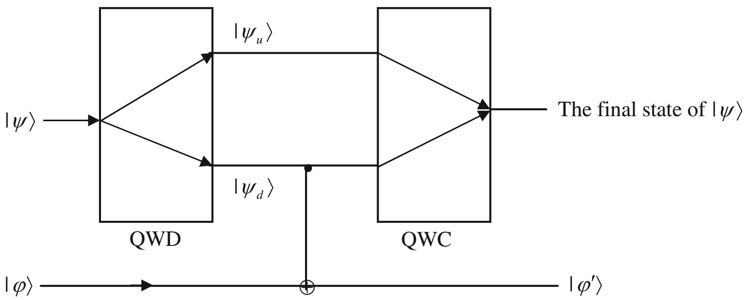

Fig. 2 The subwave is the control bit

$$D_p\,|\psi\rangle = |\psi_u\rangle \oplus |\psi_d\rangle = \left(\frac{x}{\sqrt{2}}|0\rangle + \frac{y}{\sqrt{2}}|1\rangle\right) \oplus \left(\frac{x}{\sqrt{2}}|0\rangle + \frac{y}{\sqrt{2}}|1\rangle\right), \tag{40}$$

and the global state is changed to

$$\begin{aligned}
(|\psi_u\rangle \oplus |\psi_d\rangle) \otimes |\varphi\rangle &= \left[\left(\frac{x}{\sqrt{2}}|0\rangle+\frac{y}{\sqrt{2}}|1\rangle\right) \oplus \left(\frac{x}{\sqrt{2}}|0\rangle+\frac{y}{\sqrt{2}}|1\rangle\right)\right] \otimes (a|0\rangle+b|1\rangle) \\
&= \left[\left(\frac{x}{\sqrt{2}}|0\rangle + \frac{y}{\sqrt{2}}|1\rangle\right) \otimes (a|0\rangle + b|1\rangle)\right] \\
&\quad \oplus \left[\left(\frac{x}{\sqrt{2}}|0\rangle + \frac{y}{\sqrt{2}}|1\rangle\right) \otimes (a|0\rangle + b|1\rangle)\right] \\
&= \left(\frac{ax}{\sqrt{2}}|00\rangle + \frac{bx}{\sqrt{2}}|01\rangle + \frac{ay}{\sqrt{2}}|10\rangle + \frac{by}{\sqrt{2}}|11\rangle\right) \\
&\quad \oplus \left(\frac{ax}{\sqrt{2}}|00\rangle + \frac{bx}{\sqrt{2}}|01\rangle + \frac{ay}{\sqrt{2}}|10\rangle + \frac{by}{\sqrt{2}}|11\rangle\right).
\end{aligned} \tag{41}$$

After $|\psi_d\rangle$ and $|\varphi\rangle$ are interacted by the CONT gate, the global state is changed to

$$\begin{aligned}
\mu = &\left(\frac{ax}{\sqrt{2}}|00\rangle + \frac{bx}{\sqrt{2}}|01\rangle + \frac{ay}{\sqrt{2}}|10\rangle + \frac{by}{\sqrt{2}}|11\rangle\right) \\
&\oplus \left(\frac{ax}{\sqrt{2}}|00\rangle + \frac{bx}{\sqrt{2}}|01\rangle + \frac{by}{\sqrt{2}}|10\rangle + \frac{ay}{\sqrt{2}}|11\rangle\right).
\end{aligned} \tag{42}$$

Thus, the density matrix of the global state after the CONT gate is

$$\begin{aligned}
\rho_{CNOT} = &\left(\frac{ax}{\sqrt{2}}|00\rangle + \frac{bx}{\sqrt{2}}|01\rangle + \frac{ay}{\sqrt{2}}|10\rangle + \frac{by}{\sqrt{2}}|11\rangle\right)\left(\frac{ax}{\sqrt{2}}|00\rangle + \frac{bx}{\sqrt{2}}|01\rangle\right. \\
&\left.+\frac{ay}{\sqrt{2}}|10\rangle + \frac{by}{\sqrt{2}}|11\rangle\right)^{\dagger} \oplus \left(\frac{ax}{\sqrt{2}}|00\rangle + \frac{bx}{\sqrt{2}}|01\rangle + \frac{by}{\sqrt{2}}|10\rangle + \frac{ay}{\sqrt{2}}|11\rangle\right) \\
&\times \left(\frac{ax}{\sqrt{2}}|00\rangle + \frac{bx}{\sqrt{2}}|01\rangle + \frac{by}{\sqrt{2}}|10\rangle + \frac{ay}{\sqrt{2}}|11\rangle\right)^{\dagger}.
\end{aligned} \tag{43}$$

Therefore, the final global state after the QWC is

$$\begin{aligned}
\rho_{final} = &\left(\frac{ax}{\sqrt{2}}|00\rangle + \frac{bx}{\sqrt{2}}|01\rangle + \frac{ay}{\sqrt{2}}|10\rangle + \frac{by}{\sqrt{2}}|11\rangle\right) \\
&\times \left(\frac{ax}{\sqrt{2}}|00\rangle + \frac{bx}{\sqrt{2}}|01\rangle + \frac{ay}{\sqrt{2}}|10\rangle + \frac{by}{\sqrt{2}}|11\rangle\right)^{\dagger} \\
&+ \left(\frac{ax}{\sqrt{2}}|00\rangle + \frac{bx}{\sqrt{2}}|01\rangle + \frac{by}{\sqrt{2}}|10\rangle + \frac{ay}{\sqrt{2}}|11\rangle\right) \\
&\times \left(\frac{ax}{\sqrt{2}}|00\rangle + \frac{bx}{\sqrt{2}}|01\rangle + \frac{by}{\sqrt{2}}|10\rangle + \frac{ay}{\sqrt{2}}|11\rangle\right)^{\dagger},
\end{aligned} \tag{44}$$

i.e.,

$$\rho_{final} = \frac{1}{2}\begin{pmatrix} 2|ax|^2 & 2a\overline{b}|x|^2 & |a|^2x\overline{y}+a\overline{b}x\overline{y} & |a|^2x\overline{y}+a\overline{b}x\overline{y} \\ 2\overline{a}b|x|^2 & 2|bx|^2 & \overline{a}bx\overline{y}+|b|^2x\overline{y} & \overline{a}bx\overline{y}+|b|^2x\overline{y} \\ |a|^2\overline{x}y+\overline{a}b\overline{x}y & a\overline{b}\overline{x}y+|b|^2\overline{x}y & |ay|^2+|by|^2 & a\overline{b}|y|^2+\overline{a}b|y|^2 \\ |a|^2\overline{x}y+\overline{a}b\overline{x}y & a\overline{b}\overline{x}y+|b|^2\overline{x}y & a\overline{b}|y|^2+\overline{a}b|y|^2 & |ay|^2+|by|^2 \end{pmatrix}. \tag{45}$$

Especially, if $|\varphi\rangle = |0\rangle$,

$$\rho_{final} = \rho_0 = \frac{1}{2}\begin{pmatrix} 2|x|^2 & 0 & x\overline{y} & x\overline{y} \\ 0 & 0 & 0 & 0 \\ \overline{x}y & 0 & |y|^2 & 0 \\ \overline{x}y & 0 & 0 & |y|^2 \end{pmatrix}; \tag{46}$$

if $|\varphi\rangle = |1\rangle$,

$$\rho_{final} = \rho_1 = \frac{1}{2}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 2|x|^2 & x\overline{y} & x\overline{y} \\ 0 & \overline{x}y & |y|^2 & 0 \\ 0 & \overline{x}y & 0 & |y|^2 \end{pmatrix}. \tag{47}$$

We can obtain the density matrix of the final state of $|\psi\rangle$ by taking corresponding partial trace from ρ_{final}.

Similarly, we can obtain the final global state ρ_{final} when p is another probability distribution and any one of the subwaves goes through the CNOT gate.

5 Conclusion

In this paper, we have followed Gudder's mathematical theory of the duality computers for the mixed state formalism, and his mixed state description of the duality computers may more accord with quantum mechanics principle.

First, we have pointed out two existing problems in the pure state description of the duality computers [3–5]. (1) The pure state description of the duality computers can not explain well the result of Long's duality quantum search algorithm [1] when there is not a marked state in the database. (2) The pure state description of the duality computers can not explain well the difference between the results of reusing a QWD on one of pathes of another QWD and using a single QWD.

Then, we have given the mathematical theory of the duality computers for the pure state formalism such that it accords with Gudder's mixed state description and then we have generalized the mathematical theory of the duality computers in the density matrix formalism. Also, we have discussed some basic properties of the divider operators and the combiner operators of the duality computers.

Finally, we have pointed out that each nonunitary operation can be performed on every path of the QWD in a duality computer. In addition, we have discussed in detail that a subwave interacts with environment by a CNOT gate.

From the foregoing discussion, we have known that the new mathematical theory of the duality computers has not the two problems of the pure state description of the duality computers in Refs. [3,4]. Though the duality computers will lose its super capacity in the search problem of unsorted database and other NP-complete problems mentioned in Ref. [1], we may find new capacity of the duality computers by performing nonunitary operation on subwaves.

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