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Social welfare with net utilities

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Abstract We consider a society facing a binary choice, in an environment in which differences in utility are comparable across individuals. In such an environment, net utility is the difference between the utility that an individual attains from one alternative, and the utility she attains from the other alternative. A social welfare ordering is a preference relation over net utility profiles. We show that a social welfare ordering satisfies a collection of standard normative axioms if and only if it is representable by a collective utility function defined by the sums of a given power of net individual utilities.

Keywords Cardinal utility · Social welfare · Collective utility function

JEL Classification D72

Consider a society $N \equiv \{1, 2, \dots, n\}$ with $n \in \mathbb{N}, n \geq 3$ individuals and a binary set of alternatives $\{A, B\}$. We revisit a classic question: how to determine whether A or B is socially preferable, given the attitude of each agent toward each alternative.

In Arrow's (1963) classic preference aggregation framework, information about individual attitudes is purely ordinal: a social preference aggregation rule maps a profile of individual preference relations over the set of alternatives into a complete binary relation, interpreted as the social preference. Arrow's (1963) celebrated impossibility theorem demonstrates that if the set of alternatives contains at least three elements, any

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non-dictatorial preference aggregation rule satisfying Pareto optimality and independence of irrelevant alternatives (IIA) violates transitivity.

If the set of alternatives contains only two elements, a positive result obtains: with only two alternatives, transitivity is moot, and majority rule is the only rule satisfying anonymity, neutrality and positive-responsiveness (May 1952), which together imply Arrow's non-dictatorship, IIA and Pareto optimality.

May's (1952) characterization implicitly imposes an additional condition on the rule: the aggregation cannot take into account any information about intensity of preferences. Suppose that agent 1 strictly prefers A to B, and agent 2 strictly prefers B to A. In addition, suppose we can say meaningfully that agent 1 prefers A to B *more* than agent 2 prefers B to A. In other words, suppose that differences in individual welfare are comparable. Majority rule, and any purely ordinal rule, disregards that additional information about comparable differences in individual welfare.

We follow d'Aspremont and Gevers (1977), Roberts (1980) and, in particular, Moulin (1988) to assume that differences in individual welfare are comparable, and to allow the aggregate (ordinal) social preference over alternatives to be a function of those differences in individual welfare. That is, a rule can use information not only about which alternative each agent prefers, but also information about how much each agent prefers her most preferred alternative.

For each agent $i \in N$ and each alternative $J \in \{A, B\}$, let $u_i^J \in \mathbb{R}$ be the utility that i attains if J is chosen. For each $i \in N$ define the utility pair $u_i \equiv (u_i^A, u_i^B)$, for each $J \in \{A, B\}$ define the utility vector $u^J \equiv (u_1^J, \dots, u_n^J)$ and define a utility profile $u \in \mathbb{R}^{2n}$ by the $2 \times n$ matrix of utilities $u \equiv (u_1, \dots, u_n)$. Three possible social preferences over $\{A, B\}$ are possible, namely: A strictly preferred to B (denoted $A \succ B$), indifference between A and B (denoted $A \sim B$) and B strictly preferred to A (denoted $B \succ A$).

Definition 1 A social welfare functional $f : \mathbb{R}^{2n} \longrightarrow \{A \succ B, A \sim B, B \succ A\}$ is a mapping from the set of all possible utility profiles to the set of all social preferences over $\{A, B\}$

Given a profile $u \in \mathbb{R}^{2n}$, $f(u)$ denotes the social preference over $\{A, B\}$. The intuition is that society determines its preference over $\{A, B\}$ based on whether its members collectively prefer utility vector u^A or utility vector u^B.

We assume that the particular number attached to a utility level is not substantively meaningful, but gains and losses can meaningfully be compared across agents. Sen (1970) termed that assumption "*unit comparability* "; Roberts (1980) introduces it as "*cardinal unit comparability (CUC)*". It implies Moulin's (1988) " *zero independence*" axiom. It means that if we multiply the utility profile by a common scalar, and we translate the utility levels of each agent by adding an individual-specific constant to them, the social preference remains unaltered, because neither units of utility, nor levels, are relevant for interpersonal comparisons. Only the changes in utility, in whichever (common) unit we want to measure them, are comparable across agents.

Axiom 1 (*Unit Comparability*) For any $u \in \mathbb{R}^{2n}$, for any $\lambda \in \mathbb{R}^n$ and for any $\beta \in \mathbb{R}_{++}$, define $\hat{u} \in \mathbb{R}^{2n}$ by $\hat{u}_i^J \equiv \lambda_i + \beta u_i^J$ for each $i \in N$ and each $J \in \{A, B\}$. Then for any $u \in \mathbb{R}^{2n}$, for any $\lambda \in \mathbb{R}^n$ and for any $\beta \in \mathbb{R}_{++}$, $f(\hat{u}) = f(u)$.

Under unit comparability, for any utility profile $u \in \mathbb{R}^{2n}$, we can halve the dimensionality of the welfare aggregation problem, by considering only the net utility profile $v \equiv u^A - u^B \in \mathbb{R}^n$. By unit comparability, the social preference over $\{A, B\}$ given (u^A, u^B) is the same as the social preference over $\{A, B\}$ given $\left(v, u^B - u^B\right) = (v, \mathbf{0})$: alternative A generates a net utility profile v, while choosing B generates a net utility profile $\mathbf{0} \in \mathbb{R}^n$. Hence, we can determine whether A or B is socially preferable by comparing whether v or $\mathbf{0}$ is socially preferable. We have shrunk the information necessary to draw our comparison from a $2 \times n$ matrix of utilities u, to a vector v of net utilities with n components. We still use a social preference over the set of all possible net utility profiles vectors $\mathbb{R}^n$ to draw these comparisons. Following Moulin (1988), we call such a preference relation over net utility profiles a "social welfare ordering."[1]

Definition 2 A social welfare ordering R is a complete and transitive binary relation over $\mathbb{R}^n$.

For any two net utility profiles $v \in \mathbb{R}^n$ and $v' \in \mathbb{R}^n$, we interpret vRv' to mean that profile v is socially weakly preferable to v'. Let P denote the strict preference relation over net utility profiles defined by $vPv' \iff \neg v'Rv$,[2] and I the indifference relation defined by $vIv' \iff \{vRv' \text{ and } v'Rv\}$.

We take the social welfare ordering R over net utility profiles to be a primitive, and the social preference over $\{A, B\}$ to be derived from R and from the utility profile u. A preliminary answer to our motivating question is that a society with preference R over net utility profiles deems alternative A as socially (weakly) preferred to alternative B if and only if $vR\mathbf{0}$ and B as socially (weakly) preferred if $\mathbf{0}Rv$.

Given a social welfare ordering R, we can also identify a social welfare functional uniquely associated with ordering R by defining $f(u) = A \succ B$ for any $u \in \mathbb{R}^{2n}$, such that $vP\mathbf{0}$, $f(u) = A \sim B$ for any $u \in \mathbb{R}^{2n}$ such that $vI\mathbf{0}$ and $f(u) = B \succ A$ for any $u \in \mathbb{R}^{2n}$ such that $\mathbf{0}P(v)$.

Next we build on this preliminary answer by considering a collection of axioms that we would like the social welfare ordering R to satisfy, and characterizing the set of utility representations of any ordering R that satisfies these axioms. First, to guarantee that the social welfare ordering R admits a utility representation, we assume that R is continuous. For any $n \in \mathbb{N}$, $\varepsilon \in \mathbb{R}_{++}$ and any $x \in \mathbb{R}^n$, let $N_\varepsilon(x)$ denote the open neighborhood around x.

Axiom 2 (*Continuity*) A social welfare ordering R is continuous if for any $v, v' \in \mathbb{R}^n$, where vPv', $\exists \varepsilon \in \mathbb{R}_{++}$ such that $v''Pv'''$ for any $v'' \in N_\varepsilon(v)$ and any $v''' \in N_\varepsilon(v')$.

By continuity, sufficiently small changes in individual utilities do not reverse a strictly ordered relation between two net utility profiles.

Define a "net collective utility function" as a function $W : \mathbb{R}^n \longrightarrow \mathbb{R}$ that maps any net utility vector $v \in \mathbb{R}^n$ into a real number $W(v)$ that we interpret as the net collective utility of choice A given utility profile A. For any social welfare ordering R and any collective

[1] Note, however, that our use of the term is not exactly the same as Moulin's. Moulin defines a social welfare ordering as a preference over the set of utility profiles. Our social welfare ordering is a preference over the set of net utility profiles.

[2] We use the symbol ¬ to denote the negation of a logical statement.

utility function W, we say that W represents R if for any two net utility profiles $v, v' \in \mathbb{R}^n$, $W(v) \geq W(v')$ if and only if vRv'.

If the social welfare ordering R is continuous, we can represent it by a continuous net collective utility function W (Debreu 1954).

We seek to characterize the set of net collective utility functions representing a class of social welfare orderings that is itself characterized by a collection of standard axioms.

Our axioms follow those in Roberts (1980), as surveyed and discussed in Moulin (1988). In particular, we derive a result that follows closely from Roberts' (1980) Theorem 6. The main difference is that Roberts studies utility profiles that are all strictly positive, over many alternatives, whereas we study net utilities defined with respect to a pair of alternatives, so we necessarily must consider negative values of those net utilities.

Roberts (1980) characterizes the set of collective utility functions representing social welfare functionals that satisfy a collection of axioms over a domain of positive utility profiles. We inherit his axioms, but because net utility profiles can be negative, the functional forms of our net collective utility functions comprise a strict subset of the functional forms of Roberts' collective utility functions.[3]

Axiom 3 (*Anonymity*) A social welfare ordering R is anonymous if for any $v \in \mathbb{R}^n$ and any $v' \in \mathbb{R}^n$ such that v and v' are a permutation of one another, vIv'.

Anonymity guarantees that the social welfare ordering does not care about voters' names, and pays attention only to net utility levels. Arrow (1963) uses the term "equality" to refer to the same axiom.

Axiom 4 (*Neutrality*) A social welfare ordering R is neutral if for any $v \in \mathbb{R}^n$ and any $v' \in \mathbb{R}^n$, $vRv' \iff -v'R-v$.

Neutrality guarantees that which of the two outcomes is labeled A and which one is B does not matter. If we reverse the labels so that $\hat{A} = B$ and $\hat{B} = A$, then $\hat{v} = u^{\hat{A}} - u^{\hat{B}} = u^B - u^A = -v$. So labels are irrelevant if we prefer $\hat{v}'$ to $\hat{v}$ if and only if we prefer $-v$ to $-v'$. Neutrality guarantees that irrelevance.

Together, anonymity and neutrality guarantee that the relative merit of alternative A over B depends only on the set of net utilities that the pair of alternatives generate, and not on the labels of the alternatives, nor on who receives each of those net utilities. This definition of neutrality relates to May's (1952) classical notion as follows. May's neutrality has two components, each of which could be a separate axiom: (i) label-independence, in the sense that a permutation of the names of alternatives would not alter the social preference over the alternatives; and (ii) Arrow's (1951) "independence of irrelevant alternatives" (IIA). Since in our framework only two alternatives are available, IIA lacks content, so our notion of neutrality encompasses only the first of the two substantive assumptions built into May's definition, namely: independence of the social ordering with respect to relabeling of alternatives. Our definition aligns with Sen's (1966) notion of "*strong neutrality*", renamed

[3] We use Moulin's (1988) Theorem 2.6.b, a version of Roberts' (1980) Theorem 6. Roberts (1980) imposes his conditions on social welfare functionals; Moulin (1988) reinterprets the axioms to apply them to social welfare orderings, an approach we follow. Roberts (1980) credits previous literature, citing Arrow (1965) and Hicks (1965) for the mathematical insight behind his Theorem 6. The first proof we are aware of is in Katzner (1970) in the context of consumer theory. The functions characterized by these these theorems are often called Bergson functions, in reference to Bergson (née Burk) (1936).

just " *neutrality*" by Roberts (1980). It relates as well to Arrow's (1963, p. 101) notion of neutrality applied to collective choice rules.

Axiom 5 (*Monotonicity*) A social welfare ordering R is monotonic if vRv' for any $v, v' \in \mathbb{R}^n$ such that $v - v' \in \mathbb{R}^n_+$ and vPv' for any $v, v' \in \mathbb{R}^n$, where $v - v' \in \mathbb{R}^n_{++}$.

Monotonicity guarantees that the social value of choosing A over B is increasing in individuals' net utilities. Monotonicity is Moulin's (1988) "unanimity" axiom, and it implies Roberts's (1980) "weak Pareto" axiom, which requires only vPv' for any $v, v' \in \mathbb{R}^n$ such that $v - v' \in \mathbb{R}^n_{++}$.

For any $M \subseteq N$ and any net utility profile $v \in \mathbb{R}^n$, let v_M define the net utility profile restricted to the subset of agents M, and let $(v_M, v_{N\backslash M})$ be another way to write down vector v.

Axiom 6 (*Separability*) A social welfare ordering R is separable if for any $M \subseteq N$, for any $v, v' \in \mathbb{R}^n$,

$$(v_M, v_{N\backslash M})R(v'_M, v_{N/M}) \Longleftrightarrow (v_M, v'_{N\backslash M})R(v'_M, v'_{N/M}).$$

This is Roberts's (1980) and Moulin's (1988) notion of separability. It corresponds to Debreu's (1960) notion of strong separability. It means that one can evaluate partial utility profiles restricted to a subset of agents without taking into account the utility profiles of other agents. Sen (1977) refers to this axiom as "*Separability with respect to unconcerned individuals*" and describes it as " *seemingly innocuous.*" Roberts (1980) refers to it as "*elimination of the influence of indifferent individuals.*" Separability (in conjunction with our previous axioms) allows us to represent the social welfare ordering by an additively separable collective utility function with functional form $W = \sum_{i=1}^{n} h(v_i)$, and this representation is unique only up to affine transformations.[4]

Axiom 7 (*Scale invariance*) A social welfare ordering R is scale invariant if for any $v, v' \in \mathbb{R}^n$ and for any $\lambda \in \mathbb{R}_{++}$, $\lambda v R \lambda v' \Longleftrightarrow vRv'$.

Roberts (1980) refers to scale invariance as "*cardinal ratio scale* " (CRS) comparability. Moulin (1988) calls it "independence of the common utility scale". Scale invariance ensures that the choice of units in which we measure utility levels does not affect the social welfare ordering. Scale invariance of R is consistent with the cardinal unit comparability of the social welfare functional f: Cardinal unit comparability of f implies that neither the level nor the unit of measurement of u matter for f; and scale invariance means that the units of measurement of u (and hence of v) do not matter. R cannot satisfy cardinal unit comparability because the levels of net utilities do matter: zero net utility is substantively meaningful, representing the equality $u_i^A = u_i^B$, so levels of net utility cannot be translated, unlike levels of u, which are arbitrary. Let $sgn : \mathbb{R} \longrightarrow \{-1, 1\}$ denote the sign function, so that $sgn(x) = -1$ if $x \in (-\infty, 0)$ and $sgn(x) = 1$ if $x \in [0, \infty)$.

We now arrive at our desired representation result.

[4] This implication derives from the Debreu-Gorman separability theorem (Debreu 1960; Gorman 1968). See as well Blackorby et al. (1998).

Proposition 1 *A social welfare ordering R satisfies continuity, anonymity, monotonicity, separability and scale-invariance if and only if there exist $\gamma \in \mathbb{R}_{++}$ and $\rho \in \mathbb{R}_{++}$ such that R is represented by a net collective utility function W defined by*

$$W(v) = \sum_{i \in N : v_i > 0} \left(v_i\right)^{\rho} - \gamma \sum_{i \in N : v_i < 0} |v_i|^{\rho}. \tag{1}$$

Furthermore, R satisfies continuity, anonymity, neutrality, monotonicity, separability and scale-invariance if and only if there exists $\rho \in \mathbb{R}_{++}$ such that R is represented by a net collective utility function W defined by

$$W(v) = \sum_{i \in N} sgn(v_i)|v_i|^{\rho}. \tag{2}$$

Proof Note that if $W(v) = \sum_{i \in N : v_i > 0} \left(v_i\right)^{\rho} - \gamma \sum_{i \in N : v_i < 0} |v_i|^{\rho}$ and $vRv' \iff W(v) \geq W(v')$, then R satisfies:

(i) continuity because $|\cdot|$ is a continuous function, $(\cdot)^{\rho}$ is a continuous function and addition is a continuous operation;
(ii) anonymity, because the summation includes all agents indistinctly;
(iii) monotonicity, because if v_i increases from $a \in \mathbb{R}$ to $b \in \mathbb{R}$, $b > a$, while v_{-i} stays constant, then W increases by

$$sgn(b)|b|^{\rho} - sgn(a)|a|^{\rho} > 0. \tag{3}$$

If $a, b \leq 0$, inequality 3 is equal to $-|b|^{\rho} + |a|^{\rho} > 0$; if $a \leq 0 < b$, inequality 3 is equal to $|b|^{\rho} + |a|^{\rho} > 0$, and if $0 < a < b$, inequality 3 is equal to $|b|^{\rho} - |a|^{\rho} > 0$. Aggregating across all agents then, W strictly increases.

(iv) separability because for any $M \subseteq N$, any $v_M, v'_M \in \mathbb{R}^{|M|}$ and $v_{N\setminus M}, v'_{N\setminus M} \in \mathbb{R}^{|N\setminus M|}$,

$$\begin{aligned}
(v_M, v_{N\setminus M})R(v'_M, v_{N/M}) &\iff W((v_M, v_{N\setminus M})) \geq W((v'_M, v_{N\setminus M})) \\
&\iff \sum_{i \in M : v_i > 0} (v_i)^{\rho} - \gamma \sum_{i \in M : v_i < 0} |v_i|^{\rho} \geq \sum_{i \in M : v_i > 0} (v'_i)^{\rho} - \gamma \sum_{i \in M : v_i < 0} |v'_i|^{\rho} \\
&\iff W((v_M, v'_{N\setminus M})) \geq W((v'_M, v'_{N\setminus M})) \\
&\iff (v_M, v'_{N\setminus M})R(v'_M, v'_{N/M}).
\end{aligned}$$

(v) scale invariance because

$$W(\lambda v) = \sum_{i \in N : v_i > 0} (\lambda v_i)^{\rho} - \gamma \sum_{i \in N : v_i < 0} |\lambda v_i|^{\rho} = \lambda^{\rho}\left(\sum_{i \in N : v_i > 0} (v_i)^{\rho} - \gamma \sum_{i \in N : v_i < 0} |v_i|^{\rho}\right) \quad \text{for any} \quad \lambda \in \mathbb{R}_{++},$$
$$W(\lambda v') = \sum_{i \in N : v_i > 0} (\lambda v'_i)^{\rho} - \gamma \sum_{i \in N : v_i < 0} |\lambda v'_i|^{\rho} = \lambda^{\rho}\left(\sum_{i \in N : v_i > 0} (v'_i)^{\rho} - \gamma \sum_{i \in N : v_i < 0} |v'_i|^{\rho}\right) \quad \text{for any} \quad \lambda \in \mathbb{R}_{++}.$$

So,

$$\begin{aligned}
\lambda v R \lambda v' &\iff W(\lambda v) \geq W(\lambda v') \\
&\iff \lambda^{\rho}\left(\sum_{i\in N: v_i>0} (v_i)^{\rho} - \gamma \sum_{i\in N: v_i<0} |v_i|^{\rho}\right) \geq \lambda^{\rho}\left(\sum_{i\in N: v_i>0} (v_i')^{\rho} - \gamma \sum_{i\in N: v_i<0} |v_i'|^{\rho}\right) \\
&\iff \sum_{i\in N: v_i>0} (v_i)^{\rho} - \gamma \sum_{i\in N: v_i<0} |v_i|^{\rho} \geq \sum_{i\in N: v_i>0} (v_i')^{\rho} - \gamma \sum_{i\in N: v_i<0} |v_i'|^{\rho} \\
&\iff W(v) \geq W(v') \\
&\iff vRv'.
\end{aligned}$$

Furthermore, if $\gamma = 1$, then R satisfies neutrality, because for any $v \in \mathbb{R}^n$,

$$\begin{aligned}
W(-v) &= \sum_{i\in N} sgn(-v_i)|-v_i|^{\rho} \\
&= \sum_{i\in N} -sgn(v_i)|v_i|^{\rho} = -W(v);
\end{aligned}$$

so,

$$\begin{aligned}
vRv' &\iff W(v) \geq W(v') \\
&\iff -W(v) \leq -W(v') \\
&\iff W(-v) \leq W(-v') \\
&\iff -v'R(-v).
\end{aligned}$$

We prove that if a social welfare ordering R satisfies continuity, anonymity, monotonicity, separability and scale-invariance, then it can be represented by a collective utility function with functional form (1).

Define the axiom:

Axiom 8 (*Decreasing Monotonicity*) A social welfare ordering R is decreasingly monotonic if vRv' for any $v, v' \in \mathbb{R}^n$ such that $v' - v \in \mathbb{R}^n_+$ and vPv' for any $v, v' \in \mathbb{R}^n$ such that $v' - v \in \mathbb{R}^n_{++}$.

Assume that a social welfare ordering R satisfies continuity, anonymity, monotonicity, separability and scale-invariance. By Roberts's (1980) Theorem 6, as reformulated in Theorem 2.6 by Moulin (1988), R can be represented by a net collective utility function W that takes one of the three following functional forms in the $\mathbb{R}^n_{++}$ orthant:

(i) $W(v) = \sum_{i\in N} (v_i)^{\rho}$ for some $\rho \in \mathbb{R}_{++}$,

(ii) $W(v) = \sum_{i\in N} \ln(v_i)$, or

(iii) $W(v) = \sum_{i \in N} \frac{1}{(v_i)^q}$ for some $q \in \mathbb{R}_{++}$.

By continuity at $v = \mathbf{0}$, functional forms (ii) and (iii) are ruled out and, hence, R can be represented by a net collective utility function W that takes the functional form $W(v) = \sum_{i \in N}(v_i)^\rho$ for some $\rho \in \mathbb{R}_{++}$ in the $\mathbb{R}^n_+$ orthant.

Define a second social welfare ordering R' by $vR'v' \iff v'R'v$ for any $v, v' \in \mathbb{R}^n$, with associated strict social welfare ordering P'. Note that R' satisfies continuity, anonymity, decreasing monotonicity, separability and scale-invariance (by reversing the order given by R, monotonicity transforms into decreasing monotonicity, and all other axioms are preserved). Note that the collective utility function W' defined by $W'(v) = -W(v)$ for any $v \in \mathbb{R}^n$ represents R'.

Consider a third social welfare ordering R''represented by the collective utility function W'' defined by $W''(v) = W'(-v)$ for any $v \in \mathbb{R}^n$. Note that R'' satisfies continuity, anonymity, separability and scale-invariance. We claim that R'' satisfies monotonicity as well. Indeed, since R' satisfies decreasing monotonicity, $v'R'v$ for any $v, v' \in \mathbb{R}^n$ such that $v - v' \in \mathbb{R}^n_+$, and $v'P'v$ for any $v, v' \in \mathbb{R}^n$ such that $v - v' \in \mathbb{R}^n_{++}$. Since W' represents R', it follows that $W'(v') \geq W'(v)$ for any $v, v' \in \mathbb{R}^n$ such that $v - v' \in \mathbb{R}^n_+$, and $W'(v') > W'(v)$ for any $v, v' \in \mathbb{R}^n$ such that $v - v' \in \mathbb{R}^n_{++}$. Hence, $W''(-v') \geq W''(-v)$ for any $v, v' \in \mathbb{R}^n$ such that $v - v' \in \mathbb{R}^n_+$, and $W''(-v') > W''(-v)$ for any $v, v' \in \mathbb{R}^n$ such that $v - v' \in \mathbb{R}^n_{++}$. Relabeling using $z = -v$ and $z' = -v'$, $W''(z') \geq W''(z)$ for any $z, z' \in \mathbb{R}^n$ such that $z' - z \in \mathbb{R}^n_+$, and $W''(z') > W''(z)$ for any $z, z' \in \mathbb{R}^n$ such that $z' - z \in \mathbb{R}^n_{++}$. Hence, R'' satisfies monotonicity.

Since R'' satisfies continuity, anonymity, monotonicity, separability and scale-invariance and it is represented by W'', again by Roberts's (1980) Theorem 6, R'' can be represented by a collective utility function with one of the functional forms (i), (ii) or (iii) above in $\mathbb{R}^n_{++}$, and by continuity at $v = \mathbf{0}$, it cannot be functional forms (ii) or (iii), so R'' can be represented by a collective utility function with functional form (i). By separability, other representations must be affine transformations of this one (Debreu 1960). In particular, W'' must be an affine transformation of a collective utility function with functional form i) in $\mathbb{R}^n_{++}$. Hence, for any $v \in \mathbb{R}^n_{++}$, W'' takes the form $W''(v) = a + \gamma \sum_{i \in N}(v_i)^{\rho'}$ for some $a \in \mathbb{R}$, $\gamma \in \mathbb{R}_{++}$ and $\rho' \in \mathbb{R}_{++}$.

For any v in the non-positive orthant, since $W(v) = -W''(-v)$, it follows that $W(v) = -a - \gamma \sum_{i \in N}(-v_i)^{\rho'} = -a - \gamma \sum_{i \in N} |v_i|^{\rho'}$. Continuity at $v = \mathbf{0}$ implies $a = 0$. Hence, for any v in the non-positive orthant, $W(v) = -\gamma \sum_{i \in N} |v_i|^{\rho'}$.

We next establish that $\rho' = \rho$. By separability, W is additively separable, and it thus takes the form $W(v) = \sum_{i \in N} h(v_i)$ for some function h (that function is common to all $i \in N$ by anonymity). For the case in which $\tilde{v}_j = 0$ for any $j \in N \backslash \{i\}$, we have determined in the previous step that $W(v) = -\gamma |v_i|^{\rho'}$ if $v_i < 0$ (and in previous steps that $W(v) = |v_i|^\rho$ if $v_i \geq 0$); hence, $h(v_i) = -\gamma |v_i|^{\rho'}$ if $v_i < 0$ and $h(v_i) = |v_i|^\rho$ if $v_i \geq 0$, for each $i \in N$. We want to show that, in fact, it must be $\rho' = \rho$. Suppose not. Consider v and v' such that $v_1 = x$, $v'_1 = -x$ and $v_i = v'_i = 0$ for any $i \in N \backslash \{1\}$. If $\rho' > \rho$, then for any x sufficiently large, $W(v) > W(v')$, whereas, for any $x \in \mathbb{R}_{++}$ sufficiently small, $W(v') > W(v)$, violating scale invariance. Similarly, if $\rho' < \rho$, then for any x sufficiently large, $W(v) < W(v')$, whereas, for any $x \in \mathbb{R}_{++}$ sufficiently small, $W(v') < W(v)$, violating scale invariance. Thus $\rho' = \rho$, and $h(v_i) = -\gamma |v_i|^\rho$ if $v_i < 0$ and $h(v_i) = |v_i|^\rho$ if $v_i > 0$, so aggregating across agents, $W(v) = \sum_{i \in N: v_i > 0} \left(v_i\right)^\rho - \gamma \sum_{i \in N: v_i < 0} |v_i|^\rho$.

Finally, we want to show that if, in addition, R satisfies neutrality, then $\gamma = 1$.

Assume that $(v_1, v_2) = (1, -1)$, $(v'_1, v'_2) = (-1, 1)$ and $v_j = v'_j = 0$ for any $j \in \{3, \ldots, n\}$. Then $W(v) = W(v') = 1 - \gamma$. Suppose that (absurd)$\gamma < 1$. Then $W(v) = W(v') > 0$; thus, since W represents R, $vP\mathbf{0}$ and $v'P\mathbf{0}$. But $v' = -v$; thus, by neutrality, $vP\mathbf{0} \Longleftrightarrow \mathbf{0}Pv'$, a contradiction. Similarly, if $\gamma > 1$, $W(v) = W(v') < 0$; thus, since W represents R, $\mathbf{0}Pv$ and $\mathbf{0}Pv'$ and by neutrality, $\mathbf{0}Pv \Longleftrightarrow v'P\mathbf{0}$, a contradiction. Hence, it must be that $\gamma = 1$, in which case $W(v) = W(v') = 0$ and $vIv'I\mathbf{0}$,and for any $\tilde{v} \in \mathbb{R}^n$, $W(\tilde{v}) = \sum_{i \in N} sgn(\tilde{v}_i)|\tilde{v}_i|^\rho$.

Therefore, a net collective utility function W that represents a social welfare ordering satisfying anonymity, neutrality, monotonicity, continuity, separability and scale-invariance takes the functional form $W(v) = \sum_{i \in N} sgn(v_i)|v_i|^\rho$. □

Proposition 1 allows us to best answer our motivating question in the following way: alternative A should be deemed socially preferred to B if and only if $\sum_{i \in N} sgn(v_i)|v_i|^\rho > 0$, where $\rho \in \mathbb{R}_{++}$ is a normative parameter capturing how much intensity of individual preferences should matter to society. Three values are particularly salient:

In the limit, $\rho \longrightarrow 0$ and intensity of preference does not matter; the socially preferable alternative is the one that would be selected by majority rule, as in May (1952).

With parameter value $\rho = 1$, the socially preferable alternative is the one that maximizes utilitarian efficiency, as in Maskin (1978).

In the limit, $\rho \longrightarrow \infty$ and intensity of preference is all that matters; the socially preferable alternative is the one favored by the agent who would be most affected by the social choice.

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