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The robustness of the optimal weighted majority rule to probability distortion

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Abstract This study identifies the optimal collective decision rule in a dichotomous symmetric setting, allowing for probabilities distortion as originally assumed by Tversky and Kahneman (Journal of Risk and Uncertainty 5(4):297–323, 1992). We show that previous results that identified the weighted majority rule as the optimal one, and did not consider subjective probabilities, *are robust* to such distortion in the sense that neither the rule nor the weights are changed.

Keywords Optimal rule · Subjective probabilities · Condorcet jury theorem

1 Introduction

The task of identifying the optimal decision rule (henceforth optimal rule) drew renewed attention during the last 40 years, starting with the work of Arrow (1951) and Buchanan and Tullock (1962), later followed by Rae (1969). In the context of dichotomous symmetric choice, the optimal rule was identified by Nitzan and Paroush (1982) as the weighted majority rule. More general results were obtained in Ben-Yashar and Kraus (2002), Ben-Yashar and Nitzan (1997) and Ben-Yashar and Paroush (2001). For related studies, see Ben-Yashar (2006), Ben-Yashar and Nitzan (2001), Nitzan and Paroush (1981, 1984), Sah (1991), and Sah and Stiglitz (1985, 1986, 1988).

This line of research incorporated, inevitably, some significant aspects of decision making under uncertainty. These aspects had been simultaneously and independently studied,

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starting with Kahneman and Tversky (1979) and followed by numerous others, for example by Quiggin (1991), Tversky and Kahneman (1992), and Prelec (1998). These scholars have suggested a new approach for decision-making under uncertainty, based on individuals' distortion of probabilities, i.e., assignment of subjective probabilities instead of the objective ones. In particular, these studies share a common view regarding the s-shaped decision weight function; individuals overweigh relatively small probabilities and under weigh large ones. To the best of our knowledge, these two lines of study (identifying the optimal rule and considering subjective probabilities) were never intersected. The main objective of this study is to incorporate subjective probabilities as suggested by Tversky and Kahneman (1992) into a model of collective decision-making, and testing the robustness of the result identifying the optimal decision rule to such incorporation.

Our main finding is stated in Theorem 1 that identifies the optimal rule as the weighted majority rule, allowing for probability distortion. We show that when subjective probabilities are taken into account, the optimal rule as well the optimal weights remain *unchanged* relative to the classical setting that does not allow for such probabilities distortion. The implication of this distortion is presented in the context of the Condorcet jury theorem.

2 Dichotomous symmetric choice with distorted probabilities

We assume that there are n (not necessarily homogeneous) members that are facing a common decision problem. These members have to choose one of two symmetric alternatives, denoted by 1 and -1, where one (and only one) correct alternative exists. We denote by x_i individual i's decision, where $x_i = 1$ (support alternative 1) or $x_i = -1$ (support alternative -1), and a decision profile by $x = \{x_1, \dots, x_n\}$. The collective decision is based on the decisions of the individuals. Individual i's decisional skills are represented by p_i (the probability that individual i chooses the correct alternative). Decisional skills may vary across individuals, and are assumed to be statistically independent.[1] We allow for probability distortion, such that the objective probability p_i is different than the subjective probability perceived by the social decision maker that individual i will make a correct decision, $w(p_i)$.[2] More specifically, in this study we assume a particular form of distortion, as in Tversky and Kahneman (1992), where $w(p_i)$ is given by

$$w(p_i) = \frac{p_i^{\gamma}}{(p_i^{\gamma} + (1-p_i)^{\gamma})^{(1/\gamma)}}$$

and $w(1-p_i)$ is given by

$$w(1-p_i) = \frac{(1-p_i)^{\gamma}}{((1-p_i)^{\gamma} + p_i^{\gamma})^{(1/\gamma)}}.$$

Thus, when $\gamma = 1$, $w(p_i) = p_i$. Note that we do not assume different γ values for gains (correct decision) and losses (incorrect decision). For such a symmetric setting see Tversky and Wakker (1995). In our setting the primitive probabilities p_i and $(1-p_i)$ are assumed to be distorted. In other words, the primitive probabilities are perceived as subjective ones; $(1-p_i)$ is one of these probabilities. Figure 1 plots the probability distortion function $w(p)$, assuming $\gamma = 0.61$ as in Tversky and Kahneman.

[1] The independence assumption is widely used. See, among many others, Sah and Stiglitz (1986, 1988).

[2] Assuming subjective probability for p_i and $(1-p_i)$ is plausible since we assume that individual i's decisional skills represent a lottery, in which there is a probability of p_i for making the correct decision.

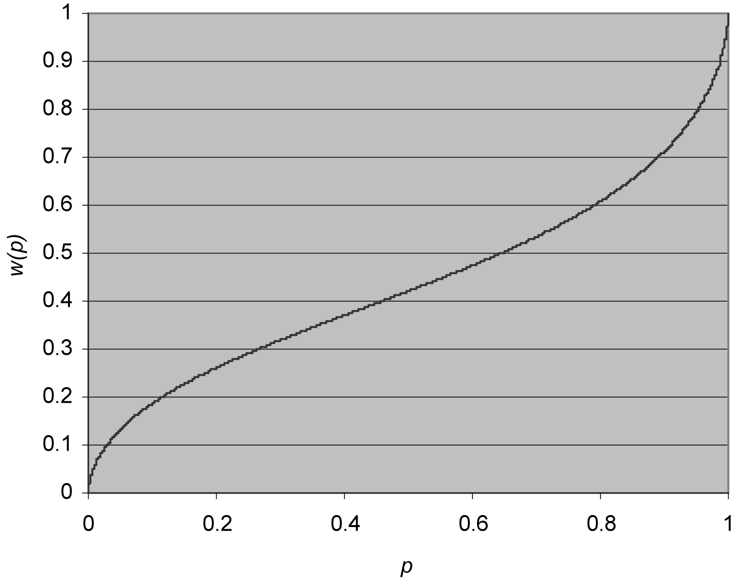

Fig. 1

A collective decision is reached by a social decision maker who applies a decisive decision rule, that is a function f that assigns 1 (choose alternative 1) or -1 (choose alternative -1) to any x in $\Omega = \{1, -1\}^n$, $f : \Omega \to \{1, -1\}$. The collective decision rule that optimizes the decision making process, i.e., maximizes the collective probability for making a correct decision, is weighted majority rule with weights of $\ln \frac{p_i}{1-p_i}$ (Nitzan and Paroush 1982). However, this rule is identified in a setting that does not take into account probability distortion. We focus on identifying the optimal rule, taking into account subjective probabilities; the social decision maker uses his or her subjective probabilities that individual i will make a correct decision, $w(p_i)$ or will make an incorrect one, $w(1 - p_i)$ (for $0 < p_i < 1$, $w(p_i) + w(1 - p_i) \neq 1$).

Let us partition the set of all decision profiles into $X(1, f)$ and $X(-1, f)$, where $X(1, f) = \{x \in \Omega : f(x) = 1\}$ and $X(-1, f) = \{x \in \Omega : f(x) = -1\}$. Denote by $T(f : 1)$ and by $T(f : -1)$ the collective probability for choosing correctly, given that the correct alternative is, respectively, 1 and -1, where $T(f : 1) = \Pr\{x \in X(1, f) : 1\}$ and $T(f : -1) = \Pr\{x \in X(-1, f) : -1\}$. Hence, we should find the collective decision rule f that solves the following maximization problem: $\text{MAX}_{f \in F} T(f : 1) + T(f : -1)$, where F is the set of all collective decision rules.[3]

3 The optimal decision rule

The optimal rule under non-distorted (objective) probabilities is a weighted majority rule with the specific weights $W_i^* = \ln \frac{p_i}{1-p_i}$, denoted as W*MR (see Nitzan and Paroush 1982).

[3]Due to the symmetry of the alternatives, it can be verified that this maximization problem is equivalent to maximizing the expected payoff. Symmetry means that the label of the alternatives does not convey any information regarding the correctness of the choice.

Theorem 1 establishes the robustness of this optimal rule to probability distortion of the sort suggested by Tversky and Kahneman (1992). That is, the optimal rule under subjective probabilities that are distorted in this way is also W*MR, using weights derived from the distorted probabilities. Formally:

Theorem 1 *The optimal rule $\hat{f}$ under subjective probabilities is the W*MR. That is,*

$$\hat{f} = \text{sign}\left(\sum_{i=1}^{n} \mathrm{W}_i^* \cdot x_i\right), \quad \textit{where } \mathrm{W}_i^* = \ln\frac{w(p_i)}{w(1-p_i)} = \ln\frac{p_i}{1-p_i},$$

and $\text{sign}(m) = 1$ *if* $m > 0$ *and* -1 *otherwise.*

Proof For any decision profile x in Ω, consider the partition of the group members into $A(x)$ and $R(x)$ such that $i \in A(x)$ if $x_i = 1$ and $i \in R(x)$ if $x_i = -1$. Denote by $g(x:1)$ and $g(x:-1)$ the probabilities to obtain x given that the correct alternative is 1 or -1, respectively. Formally:

$$g(x:1) = \prod_{i\in A(x)} w(p_i) \prod_{i\in R(x)} w(1-p_i) \quad \text{and} \quad g(x:-1) = \prod_{i\in R(x)} w(p_i) \prod_{i\in A(x)} w(1-p_i).$$

Hence,

$$T(f:1) = \sum_{x\in X(1,f)} g(x:1) \quad \text{and} \quad T(f:-1) = \sum_{x\in X(-1,f)} g(x:-1).$$

Thus, the maximization problem on which we focus can be presented as follows:

$$\operatorname*{Max}_{f} \sum_{x\in X(1,f)} g(x:1) + \sum_{x\in X(-1,f)} g(x:-1).$$

Consider the following condition: for any profile of decisions $x, x \in \Omega$:

$$f(x) = \begin{cases} 1 & \text{if } g(x:1) > g(x:-1), \\ -1 & \text{otherwise.}^{4} \end{cases}$$

This condition guarantees that f maximizes the collective subjective probability for making a correct decision. Put differently, a sufficient condition for identifying the optimal rule, $\hat{f}$, is that it partitions Ω such that:

$$X(1,\hat{f}) = \{x \in \Omega : \hat{f}(x) = 1\} = \{x : x \in \Omega \text{ and } g(x:1) > g(x:-1)\}.$$

Recall that $g(\cdot)$ uses subjective probabilities. Hence, in seeking to implement the condition above, the social decision maker will apply the following condition:

$$\left\{x \in \Omega \text{ and } \prod_{i\in A(x)} w(p_i) \prod_{i\in R(x)} w(1-p_i) > \prod_{i\in R(x)} w(p_i) \prod_{i\in A(x)} w(1-p_i)\right\}$$

[4] Due to monotonicity of $w(\cdot)$, whenever this condition is satisfied for $g(\cdot)$ it is satisfied for $w(g(\cdot))$ as well, that is $g(x:1) > g(x:-1) \Leftrightarrow w(g(x:1)) > w(g(x:-1))$.

$$= \left\{ x \in \Omega \text{ and } \sum_{i \in A(x)} \ln \frac{w(p_i)}{w(1-p_i)} > \sum_{i \in R(x)} \ln \frac{w(p_i)}{w(1-p_i)} \right\}$$

$$= \left\{ x \in \Omega \text{ and } \sum_{i=1}^{n} \left(\ln \frac{w(p_i)}{w(1-p_i)} \right) \frac{x_i + 1}{2} > \sum_{i=1}^{n} \left(\ln \frac{w(p_i)}{w(1-p_i)} \right) \frac{1 - x_i}{2} \right\}$$

$$= \left\{ x \in \Omega \text{ and } \sum_{i=1}^{n} \left(\ln \frac{w(p_i)}{w(1-p_i)} \right) x_i > 0 \right\}$$

$$= \left\{ x \in \Omega \text{ and } \sum_{i=1}^{n} \left(\ln \frac{p_i}{1-p_i} \right) x_i > 0 \right\},$$

because the definition of $w(\cdot)$ implies that

$$\ln \frac{w(p_i)}{w(1-p_i)} = \gamma \ln \frac{p_i}{1-p_i}.$$

This implies that:

$$X(1, \hat{f}) = \left\{ x \in \Omega \text{ and } \sum_{i=1}^{n} \left(\ln \frac{p_i}{1-p_i} \right) x_i > 0 \right\} \quad \text{and}$$

$$X(-1, \hat{f}) = \left\{ x \in \Omega \text{ and } \sum_{i=1}^{n} \left(\ln \frac{p_i}{1-p_i} \right) x_i < 0 \right\}.$$

That is,[5]

$$\hat{f} = \text{sign}\left(\sum_{i=1}^{n} \left(\ln \frac{p_i}{1-p_i} \right) x_i \right).$$

□

Interestingly, the optimal weighted majority rule identified in Theorem 1, in which *subjective* probabilities are taken into account, is identical to the one obtained by Nitzan and Paroush (1982), who considered *objective* probabilities.

The optimal weight assigned to individual i's decision, $\ln \frac{p_i}{1-p_i}$, is positive under the assumption of $w(p_i) > w(1-p_i)$, i.e., the subjective probability of success is greater than the subjective probability of failure.[6] This explains why the assigned weights might remain strictly positive, although $w(p_i)$ and $w(1-p_i)$ may be less than $\frac{1}{2}$ (for example, suppose that $p_i = 0.55$. In such a case, $\frac{1}{2} > w(p_i) > w(1-p_i)$). Due to monotonicity of the weighting function $w(p_i)$, when $w(p_i) > w(1-p_i)$, $p_i > (1-p_i)$, which implies that $p_i > \frac{1}{2}$. Specifically,

$$\ln \frac{p_i}{1-p_i} \gtreqless 0 \iff w(p_i) \gtreqless w(1-p_i) \iff p_i \gtreqless \frac{1}{2}.$$

[5]The weight assigned to individual i's decision remains unchanged since $\frac{w(p_i)}{w(1-p_i)} = (\frac{p_i}{1-p_i})^\gamma$.

[6]Note that when considering objective probabilities, the requirement for positive optimal weights assigned to individual i's decision is that $p_i > 1 - p_i$, that is $p_i > \frac{1}{2}$.

Theorem 1 clearly focuses on a specific form of probability distortion, that is the one suggested by Tversky and Kahneman (1992). Nonetheless, it can be generalized to a family of probability distortion functions. For example, the functions that share a common property of symmetry, that is where $w(p_i) = p_i^\gamma \cdot f(p_i, 1-p_i)$, $\gamma > 0$ and f is symmetric in the following sense: $f(p_i, 1-p_i) = f(1-p_i, p_i)$.[7]

4 Discussion

We have shown that, somewhat surprisingly, the optimal (weighted majority) rule is robust to distortion of probabilities that represent decisional skills. This result is strongly related to the very well-known Condorcet Jury Theorem (CJT). As attributed to Condorcet (1785) by Ben-Yashar and Paroush (2000), the CJT consists of the formal sufficient conditions for which the following statements are satisfied:

1. A group of individuals who use the simple majority rule is more likely than any single member of this group, to select the better among two alternatives, when there is uncertainty regarding which alternative is preferred.
2. When using the simple majority rule, the better alternative will be chosen with a probability of 1 when the group's size tends to infinity.

The most popular CJT is referring to a special case of our model, in which individuals' decisional skills, denoted by p, are identical and pass a threshold of $\frac{1}{2}$ (skilled individuals, $p > \frac{1}{2}$). It can be verified that in such a case the optimal rule is the simple majority rule, independent of the assumed probabilities, distorted or not.

Nevertheless, applying the CJT might lead to the use of a non-optimal decision rule, to the extent that flipping a fair coin might be preferred to consulting a group of individuals. This case is possible when the individuals' decisional skills are mistakenly perceived as smaller than $\frac{1}{2}$, that is, $w(p) < \frac{1}{2}$ whereas $p > \frac{1}{2}$.[8] Thus, in the context of distorted probabilities, the social decision maker should adopt 0.4206 (that is $w(\frac{1}{2})$), and not $\frac{1}{2}$, as the threshold that discriminates skilled and unskilled individuals. Note that this threshold is irrelevant when weights are incorporated into the model, as is in Theorem 1, since individuals with decisional skills that satisfy $p > \frac{1}{2}$ ($p < \frac{1}{2}$) are assigned a positive (negative) weight.

References

Arrow, K. J. (1951). *Individual values and social choice*. New York: Wiley.

Ben-Yashar, R. (2006). Information is important to Condorcet jurors. *Public Choice*, *127*, 313–27.

Ben-Yashar, R., & Kraus, S. (2002). Optimal collective dichotomous choice under quota constraints. *Economic Theory*, *19*(4), 839–52.

Ben-Yashar, R., & Nitzan, S. (1997). The optimal decision rule for fixed-size committees in dichotomous choice situations: the general result. *International Economic Review*, *38*(1), 175–86.

Ben-Yashar, R., & Nitzan, S. (2001). Investment criteria in single- and multi-member economic organizations. *Public Choice*, *109*, 1–13.

Ben-Yashar, R., & Paroush, J. (2000). A nonasymptotic Condorcet jury theorem. *Social Choice and Welfare*, *17*, 189–99.

[7] We thank Jacob Paroush for this insightful comment.

[8] Assuming Tversky and Kahneman's (1992) parameter estimations, these are the cases where $0.5 < p < 0.645$. All the numerical values are based on Tversky and Kahneman. Clearly, other parameter estimations (such as in Camerer and Ho 1994) yield different values.

Ben-Yashar, R., & Paroush, J. (2001). Optimal decision rules for fixed-size committees in polychotomous choice. *Social Choice and Welfare*, *18*(4), 737–46.

Buchanan, J. M., & Tullock, G. (1962). *The calculus of consent*. Ann Arbor: University of Michigan Press.

Camerer, C. F., & Ho, T. H. (1994). Violations of the betweenness axiom and nonlinearity in probability. *Journal of Risk and Uncertainty*, *8*, 167–96.

de Condorcet, N. C. (1785). *Essai sur l'application de l'analyse à la probabilité des décisions rendues à la pluralité des voix*. Paris. Translated by I. McLean and F. Hewitt (1994).

Kahneman, D., & Tversky, A. (1979). Prospect theory: an analysis of decision under risk. *Econometrica*, *47*(2), 263–91.

Nitzan, S., & Paroush, J. (1981). The characterization of decisive weighted majority rules. *Economics Letters*, *7*, 119–124.

Nitzan, S., & Paroush, J. (1982). Optimal decision rules in uncertain dichotomous choice situations. *International Economic Review*, *23*(2), 289–97.

Nitzan, S., & Paroush, J. (1984). Are qualified majority rules special? *Public Choice*, *42*, 257–72.

Prelec, D. (1998). The probability weighting function. *Econometrica*, *66*(3), 497–527.

Quiggin, J. (1991). Comparative statics for rank-dependent expected utility theory. *Journal of Risk and Uncertainty*, *4*(4), 339–50.

Rae, D. W. (1969). Decision rules and individual values in constitutional choice. *American Political Science Review*, *63*, 40–56.

Sah, R. K. (1991). Fallibility in human organizations and political systems. *The Journal of Economic Perspectives*, *5*, 67–88.

Sah, R. K., & Stiglitz, J. E. (1985). The theory of economic organizations, human fallibility and economic organizations. *American Economic Review, Papers and Proceedings*, *75*, 292–97.

Sah, R. K., & Stiglitz, J. E. (1986). The architecture of economic systems: hierarchies and polyarchies. *American Economic Review*, *76*, 716–27.

Sah, R. K., & Stiglitz, J. E. (1988). Committees, hierarchies and polyarchies. *The Economic Journal*, *98*, 451–70.

Tversky, A., & Kahneman, D. (1992). Advances in prospect theory: cumulative representation of uncertainty. *Journal of Risk and Uncertainty*, *5*(4), 297–323.

Tversky, A., & Wakker, P. (1995). Risk attitudes and decision weights. *Econometrica*, *63*(6), 1255–1280.