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A critical reappraisal of some voting power paradoxes*

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Abstract. Power indices are meant to assess the power that a voting rule confers *a priori* to each of the decision makers who use it. In order to test and compare them, some authors have proposed 'natural' postulates that a measure of *a priori* voting power 'should' satisfy, the violations of which are called 'voting power paradoxes.' In this paper two general measures of success and decisiveness based on the voting rule and voters' behavior and some of these postulates/paradoxes test each other. As a result serious doubts are cast on the discriminating power of most voting power postulates.

Introduction

Different power indices have been proposed to assess the *a priori* distribution of power among voters for a given voting rule. Since the only recently vindicated Penrose (1946) and the later but much more popular Shapley and Shubik's (1954) and Banzhaf's (1965) indices, other power indices have been proposed: Coleman's (1971, 1986) indices, Deegan and Packel's (1978) index, Johnston's (1978) index, and Holler and Packel's (1983) index. Some solution concepts can also be found in cooperative game theoretic literature, such as semivalues (Weber, 1979, see also Dubey et al. (1981)) that can be seen as generalizations of the concept of power index when restricted to simple games (see e.g., Laruelle & Valenciano, 2002, 2003; Carreras et al., 2003).

These indices sometimes display undesirable properties, referred to somewhat exaggeratedly as 'paradoxes' in the literature on power indices, where they have been widely discussed. Recently, Felsenthal and Machover (1995, 1998) critically discussed them, dismissing some of them as trivial, refining

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the formulation of others, and proposing some new ones. They consider that, in view of the lack of conclusive arguments from the axiomatic point of view, some paradoxes (i.e., the violation of some reasonable postulates) can be used to judge and filter power indices. This methodology and the distinction between two notions of power, 'the power to influence' (or 'I-power') and the 'power to share a purse' (or 'P-power'), lead them to disqualify some power indices as unreasonable.

Brams (1975) was the first to point out 'paradoxical behavior' of some power indices. He claims that if two voters decide to form a kind of indissoluble 'bloc', the power of the bloc cannot be smaller than the sum of the power of its components. The *paradox of size* occurs when this property is not satisfied. Felsenthal and Machover (1995) consider that the paradox of size is not that surprising, and claim that what should be expected is that the power of a bloc should be at least as great as the power of the most powerful of its component parts. They refer to the violation of this property as the *bloc paradox*. The *paradox of new members* (Brams, 1975; Brams & Affuso, 1976, 1985a, 1985b) occurs when the addition of a new member to a weighted body increases the power of some of the old members, despite the fact that their share of votes constitutes a smaller proportion of the total number of votes. Felsenthal and Machover (1998) consider that this phenomenon is not paradoxical, and suggest that what should be expected is that a voter with a veto right should get at least as much power as any other voter. They refer to this property as the *preference for blocker postulate*. Brams (1975) and Kilgour (1974) introduce the *quarrelling paradox*, which occurs when it is beneficial (according to some power indices) for voters to quarrel or refuse to vote together. Straffin (1982) and Felsenthal and Machover (1998) raise some doubts concerning the statement of this paradox: in their view, the model does not permit this modification of voters' behavior to be captured. Deegan and Packel (1982) show that in weighted majorities, some indices do not satisfy the 'larger the weight, the more the power' principle. They refer to the violation of this principle as the *paradox of weighted voting*. This principle is generalized by Felsenthal and Machover (1995) to arbitrary voting rules as the *dominance postulate*. Fisher and Schotter (1978), and Dreyer and Schotter (1980) present the *paradox of redistribution*. They consider a weighted majority where weights are redistributed, but the total weight and the quota remain identical. The paradox is said to occur when a voter loses weight but increases his or her voting power according to some power indices. Felsenthal and Machover (1995) argue that only when a single transfer of weight occurs is it paradoxical that the receiver's power should decrease, which they refer to as the *donation paradox*. The *bicameral paradox* (Felsenthal et al., 1998) occurs when the ranking of power is reversed from one chamber to a bicameral system. Saari and Sieberg (2000) show that different semivalues, which can be seen as a generalization of the concept of power index, rank voters differently. Recently, van Deemen and Rusinowska (2003) tested the occurrence of paradoxes in the Dutch Parliament.

All the variations of the traditional power index notion alluded to in the first paragraph which display one or other of these paradoxes formally take the voting rule as the only explicit input for the assessment of power. That is to say, traditional power indices map voting rules, usually modeled as simple games, onto vectors whose coordinates are interpreted as the 'power' of the corresponding voter. These power measures leave aside the voters' voting behavior and whatever might condition it, such as their preferences over the issues, their interpersonal relations or any contextual information. Consequently, the lack of basis for a positive or descriptive interpretation of these indices has been pointed out by some authors, such as Garrett and Tsebelis (1999, 2001), because no information about the voters' behavior enters the model.

In order to provide a richer, clearer conceptual framework for dealing with the foundations of voting power theory, Laruelle and Valenciano (2005) summarize the voting behavior of voters by a probability distribution over the vote configurations and include it as a second independent ingredient in the model. Voting power depends then on two independent inputs: the voting rule and voting behavior. The measure of *success* is defined as the probability of getting the final outcome that one wants, and the measure of *decisiveness* as the probability of being successful and crucial for this. Most power indices appear as measures of success or decisiveness for special voting behaviors.

In this paper we carry out a reciprocal test between some of the best-established voting power postulates/paradoxes and the general measures of decisiveness and success introduced in Laruelle and Valenciano (2005). What is the purpose of testing the behavior of these measures that take into account voters' behavior, against postulates/paradoxes intended for *a priori* measures of voting power (related to decisiveness) that ignore voters' behavior? As will be shown, this reciprocal test sheds some light on the meaning of these so-called paradoxes and helps us to understand better the concept of power as decisiveness and the differences with the notion of success in voting situations. In particular it shows explicitly how voters' behavior influences their success and decisiveness, and within what limits voting behavior is compatible with the postulates. Surprisingly enough, in spite of the selecting aim of these postulates in order to discard 'bad' *a priori* power measures, it turns out that these measures never violate some postulates (such as the 'donation' and 'bloc' postulates), while in others no violation occurs for a wide family of behaviors exhibiting a certain level of symmetry. Moreover success, unavoidably intermingled with decisiveness in any pre-conceptual notion of voting power, behaves even better with respect to some postulates in principle intended for decisiveness. On the other hand, the explicit consideration of behavior in the approach shows the lack of consistency in the formulation of certain paradoxes/postulates (the ‘quarrel’ paradox and ‘bloc’ postulates) related to a change of behavior and treated as changes of voting rule within the limitations of the traditional framework. Finally the coherence of the aforesaid general notions of success and decisiveness is ratified by this test, as no ‘paradox’ fails to be explained in plain terms consistent with real life experience. In brief this paper shows that a deeper understanding and a precise formulation of what is to be measured dispels confusion about the expectation of how the measure should behave.

The rest of the paper is organized as follows. Section on ‘Voting rules and power indices’ contains the basic framework concerning voting rules, and main classical power indices. In section on ‘Voting situations, success and decisiveness’ the measures of success and decisiveness based on the voting rule and the voting behavior introduced in Laruelle and Valenciano (2005) are presented. Section on ‘Some paradoxes reexamined’ examines the behavior of the measures of success and decisiveness with respect of some postulates/paradoxes. Subsection on ‘The better the seat, the more the power?’ deals with the ‘dominance paradox’ and the ‘preference for blocker paradox’. Section on ‘Transferring weight to gain power’ deals with the effect of transferring weight in weighted majorities from one voter to another (the ‘donation paradox’). Section on ‘Joining to harm? Quarrelling to help?’ deals with the ‘paradox of quarrelling members’ and the ‘bloc paradox’. New ‘behavioral’ versions of these paradoxes are also proposed. Section on ‘Bicameral paradox?’ deals with the ‘bicameral paradox’. Finally, section on ‘Conclusion’ sums up with some concluding comments.

Voting Rules and Power Indices

Let $N = \{1, \ldots, n\}$ denote the set of *seats*. A *vote configuration* is a conceivable result of a vote, listing the votes cast from the different seats. If we consider only voting rules that assimilate any vote different from ‘yes’ into ‘no’,[1] there are 2^n possible vote configurations, and each configuration can be represented by the set of seats from which a ‘yes’ vote is cast. An *N-voting rule* specifies when a proposal is accepted, and it can be fully represented by the set of *winning* vote configurations, i.e. those that lead to the acceptance of a proposal. In what follows W denotes the set of winning configurations representing an N-voting rule. It is assumed that an N-voting rule satisfies: (i) $N \in W$, $\emptyset \notin W$, and (ii) for all $S, T \subseteq N$, $(S \subseteq T$ and $S \in W) \Rightarrow T \in W$. Let VR_N denote the set of all such N-voting rules,[2] and for any set A, a will denote its cardinal. We drop i’s brackets in $S \backslash \{i\}$ and $S \cup \{i\}$.

Some particular voting rules that will be considered later are the following. The *dictatorship* of seat i is the voting rule $W = \{S \subseteq N : i \in S\}$. In this rule the decision always coincides with the vote of voter i, called the *dictator*. In a *weighted majority* rule, a 'weight' $w_i \geq 0$ is associated with each seat i, and a certain 'quota' $Q > 0$, such that $\frac{1}{2}\sum_{i \in N} w_i < Q \leq \sum_{i \in N} w_i$, is given. After a vote, the proposal is passed if the sum of the weights of the seats where 'yes' votes were cast is greater than or equal to the quota. The voting rule is thus specified by the quota Q and the vector $w = (w_i)_{i \in N}$

$$W(Q; w) = \left\{ S \subseteq N : \sum_{i \in S} w_i \geq Q \right\}.$$

If one can choose between two seats, the seat with the larger weight seems better. This idea is formalized (and generalized) as follows. In voting rule W, seat j (*weakly*) *dominates* seat i (denoted $j \succeq_W i$) if for any configuration of votes S such that $i, j \notin S$,

$$S \cup i \in W \Rightarrow S \cup j \in W.$$

If j strictly dominates i ($j \succ_W i$), then j is said to be *more desirable than* i (Isbell, 1958). In a voting rule W, seat i is a *seat with veto* if for any $S \in W$, $i \in S$. Obviously a seat with veto dominates any other seat.

A *power index* is a function $\phi : \mathrm{VR}_N \rightarrow R^n$, that associates with each voting rule W a vector whose ith component is interpreted as a measure of the power that the voting rule W confers to voter i. For evaluating the distribution of power among voters the two best-known power indices are the Shapley–Shubik (1954) index and the Banzhaf (1965) index. For a voting rule W, voter i's Shapley–Shubik index is given by

$$Sh_i(W) = \sum_{\substack{S: i \in S \in W \\ S \setminus i \notin W}} \frac{(s-1)!(n-s)!}{n!},$$

while voter i's (non normalized) Banzhaf index is given by

$$Bz_i(W) = \sum_{\substack{S: i \in S \in W \\ S \setminus i \notin W}} \frac{1}{2^{n-1}}.$$

These two power indices are the most distinguished members of the family of *semivalues* (see Weber, 1979; Einy, 1987; Laruelle & Valenciano, 2003), which can be seen as an extension of the notion of power index. In our setting *semivalues* are maps $\varphi : \mathrm{VR}_N \rightarrow R^n$, given by

$$\varphi_i(W) = \sum_{\substack{S: i \in S \in W \\ S \setminus i \notin W}} p_s, \quad i = 1, \ldots, n,$$

where $(p_s)_{s=1,2,\ldots,n}$ are such that $p_s \geq 0$, and $\sum_{S: i \in S} p_s = \sum_{s=1}^{n} \binom{n-1}{s-1} p_s = 1$.

Voting Situations, Success and Decisiveness

In any real world *voting situation* a group of voters makes decisions by means of a voting rule. The voting rule is modeled as above, and the voters are labeled by attaching to each of them the label of the seat he or she occupies. As in Laruelle and Valenciano (2005), their behavior is summarized by a probability distribution over the set of vote configurations: $p : 2^N \rightarrow R$ which associates with each vote configuration S its probability of occurrence $p(S)$, where $0 \leq p(S) \leq 1$ for any $S \subseteq N$, and $\sum_{S \subseteq N} p(S) = 1$. That is, $p(S)$ gives the probability that voters in S and only these voters will vote 'yes'. Given this distribution of probability, let $\gamma_i(p)$ denote the probability that voter i votes 'yes':

$$\gamma_i(p) = Prob(i \text{ votes 'yes'}) = \sum_{S:i \in S} p(S),$$

and $\bar{\gamma}_i(p)$ denotes the probability that voter i votes 'no': $\bar{\gamma}_i(p) = 1 - \gamma_i(p)$. $\mathfrak{P}_N$ will denote the set of all maps representing such probability distributions over 2^N. This set can be interpreted as the set of all conceivable voting behaviors of n voters within this setting.

The notion of success and decisiveness are grounded *ex post*, that is, once a proposal has been submitted to a vote, the vote configuration has emerged and the final outcome of passage or rejection is known. Once the resulting vote configuration S is known, voter i is said to have been *successful*[3] if his or her vote coincides with the decision that has been made. That is, if

$$(i \in S \in W) \quad \text{or} \quad (i \notin S \notin W).$$

and voter i is said to have been *decisive*, the basic notion behind several concepts of 'voting power', if

$$(i \in S \in W \text{ and } S \backslash i \notin W) \quad \text{or} \quad (i \notin S \notin W \text{ and } S \cup i \in W).$$

In a voting situation (W, p), *ex ante*, that is, once voters occupy their seats, but before voters cast their vote, decisiveness and success can be defined in probabilistic terms:

Definition 1 (Laruelle & Valenciano, 2005). For any N-voting rule $W \in \mathrm{VR}_N$ and any probability distribution $p \in \mathfrak{P}_N$ over the vote configurations:

(i) Voter i's measure of success in voting situation (W, p) is given by

$$\begin{aligned} \Omega_i(W, p) &:= P \text{ (the decision coincides with i's vote)} \\ &= \sum_{S:i \in S \in W} p(S) + \sum_{S:i \notin S \notin W} p(S). \end{aligned} \tag{1}$$

(ii) voter i's measure of decisiveness in voting situation (W, p) is given by

$$\Phi_i(W, p) := P\,(i \text{ is decisive}) = \sum_{\substack{S: i \in S \in W \\ S \setminus i \notin W}} p(S) + \sum_{\substack{S: i \notin S \notin W \\ S \cup i \in W}} p(S). \quad (2)$$

In the following we will sometimes find the following decompositions useful:

$$\Omega_i(W, p) = \Omega_i^+(W, p) + \Omega_i^-(W, p),$$

where $\Omega_i^+(W, p) := P(i$ is successful & i votes 'yes'), $\Omega_i^-(W, p) := P(i$ is successful & i votes 'no'), and

$$\Phi_i(W, p) = \Phi_i^+(W, p) + \Phi_i^-(W, p),$$

where $\Phi_i^+(W, p) := P(i$ is decisive & i votes 'yes'), and $\Phi_i^-(W, p) := P(i$ is decisive & i votes 'no').

Most well-known power indices are special cases of these general measures. In particular, the Rae (1969) index (or rather the generalization proposed by Dubey and Shapley (1979)) is the measure of success for p^* such that $p^*(S) = \frac{1}{2^n}$ for all $S \subseteq N$. Namely, for all $W \in \mathrm{VR}_N$,

$$Rae_i(W) = \sum_{S: i \in S \in W} \frac{1}{2^{n-1}} = \Omega_i(W, p^*).$$

The Banzhaf index and the Shapley–Shubik index are measures of decisiveness.[4] More precisely, for p^* such that $p^*(S) = \frac{1}{2^n}$ for all $S \subseteq N$, and all $W \in \mathrm{VR}_N$,

$$\Phi_i(W, p^*) = Bz_i(W),$$

while for p^{Sh} such that $p^{\mathrm{Sh}}(S) = \frac{1}{(n+1)\binom{n}{s}}$ for all $S \subseteq N$, and all $W \in \mathrm{VR}_N$,

$$\Phi_i(W, p^{\mathrm{Sh}}) = Sh_i(W).$$

Finally, we have the following relation between decisiveness and semivalues:

Proposition 1. For all $p \in \mathfrak{P}_n$ that assign the same probability to any two vote configurations with the same number of 'yes' voters, the measure of decisiveness $\Phi(-, p)$ becomes a semivalue.

Proof. Let $p \in \mathfrak{P}_n$ such that $p(S) = p(T)$ whenever $s = t$. For any $W \in \mathrm{VR}_N$,

$$\Phi_i(W, p) := \sum_{\substack{S:i \in S \in W \\ S \setminus i \notin W}} p(S) + \sum_{\substack{S:i \notin S \notin W \\ S \cup i \in W}} p(S) = \sum_{\substack{S:i \in S \in W \\ S \setminus i \notin W}} (p(S) + p(S \setminus i)).$$

Now as $p(S)$ depends only on the size of S, calling $p_s := p(S) + p(S \setminus i)$, for all $s = 1, \dots, n$, we have:

$$\Phi_i(W, p) := \sum_{\substack{S:i \in S \in W \\ S \setminus i \notin W}} p_s,$$

where the p_ss verify

$$\sum_{S:i \in S} p_s = \sum_{S:i \in S} (p(S) + p(S \setminus i)) = \sum_{S \subseteq N} p(S) = 1,$$

and consequently $\Phi(-, p)$ is a semivalue. □

Some Paradoxes Reexamined

In the traditional power indices setting only the simple game describing the voting rule enters the picture, and consequently all the 'paradoxes' briefly reviewed in the introduction were originally stated for 'power indices' or maps $\phi : \mathrm{VR}_N \to R^N$, while now they must be adequately re-stated in terms of a map $\Psi : \mathrm{VR}_N \times \mathfrak{P}_N \to R^N$. This is easily achieved taking into account that with any such map Ψ and each $p \in \mathfrak{P}_N$, one can associate a map or 'power index' $\Psi(-, p) : \mathrm{VR}_N \to R^N$, which associates with each voting rule W the vector $\Psi(W, p)$, interpretable as the power profile corresponding to rule W under behavior p. We will refer to such maps generically as 'power measures', leaving the meaning of this 'power' deliberately unspecified, so that *both success and decisiveness* (*as given by* (1) *and* (2)) *are included*. Thus, we will say that success (or decisiveness) displays or not a given paradox[5] for a certain $p \in \mathfrak{P}_N$, if $\Omega(-, p)$ (or $\Phi(-, p)$) (see Definition 1) displays it.

The better the seat, the more the power?

The paradoxes that we consider in this section refer to the conflict between the ranking of voters' power provided by a measure for a given voting rule and variations of the principle 'the better the seat, the more the power'. For some power measures it may happen that one voter occupies a 'better' seat than another but has less power. There are several paradoxes of this type that result from different specifications of when one seat is considered 'better' than another. The first one concerns weighted majority rules, where it seems clear that 'the larger the weight, the more the power'. Nevertheless not all power measures satisfy this property. Deegan and Packel (1982) show that their index does not satisfy it, and refer to this failure as the 'paradox of weighted voting'. According to Felsenthal and Machover (1995), a valid measure of *a priori* power should not display this paradox. They even go further, proposing the 'dominance' postulate that states that the more desirable (as defined in section on 'Voting rules and power indices') the seat, the more the power ought to be. We will refer to the violation of this property as the 'dominance paradox', which can be restated as follows in our setting:

Dominance paradox: A power measure $\Psi : \mathrm{VR}_N \times \mathfrak{P}_N \to R^N$ is said to display the *dominance paradox* for a given $p \in \mathfrak{P}_N$, if there exists an N-voting rule W, such that $\Psi_j(W, p) < \Psi_i(W, p)$ although $j \succ_W i$.

A weaker form of the same principle is to require that a 'blocker' (that is, a seat with veto) has at least as much power as any other voter. The violation of this property is referred to by Felsenthal and Machover as the 'preference for blocker paradox', and can be reformulated as follows:

Preference for blocker paradox: A power measure Ψ is said to display the *preference for blocker paradox* for a given $p \in \mathfrak{P}_N$, if there exists an N-voting rule W, such that $\Psi_j(W, p) < \Psi_i(W, p)$ although j has a veto and i has not.

Is it reasonable to expect that 'the better the seat, the more the power'? Now the probabilities of the vote configurations also matter. Therefore it may happen that a voter sitting in a more desirable seat has less chances of being decisive/successful because the distribution of probability over the vote configurations over-compensates the voter in the worse seat. The following example illustrates this intuitively plausible possibility.

Example 1. In the four-person voting rule

$$W = \{\{1, 4\}, \{2, 4\}, \{3, 4\}, \{1, 2, 3\}, \{1, 2, 4\}, \{1, 3, 4\}, \{2, 3, 4\}, \{1, 2, 3, 4\}\},$$

seat 4 is more desirable than any other seat. Nevertheless, for the probability distribution over vote configurations

$$p(S) = \begin{cases} 1/2, & \text{if } S = \{1, 2, 3\} \text{ or } \{4\} \\ 0, & \text{otherwise,} \end{cases}$$

we obtain $\Phi_4(W, p) < \Phi_i(W, p)$ and $\Omega_4(W, p) < \Omega_i(W, p)$, for $i = 1, 2, 3$. This could be a stylized model for a four-party parliament, with three small left-wing parties (1, 2, and 3) and a large right-wing party 4. The large party has a smaller probability of exerting power than any of the small parties because these parties have similar (in the example identical) behaviors, far different from the right-wing party's behavior.

Thus, many violations of the dominance postulate for many behaviors may be expected. Notwithstanding, the dominance paradox never occurs for distributions of probability over vote configurations that exhibit a strong degree of symmetry. Namely, if the probability of a vote configuration only depends on the number of its 'yes' voters that is when, according to Proposition 1, $\Phi(-, p)$ becomes a semivalue (as is the case, for example, for the Shapley–Shubik and Banzhaf indices), the dominance postulate is preserved. This sets a limit on the possibility of occurrence of the dominance paradox (and therefore on the preference for blocker paradox).

Proposition 2. Neither the measure of success (1), nor the measure of decisiveness (2) displays the dominance paradox when the probability of a vote configuration only depends on the number of 'yes'-voters.

Proof. Let W be an N-voting rule, and $i, j \in N$, s.t., $j \succeq_W i$, that is, $S \cup i \in W \Rightarrow S \cup j \in W$, for any $S \subseteq N\backslash\{i, j\}$. Therefore $S\backslash i \notin W \Rightarrow S\backslash j \notin W$, for any S containing i and j. Then for any $p \in \mathfrak{P}_N$ we have

$$\Phi_i^+(W, p) = \sum_{\substack{S:i\in S\in W\\ S\backslash i\notin W}} p(S) = \sum_{\substack{S:i,j\in S\in W\\ S\backslash i\notin W}} p(S) + \sum_{\substack{S:i,j\notin S\notin W\\ S\cup i\in W}} p(S\cup i),$$

$$\Phi_j^+(W, p) = \sum_{\substack{S:i,j\in S\in W\\ S\backslash i\notin W}} p(S) + \sum_{\substack{S:i,j\in S\in W\\ S\backslash i\in W\\ S\backslash j\notin W}} p(S) + \sum_{\substack{S:i,j\notin S\notin W\\ S\cup i\in W}} p(S\cup j)$$
$$+ \sum_{\substack{S:i,j\notin S\notin W\\ S\cup i\notin W\\ S\cup j\in W}} p(S\cup j).$$

If $p(S) = p(T)$ whenever $s = t$, we have $p(S \cup i) = p(S \cup j)$ for all $S \subseteq N\backslash\{i, j\}$, which yields $\Phi_i^+(W, p) \leq \Phi_j^+(W, p)$. Similarly for $\Phi_i^-(W, p)$ we have:

$$\Phi_i^-(W, p) = \sum_{\substack{S:i\notin S\notin W\\ S\cup i\in W}} p(S) = \sum_{\substack{S:i,j\notin S\notin W\\ S\cup i\in W}} p(S) + \sum_{\substack{S:i,j\in S\in W\\ S\backslash i\notin W}} p(S\backslash i)$$

$$\Phi_j^-(W, p) = \sum_{\substack{S:i,j\notin S\notin W\\ S\cup i\in W}} p(S) + \sum_{\substack{S:i,j\notin S\notin W\\ S\cup i\notin W\\ S\cup j\in W}} p(S) + \sum_{\substack{S:i,j\in S\in W\\ S\backslash i\notin W}} p(S\backslash j)$$
$$+ \sum_{\substack{S:i,j\in S\in W\\ S\backslash i\in W\\ S\backslash j\notin W}} p(S\backslash j).$$

Thus $\Phi_i^-(W, p) \leq \Phi_j^-(W, p)$. Finally, as $\Phi_i(W, p) = \Phi_i^+(W, p) + \Phi_i^-(W, p)$, we also have $\Phi_i(W, p) \leq \Phi_j(W, p)$. The proof of $\Omega_j(W, p) \geq \Omega_i(W, p)$ is similar. □

Finally, we have a weaker condition limiting the possibility of occurrence of the 'preference for blocker paradox' for success.

Proposition 3. The measure of success (1) does not display the preference for blocker paradox when for any two voters the probability of voting 'yes' is the same.

Proof. Let W be an N-voting rule in which j has a veto. Then $S \in W \Rightarrow j \in S$, and the probability of a successful negative vote by j equals j's probability of voting 'no', that is, $\Omega_j^-(W, p) = \bar{\gamma}_j(p)$. Then we have for any $p \in \mathfrak{P}_N$, such that $\gamma_i(p) = \gamma_k(p)$ for any i, k,

$$\Omega_i(W, p) = \sum_{S:i\in S\in W} p(S) + \sum_{S:i\notin S\notin W} p(S) \leq \sum_{S:j\in S\in W} p(S) + \bar{\gamma}_i(p)$$
$$= \Omega_j^+(W, p) + \bar{\gamma}_j(p) = \Omega_j^+(W, p) + \Omega_j^-(W, p) = \Omega_j(W, p).$$

Thus, $\Omega_j(W, p) \geq \Omega_i(W, p)$ for all i. □

The following example shows how the decisiveness may display the preference for blocker paradox even if all voters have the same probability of voting 'yes'.

Example 2. In the four-person voting rule $W = \{\{1, 2, 3\}, \{1, 2, 4\}, \{1, 2, 3, 4\}\}$, voters 1 and 2 have a veto. Suppose that the vote configurations have the following probabilities:

$$p(S) := \begin{cases} 9/32, & \text{if } S = \{1, 2\} \text{ or } \{3, 4\} \\ 1/32, & \text{otherwise.} \end{cases}$$

A simple calculation shows that $\Phi_1(W, p) < \Phi_3(W, p)$. Note that all voters have the same probability of voting 'yes': $\gamma_i(p) = \frac{1}{2}$, for $i = 1, \ldots, 4$.

In sum the 'paradox of dominance' is not that paradoxical after all, although it never occurs when the probability of a vote configuration only depends on its number of 'yes'-voters, something not to be expected in real-world situations in general, but a condition which is satisfied, for instance, by the family of semivalues. Success does not display the preference for blocker paradox under even more general conditions, as it suffices for all voters to have the same probability of voting 'yes'.

Transferring weight to gain power?

The paradox considered in this section concerns weighted majorities. The principle at stake is that a voter should not gain power when part or all of his or her weight is transferred to another voter. Dreyer and Schotter (1980) consider a weighted majority where weights are redistributed, but total weight and quota remain identical. They show that it may happen that a voter loses weight but increases his or her voting power according to some power indices. They refer to this phenomenon as the 'paradox of redistribution'. But as Felsenthal and Machover (1995) rightly argue in the context of traditional power indices, the transfer of weight between two voters will affect the other voters. Therefore if there is more than one transfer of weight, the fact that a 'donor' gains power is not paradoxical because it might be due to the transfers that have occurred among other voters. But if there is just one transfer between two voters: 'We surely ought to expect that donating weight may if anything cause a reduction in the donor's power' (Felsenthal & Machover, 1998, p. 215). The violation of this principle is called the 'donation paradox'.

It is worth remarking that strictly speaking, in spite of the term 'donation' conveying the idea of a certain behavior on the part of voters (a voter giving part of his or her weight to another voter), the formal statement of this paradox entails just a change of voting rule. It could not be otherwise in a setting in which the only ingredient is the voting rule![6] In our setting the question is whether just one such transfer may increase the power of the 'donor' assuming that the change of rule does not modify the voters' voting behavior:

Donation paradox: A power measure Ψ is said to display the *donation paradox* for a given $p \in \mathfrak{P}_N$, if there exist two weighted majority rules with the same quota, $W = W(Q; w)$ and $W' = W(Q; w')$, such that

$$w'_k = \begin{cases} w_i - \lambda & \text{if } k = i \\ w_j + \lambda & \text{if } k = j \\ w_k, & \text{if } k \neq i, j, \end{cases} \tag{3}$$

for some λ $(0 < \lambda \leq w)$, and such that

$$\Psi_i(W', p) > \Psi_i(W, p).$$

The following result shows that neither success nor decisiveness exhibits this paradox *whatever the voters' behavior*.

Proposition 4. Whatever the voters' behavior, neither the measure of success, nor the measure of decisiveness displays the donation paradox.

Proof. Let W and W' be two N-weighted majority rules with the same quota, $W = W(Q; w)$ and $W' = W(Q; w')$, and (3) for some $0 < \lambda \leq w_i$. Then, $w'(S) = w(S)$ for all S s.t. $i, j \in S$, and $w'(S) = w(S) - \lambda$ for all S s.t. $i \in S$ and $j \notin S$. Therefore for any probability distribution over vote configurations $p \in \mathfrak{P}_N$, it holds that:

$$\Phi_i^+(W, p) = \sum_{\substack{S: i \in S \in W \\ S \setminus i \notin W}} p(S) = \sum_{\substack{S: i, j \in S \\ w(S) \geq Q \\ w(S) - w_i < Q}} p(S) + \sum_{\substack{S: i \in S, j \notin S \\ w(S) \geq Q \\ w(S) - w_i < Q}} p(S),$$

$$\begin{aligned} \Phi_i^+(W', p) &= \sum_{\substack{S: i \in S \in W' \\ S \setminus i \notin W'}} p(S) = \sum_{\substack{S: i, j \in S \\ w'(S) \geq Q \\ w'(S) - w'_i < Q}} p(S) + \sum_{\substack{S: i \in S, j \notin S \\ w'(S) \geq Q \\ w'(S) - w'_i < Q}} p(S) \\ &= \sum_{\substack{S: i, j \in S \\ w(S) \geq Q \\ w(S) - w_i + \lambda < Q}} p(S) + \sum_{\substack{S: i \in S, j \notin S \\ w(S) - \lambda \geq Q \\ w(S) - w_i < Q}} p(S), \end{aligned}$$

which entails $\Phi_i^+(W', p) \leq \Phi_i^+(W, p)$. The same inequality for Φ_i^- is derived similarly, and as a consequence it also holds for Φ_i. The proof for Ω_i is entirely similar. □

Joining to harm? Quarrelling to help?

The paradoxes considered in this section concern the effect on voters' power of the formation of a 'bloc', or its opposite, that is, the effect of a 'quarrel'. Brams (1975) considers weighted rules where two voters decide to form a kind of indissoluble 'bloc'. The 'paradox of size' occurs when the power of the bloc is strictly smaller than the sum of the power of its components. Felsenthal and Machover (1998, p. 226) criticize this paradox: "The 'conventional wisdom' that the *whole is greater than – or at least equal to – the sum of its parts* is no argument at all but a mere saying". But in their view, "There are indeed very good common-sense arguments suggesting that the power of a bloc ought to be at least as great as the power of *the most powerful* of its component parts". The violation of this principle is called the 'bloc paradox'.

Again, in spite of the behavioral flavor of the preceding terms and stories, the traditional setting forces their formalization *as a change of voting rule.* For any N-voting rule W, and any two seats $i, j \in N$, the formation of *a bloc by j's annexation of i*, is modeled (Felsenthal & Machover, 1998, p. 254) by the N-voting rule $W_B^{i,j}$ where

$$S \in W_B^{i,j} \Leftrightarrow S \cup i \in W \quad \text{(for any } S \text{ containing } j),$$
$$S \in W_B^{i,j} \Leftrightarrow S \setminus i \in W \quad \text{(for any } S \text{ not containing } j).$$

The 'bloc paradox' occurs when voter j's power in the new rule is strictly smaller than his or her power in the original rule (as far as i is not a null seat[7] in the original rule).

Bloc paradox: A power measure Ψ is said to display the *bloc paradox* for a given $p \in \mathfrak{P}_N$, if for some N-voting rule W, some $i, j \in N$, and $W_{\mathrm{B}}^{i,j}$ as defined above,

$$\Psi_j\big(W_{\mathrm{B}}^{i,j}, p\big) < \Psi_j(W, p).$$

A symmetrically opposite situation occurs when two voters 'refuse to join together to help forming a winning coalition' (Brams, 1975, p. 181): this ought not to benefit either of these two voters. The 'paradox of quarrelling members' occurs when this principle is not satisfied. Felsenthal and Machover (1995) note that the original formulation, which consists of deleting those including the quarrelling members from the list of winning configuration, does not always lead to a voting rule. But an alternative formulation, entirely similar to that of Felsenthal and Machover's bloc paradox, as a change of voting rule is possible. Namely, given an N-voting rule W, and any two seats $i, j \in N$ (where i is not a null seat in the original rule), the *quarrel of i against j* is modeled by the N-voting $W_{\mathrm{Q}}^{i,j}$ where

$$S \in W_{\mathrm{Q}}^{i,j} \Leftrightarrow S \setminus i \in W \quad (\text{for any } S \text{ containing } j),$$
$$S \in W_{\mathrm{Q}}^{i,j} \Leftrightarrow S \cup i \in W \quad (\text{for any } S \text{ not containing } j).$$

The 'quarrel paradox' occurs when voter j's power in the new rule is strictly larger than his or her power in the original rule.

Quarrel paradox: A power measure Ψ is said to display the *quarrel paradox* for a given $p \in \mathfrak{P}_N$, if for some N-voting rule W, some $i, j \in N$, and $W_{\mathrm{Q}}^{i,j}$ as defined above,

$$\Psi_j\big(W_{\mathrm{Q}}^{i,j}, p\big) > \Psi_j(W, p).$$

We have the following result:

Proposition 5. Whatever the voters' behavior, neither the measure of success nor the measure of decisiveness displays the bloc paradox or the quarrel paradox.

Proof. Let W be an N-voting rule, and $i, j \in N$. Let us consider the case of a bloc $W_{\mathrm{B}}^{i,j}$. For any vote configuration S such that $j \in S$, if $S \in W$ then

$S \in W_B^{i,j}$, and if $S\backslash j \notin W$ then $S\backslash j \notin W_B^{i,j}$. Thus for any $p \in \mathfrak{P}_N$,

$$\Phi_j^+\big(W_B^{i,j}, p\big) = \sum_{\substack{S:j\in S\in W_B^{i,j} \\ S\backslash j\notin W_B^{i,j}}} p(S) \geq \sum_{\substack{S:j\in S\in W \\ S\backslash j\notin W}} p(S) = \Phi_j^+(W, p),$$

$$\Phi_j^-\big(W_B^{i,j}, p\big) = \sum_{\substack{S:j\notin S\notin W_B^{i,j} \\ S\cup j\in W_B^{i,j}}} p(S) \geq \sum_{\substack{S:j\notin S\notin W \\ S\cup j\in W}} p(S) = \Phi_j^-(W, p).$$

Therefore the $\Phi_j(W_B^{i,j}, p) \geq \Phi_j(W, p)$. The same inequality for Ω_j is derived similarly. Finally, the reverse inequalities for Φ and Ω for the case of a quarrel are obtained similarly. □

As already commented, in spite of the verbal 'dramatization' of the formation of a bloc (or a quarrel) in terms of a change of behavior, the above formulations are the only ones feasible in the traditional setting: that is, as changes of voting rule.[8] But if a voter's voting behavior changes so as to always vote with (or against) another voter, such a change concerns the *voting behavior of the voters, not the voting rule.* This is possible in our setting, if the starting point is a voting situation (W, p) we can keep the voting rule W unchanged and modify the voting behavior represented by p.

First, consider the case of a 'bloc'. If voter i changes his or her behavior to vote always as voter j, we will say that 'i switches in favor of j'. Similarly, in the case of a 'quarrel' between i and j, we will say 'i switches against j' to mean that voter i decides to vote always the opposite to voter j. Thus we consider two similar and opposed changes *affecting only voter i's behavior,* from a previous voting situation described by a probability distribution p. The changes induced in the distribution of probability when i switches in favor of j are (i) the probability of a vote configuration where i and j vote differently becomes zero, (ii) the probability of a vote configuration S where i and j both vote 'yes' is increased by the previous probability of the vote configuration $S \setminus i$, and (iii) the probability of a vote configuration S where i and j both vote 'no' is increased by the former probability of the vote configuration $S \cup i$. Denoting by p_B^{ij} the probability distribution resulting from p by the 'bloc' resulting from i switching in favor of j we have

$$p_B^{ij}(S) := \begin{cases} p(S) + p(S \setminus i), & \text{if } i, j \in S \\ p(S) + p(S \cup i), & \text{if } i, j \notin S \\ 0, & \text{otherwise.} \end{cases}$$

In the 'quarrel' case, when voter 'i switches against j', the resulting probability distribution p^{ij}_{Q} from p, can be similarly derived:

$$p^{ij}_{\mathrm{Q}}(S) := \begin{cases} p(S) + p(S \cup i), & \text{if } j \in S \text{ and } i \notin S \\ p(S) + p(S \setminus i), & \text{if } j \notin S \text{ and } i \in S \\ 0, & \text{otherwise.} \end{cases}$$

It seems reasonable to expect that if voter i gives his or her vote to voter j this would not harm voter j. Similarly, if voter i switches to oppose j's vote permanently this would not benefit voter j. The violation of these properties gives rise to the following 'paradoxes' in terms of our power measures.

Behavioral bloc (i switching in favor of j) paradox: A power measure Ψ is said to display the *behavioral bloc* (i *switching in favor of* j) *paradox* for a given $p \in \mathfrak{P}_N$, if there exists an N-voting rule W, such that for some $i, j \in N$, $\Psi_j(W, p^{ij}_{\mathrm{B}}) < \Psi_j(W, p)$.

Behavioral quarrel (i switching against j) paradox: A power measure Ψ is said to display the *behavioral quarrel* (i *switching against* j) *paradox* for a given $p \in \mathfrak{P}_N$, if there exists an N-voting rule W, such that for some $i, j \in N$, $\Psi_j(W, p^{ij}_{\mathrm{Q}}) > \Psi_j(W, p)$.

The following result, whose simple proof we omit, confirms the intuition for the measure of success.

Proposition 6. The measure of success never displays the behavioral bloc paradox nor the behavioral quarrel paradox.

But the result does *not* hold for the measure of decisiveness. Although surprising at first sight, this is not paradoxical as shown by the following reasoning for the behavioral bloc paradox (similar considerations apply to the quarrel paradox). When voter i switches in favor of voter j, the change of voting behavior has two opposite effects on j's decisiveness. On the one hand, the probability of those winning configurations S containing i and j (respectively, containing neither i nor j) in which j is decisive increases in $p(S \setminus i)$ (respectively, $p(S \cup i)$), which increases j's decisiveness. But on the other hand, the probability of those winning configurations S containing j but not i (respectively, i but not j) in which j is decisive become 0, which diminishes j's decisiveness. The net effect is thus uncertain: when the second effect is more important, we will have the paradox, as illustrated in the following example.

Example 3. Consider the voting situation given by the three-person majority rule $W = \{\{1, 2\}, \{1, 3\}, \{2, 3\}, \{1, 2, 3\}\}$, and the following probability distribution over vote configurations:

$$p(S) = \begin{cases} 9/16, & \text{if } S = \{1, 3\} \\ 1/16, & \text{otherwise.} \end{cases}$$

Assume voter 2 switches in favor of voter 1. Then

$$p_{\mathrm{B}}^{21}(S) = \begin{cases} 5/8, & \text{if } S = \{1, 2, 3\} \\ 1/8, & \text{if } S = \emptyset, \{3\}, \text{ or } \{1, 2\} \\ 0, & \text{otherwise.} \end{cases}$$

For this voting rule voter 1 is decisive in configurations $\{1, 2\}$, $\{1, 3\}$, $\{2\}$ and $\{3\}$. After voter 2 joins voter 1, it is easy to check that $\Phi_1(W, p_{\mathrm{B}}^{21}) = 1/4 < 3/4 = \Phi_1(W, p)$.

The fact that the measure of success and the measure of decisiveness exhibit different properties underline that these are two different notions. The difference has perhaps been overlooked in the literature, possibly due to unawareness of the fact that the linear relationship between these two notions for the particular behavior p^* (as pointed out by Dubey and Shapley (1979) for the Banzhaf and Rae indices) does not hold in general. As a result too little attention has been paid to the measure of success.

Bicameral paradox?

Felsenthal et al. (1998) consider a bicameral system, where a bill requires the approval of two separate chambers to be passed. Let N_1 and N_2 denote the seats in the two chambers ($N_1 \cap N_2 = \emptyset$), and let W_{N_1} and W_{N_2} denote the voting rules used by each chamber. Then a bicameral rule based on these rules is defined by the N-voting rule W_N, with $N = N_1 \cup N_2$, where

$$W_N = \left\{ S \subseteq N : S \cap N_1 \in W_{N_1} \text{ and } S \cap N_2 \in W_{N_2} \right\}.$$

They argue that it would be unreasonable for the ranking of power between two voters to be reversed from one chamber to the bicameral system: if one voter has more power in one chamber than another voter then he or she should also be more powerful in the bicameral system. If this is not so, the index would display the ‘bicameral paradox’.[9]

The formulation of this paradox in our setting must specify the voting behavior for all the three voting rules. But as N_1 and N_2 are subsets of N, we are speaking of a single set of voters. Thus, the voting behavior of N in the bicameral system (p_N) specifies in particular the voting behavior of both subsets of voters (p_{N_1} and p_{N_2}). Namely,

$$p_{N_1}(S) = \sum_{\substack{R \subset N \\ R \cap N_1 = S}} p_N(R) = \sum_{T \subset N_2} p_N(S \cup T) \quad \text{for any } S \subset N_1,$$

$$p_{N_2}(S) = \sum_{\substack{R \subset N \\ R \cap N_2 = S}} p_N(R) = \sum_{T \subset N_1} p_N(S \cup T) \quad \text{for any } S \subset N_2.$$

Thus, p_{N_1} and p_{N_2} are fully determined by p_N, while the voting rule W_N is fully determined by W_{N_1} and W_{N_2}. Note that for any i in chamber $k = 1, 2$, i.e., for all $i \in N_k$, $\gamma_i(p_{N_k}) = \gamma_i(p_N)$. Then the bicameral paradox can be formulated for our general measures.

Bicameral paradox: A power measure Ψ displays the *bicameral paradox* for some $p_N \in \mathfrak{P}_N$, if for some bicameral system W_N based on W_{N_1} and W_{N_2}, the following property is not satisfied for any pair of voters i and j from the first chamber:

$$\Psi_i\left(W_{N_1}, p_{N_1}\right) < \Psi_j\left(W_{N_1}, p_{N_1}\right) \Leftrightarrow \Psi_i(W_N, p_N) < \Psi_j(W_N, p_N).$$

It is easy to provide examples showing that both measures (success and decisiveness) display this paradox. Is the violation of this property for these measures as happens to be the case paradoxical? Not really. As an extreme example, consider a bicameral system in which decisions are made by simple majority in both chambers. Imagine that in the first chamber all voters independently toss a coin to vote 'yes' or 'no', while in the second chamber all voters blindly vote as a particular voter from the first chamber does. Then, while in the first chamber all voters will have identical chances of success and decisiveness, in the bicameral system the voter whose vote is always followed by the members of the second chamber will have more chances than any other from the first chamber.

Notwithstanding, it is possible to set a clear limit on the occurrence of this paradox. Consider a voting situation consisting of a bicameral system in which the voting behavior of the voters in one chamber is *independent* from that of voters in the other, that is, we have:

$$p_N(R) = p_{N_1}(R \cap N_1) p_{N_2}(R \cap N_2) \quad \text{for all } R \subseteq N. \tag{4}$$

In this case we have the following result:

Proposition 7. In a bicameral system in which the voting behavior in one chamber is independent from the behavior in the other, and the probability of passing a decision is not zero in either chamber:

(i) The measure of decisiveness never displays the bicameral paradox.
(ii) The measure of success does not display the bicameral paradox if, in addition, the probability of voting ‘yes’ is the same for any two voters in the same chamber.

Proof.

(i) Let W_N be a bicameral system based on W_{N_1} and W_{N_2}. Let $A(W_{N_2}, p_{N_2})$ denote the probability of chamber 2 accepting the proposal (and $\bar{A}(W_{N_2}, p_{N_2}) = 1 - A(W_{N_2}, p_{N_2})$), that is

$$A\big(W_{N_2}, p_{N_2}\big) := P\,(\text{Chamber 2 accepts the proposal}) = \sum_{T \in W_{N_2}} p_{N_2}(T).$$

For any voter i, the probability of being decisive in the bicameral system, $\Phi_i(W_N, p_N)$, is the probability of i being decisive in the chamber to which the voter belongs *and* a winning vote configuration occurring in the other chamber. That is, if $i \in N_1$, with the behavior in one chamber being independent from that in the other (i.e., assuming $p_N \in \mathfrak{P}_N$ satisfies (4)), we have

$$\Phi_i(W_N, p_N) = \Phi_i\big(W_{N_1}, p_{N_1}\big)A\big(W_{N_2}, p_{N_2}\big).$$

Then, as $A(W_{N_2}, p_{N_2}) > 0$, the measure of decisiveness will never display the bicameral paradox.

(ii) Now for success we have that a voter i will be successful if he or she either votes ‘yes’ and in both chambers the result is approval, or votes ‘no’ and at least in one chamber the result is rejection. That is, denoting $\gamma_i(p) := \gamma_i(p_N) = \gamma_i(p_{N_1})$, if $i \in N_1$,

$$\begin{aligned}\Omega_i(W_N, p_N) = {} & \Omega_i^+\big(W_{N_1}, p_{N_1}\big)A\big(W_{N_2}, p_{N_2}\big) + \Omega_i^-\big(W_{N_1}, p_{N_1}\big) \\ & + \big(\bar{\gamma}_i(p) - \Omega_i^-\big(W_{N_1}, p_{N_1}\big)\big)\bar{A}\big(W_{N_2}, p_{N_2}\big),\end{aligned}$$

where the last summand can be rewritten as

$$\begin{aligned}&\big(\bar{\gamma}_i(p) - \Omega_i^-\big(W_{N_1}, p_{N_1}\big)\big)\bar{A}\big(W_{N_2}, p_{N_2}\big) \\ &\quad = \bar{\gamma}_i(p)\bar{A}\big(W_{N_2}, p_{N_2}\big) - \Omega_i^-\big(W_{N_1}, p_{N_1}\big)\bar{A}\big(W_{N_2}, p_{N_2}\big) \\ &\quad = \bar{\gamma}_i(p)\bar{A}\big(W_{N_2}, p_{N_2}\big) - \Omega_i^-\big(W_{N_1}, p_{N_1}\big)\big(1 - A\big(W_{N_2}, p_{N_2}\big)\big) \\ &\quad = \bar{\gamma}_i(p)\bar{A}\big(W_{N_2}, p_{N_2}\big) - \Omega_i^-\big(W_{N_1}, p_{N_1}\big) + \Omega_i^-\big(W_{N_1}, p_{N_1}\big)A\big(W_{N_2}, p_{N_2}\big).\end{aligned}$$

Substituting we have

$$\begin{aligned}\Omega_i(W_N, p_N) &= \Omega_i^+\big(W_{N_1}, p_{N_1}\big)A\big(W_{N_2}, p_{N_2}\big)\\ &\quad + \bar{\gamma}_i(p)\bar{A}\big(W_{N_2}, p_{N_2}\big) + \Omega_i^-\big(W_{N_1}, p_{N_1}\big)A\big(W_{N_2}, p_{N_2}\big)\big)\\ &= \Omega_i\big(W_{N_1}, p_{N_1}\big)A\big(W_{N_2}, p_{N_2}\big) + \bar{\gamma}_i(p)\bar{A}\big(W_{N_2}, p_{N_2}\big).\end{aligned}$$

Then, as $A(W_{N_2}, p_{N_2}) > 0$ and $\gamma_i(p)$ is the same for all the voters in the same chamber, the measure of success does not display the bicameral paradox. □

Thus, this simple result provides a clear-cut class of examples of bicameral situations (wider for decisiveness) in which the bicameral paradox does not occur for success or for decisiveness.[10] In particular, the Banzhaf index does not display the paradox, because it gives the decisiveness of voters when every voter *independently* votes 'yes' with probability $1/2$. Not surprisingly, the Shapley–Shubik index, for which the independence condition does not hold, does display the bicameral paradox, as is well known. As to real-world bicameral situations, the voting behavior in both chambers is not usually independent and occurrences of the bicameral paradox are not surprising.

Conclusion

We have tested some of the best known voting power postulates and paradoxes, and the general measures of success and decisiveness introduced in Laruelle and Valenciano (2005) 'against each other'. Table 1 summarizes the result of the test.

In summary we consider the following facts to be worth remarking. (i) Some paradoxes (donation, bloc and quarrel) *never* occur either for the measure of decisiveness or for the measure of success. (ii) The measure of success

Table 1. Testing the measures of success and decisiveness

Paradox\measure	Decisiveness	Success
Dominance	Not if $p(S)$ dependent on s	Not if $p(S)$ dependent on s
Preference for blocker	Not if $p(S)$ dependent on s	Not if $\gamma_i = \gamma_j$ all i, j
Donation	Never	Never
Quarrel	Never	Never
Bloc	Never	Never
Behavioral quarrel	May occur	Never
Behavioral bloc	May occur	Never
Bicameral	Not if independence	Not if independence and $\gamma_i = \gamma_j$ all i, j

behaves better than that of decisiveness for the 'behavioral' versions of the bloc and quarrel paradoxes (as well as for the preference for blocker paradox). (iii) A condition of symmetry on the probability distribution (probability dependent exclusively on the number of 'yes' voters) suffices to avoid some 'paradoxes' (dominance and preference for blocker). (iv) Only for the bicameral postulate does decisiveness do better than success: independence of behavior of the two chambers suffices to prevent the paradox from occurring for the measure of decisiveness.

These results are even more remarkable taking into account that the postulates on which these paradoxes are based were intended for '*a priori*' measures of power that disregard any information about the voters' behavior. As a result this test yields some conclusions about the measures considered here and some conclusions concerning the paradoxes/postulates.

A general conclusion concerning these measures of success and decisiveness is that their conceptual coherence challenges these so-called paradoxes. In all cases in which a 'paradox' may occur, the situation can be explained in clear and simple terms consistent with real-world experience, so that the paradoxes dissipate as such. A side result of this analysis is to underline the difference in behavior between measures of success and decisiveness. This difference in behavior permits us to stress the distinction between these two notions, a fact that has perhaps been overlooked in the literature (see however Brams and Lake (1978), Barry (1980) or Straffin et al. (1981)). In our view too little attention has been paid to the measure of success, which is possibly more important than decisiveness from the point of view of the voters.

As to the postulates whose violation give rise to the paradoxes considered here, this test yields also some conclusions. The ample variety of 'indices' (in a general sense, i.e., maps $\phi : \mathrm{VR}_N \to R^n$),[11] even with completely different meanings (measures of success or decisiveness, *a priori* or not), which satisfy each of these postulates provides twofold conclusions, which are the two faces of a single fact. (i) On the one hand, the '*solidity*' of the postulates in general: they are not totally arbitrary requirements. (ii) On the other hand, the *weakness* of these postulates, which are at the bases of this solidity: they are undemanding. Thus, although they were intended for *a priori* measures of power, it turns out that these measures of power (even of success) meet them always or in many cases. As a consequence, their lack of filtering or selecting power is the most obvious conclusion: only the bicameral postulate has some discriminating power in favor of decisiveness and beyond semivalues. The reformulation of these paradoxes/postulates within our setting has also disclosed some internal difficulties in the formulation of some of them within the traditional setting. The rigidity of a setting in which the voting rule is the only input on which to found a notion of power, forces the inconsistency of formalizing as a change of voting rule something that, according to the interpretation (perceptible even in the denomination of some paradoxes in the classical setting: ‘donation’, ‘bloc’, ‘quarrel’), is actually a change of behavior.

Finally, the lack of justification for speaking of ‘voting power paradoxes’ anymore, at least in connection with those considered here,[12] seems a clear outcome, beyond the ‘deeper insight into the true nature of voting power’ (Felsenthal & Machover, 1998, p. 276) their discussion helps to gain. Some authors seem to endorse the use of postulates/paradoxes to select the ‘best’ power measure. The problem then is: among which measures? Testing measures of insufficiently specified notions by imposing ‘postulates’ seems to us a very dubious (not to say metaphysical) methodology. In fact, the results presented in this paper show that such a methodology may only apparently work when only a few disperse and heterogeneous notions to be found in the literature under the name of ‘power indices’ are submitted to the test. It is our humble opinion that before rushing to raise expectations about how the measure of something should behave, it is wisest to obtain an in-depth understanding and consistent formulation of what one is talking about. In this case ‘power’, a notion whose complexity[13] is its only obvious feature.

Notes

1. See Freixas and Zwicker (2003) for a more general notion of a voting rule that admits vote configurations with ‘different levels of approval’.
2. As is well known a voting rule W can also be represented by the simple game $v : 2^N \to R$, such that $v(S) = 1$ if $S \in W$, and $v(S) = 0$ if $S \notin W$. But we prefer this presentation because strictly speaking the specification of a voting rule does not involve the voters.
3. The term ‘success’ is due to Barry (1980), but these notions can be traced back under different names at least to Rae (1969) (see also Brams & Lake, 1978; Straffin et al., 1981).
4. Coleman’s (1971) power to initiate and to prevent action can also be seen as probabilities of being decisive, while the Deegan and Packel (1978), Johnston (1978) and Holler and Packel (1983) indices cannot. For details, see Laruelle and Valenciano (2005).
5. We use the term ‘paradox,’ common in the voting power literature to refer to the violation of some property considered desirable for an *a priori* measure, but we do not attach to it any positive nor negative value, we just study the conditions and explanation of their occurrence. In fact, the absence of anything paradoxical in case of their occurrence in this setting is one of the obvious outcomes of this study.
6. In our setting, in addition to the voting rule, voters behavior (in probabilistic terms) enters the picture, thus a ‘behaviorial’ formulation is possible for some paradoxes, such as the ‘bloc paradox’ considered in the next subsection, which is a generalization of the particular case of the donation paradox in which all the weight is transferred from one voter to another.
7. A ‘null seat’ in a voting rule is a seat such that the result of a vote is never influenced by the vote cast from that seat. That is, i is a null seat in rule W, if $S \in W \Leftrightarrow S \setminus i \in W$.
8. Similar doubts were already raised by Straffin (1982) and Felsenthal and Machover (1998) concerning the paradox of quarrelling members in its original formulation.

9. A weaker bicameral paradox (or violation of a stronger principle) occurs when the ratio of power between any two voters is not the same in the bicameral rule and in one of the two chambers.
10. Under the same assumptions, decisiveness will never display the *weak* bicameral paradox alluded to in footnote (8), while success may display it.
11. Recall that for each p, $\Omega(-, p)$ and $\Phi(-, p)$ are two such maps, and when p is symmetric $\Phi(-, p)$ is a semivalue, which satisfies all postulates but the bicameral one. Still Saari and Sieberg (2000) present as paradoxical the fact that different semivalues (considered as a general notion of power index) may generate different rankings of the players in the same game. But when considered from the point of view provided by the model based on two inputs, rule and behavior, only misunderstanding can account for expecting otherwise. By now it is clear that behavior influences decisiveness, even for the highly symmetrical kind of behavior represented by semivalues. In fact the different rankings provided by the two most popular semivalues, the Shapley–Shubik and Banzhaf indices in many cases have long been well known. Laruelle and Merlin (2002) obtain similar results, but show that all semivalues rank voters identically in any weighted majority rule.
12. Nevertheless, other notions of power, rooted in noncooperative game theory, still have some paradoxical aspects. See, e.g., Farquharson (1969) and Brams (1994).
13. Harsanyi (1962) adds two to the five ‘constituents of the power relation’ already distinguished by Dahl (1957).

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