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GHZ States, Almost-Complex Structure and Yang–Baxter Equation

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Recent research suggests that there are natural connections between quantum information theory and the Yang–Baxter equation. In this paper, in terms of the almost-complex structure and with the help of its algebra, we define the Bell matrix to yield all the Greenberger–Horne–Zeilinger (GHZ) states from the product basis, prove it to form a unitary braid representation and presents a new type of solution of the quantum Yang–Baxter equation. We also study Yang–Baxterization, Hamiltonian, projectors, diagonalization, noncommutative geometry, quantum algebra and FRT dual algebra associated with this generalized Bell matrix.

KEY WORDS: GHZ state; Yang–Baxter; almost-complex structure; FRT.

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1. INTRODUCTION

Recently, a series of papers(1–10) have suggested there are natural and deep connections between quantum information theory(11) and the Yang–Baxter equation (YBE).(12,13) Unitary solutions of the braided YBE (i.e., the braid group relation)(1,2) as well as unitary solutions of the quantum Yang–Baxter equation (QYBE)(3,4) can be often identified with universal quantum gates.(14) Yang–Baxterization(15) is exploited to set up the Schrödinger equation determining the unitary evolution of a unitary braid gate.(3,4) Furthermore, the Werner state(16) is viewed as a rational solution of the QYBE and the isotropic state(17) with a specific parameter forms a

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braid representation, see Refs. 7, 8. More interestingly, the Temperley–Lieb algebra[18] deriving a braid representation in the state model for the Jones polynomial[19] is found to present a suitable mathematical framework for a unified description of various quantum teleportation phenomena,[20] see Refs. 9, 10.

Based on previous research work[3,4,6] in which the Bell matrix has been recognized to form a unitary braid representation and generate all the Bell states from the product basis, in this paper, a high dimensional unitary braid representation called the Bell matrix is obtained to create all the Greenberger–Horne–Zeilinger states (GHZ states) from the product basis. The GHZ states are maximally multipartite entangled states and play important roles in the study of quantum information phenomena.[21–23] More importantly, this Bell matrix has a form in terms of the almost-complex structure which is fundamental for complex and Kähler geometry and symplectic geometry. Therefore, our paper is building heuristic connections among quantum information theory, the Yang–Baxter equation and differential geometry.

We hereby summarize our main result which is new to our knowledge.

1. We define the Bell matrix to produce all the GHZ states from the product basis, prove it to be a unitary braid representation, and derive the Hamiltonian to determine the unitary evolution of the GHZ states.
2. We recognize the almost-complex structure in the formulation of the Bell matrix and refine its algebra in the proof for the Bell matrix satisfying the braided YBE, while we represent a new type of the solution of the QYBE in terms of the almost-complex structure.
3. We study topics associated with the generalized Bell matrix which include Yang–Baxterization, diagonalization, noncommutative geometry, quantum algebra via the RTT relation and standard FRT procedure.[24,25]

We focus on the Bell matrix of the type $2^{2n} \times 2^{2n}$ related to the GHZ states of an even number of objects, and submit our result on the Bell matrix of the type $2^{2n+1} \times 2^{2n+1}$ elsewhere.

The plan of this paper is organized as follows. Section 2 sketches the definition of the GHZ states and represent the Bell matrix in terms of the almost-complex structure. Section 3 introduces the generalized Bell matrix and presents a new type of solution of the QYBE in terms of the almost-complex structure. Sections 4 and 5 deal with various topics around the Bell matrix: projectors, diagonalization, noncommutative geometry, quantum algebra and FRT dual algebra. The last section concludes with worthwhile problems for further research.

2. GHZ STATES, BELL MATRIX AND HAMILTONIAN

This section is devised to set up a simplest example to be appreciated by readers mostly interested in quantum information and physics, and it explains how to observe the Bell matrix from the formulation of the GHZ states (as well as the almost-complex structure from the Bell matrix) and how to derive Hamiltonians to determine the unitary evolution of the GHZ states.

2.1. GHZ States, Bell Matrix and Almost-complex Structure

In the 2^N-dimensional Hilbert space with the basis denoted by the Dirac kets $|m_1, m_2, \ldots, m_N\rangle$, $m_1, \ldots, m_N = \pm\frac{1}{2}$, there are 2^N linearly independent GHZ states of N-objects having the form

$$\frac{1}{\sqrt{2}}(|m_1, m_2, \ldots, m_N\rangle \pm |-m_1, -m_2, \ldots, -m_N\rangle) \tag{1}$$

which are maximally entangled states in quantum information theory.[11] In this paper, all the GHZ states are found to be generated by the Bell matrix acting on the chosen product basis,

$$|\Phi_k\rangle = |m_1, m_2, \ldots, m_N\rangle, \qquad |\Phi_{2^N-k+1}\rangle = |-m_1, -m_2, \ldots, -m_N\rangle, \tag{2}$$

where $1 \le k \le 2^{N-1}$. One can take a notation similar to,[26,27]

$$k[m_1, \ldots, m_N] = 2^{N-1} + \frac{1}{2} - \sum_{i=1}^{N} 2^{N-i}\, m_i \tag{3}$$

which has the result at $N=2$, for example,

$$k\left[\frac{1}{2}, \frac{1}{2}\right] = 1, \quad k\left[\frac{1}{2}, -\frac{1}{2}\right] = 2, \quad k\left[-\frac{1}{2}, \frac{1}{2}\right] = 3, \quad k\left[-\frac{1}{2}, -\frac{1}{2}\right] = 4, \tag{4}$$

assigned to label the GHZ states of two objects (the well known Bell states).

The 4×4 Bell matrix B_4 acts on the product basis $|\frac{1}{2}\frac{1}{2}\rangle$, $|\frac{1}{2}\frac{-1}{2}\rangle$ and $|\frac{-1}{2}\frac{1}{2}\rangle$, $|\frac{-1}{2}\frac{-1}{2}\rangle$ to yield the Bell states, and it has a known form,[1–4,6]

$$B_4 = (B_{kn,lm})_4 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & -1 & 1 & 0 \\ -1 & 0 & 0 & 1 \end{pmatrix}, \quad k, n, l, m = \frac{1}{2}, -\frac{1}{2}, \tag{5}$$

and the 8×8 Bell matrix B_8 given by

$$B_8 \equiv (B_{\alpha l,\,\beta m})_8 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

$$\alpha, \beta = \frac{3}{2}, \frac{1}{2}, -\frac{1}{2}, -\frac{3}{2}, \quad l, m = \frac{1}{2}, -\frac{1}{2} \tag{6}$$

creates the GHZ states of three objects by acting on $|\Phi_k\rangle$, $1 \leq k \leq 8$.

The $2^N \times 2^N$ Bell matrix generating the GHZ states of N-objects from the product basis $|\Phi_k\rangle$, $1 \leq k \leq 2^N$, has a form in terms of the almost-complex structure[4] denoted by M,

$$B = \frac{1}{\sqrt{2}}(1\!\!1 + M), \quad B_{ij,kl} \equiv B_{ij}^{kl} = \frac{1}{\sqrt{2}}(\delta_i^k \delta_j^l + M_{ij}^{kl}) \tag{7}$$

where $1\!\!1$ denotes the identity matrix, the lower index of B_{2^N} is suppressed for convenience, δ_i^j is the Kronecker function of two variables i, j which is 1 if $i = j$ and 0 otherwise, and the almost-complex structure M has the component formalism involving the step function $\epsilon(i)$,

$$M_{ij,kl} \equiv M_{ij}^{kl} = \epsilon(i) \delta_i^{-k} \delta_j^{-l}, \quad \epsilon(i) = 1, i \geq 0; \quad \epsilon(i) = -1, i < 0, \tag{8}$$

which satisfies $M^2 = -1\!\!1$. In terms of the tensor product of the Pauli matrices, the Bell matrix B and the almost complex structure M for N-objects have the forms given by

$$B = e^{\frac{\pi}{4} M}, \quad M = \sqrt{-1} \sigma_y \otimes (\sigma_x)^{\otimes(N-1)}, \quad (\sigma_x)^{\otimes(N-1)} = \underbrace{\sigma_x \otimes \cdots \otimes \sigma_x}_{N-1}. \tag{9}$$

Note that there exist other interesting matrices to produce all the GHZ states from the product basis, for example, one can choose matrix entries $\epsilon(i) B_{ij,kl}$ for a new matrix. But so far as the authors know, only the Bell matrix is found to form a unitary braid representation.

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[4]The almost-complex structure is usually denoted by the symbol J in the literature and it is a linear map from a real vector space to itself satisfying $J^2 = -1$. More details on geometry underlying what we are presenting here will be discussed elsewhere.

2.2. Yang–Baxterization and Hamiltonian

The Bell matrix B satisfies the characteristic equation

$$(B - e^{i\frac{\pi}{4}} 1\!\!1)(B - e^{-i\frac{\pi}{4}} 1\!\!1) = 0 \tag{10}$$

and so it has two distinct eigenvalues $e^{\pm i\frac{\pi}{4}}$. Using Yang–Baxterization,[5] a solution of the QYBE with the Bell matrix as its asymptotic limit, is obtained to be

$$\check{R}(x) = B + xB^{-1} = \frac{1}{\sqrt{2}}(1+x)1\!\!1 + \frac{1}{\sqrt{2}}(1-x)M. \tag{11}$$

As this solution $\check{R}(x)$ is required to be unitary, it needs a normalization factor ρ with a real spectral parameter x,

$$B(x) = \rho^{-\frac{1}{2}}\check{R}(x), \quad \rho = 1 + x^2, \quad x \in \mathbb{R}. \tag{12}$$

As the real spectral parameter x plays the role of the time variable, the Schrödinger equation describing the unitary evolution of a state $\psi(0)$ determined by the $B(x)$ matrix, i.e., $\psi(x) = B(x)\psi(0)$, has the form

$$\sqrt{-1}\frac{\partial}{\partial x}\psi(x) = H(x)\psi(x), \quad H(x) \equiv \sqrt{-1}\frac{\partial B(x)}{\partial x}B^{-1}(x), \tag{13}$$

where the time-dependent Hamiltonian $H(x)$ is given by

$$H(x) = \sqrt{-1}\frac{\partial}{\partial x}(\rho^{-\frac{1}{2}}\check{R}(x))(\rho^{-\frac{1}{2}}\check{R}(x))^{-1} = -\sqrt{-1}\rho^{-1}M. \tag{14}$$

To construct the time-independent Hamiltonian, a new time variable θ instead of the spectral parameter x is introduced in the way

$$\cos\theta = \frac{1}{\sqrt{1+x^2}}, \quad \sin\theta = \frac{x}{\sqrt{1+x^2}}, \tag{15}$$

so that the Bell matrix $B(x)$ has a new formulation as a function of θ,

$$B(\theta) = \cos\theta B + \sin\theta B^{-1} = e^{(\frac{\pi}{4}-\theta)M}, \tag{16}$$

[5]See Subsects. 3.1 and 4.1 or Ref. 4 for details. Yang–Baxterization is applied to the Bell matrix of the type $2^{2n} \times 2^{2n}$, while Yang–Baxterization of the Bell matrix of the type $2^{2n+1} \times 2^{2n+1}$ is to be presented elsewhere.

and hence the Schrödinger equation for the time evolution $\psi(\theta) = B(\theta)\psi(0)$ has the form

$$\sqrt{-1}\frac{\partial}{\partial\theta}\psi(\theta) = H\psi(\theta), \quad H \equiv \sqrt{-1}\frac{\partial B(\theta)}{\partial\theta}B^{-1}(\theta) = -\sqrt{-1}M, \tag{17}$$

where the Hamiltonian[6] is time-independent and Hermitian together with the unitary evolution operator $U(\theta) = e^{-M\theta}$.

3. GENERALIZED BELL MATRIX AND YBE

This section proves the generalized[7] Bell matrix $\widetilde{B}$ of the type $2^{2n} \times 2^{2n}$ to form a unitary braid representation with the help of the algebra of the almost-complex structure $\widetilde{M}$, and presents a new type of solution of the QYBE in terms of $\widetilde{M}$.

3.1. YBE and Yang–Baxterization

In this paper, the braid group representation σ-matrix and the QYBE solution $\check{R}(x)$-matrix are $d^2 \times d^2$ matrices acting on $V \otimes V$ where V is a d-dimensional complex vector space. As σ and $\check{R}$ act on the tensor product $V_i \otimes V_{i+1}$, they are denoted by σ_i and $\check{R}_i$, respectively.

The generators σ_i of the braid group B_n satisfy the algebraic relation called the braid group relation,

$$\begin{aligned} \sigma_i\sigma_{i+1}\sigma_i &= \sigma_{i+1}\sigma_i\sigma_{i+1}, \quad 1 \le i \le n-1, \\ \sigma_i\sigma_j &= \sigma_j\sigma_i, \quad |i-j| > 1. \end{aligned} \tag{18}$$

while the quantum Yang–Baxter equation (QYBE) has the form

$$\check{R}_i(x)\,\check{R}_{i+1}(xy)\,\check{R}_i(y) = \check{R}_{i+1}(y)\,\check{R}_i(xy)\,\check{R}_{i+1}(x) \tag{19}$$

with the spectral parameters x and y. In addition, the component formalism the QYBE (or the braid group relation) can be shown in terms of matrix entries,

$$\check{R}(x)_{i_1 j_1}^{i' j'}\check{R}(xy)_{j' k_1}^{k' k_2}\check{R}(y)_{i' k'}^{i_2 j_2} = \check{R}(y)_{j_1 k_1}^{j' k'}\check{R}(xy)_{i_1 j'}^{i_2 i'}\check{R}(x)_{i' k'}^{j_2 k_2}. \tag{20}$$

---

[6]The Hamiltonian[(3,4)] has an additional numerical factor $\frac{1}{2}$ compared to (17), which makes it possible that the action of $\exp\frac{1}{2}\theta M$ on the product basis is equivalent to a product of two unitary rotations of Wigner functions of the Bell states. Note that no boundary conditions are imposed on the Schrödinger equations (13) or (17).

[7]Here "generalized" means that the object has deformation parameters.

In view of the fact that $\check{R}(x=0)$ forms a braid representation, the braid group relation is also called the braided YBE. Concerning relations between braid representations and x-dependent solutions of the QYBE (19), the procedure of constructing the $\check{R}(x)$-matrix from a given braid representation σ-matrix is called Yang–Baxterization.[15] For a braid representation σ with two distinct eigenvalues λ_1 and λ_2, the corresponding $\check{R}(x)$-matrix obtained via Yang–Baxterization has the form

$$\check{R}(x) = \sigma + x\lambda_1\lambda_2\sigma^{-1} \tag{21}$$

which has been exploited in Subsection 2.2, see (11).

3.2. Generalized Bell Matrix as a Solution of YBE

The generalized Bell matrix $\widetilde{B}$ has the form by the almost-complex structure $\widetilde{M}$ with deformation parameters q_{ij},

$$\widetilde{B}_{ij}^{kl} = \frac{1}{\sqrt{2}}(\delta_i^k\delta_j^l + \widetilde{M}_{ij}^{kl}), \quad \widetilde{M}_{ij}^{kl} = \epsilon(i)q_{ij}\delta_i^{-k}\delta_j^{-l}, \tag{22}$$

where $q_{ij}q_{-i-j} = 1$ is required for $\widetilde{M}^2 = -1\!\!1$ and the generalized step function $\epsilon(i)$ is defined by

$$\epsilon(i)\epsilon(i) = 1, \quad \epsilon(i)\epsilon(-i) = -1, \tag{23}$$

which has solutions $\epsilon(i) = \pm 1$, $\epsilon(-i) = \mp 1$.

We label $\widetilde{B}$ by familiar indices for the angular momentum theory in quantum mechanics,

$$(\widetilde{B}^{J_1J_2})_{\mu a}^{b\nu}, \quad \mu, \nu = J_1, J_1-1, \ldots, -J_1, \quad a, b = J_2, J_2-1, \cdots, -J_2, \tag{24}$$

where $\widetilde{B}^{JJ}$ denotes the generalized Bell matrix $\widetilde{B}$ associated with the GHZ states of an even number of objects, for example,

$$\widetilde{B}_4 = \widetilde{B}^{\frac{1}{2}\frac{1}{2}}, \quad \widetilde{B}_{16} = \widetilde{B}^{\frac{3}{2}\frac{3}{2}}, \quad \widetilde{B}_{64} = \widetilde{B}^{\frac{7}{2}\frac{7}{2}}, \tag{25}$$

but the same type of generalized Bell matrix may be labeled differently, for example, both $\widetilde{B}^{\frac{1}{2}\frac{3}{2}}$ and $\widetilde{B}^{\frac{3}{2}\frac{1}{2}}$ belong to the same type of $\widetilde{B}_8$.

In the following, we study the generalized Bell matrix of the type $\widetilde{B}^{JJ}$ denoted by $\widetilde{B}$, and leave our result on $\widetilde{B}^{J_1J_2}$, $J_1 \neq J_2$ elsewhere.

In the proof for $\widetilde{B}^{JJ}$ forming a braid representation (18) in terms of its component formalism (22), deformation parameters q_{ij} are found to satisfy

$$
\begin{aligned}
&q_{i_1 j_1} q_{-i_1 -j_1} = q_{j_1 k_1} q_{-j_1 -k_1}, \quad i_1, j_1, k_1 = J, J-1, \ldots, -J, \\
&q_{j_1 k_1} = q_{i_1 j_1} q_{-j_1 k_1} q_{-i_1 j_1}, \quad q_{i_1 j_1} = q_{j_1 k_1} q_{i_1 - j_1} q_{j_1 - k_1},
\end{aligned} \tag{26}
$$

where no summation is imposed between same lower indices and which is simplified by $q_{i_1 j_1} q_{-i_1 -j_1} = 1$. Furthermore, the unitarity of $\widetilde{B}$ leads to a constraint on $\widetilde{M}$, namely,

$$
\widetilde{M}^{\dagger} \equiv \widetilde{M}^{*T} = \widetilde{M}^{-1} = -\widetilde{M} \Rightarrow q_{ij}^{*} q_{ij} = 1, \tag{27}
$$

where the superscript $*$ denotes the complex conjugation and the symbol T denotes the matrix transpose operation.

As J is a half-integer, we obtain solutions for Eqs. (26) and (27) by $(J+\frac{1}{2})$-number of independent angle parameters φ_J, φ_{J-1}, ..., $\varphi_{\frac{1}{2}}$,

$$
q_{lm} = e^{i\frac{\varphi_l + \varphi_m}{2}}, \quad \varphi_{-l} = -\varphi_l, \quad 0 \leq l \leq J, \tag{28}
$$

where the method of separation of variables has been used by choosing $q_l = e^{i\varphi_l}$ and then deriving $q_{lm} = q_l q_m$.

For example, deformation parameters for the generalized Bell matrix $\widetilde{B}^{\frac{1}{2}\frac{1}{2}}$ have the form

$$
q_{\frac{1}{2}\frac{1}{2}} = e^{i\varphi}, \quad q_{-\frac{1}{2}-\frac{1}{2}} = e^{-i\varphi}, \quad q_{\frac{1}{2}-\frac{1}{2}} = q_{-\frac{1}{2}\frac{1}{2}} = 1, \tag{29}
$$

which are the same as those,[3,4,6] and deformation parameters of the generalized Bell matrix $\widetilde{B}^{\frac{3}{2}\frac{3}{2}}$ are given by

$$
\begin{aligned}
&q_{\frac{3}{2}\frac{3}{2}} = e^{i\varphi_1}, \quad q_{\frac{3}{2}\frac{1}{2}} = e^{i\frac{\varphi_1+\varphi_2}{2}}, \quad q_{\frac{3}{2}-\frac{1}{2}} = e^{i\frac{\varphi_1-\varphi_2}{2}}, \quad q_{\frac{3}{2}-\frac{3}{2}} = 1, \\
&q_{\frac{1}{2}\frac{3}{2}} = e^{i\frac{\varphi_1+\varphi_2}{2}}, \quad q_{\frac{1}{2}\frac{1}{2}} = e^{i\varphi_2}, \quad q_{\frac{1}{2}-\frac{1}{2}} = 1, \quad q_{\frac{1}{2}-\frac{3}{2}} = e^{i\frac{\varphi_2-\varphi_1}{2}}.
\end{aligned} \tag{30}
$$

In the 2^{2n}-dimensional[8] compelx vector space, the almost-complex structure $\widetilde{M}$ is found to satisfy the algebraic relations,

$$
\begin{aligned}
&\widetilde{M}^2 = -1\!\!1, \qquad \widetilde{M}_{i\pm1}\widetilde{M}_i = -\widetilde{M}_i\widetilde{M}_{i\pm1}, \\
&\widetilde{M}_i\widetilde{M}_j = \widetilde{M}_j\widetilde{M}_i, \qquad |i-j| \geq 2,\ i, j \in \mathbb{N},
\end{aligned} \tag{31}
$$

which define an algebra different from the Temperley–Lieb algebra[18] or the symmetric group algebra and where deformation parameters q_{ij} have to satisfy

$$
q_{ij} q_{-i-j} = 1, \quad q_{ij} q_{-ij} = q_{jk} q_{j-k}. \tag{32}
$$

---

[8] Here we have $2^{2n} = (2J+1)^2$, for example, $n=1$, $J=\frac{1}{2}$ and $n=2$, $J=\frac{3}{2}$, see Ref. 25.

With the help of this algebra (31), the generalized Bell matrix $\widetilde{B}$ can be easily proved to satisfy the braided YBE (18) in the way

$$\widetilde{B}_i \widetilde{B}_{i+1} \widetilde{B}_i = 2\widetilde{M}_i + 2\widetilde{M}_{i+1} + \widetilde{M}_i \widetilde{M}_{i+1} + \widetilde{M}_{i+1} \widetilde{M}_i = \widetilde{B}_{i+1} \widetilde{B}_i \widetilde{B}_{i+1}. \quad (33)$$

Additionally, the almost-complex structure $\widetilde{M}$ and the permutation operator P satisfy the following algebraic relation

$$P_i \widetilde{M}_{i+1} P_i = P_{i+1} \widetilde{M}_i P_i, \quad P = \sum_{ij} |ij\rangle\langle ji|, \quad (34)$$

which is underlying algebraic relations of the virtual braid group, i.e., the braid $\widetilde{B}$ and permutation P forming a unitary virtual braid representation, see Refs. 7, 8.

3.3. New Type of Solution of QYBE Via Reparameterization

Similar to the rational solution of the QYBE (19),

$$\check{R}_{\text{rational}}(u) = \mathbb{1} + uP, \quad P^2 = \mathbb{1} \quad (35)$$

with the permutation matrix P, we have a solution of the QYBE in terms of the almost-complex structure,

$$\widetilde{\check{R}}(u) = \mathbb{1} + u\widetilde{M} \quad (36)$$

satisfying the equation of Yang–Baxter type,

$$\check{R}_i(u)\, \check{R}_{i+1}\left(\frac{u+v}{1+uv}\right) \check{R}_i(v) = \check{R}_{i+1}(v)\, \check{R}_i\left(\frac{u+v}{1+uv}\right) \check{R}_{i+1}(u), \quad (37)$$

which has been firstly exploited[(4)] and where new spectral parameters u, v are related to original spectral parameters x, y in the way

$$u = \frac{1-x}{1+x}, \qquad v = \frac{1-y}{1+y}, \qquad \frac{1-xy}{1+xy} = \frac{u+v}{1+uv}. \quad (38)$$

Using reparametrization of u, v in terms of angle variables Θ_1, Θ_2,

$$u = -\sqrt{-1}\tan\Theta_1, \quad v = -\sqrt{-1}\tan\Theta_2, \quad \frac{u+v}{1+uv} = -\sqrt{-1}\tan(\Theta_1+\Theta_2), \quad (39)$$

the modified Yang–Baxter equation (37) has the ordinary form

$$\check{R}_i(\Theta_1)\, \check{R}_{i+1}(\Theta_1+\Theta_2)\, \check{R}_i(\Theta_2) = \check{R}_{i+1}(\Theta_2)\, \check{R}_i(\Theta_1+\Theta_2)\, \check{R}_{i+1}(\Theta_1). \quad (40)$$

with the solution given by

$$\tilde{\check{R}}(\Theta) = 1\!\!1 - \sqrt{-1}\tan\Theta\widetilde{M}, \quad \text{or} \quad \tilde{\check{R}}(\Theta') = 1\!\!1 + \tanh\Theta'\widetilde{M}. \tag{41}$$

Note that physical models underlying this type of solution of QYBE will be explored elsewhere.

4. PROJECTORS, DIAGONALIZATION AND GEOMETRY

This section and the next one are aimed at introducing selective topics related to the generalized Bell matrix (i.e., the almost-complex structure), for example, associated noncommutative geometry, quantum algebra and FRT dual algebra.

4.1. Projectors and Yang–Baxterization

In terms of $\widetilde{M}$, two projectors $\widetilde{P}_+$ and $\widetilde{P}_-$ defined by

$$\widetilde{P}_+ = \frac{1}{2}(1 + \sqrt{-1}\widetilde{M}), \quad \widetilde{P}_- = \frac{1}{2}(1 - \sqrt{-1}\widetilde{M}) \tag{42}$$

satisfy properties of two mutually orthogonal projectors,

$$\widetilde{P}_+ + \widetilde{P}_- = 1\!\!1, \quad \widetilde{P}_\pm^2 = \widetilde{P}_\pm, \quad \widetilde{P}_+\widetilde{P}_- = 0. \tag{43}$$

The generalized Bell matrix $\widetilde{B}$ has two distinct eigenvalues $e^{\pm i\frac{\pi}{4}}$ and it satisfies the same characteristic equation as (10),

$$(\widetilde{B} - \lambda_- 1\!\!1)(\widetilde{B} - \lambda_+ 1\!\!1) = 0, \quad \lambda_+ = e^{-i\frac{\pi}{4}}, \quad \lambda_- = e^{i\frac{\pi}{4}}. \tag{44}$$

With the projectors $\widetilde{P}_\pm$ and eigenvalues $\lambda_\pm$, the generalized Bell matrix and its inverse have the form

$$\widetilde{B} = \lambda_+\widetilde{P}_+ + \lambda_-\widetilde{P}_-, \quad \widetilde{B}^{-1} = \lambda_-\widetilde{P}_+ + \lambda_+\widetilde{P}_-. \tag{45}$$

Using Yang–Baxterization,[4] the $\check{R}(x)$-matrix as a solution of the QYBE (19) has a form similar to (11),

$$\tilde{\check{R}}(x) = (\lambda_+ + \lambda_- x)\widetilde{P}_+ + (\lambda_- + \lambda_+ x)\widetilde{P}_- = \widetilde{B} + x\widetilde{B}^{-1}, \tag{46}$$

and hence the Schrodinger equation has a form similar to (13) or (17) except that the Hamiltonian is determined by $\widetilde{M}$ instead of M.

4.2. Diagonalization of $\widetilde{B}$

The diagonalization of the generalized Bell matrix $\widetilde{B}$ is performed by a unitary matrix D via the unitary transformation,

$$D\widetilde{B}D^{\dagger}=\frac{1}{\sqrt{2}}Diag(1+\sqrt{-1},\ldots,1-\sqrt{-1}) \tag{47}$$

where the diagonal matrix $Diag$ has the same number of matrix entries $1+\sqrt{-1}$ as $1-\sqrt{-1}$. Assume D to have the form by a Hermitian matrix N,

$$D=\frac{1}{\sqrt{2}}(1\!\!1+\sqrt{-1}N), \quad N^{\dagger}=N, \quad N^2=1\!\!1 \tag{48}$$

and then this N is found to satisfy an additional constraint,

$$N\widetilde{M}=-\widetilde{M}N=Diag(1,-1,\ldots,1,-1), \tag{49}$$

where the diagonal matrix $Diag$ has the same number of matrix entries 1 as -1 but the ordering between 1 and -1 is flexible.

After some algebra, a formalism of the N is given by

$$N_{ij}^{kl}=f(i)q_{ij}\delta_i^{-k}\delta_j^{-l}, \quad f(i)f(i)=1, \quad f(i)=f(-i)=f^*(i) \tag{50}$$

where q_{ij} are the same as unitary deformation parameters q_{ij} in the generalized Bell matrix $\widetilde{B}$. It gives rise to the diagonalization form of $\widetilde{B}$,

$$(D\widetilde{B}D^{\dagger})_{ij}^{mn}=\frac{1}{\sqrt{2}}(1+\sqrt{-1}f(i)\epsilon(-i))\delta_i^m\delta_j^n, \tag{51}$$

where $f(i)=\epsilon(-i)$, $i>0$ and $f(i)=\epsilon(i)$, $i<0$ leads to

$$D\widetilde{B}D^{\dagger}=\frac{1}{\sqrt{2}}Diag(\underbrace{1+\sqrt{-1},\ldots,1+\sqrt{-1}}_{2^{N-1}},\underbrace{1-\sqrt{-1},\ldots,1-\sqrt{-1}}_{2^{N-1}}). \tag{52}$$

For example, the Bell matrix B_4 is diagonalized in the way

$$D_4B_4D_4^{\dagger}=\frac{1}{\sqrt{2}}Diag(1-\sqrt{-1},1+\sqrt{-1},1-\sqrt{-1},1+\sqrt{-1}),\ N_4=-\sigma_y\otimes\sigma_y, \tag{53}$$

where B_4 can be also diagonalized by unitary transformations of the Malkline matrix (or the magic matrix),[26–28] or the diagonaliser,[29] and the generalized Bell matrix $\widetilde{B}_4$ can be diagonalized with a given $N_{4,1}$,

$$\widetilde{B}_4 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 0 & 0 & q \\ 0 & 1 & 1 & 0 \\ 0 & -1 & 1 & 0 \\ -q^{-1} & 0 & 0 & 1 \end{pmatrix}, \quad N_{4,1} = \begin{pmatrix} 0 & 0 & 0 & -q \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ q^{-1} & 0 & 0 & 0 \end{pmatrix},$$

$$D_{4,1}\widetilde{B}_4 D_{4,1} = \frac{1}{\sqrt{2}} Diag(1+\sqrt{-1}, 1+\sqrt{-1}, 1-\sqrt{-1}, 1-\sqrt{-1}). \quad (54)$$

As a remark, calculation for noncommutative geometry and quantum algebra associated with the generalized Bell matrix can be greatly simplified once the above diagonalization procedure is exploited.

4.3. Associated Noncommutative Geometry

In view of the standard procedure of setting up associated noncommutative geometry with a given braid group representation,[30,31] we denote coordinate operators X and differential operators ξ in the way

$$X^T = (x_1, x_2, \ldots, x_{2^N}), \quad \xi^T = (\xi_1, \xi_2, \ldots, \xi_{2^N}) \quad (55)$$

and demand them to satisfy constraint equations,

$$\widetilde{P}_-(X \otimes X) = 0, \quad \widetilde{P}_+(\xi \otimes \xi) = 0, \quad X \otimes \xi = (\mu \widetilde{P}_+ - 1\!\!1)(\xi \otimes X) \quad (56)$$

where μ is a free parameter. Note that these equations can be chosen in the other way by exchanging $\widetilde{P}_+$ with $\widetilde{P}_-$.

More essentially, noncommutative differential geometry generated by X and ξ is determined by the equations in terms of the almost-complex structure $\widetilde{M}$,

$$\begin{aligned} &X \otimes X = \sqrt{-1}\widetilde{M}(X \otimes X), \quad \xi \otimes \xi = -\sqrt{-1}\widetilde{M}(\xi \otimes \xi), \\ &X \otimes \xi = \left(\tfrac{\mu}{2} - 1\right) \xi \otimes X + \tfrac{\mu}{2}\sqrt{-1}\widetilde{M}(\xi \otimes X), \end{aligned} \quad (57)$$

which have the formalism of component,

$$\begin{aligned} &x_i x_j = \sqrt{-1}\epsilon(i) q_{ij} x_{-i} x_{-j}, \quad \xi_i \xi_j = -\sqrt{-1}\epsilon(i) q_{ij} \xi_{-i} \xi_{-j}, \\ &x_i \xi_j = \left(\tfrac{\mu}{2} - 1\right) \xi_i x_j + \tfrac{\mu}{2}\sqrt{-1}\epsilon(i) q_{ij} \xi_{-i} x_{-j}. \end{aligned} \quad (58)$$

with the significant geometry at $\mu = 2$. Note that noncommutative plane related to the Bell matrix B_4 has been already briefly discussed.[32]

5. QUANTUM ALGEBRA VIA THE FRT PROCEDURE

For a given solution $\check{R}$ of the braided YBE (18), there exists a standard procedure(24,25) using the $\check{R}TT$ relation and $\check{R}LL$ relations to respectively define associated quantum algebra and FRT dual algebra. In this section, we present quantum algebra and FRT dual algebra specified by the $\widetilde{B}TT$ relation and $\widetilde{B}LL$ relations.

5.1. Quantum Algebra Using the $\widetilde{M}TT$ Relation

In the well known $\check{R}TT$ relation: $\check{R}(T\otimes T)=(T\otimes T)\check{R}$, all matrix entries of the T-matrix are assumed to be noncommutative operators. As the $\check{R}$-matrix is the generalized Bell matrix $\widetilde{B}$, the $\widetilde{B}TT$ relation is essentially the $\widetilde{M}TT$ relation,

$$\widetilde{B}(T\otimes T)=(T\otimes T)\widetilde{B}\Rightarrow \widetilde{M}(T\otimes T)=(T\otimes T)\widetilde{M}, \tag{59}$$

where $\widetilde{M}$ is a $2^{2n}\times 2^{2n}$ matrix and T is a $2^{2n-1}\times 2^{2n-1}$ matrix. By matrix entries of $\widetilde{M}$, T and the convention $(A\otimes B)_{ij,kl}\equiv A_{ik}B_{jl}$, the $\widetilde{M}TT$ relation has the component formalism,

$$\begin{aligned}&T_{i_1-i_2}T_{j_1-j_2}+\epsilon(i_1)\epsilon(i_2)q_{i_1j_1}q_{i_2j_2}T_{-i_1i_2}T_{-j_1j_2}=0,\\&i_1,i_2,j_1,j_2=J,J-1,\ldots,-J.\end{aligned} \tag{60}$$

Note that this $\widetilde{M}TT$ relation (60) lead to eight simplified equations,

$$\begin{aligned}&T_{ii}T_{ii}=T_{-i-i}T_{-i-i}, && T_{ii}T_{-i-i}=T_{-i-i}T_{ii},\\&T_{i-i}T_{i-i}=-q_{ii}^2T_{-ii}T_{-ii}, && T_{i-i}T_{-ii}=-T_{-ii}T_{i-i},\\&T_{ii}T_{i-i}=q_{ii}T_{-i-i}T_{-ii}, && T_{ii}T_{-ii}=q_{-i-i}T_{-i-i}T_{i-i},\\&T_{i-i}T_{ii}=-q_{ii}T_{-ii}T_{-i-i}, && T_{i-i}T_{-i-i}=-q_{ii}T_{-ii}T_{ii}\end{aligned} \tag{61}$$

which completely determine the quantum algebra related to $\widetilde{B}_4$.

With the help of a new $\widetilde{T}$-matrix given by

$$\widetilde{T}_{ij}=\epsilon(i)T_{ij}+T_{-i-j},\quad \widetilde{T}_{-i-j}=-\epsilon(i)T_{-i-j}+T_{ij}, \tag{62}$$

where q_{ij} is chosen to be unit for convenience, the $\widetilde{M}TT$ relation (60) is replaced by the $\widetilde{M}\widetilde{T}\widetilde{T}$ relation having the algebraic relations,

$$\begin{aligned}&\widetilde{T}_{i_1-i_2}\widetilde{T}_{j_1-j_2}=-\widetilde{T}_{-i_1i_2}\widetilde{T}_{-j_1j_2}, && \text{if } \epsilon(i_1)\epsilon(i_2)=1, && \epsilon(i_2)\epsilon(j_1)=1,\\&\widetilde{T}_{i_1-i_2}\widetilde{T}_{-j_1j_2}=\widetilde{T}_{-i_1i_2}\widetilde{T}_{j_1-j_2}, && \text{if } \epsilon(i_1)\epsilon(i_2)=1, && \epsilon(i_2)\epsilon(j_1)=-1,\\&\widetilde{T}_{i_1-i_2}\widetilde{T}_{j_1-j_2}=\widetilde{T}_{-i_1i_2}\widetilde{T}_{-j_1j_2}, && \text{if } \epsilon(i_1)\epsilon(i_2)=-1, && \epsilon(i_2)\epsilon(j_1)=1,\\&\widetilde{T}_{i_1-i_2}\widetilde{T}_{-j_1j_2}=-\widetilde{T}_{-i_1i_2}\widetilde{T}_{j_1-j_2}, && \text{if } \epsilon(i_1)\epsilon(i_2)=-1, && \epsilon(i_2)\epsilon(j_1)=-1,\end{aligned} \tag{63}$$

which give rise to four simplest algebraic relations,

$$\widetilde{T}^2_{i-i} = 0, \quad \widetilde{T}_{ii}\widetilde{T}_{-i-i} = 0, \quad \widetilde{T}_{ii}\widetilde{T}_{-ii} = 0, \quad \widetilde{T}_{i-i}\widetilde{T}_{ii} = 0. \tag{64}$$

5.2. Example: The Quantum Algebra from the $\widetilde{B}_4TT$ Relation

The $\widetilde{B}_4$-matrix and T-matrix take the forms,

$$\widetilde{B}_4 = \begin{pmatrix} 1 & 0 & 0 & q \\ 0 & 1 & 1 & 0 \\ 0 & -1 & 1 & 0 \\ -q^{-1} & 0 & 0 & 1 \end{pmatrix}, \quad T = \begin{pmatrix} \hat{a} & \hat{b} \\ \hat{c} & \hat{d} \end{pmatrix}, \tag{65}$$

and the $\widetilde{B}_4TT$ relation leads to the quantum algebra generated by $\hat{a}, \hat{b}, \hat{c}, \hat{d}$ satisfying algebraic relations,

$$\begin{aligned} \hat{a}\hat{a} &= \hat{d}\hat{d}, \quad \hat{a}\hat{b} = q\hat{d}\hat{c}, \quad \hat{b}\hat{b} = -q^2\hat{c}\hat{c}, \quad \hat{a}\hat{c} = q^{-1}\hat{d}\hat{b}, \\ \hat{a}\hat{d} &= \hat{d}\hat{a}, \quad \hat{b}\hat{a} = -q\hat{c}\hat{d}, \quad \hat{b}\hat{c} = -\hat{c}\hat{b}, \quad \hat{c}\hat{a} = -q^{-1}\hat{b}\hat{d}, \end{aligned} \tag{66}$$

where the deformation parameter q can be absorbed into the generator $\hat{c}$ by a rescaling transformation. With the new operators $\widetilde{\hat{a}}, \widetilde{\hat{b}}, \widetilde{\hat{c}}, \widetilde{\hat{d}}$(33) specified by

$$\widetilde{\hat{a}} = \hat{a} + \hat{d}, \quad \widetilde{\hat{b}} = \hat{b} + \hat{c}, \quad \widetilde{\hat{c}} = \hat{b} - \hat{c}, \quad \widetilde{\hat{c}} = \hat{a} - \hat{d}, \tag{67}$$

the above algebraic relations have a very simplified formalism,

$$\widetilde{\hat{a}}\widetilde{\hat{d}} = \widetilde{\hat{d}}\widetilde{\hat{a}} = 0, \quad \widetilde{\hat{b}}\widetilde{\hat{b}} = \widetilde{\hat{c}}\widetilde{\hat{c}} = 0, \quad \widetilde{\hat{a}}\widetilde{\hat{c}} = \widetilde{\hat{d}}\widetilde{\hat{b}} = 0, \quad \widetilde{\hat{b}}\widetilde{\hat{a}} = \widetilde{\hat{c}}\widetilde{\hat{d}} = 0. \tag{68}$$

Note that the quantum algebra from the B_4TT relation and its representation theory has been presented,(33) while the same quantum algebra from the $\widetilde{B}_4TT$ relation, interesting algebraic structures underlying its representation and its natural connection to quantum information theory has been explored.(6) Remarkably, quantum algebra obtained from $\widetilde{B}TT$ relation may be higher-dimensional representations of that algebra given by $\widetilde{B}_4TT$ relation.

5.3. FRT Dual Algebra Using the $\widetilde{M}LL$ Relations

The $\check{R}LL$ relations determining the FRT dual algebra can be derived from the generalized $\check{R}TT$ relation which relies on the spectral parameter,

$$\check{R}(xy^{-1})(L(x) \otimes L(y)) = (L(y) \otimes L(x))\check{R}(xy^{-1}). \tag{69}$$

Assume the $L(x)$-matrix to have a similar form to $\widetilde{B}(x)$,

$$L(x)=L^{+}+x\,L^{-},\quad \widetilde{B}(x)=\widetilde{B}+x\widetilde{B}^{-1}, \tag{70}$$

and this leads to the $\widetilde{B}LL$ relations for the FRT dual algebra,

$$\widetilde{B}(L^{\pm}\otimes L^{\pm})=(L^{\pm}\otimes L^{\pm})\widetilde{B},\quad \widetilde{B}(L^{+}\otimes L^{-})=(L^{-}\otimes L^{+})\widetilde{B}, \tag{71}$$

where matrix entries of $L^{\pm}$ are noncommutative operators. These $\widetilde{B}LL$ relations, i.e., the $\widetilde{M}LL$ relations,

$$\begin{aligned}&\widetilde{M}(L^{\pm}\otimes L^{\pm})=(L^{\pm}\otimes L^{\pm})\widetilde{M},\\ &L^{+}\otimes L^{-}-L^{-}\otimes L^{+}+\widetilde{M}(L^{+}\otimes L^{-})-(L^{-}\otimes L^{+})\widetilde{M}=0,\end{aligned} \tag{72}$$

have the component formalism,

$$\begin{aligned}&L^{\pm}_{i_1-i_2}L^{\pm}_{j_1-j_2}+\epsilon(i_1)\epsilon(i_2)q_{i_1j_1}q_{i_2j_2}L^{\pm}_{-i_1i_2}L^{\pm}_{-j_1j_2}=0,\\ &L^{+}_{i_1i_2}L^{-}_{j_1j_2}-L^{-}_{i_1i_2}L^{+}_{j_1j_2}+\epsilon(i_1)q_{i_1j_1}L^{+}_{-i_1i_2}L^{-}_{-j_1j_2}\\ &+\epsilon(i_2)q_{-i_2-j_2}L^{-}_{i_1-i_2}L^{+}_{j_1-j_2}=0,\quad i_1,i_2,j_1,j_2=J,J-1,\ldots,-J.\end{aligned} \tag{73}$$

Remark that the FRT dual algebra for the Bell matrix B_4 firstly given[(29)] and its quotient algebra with the condition $L^{+}\otimes L^{-}=L^{-}\otimes L^{+}$ presented.[(6)] Also, in view of Refs. 6,29,33, further research is needed to set up representation theories for these quantum algebra and FRT dual algebra.

6. CONCLUDING REMARKS AND OUTLOOKS

This paper is motivated by recent work,[(1–4,6)] and it sheds a light on further research for unraveling deep connections among quantum information theory, Yang–Baxter equation and complex geometry. We find that the GHZ states can be yielded by the high dimensional Bell matrix on the product basis and prove that the generalized Bell matrix of the type $2^{2n}\times 2^{2n}$ forms a unitary braid representation with the help of the algebra generated by the almost-complex structure. The algebraic and diagrammatic proofs for the generalized Bell matrix of the type $2^{2n+1}\times 2^{2n+1}$ satisfying the braided YBE together with other interesting result will be submitted elsewhere.

Besides what we have done in the present paper, there still remain many meaningful topics worthwhile to be explored. For example, almost-complex structure, classical YBE and symplectic geometry; construction of a universal R-matrix[34] in terms of the generators of the algebra from the $\widetilde{B}_4 TT$ relation; Yangian, Yang–Baxter equation and quantum information; new quantum algebra obtained by exploiting methodologies for the Sklyanin algebra[35,36] to the generalized Bell matrix. The most important thing (at least for the authors) is still to look for further connections among physics, quantum information and the YBE.

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