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Geometrical Structure of Entangled States and the Secant Variety

Hoshang Heydari[1]

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We investigate the geometrical structure of entangled and separable bipartite and multipartite states based on the secant variety of the Segre variety. We show that the Segre variety coincides with the space of separable multipartite state and the higher secant variety of the Segre variety coincides with the space of entangled multipartite states.

KEY WORDS: multipartite quantum system; quantum entanglement; complex projective variety; secant variety of the serge variety.

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1. INTRODUCTION

Recently, the geometry and topology of entanglement has got more attention and we know more about the geometrical structure of pure multipartite entangled quantum states. We have also managed to construct some useful measures of entanglement based on these underlying geometrical structures. However, we know less about the geometrical structure of an arbitrary multipartite quantum state and there is a need for further investigation on these states. Concurrence is a measure of entanglement which is directly related to the entanglement of formation.(1) Its geometrical structure is hidden in a map called Segre embedding.(2–5) The Segre variety is generated by the quadratic polynomials that correspond to the separable set of pure multipartite states. We can construct a measure of entanglement for bipartite and three-partite states based on the Segre variety.(4) We can also construct a measure of entanglement for general pure multipartite states based on a modification of the Segre variety by adding similar

[1] Institute of Quantum Science, Nihon University, 1-8 Kanda-Surugadai, Chiyoda-ku, Tokyo 101-8308, Japan. E-mail: hoshang@edu.cst.nihon-u.ac.jp

quadratic polynomials.[5] In this paper, we will establish a relation between the higher secant variety of the Segre variety and multipartite states. We show that the secant variety of the Segre variety can describes the geometry of entangled and separable general bipartite and multipartite quantum systems. In Sect. 2, we will define the complex projective variety and introduce the Segre embedding and the Segre variety for general pure multipartite states. We will also define and discuss the secant variety of the Segre variety, which is of central importance in this paper. In Sect. 3 we investigate relation and relevance of the secant variety of the Segre variety as geometrical structure of entangled and separable states. As usual, we denote a general, composite quantum system with m subsystems as $\mathcal{Q}=\mathcal{Q}_m(N_1,N_2,\ldots,N_m)=\mathcal{Q}_1\mathcal{Q}_2\ldots\mathcal{Q}_m$, consisting of the purestates $|\Psi\rangle=\sum_{k_1=1}^{N_1}\sum_{k_2=1}^{N_2}\cdots\sum_{k_m=1}^{N_m}\alpha_{k_1,k_2,\ldots,k_m}|k_1,k_2,\ldots,k_m\rangle$ and corresponding to the Hilbert space $\mathcal{H}_{\mathcal{Q}}=\mathcal{H}_{\mathcal{Q}_1}\otimes\mathcal{H}_{\mathcal{Q}_2}\otimes\cdots\otimes\mathcal{H}_{\mathcal{Q}_m}$, where $N_j=\dim(\mathcal{H}_{\mathcal{Q}_j})$ is the dimension of the jth Hilbert space.

2. COMPLEX PROJECTIVE VARIETY, SEGRE VARIETY, AND SECANT VARIETY

In this section, we review some basic definition of complex projective variety. The general reference on projective algebraic geometry can be found in Refs. 6–8. Then, we will construct the Segre variety for complex multi-projective space. Next, we will define the joint and secant variety of a projective variety. We will also discuss the rank of a tensor in relation with the Segre variety and the secant variety.

2.1. Complex Projective Variety

Let $C[z]=C[z_1,z_2,\ldots,z_n]$ denote the polynomial algebra in n variables with complex coefficients. Then, given a set of q polynomials $\{h_1,h_2,\ldots,h_q\}$ with $h_i\in C[z]$, we define a complex affine variety as

$$\mathcal{V}_{\mathbb{C}}(h_1,h_2,\ldots,h_q)=\{P\in\mathbb{C}^n\colon h_i(P)=0\forall 1\leq i\leq q\}, \tag{2.1.1}$$

where $P=(a_1,a_2,\ldots,a_n)$ is called a point of $\mathbb{C}^n$ and the a_i are called the coordinates of P. A complex projective space $\mathbb{CP}^n$ is defined to be the set of lines through the origin in $\mathbb{C}^{n+1}$, that is, $\mathbb{CP}^n=\mathbb{C}^{n+1}-0/\sim$, where the equivalence relation $\sim$ is defined as follow; $(x_1,\ldots,x_{n+1})\sim(y_1,\ldots,y_{n+1})$ for $\lambda\in\mathbb{C}-0$, where $y_i=\lambda x_i$ for all $0\leq i\leq n+1$. Given a set of homogeneous polynomials $\{h_1,h_2,\ldots,h_q\}$ with $h_i\in C[z]$, we define a complex projective variety as

$$\mathcal{V}(h_1, \dots, h_q) = \{O \in \mathbb{CP}^n : h_i(O) = 0 \quad \forall\ 1 \leq i \leq q\}, \tag{2.1.2}$$

where $O = (a_1, a_2, \dots, a_{n+1})$ denotes the equivalent class of points $\{\alpha_1, \alpha_2, \dots, \alpha_{n+1}\} \in \mathbb{C}^{n+1}$. We can view the affine complex variety as a complex cone over the complex projective variety.

2.2. Segre Variety

As an important example of projective variety we will discuss the Segre variety.(4) For a multipartite quantum system $\mathcal{Q}(N_1, \dots, N_m)$, let $\overline{N} = (N_1, \dots, N_m)$ and $V_1, V_2, \dots, V_m$ be vector spaces over the field of complex numbers $\mathbb{C}$, where $\dim V_j = N_j$. That is, we have $\mathbb{CP}^{N_j-1} = \mathbb{CP}(V_j)$ for all j. Then we define a Segre map by

$$\mathcal{S}_{N_1, N_2, \dots, N_m} : \mathbb{CP}^{N_1-1} \times \mathbb{CP}^{N_2-1} \times \cdots \times \mathbb{CP}^{N_m-1} \longrightarrow \mathbb{CP}^{\mathcal{N}-1}, \tag{2.2.1}$$

where $\mathcal{N} = \prod_{j=1}^{m} N_j$. This map is based on the canonical multilinear map

$$\begin{aligned} V_1 \times V_2 \times \cdots \times V_m &\rightarrow V_1 \otimes V_2 \otimes \cdots \otimes V_m \\ v_1 \times v_2 \times \cdots \times v_m &\mapsto v_1 \otimes v_2 \otimes \cdots \otimes v_m \end{aligned} \tag{2.2.2}$$

Thus, we have $\mathbb{CP}^{\mathcal{N}-1} = \mathbb{CP}(V_1, \dots, V_m)$. The Segre variety $\mathfrak{S}_{\overline{N}} = \mathrm{Im}(\mathcal{S}_{N_1, N_2, \dots, N_m})$ is defined to be the image of the Segre embedding. By definition, the Segre variety is formed by the set of all classes of decomposable tensors in $\mathbb{CP}^{\mathcal{N}-1}$. For a quantum system $\mathcal{Q}(N_1, \dots, N_m)$, the Segre variety is given by

$$\begin{aligned} \mathfrak{S}_{\overline{N}} = \bigcap_{\forall j} \mathcal{V}(&\alpha_{k_1, k_2, \dots, k_m} \alpha_{l_1, l_2, \dots, l_m} \\ &- \alpha_{k_1, k_2, \dots, k_{j-1}, l_j, k_{j+1}, \dots, k_m} \alpha_{l_1, l_2, \dots, l_{j-1}, k_j, l_{j+1}, \dots, l_m}). \end{aligned} \tag{2.2.3}$$

Let $\mathbb{X} \subset \mathbb{CP}^N$. Then, there are two important subsets of $\mathbb{CP}^N$; the secant variety $\mathfrak{Sec}(\mathbb{X})$, which is defined to be the closure of the set of point lying on secant $\overline{x_1 x_2}$, where x_1 and x_2 are distinct points of $\mathbb{X}$. The second one $\mathfrak{Tan}(\mathbb{X})$ is the union of the projective tangent spaces. In the next section we will discuss the secant variety of a projective variety.

2.3. Secant Variety

The secant variety of a projective variety has been studied in algebraic geometry and some recent references include.(9,10) Let $\mathbb{Y}, \mathbb{Z} \subset \mathbb{CP}(V)$

be varieties and for $y, z \in \mathbb{CP}(V)$, such that $y \neq z$ let $\mathbb{CP}^1_{yz}$ denote the projective line containing y and z. Then we define the join of $\mathbb{Y}$ and $\mathbb{Z}$ by

$$\mathfrak{J}(\mathbb{Y}, \mathbb{Z}) = \overline{\bigcup_{y \in \mathbb{Y}, z \in \mathbb{Z}} \mathbb{CP}^1_{yz}}. \tag{2.3.1}$$

If $\mathbb{Z} = \mathbb{CP}^k$ is a k-plane, we call $\mathfrak{J}(\mathbb{Y}, \mathbb{CP}^k)$ the cone over $\mathbb{Y}$ with vertex $\mathbb{CP}^k$. Moreover, if $\mathbb{Y} = \mathbb{Z}$, we call $\mathfrak{Sec}(\mathbb{Y}) = \mathfrak{J}(\mathbb{Y}, \mathbb{Y})$ the secant variety of $\mathbb{Y}$. We also define the join of k varieties to be the union of the corresponding $\mathbb{CP}^{k-1}$'s which is by definition $\mathfrak{J}(\mathbb{Y}_1, \ldots, \mathbb{Y}_k) = \mathfrak{J}(\mathbb{Y}_1, \mathfrak{J}(\mathbb{Y}_2, \ldots, \mathbb{Y}_k))$. Furthermore, we let $\mathfrak{Sec}_k(\mathbb{Y}) = \mathfrak{J}(\mathbb{Y}, \ldots, \mathbb{Y})$ be the join of k copies of $\mathbb{Y}$ which we call the k-th secant variety of $\mathbb{Y}$. Thus, the k-th secant variety $\mathfrak{Sec}_k(\mathbb{X})$ of $\mathbb{X} \subset \mathbb{CP}^M$ with $\dim \mathbb{X} = d$ is defined to be the closure of the union of k-dimensional linear subspaces of $\mathbb{CP}^M$ determined by general k points on $\mathbb{X}$

$$\begin{aligned} \mathfrak{Sec}_k(\mathbb{X}) &= \overline{\bigcup \{\text{all secant } \mathbb{CP}^{k-1}\text{'s to } \mathbb{X}\}} \\ &= \overline{\bigcup_{x_1, x_2, \ldots, x_k \in \mathbb{X}} \mathbb{CP}^{k-1}_{x_1 x_2 \ldots x_k}}, \end{aligned} \tag{2.3.2}$$

where for $\mathbb{CP}^{k-1}_{x_1 x_2 \cdots x_k}$, denotes the linear space spanned by $x_1, x_2, \ldots, x_k$, e.g., $\mathbb{CP}^{k-1}$. Moreover, the dimension of $\mathfrak{Sec}_k(\mathbb{X})$ satisfies

$$\dim \mathfrak{Sec}_k(\mathbb{X}) \leq \min\{M', k(d+1) - 1\}, \tag{2.3.3}$$

where M' is the dimension of the linear subspace spanned by $\mathbb{X}$. The subvariety $\mathbb{X}$ is called k-defect when $\dim \mathfrak{Sec}_k(\mathbb{X}) < \min\{M', k(d+1) - 1\}$. For example, the secant variety of Segre variety $\mathfrak{Sec}_k(\mathfrak{S}_{\overline{N}})$ is the closure of the set of classes of those tensor products which can be written as the sum of at most $k+1$ decomposable tensor products. Thus, the secant variety of Segre variety $\mathfrak{Sec}_k(\mathfrak{S}_{\overline{N}})$ gives some useful information about the geometry of entangled and separable mixed multipartite states.

2.4. Rank of Tensor and Secant Variety

The rank $R(t)$ of a tensor $U \otimes V \otimes W$ is defined as the minimum number r of triad $u_i \otimes v_i \otimes w_i$ such that t can be represented as $t = \sum_{i=1}^r u_i \otimes v_i \otimes w_i$. Let $\overline{N} = (N_1, N_2, N_3)$ which is also called a format.[10] Then, to this format we can assign the tensor product $T(\overline{N}) = U_1 \otimes U_2 \otimes U_3$, where $U_j = \mathbb{C}^{N_j}$. Moreover, the subset of nonzero triads $S(\overline{N}) = \{u_1 \otimes u_2 \otimes u_3 : u_j \in U_j\} \backslash 0$ is a smooth and irreducible Zariski closed subset of $T(\overline{N})$ of dimension $\sum_{j=1}^3 N_j - 2$ and its tangent space at point $t = u_1 \otimes u_2 \otimes u_3 \in S(\overline{N})$ is given by

$$\mathfrak{Tan}(\overline{N}) = U_1 \otimes u_2 \otimes u_3 + u_1 \otimes U_2 \otimes u_3 + u_1 \otimes u_2 \otimes U_3. \tag{2.4.1}$$

Note that by construction we have $S(\overline{N}) = \mathfrak{S}_{\overline{N}}$. Now, the image of the summation map

$$\sigma_k(\overline{N}) : S(\overline{N})^k \longrightarrow T(\overline{N}), \tag{2.4.2}$$

defined by $(t_1, t_2, \ldots, t_k) \longmapsto \sum_{i=1}^{k} t_i$ consists of the tensors in $T(\overline{N})$ of rank $\leq k$, which we also denote by $\mathfrak{Sec}_k(S(\overline{N}))$ and the Zariski closure of $\mathfrak{Sec}_k(S(\overline{N}))$ is called the $(k-1)$th secant variety of the format $\overline{N}$.

3. GEOMETRY OF ENTANGLED STATES AND SECANT VARIETY

In this section we will apply above mathematical tools to find out more about geometrical structure of entangled and separable bipartite and multipartite quantum system. First we will consider a general bipartite quantum system and show that the secant variety of the Segre variety fills the enveloping space of the Segre embedding. Then we will discuss geometrical structure of multipartite quantum system.

3.1. General Bipartite State and Secant Variety of the Segre Variety

For a bipartite quantum system $\mathcal{Q}(N_1, N_2)$, the Segre variety $\mathfrak{S}_{\overline{N}}$ is the variety of $N_1 \times N_2$ matrices of rank 1. Thus the secant variety $\mathfrak{Sec}_k(\mathfrak{S}_{N_1,N_2})$ is the matrices of rank less than k and $k = N_1$ is the least integer for which

$$\mathfrak{Sec}_k(\mathfrak{S}_{N_1,N_2}) = \mathbb{CP}^{N_1 N_2 - 1}. \tag{3.1.1}$$

In this case the Segre variety has two rulings by the families of linear spaces $v \otimes \mathbb{CP}(W)$ and $\mathbb{CP}(V) \otimes w$ for all $v \in V$ and $w \in W$. The Segre variety can be seen as decomposable tensors in $\mathbb{CP}(V) \otimes \mathbb{CP}(W)$. The k-fold secant plane to the Segre variety is given by the tensor of rank k. For example, a tensor which can be written as $\sum_{i=1}^{k} v_i \otimes w_i = v_1 \otimes w_1 + v_2 \otimes w_2 + \cdots + v_k \otimes w_k$. For a bipartite quantum system $\mathcal{Q}(N_1, N_2)$, the first secant variety of the Segre variety $\mathfrak{Sec}_1(\mathfrak{S}_{N_1,N_2}) = \mathfrak{S}_{\overline{N}}$ which coincides with the Segre variety is the space of separable states. Moreover, for the higher secant variety of the Segre variety, $k \geq 2$, we have $\mathfrak{Sec}_k(\mathfrak{S}_{N_1,N_2}) = \mathbb{CP}^{N_1 N_2 - 1}$ for $k \leq N_1$ and the space of an entangled bipartite state is given by

$$\mathfrak{Sec}_k(\mathfrak{S}_{N_1,N_2}) \backslash \mathfrak{Sec}_1(\mathfrak{S}_{N_1,N_2}) = \mathbb{CP}^{N_1 N_2 - 1} \backslash \mathfrak{S}_{\overline{N}}. \tag{3.1.2}$$

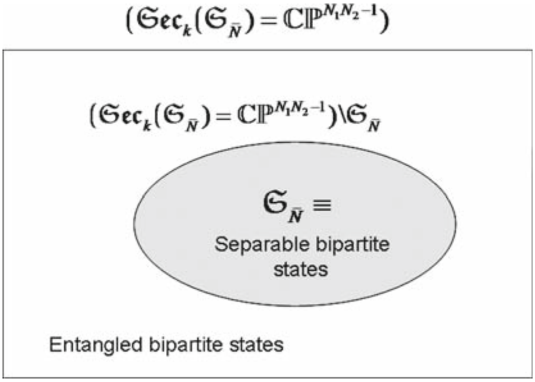

Fig. 1. Schematic structure of entangled and separable general bipartite states based on the secant variety of the Segre variety. The space of pure separable state is defined by the Segre variety $\mathfrak{S}_{\overline{N}}$ and the space of entangled state is defined by $\mathfrak{Sec}_k(\mathfrak{S}_{\overline{N}})\backslash\mathfrak{S}_{\overline{N}}$.

This follows from the construction of the Segre variety and definition of the secant variety of the Segre variety. Since the Segre variety is the space of completely decomposable tensor and the secant variety fills the enveloping space under the Segre embedding. See also the Fig. 1. For bipartite systems, if we assume that $N_1 < N_2$, then for all $1 \leq k < N_1$ the secant variety $\mathfrak{Sec}_k(\mathfrak{S}_{\overline{N}})$ has dimension less than the expected dimension and the least k for which $\mathfrak{Sec}_k(\mathfrak{S}_{\overline{N}})$ fills its enveloping space is $k = N_1$.

4. SECANT VARIETY AS THE SPACE OF MULTIPARTITE ENTANGLED STATES

Now, we will show that the space of multipartite entangled states is given by the higher secant variety of the Segre variety. For example this geometrical structure can be seen by looking at the relation between perfect codes and secant variety of the Segre variety. The existence of perfect codes can be proved based on finite fields with M elements. The perfect code exist only for the following parameters: M is a prime power, $m = \frac{M^l-1}{M-1}$ for $l \geq 2$ and $k = M^{m-l}$. Let us look at some examples of this kind. Let $M = 2$, $m = 2^l - 1$, and $k = 2^{m-l}$, where l is a positive number. Then, for the Segre embedding

$$\mathcal{S}_{2,2,\ldots,2}\colon \overbrace{\mathbb{CP}^1 \times \mathbb{CP}^1 \times \cdots \times \mathbb{CP}^1}^{m\text{-times}} \longrightarrow \mathbb{CP}^{2^m-1}, \tag{4.0.1}$$

the secant variety of the corresponding Segre variety $\mathfrak{Sec}_k(\mathfrak{S}_{\overline{N}}) = \mathbb{CP}^{2^m-1}$ which fits exactly into its enveloping space and all $\mathfrak{Sec}_k(\mathfrak{S}_{\overline{N}})$ have the expected dimension. Thus for a multi-qubit quantum system $\mathcal{Q}(2, 2, \ldots, 2)$, the first secant variety of the Segre variety $\mathfrak{Sec}_1(\mathfrak{S}_{2,2,\ldots,2}) = \mathfrak{S}_{2,2,\ldots,2}$ which coincides with the Segre variety is the space of separable states. Moreover, for the higher secant variety of the Segre variety, $k \geq 2$, we have $\mathfrak{Sec}_k(\mathfrak{S}_{2,2,\ldots,2}) = \mathbb{CP}^{2^m-1}$, and the space of an entangled multipartite state is given by

$$\mathfrak{Sec}_k(\mathfrak{S}_{2,2,\ldots,2}) \backslash \mathfrak{Sec}_1(\mathfrak{S}_{2,2,\ldots,2}) = \mathbb{CP}^{2^m-1} \backslash \mathfrak{S}_{2,2,\ldots,2}. \tag{4.0.2}$$

This also follows from the construction of the Segre variety and definition of the secant variety of the Segre variety. See also the Fig. 2. Next, let M be a prime power. Then for any $l \geq 1, m = \frac{M^l-1}{M-1}$, we have the Segre embedding

$$\mathcal{S}_{M,M,\ldots,M} : \overbrace{\mathbb{CP}^{M-1} \times \mathbb{CP}^{M-1} \times \cdots \times \mathbb{CP}^{M-1}}^{m\text{-times}} \longrightarrow \mathbb{CP}^{M^m-1}. \tag{4.0.3}$$

The secant variety of this Segre variety $\mathfrak{Sec}_k(\mathfrak{S}_{M,M,\ldots,M}) = \mathbb{CP}^{M^m-1}$ gives useful information about the geometry of multipartite quantum system which we also summarize as follows. For a quantum system $\mathcal{Q}(M, M, \ldots, M)$, the first secant variety of the Segre variety $\mathfrak{Sec}_1(\mathfrak{S}_{M,M,\ldots,M}) = \mathfrak{S}_{M,M,\ldots,M}$ which coincides with the Segre variety is the space of separable states. Moreover, for the higher secant variety of the Segre variety

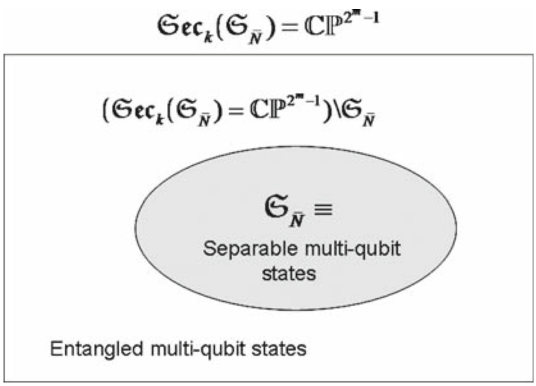

Fig. 2. Schematic structure of entangled and separable general multi-qubit states based on the secant variety of the Segre variety. The space of pure separable state is defined by the Segre variety $\mathfrak{S}_{2,2,\ldots,2}$ and the space of entangled state is defined by $\mathfrak{Sec}_k(\mathfrak{S}_{2,2,\ldots,2}) \backslash \mathfrak{S}_{2,2,\ldots,2}$.

we have $\mathfrak{Sec}_k(\mathfrak{S}_{M,M,\ldots,M}) = \mathbb{CP}^{M^m-1}$. Thus the space of an entangled multipartite state is given by

$$\mathfrak{Sec}_k(\mathfrak{S}_{M,M,\ldots,M}) \backslash \mathfrak{Sec}_1(\mathfrak{S}_{M,M,\ldots,M}) = \mathbb{CP}^{M^m-1} \backslash \mathfrak{S}_{M,M,\ldots,M}. \quad (4.0.4)$$

We have established a connection between pure mathematics and fundamental quantum mechanics with some applications in the field of quantum information and computation. We have introduced and discussed the secant variety of the Segre variety. We have shown that geometrical structure of entangled and separable multipartite states are given by the secant variety of the Segre variety. But the secant varieties are still subject of research in algebraic geometry. For example, there are still many fundamental open questions about the secant variety of the Segre variety. However, we hope that this geometrical structure may give us some hint to how to solve the problem of quantifying entanglement of an arbitrary multipartite system.

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