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The structure of production technologies with ratio inputs and outputs

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Abstract
Applications of efficiency and productivity analysis in which some inputs and outputs are given in the form of percentages, averages and other types of ratio measures are sufficiently common in the literature. In two recent papers, the authors developed the variable and constant returns-to-scale technologies with both volume and ratio types of inputs and outputs, referred to as the R-VRS and R-CRS technologies. These technologies are generally nonconvex and have a complex structure. In this paper we explore this in detail. We show that the R-VRS technology can be stated as the union of a finite number of specially constructed standard VRS technologies. Similarly, the R-CRS technology in which all ratio inputs and outputs are of the fixed type, which are typically used to represent environmental and quality factors, can be stated as the union of a finite number of partial polyhedral cones. We show that these results have important theoretical, including conceptual, implications.

JEL C14 · C61 · C67 · D24

Keywords Data envelopment analysis · Production technology · Polyhedral technology · Ratio inputs and outputs

1 Introduction

In the literature on data envelopment analysis (DEA), it has long been realized that the standard constant and variable returns-to-scale (CRS and VRS) production technologies of Banker (1984) and Charnes et al. (1978) are not suitable for the situations in which some inputs and outputs are represented by ratio measures—see, e.g., Dyson et al. (2001). Ratio inputs and outputs typically represent ratios of volume measures expressed in the form of percentages and averages. The reason for this is that, in contrast with volume measures, the use of ratio inputs and outputs is generally inconsistent with the axiom of convexity assumed in both VRS and CRS technologies. In the case of CRS, the use of ratio measures generally violates the additional assumption that decision making units (DMUs) can be proportionally scaled by a nonnegative scaling factor.

To illustrate this problem, consider the following example.

Example 1 Suppose that, in the assessment of efficiency of schools, output y_1 represents all students and output y_2 represents students with good academic achievements in the school-leaving exams. Let the additional output y_3 represent the percentage of students with good academic achievements, calculated based on the values of the volume outputs y_1 and y_2.

Table 1 shows the above three outputs for the two observed schools A and B. Consider only the volume outputs y_1 and y_2 and assume that schools A and B operate in either VRS or CRS technology. Then any convex combination of these schools is also in the corresponding technology. For example, the simple average of schools A and B, shown as the hypothetical (unobserved) school C in Table 1, is included in both technologies. The correct percentage of students with good academic achievements is 45/150, or 30%.

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Table 1 Schools in Example 1

School	Output y_1	Output y_2	Percentage y_3
A	200	20	10
B	100	70	70
$C = 0.5A + 0.5B$	150	45	40 (incorrect), 30 (correct)

Note that if we simply include the ratio output y_3 in the VRS or CRS technology, then, by averaging this output of schools A and B, we obtain an overestimated, and hence unsubstantiated, value of 40%.

This example shows that, if we include ratio outputs (and, similarly, inputs) in the standard VRS or CRS technology, then such measures in the hypothetical DMUs will generally be incorrect.

A special case arises in which the analyst has access to the volume measures used in the numerator and denominator of the ratio inputs and outputs. For this scenario, Emrouznejad and Amin (2009) developed an approach in which the assumption of convexity applies separately to the known numerators and denominators used in the ratio measures. However, this approach cannot be employed when the volume measures used as the numerator and denominator of the ratio input or output are unavailable.

In two recent papers, Olesen et al. (2015, 2017) developed two different models of technology, referred to as the ratio-VRS (R-VRS) and ratio-CRS (R-CRS) technologies, in which ratio inputs and outputs are used "as is", without any decomposition to the underlying volume numerator and denominator. Both technologies satisfy a generalized variant of the standard axiom of convexity known as *selective convexity* (Podinovski 2005). The R-VRS and R-CRS technologies allow conventional convex combinations of the volume inputs and outputs, while at the same time taking the ratio inputs at their maximal level, and ratio outputs at the minimal level, across all combined DMUs.

The R-CRS technology also uses a generalization of the standard axiom of scalability, referred to as "ray unboundedness" by Banker et al. (1984), according to which the volume inputs and outputs are scaled with a nonnegative parameter in the conventional way, while the treatment of ratio measures is different and depends on their assumed type. Of particular practical significance is the R-CRS technology in which all ratio inputs and outputs are of the fixed type. Such technology is suitable in situations in which ratio inputs and outputs represent environmental and other exogenously fixed factors (e.g., in applications to school performance, these could be the proportion of households with high income or percentage of students with special needs) or quality of products of goods and services (e.g., the percentage of school leavers achieving good results in exams or success rate of treatments in a hospital). In all described situations, the R-CRS model with fixed ratio inputs and outputs assumes that it is possible to scale the volume inputs and outputs while keeping the environmental and quality measures constant.[1]

Although the R-VRS and R-CRS technologies are based on transparent sets of axioms, their explicit mathematical statements are less intuitive. Olesen et al. (2015, 2017) establish several properties of these technologies and the models assessing the efficiency of DMUs based on them. However, the overall geometric structure of the R-VRS and R-CRS technologies has so far remained unclear. This in turn made some theoretical results (for example, the fact that both technologies are closed sets, which is one of the basic assumptions in production theory) formally correct but not sufficiently intuitive.

In this paper, we explore the structure of the R-VRS and R-CRS technologies, the latter assuming that all ratio measures are of the fixed type. We prove that both technologies are the unions of a finite number of polyhedral sets. More precisely, in the case of R-VRS, these sets are the standard VRS technologies generated by different subsets of observed DMUs whose ratio inputs and outputs are modified in a particular way. In the case of R-CRS, these polyhedral sets are the partial cones obtained by scaling only volume measures of the similarly modified observed DMUs.

We show that the new results obtained in this paper have various implications. In particular, these results should be useful for the exploration of the R-VRS and R-CRS technologies and their efficient frontiers. This includes the issues of classification of efficient DMUs, and sensitivity and stability of efficiency measures. The representations of the R-VRS and R-CRS technologies obtained in this paper are also useful as theoretical foundations for the correct definition of various frontier characteristics such as the most productive scale size, marginal rates and local returns to scale (RTS).

We proceed as follows. In Section 2, we introduce basic terminology and notation. In Section 3, we discuss production assumptions on which the R-VRS and R-CRS technologies are based and give statements of these technologies. In Sections 4 and 5 we obtain the main results that reveal the structure of the R-VRS and R-CRS technologies

[1] The incorporation of exogenous and quality-related measures in efficiency analysis has been extensively discussed in the literature, and different approaches have been developed based on different sets of deterministic and probabilistic assumptions. Examples include models with fixed exogenous factors of Banker & Morey (1986), the model of production technology conditional on the level of the exogenous input of Ruggiero (1996) and the conditional probabilistic approach of Daraio and Simar (2005).

as finite unions of polyhedral sets. In Section 6, we outline some theoretical implications of our results. In Section 7, we provide concluding remarks. The proofs of all mathematical statements are given in the Appendix.

2 Preliminaries

Let $\mathrm{I} = \{1, \ldots, m\}$ and $\mathrm{O} = \{1, \ldots, s\}$ be, respectively, the sets of nonnegative inputs and outputs of technology $T \subset \mathbb{R}^m_+ \times \mathbb{R}^s_+$. The subsets $\mathrm{I}^V \subseteq \mathrm{I}$ and $\mathrm{O}^V \subseteq \mathrm{O}$ include inputs and outputs given by volume measures. Their complementary subsets $\mathrm{I}^R = \mathrm{I} \backslash \mathrm{I}^V$ and $\mathrm{O}^R = \mathrm{O} \backslash \mathrm{O}^V$ include ratio inputs and outputs. We assume that there is at least one input and at least one output, i.e., that the sets $\mathrm{I} \neq \varnothing$ and $\mathrm{O} \neq \varnothing$, but allow any of their subsets of volume or ratio measures to be empty.

In line with the given notation, we define a decision making unit (DMU) as a member of technology T stated as follows:

$$(X, Y) = (X^V, X^R, Y^V, Y^R),$$

where $X \in \mathbb{R}^m_+$ and $Y \in \mathbb{R}^s_+$ are the vectors of inputs and outputs, respectively. The subvectors X^V, X^R, Y^V and Y^R consist of the inputs and outputs in the corresponding subsets I^V, I^R, O^V and O^R.

Following Olesen et al. (2015), we allow a situation in which some or all ratio inputs and outputs have specified upper bounds, for example, 1 or 100%. Such bounds are optional and, as shown by Olesen et al. (2015), in most (but not all) cases, their specification does not affect the input or output radial efficiency of the DMUs. However, for a rigorous definition of technology, such bounds need to be taken into account. We state the bounds as follows:

$$X^R \leq \overline{X}^R \quad \text{and} \quad Y^R \leq \overline{Y}^R. \tag{1}$$

Each component of vectors $\overline{X}^R$ and $\overline{Y}^R$ can be finite or formally taken as $+\infty$. The latter case is equivalent to the situation in which no upper bound is specified on the respective ratio input or output.[2]

Suppose that there are n observed DMUs (X_j, Y_j), where $j \in J = \{1, \ldots, n\}$. We assume that each observed DMU has at least one strictly positive input and output, i.e., X_j and Y_j are nonzero vectors, and that the ratio inputs and outputs of these vectors are within the bounds specified by the inequalities (1), for all $j \in J$.

[2] In this paper, vector inequalities and equalities mean that the specified relation is true on the component-wise basis, e.g., $X^R \leq \overline{X}^R$ means that $X_i^R \leq \overline{X}_i^R$, for all $i \in \mathrm{I}^R$.

3 Technologies with ratio inputs and outputs

Olesen et al. (2015) use the axiomatic approach for the definition of the R-VRS and R-CRS technologies. This approach defines a production technology based on the explicitly stated production assumptions that this technology is deemed to satisfy (Banker et al. 1984, Färe et al. 1985).

3.1 The R-VRS technology

Olesen et al. (2015) state the following three axioms of a production technology T with volume and ratio inputs.

Axiom 1 (*Feasibility of observed data*). For any $j \in J$, $(X_j, Y_j) \in T$.

Axiom 2 (*Free disposability*). Let $(X, Y) \in T$. Consider any $(\tilde{X}, \tilde{Y}) = (\tilde{X}^V, \tilde{X}^R, \tilde{Y}^V, \tilde{Y}^R) \in \mathbb{R}^m_+ \times \mathbb{R}^s_+$ such that $\tilde{Y} \leq Y$, $\tilde{X} \geq X$, $\tilde{X}^R \leq \overline{X}^R$ and $\tilde{Y}^R \leq \overline{Y}^R$, where $\overline{X}^R$ and $\overline{Y}^R$ are the vectors of upper input and output bounds from (1). Then $(\tilde{X}, \tilde{Y}) \in T$.

Axiom 3 (*Selective convexity*). Let $(\tilde{X}, \tilde{Y}) \in T$ and $(\hat{X}, \hat{Y}) \in T$. Assume that $\tilde{X}^R = \hat{X}^R$ and $\tilde{Y}^R = \hat{Y}^R$. Then $\gamma(\tilde{X}, \tilde{Y}) + (1 - \gamma)(\hat{X}, \hat{Y}) \in T$, for any $\gamma \in [0, 1]$.

Note that Axiom 2 is a simple variation of the standard axiom of free (or strong) disposability of all inputs and outputs, adjusted for the requirement that the ratio inputs and outputs must be within the bounds (1).

Axiom 3 of selective convexity was first stated in a more general setting in Podinovski (2005). This axiom allows convex combinations of volume inputs and outputs of DMUs $(\tilde{X}, \tilde{Y})$ and $(\hat{X}, \hat{Y})$ provided their ratio inputs and outputs are identical. Furthermore, as proved by Podinovski (2005), Axiom 3, taken together with Axiom 2 of free disposability, allows convex combinations of volume inputs and outputs of any DMUs $(\tilde{X}, \tilde{Y})$ and $(\hat{X}, \hat{Y})$, even if their ratio inputs and outputs are different. In this case, each ratio input of the resulting DMU is taken as the *maximum* of this input among the two combined DMUs, and the ratio output is taken equal to the *minimum* of this output among the two combined DMUs. Olesen et al. (2017) call the resulting DMU obtained in this way the *ratio-convex* (R-convex) combination of the two DMUs. Therefore, if Axiom 2 is assumed, Axiom 3 can be replaced by the equivalent in this case assumption that all R-convex combinations of any two DMUs $(\tilde{X}, \tilde{Y}) \in T$ and $(\hat{X}, \hat{Y}) \in T$ are also in T.

Applying the minimum extrapolation principle first used in DEA by Banker et al. (1984), Olesen et al. (2015) define the R-VRS technology as follows.

Definition 1 The R-VRS technology $T^{\mathrm{R}}_{\mathrm{VRS}}$ is the intersection of all technologies $T \subset \mathbb{R}^m_+ \times \mathbb{R}^s_+$ that satisfy Axioms 1–3.

Olesen et al. (2015) further obtain an operational statement of the R-VRS technology.[3] Namely, technology $T^{\mathrm{R}}_{\mathrm{VRS}}$ is the set of all DMUs $(X, Y) \in \mathbb{R}^m_+ \times \mathbb{R}^s_+$ for which there exists a vector $\lambda \in \mathbb{R}^n$ such that the following conditions are true:

$$\sum_{j\in J} \lambda_j Y^V_j \geq Y^V, \tag{2a}$$

$$\sum_{j\in J} \lambda_j X^V_j \leq X^V, \tag{2b}$$

$$\lambda_j \left(Y^R_j - Y^R \right) \geq \mathbf{0}, \quad \forall j \in J, \tag{2c}$$

$$\lambda_j \left(X^R_j - X^R \right) \leq \mathbf{0}, \quad \forall j \in J, \tag{2d}$$

$$\sum_{j\in J} \lambda_j = 1, \tag{2e}$$

$$X^R \leq \overline{X}^R, \tag{2f}$$

$$Y^R \leq \overline{Y}^R, \tag{2g}$$

$$\lambda \geq \mathbf{0}. \tag{2h}$$

To see the meaning of the above conditions, note that the vectors X^V_j and Y^V_j of volume inputs and outputs of the observed DMUs enter convex combinations in conditions (2a) and (2b) with the nonnegative weights λ_j, for all $j \in J$. Furthermore, if $\lambda_j > 0$, the inequalities (2c) imply $Y^R_j \geq Y^R$, and the inequalities (2d) imply $X^R_j \leq X^R$. Therefore, conditions (2c) and (2d) mean that the observed DMUs (X_j, Y_j) that enter convex combinations with a positive λ_j are not worse than the DMU (X, Y) on all ratio inputs and outputs.

Remark 1 Two special cases of technology $T^{\mathrm{R}}_{\mathrm{VRS}}$ are worth highlighting. First, assume that there are no ratio inputs and outputs. Then conditions (2c), (2d), (2f) and (2g) are omitted and the resulting statement (2a)–(2h) defines the conventional VRS technology of Banker et al. (1984). Alternatively, suppose that there are no volume measures and all inputs and outputs are ratios. In this case conditions (2a) and (2b) are removed. It is straightforward to prove that the remaining conditions (2c)–(2h) define free disposal hull of Deprins et al. (1984), with the additional upper limits on the inputs and outputs (2f) and (2g).

[3] In the mathematical statements in this paper, we use bold notation $\mathbf{0}$ and $\mathbf{1}$ for the vectors of zeros and ones, respectively. The dimensions of both vectors are clear from the context in which they are used. The superscript $^\top$ means transposition.

3.2 The R-CRS technology with fixed ratio inputs and outputs

The idea of the R-CRS technology of Olesen et al. (2015) is to allow the scaling of the *volume* inputs and outputs of any DMU by any factor $\alpha \geq 0$, while allowing the ratio inputs and outputs to change (or remain constant) with respect to such scaling. Olesen et al. (2015) identify four different types of ratio measures, including the fixed and proportional types of such measures. For example, as the name suggests, the inputs and outputs of the proportional type are scaled in the same proportion α as the ratio measures. However, it is the fixed type that appears to be the most common in applications.

The fixed ratio inputs and outputs typically represent either environmental or other exogenously fixed measures that do not change as the volume of operations represented by the volume inputs and outputs is scaled up or down. For example, we would often consider ratio inputs and outputs representing the socio-economic environment, such as the proportion of population with higher socio-economic status or percentage of population above certain age, as fixed ratio measures. Similarly, fixed ratio measures may represent the quality of the inputs or outputs that are not assumed to change with the change of the volume, and may be discretionary or non-discretionary. Examples include the average attainment or percentage of pupils with good grades on entry to, and on exit from, a secondary school. We would normally assume that these quality inputs and outputs remain constant while the physical volume of operations of the school, including the number of teachers, expenses and pupils, could be scaled up or down.

The results obtained in this paper concern only the R-CRS technology with the fixed type of ratio inputs and outputs. This technology is defined by Axioms 1–3 stated above and the additional axiom that allows the scaling of volume measures while keeping all ratio inputs and outputs constant. To state this assumption formally, we use a variant of the axioms of scaling employed by Olesen et al. (2015) in which, out of the four types of ratio measures, we leave only the fixed type.[4]

Axiom 4 (*Scaling with fixed ratio inputs and outputs*). Let $(X, Y) = (X^V, X^R, Y^V, Y^R) \in T$. Then, for all $\alpha \geq 0$, the DMU $(\alpha X^V, X^R, \alpha Y^V, Y^R) \in T$.

The R-CRS technology $T^{\mathrm{R}}_{\mathrm{CRS}}$ with fixed ratio inputs and outputs is denoted $T^{\mathrm{F}}_{\mathrm{CRS}}$. Using the minimum extrapolation principle, it is defined as follows. (This is a special case of the general definition of the R-CRS technology given by Olesen et al. (2015) which allows specification of different types of ratio inputs and outputs).

[4] Axiom 4 corresponds to Axioms 4 and 5 (Olesen et al. 2015) in which all not fixed types of ratio measures (proportional, downward-proportional and upward-proportional) are ignored.

Definition 2 The R-CRS technology $T^{\mathrm{F}}_{\mathrm{CRS}}$ with fixed ratio inputs and outputs is the intersection of all technologies $T \subset \mathbb{R}^m_+ \times \mathbb{R}^s_+$ that satisfy Axioms 1–4.

Under a mild additional assumption that $\mathrm{I}^V \neq \emptyset$ and $X^V_j \neq \mathbf{0}$, for all $j \in J$, which should clearly be true in all meaningful applications in which CRS is assumed, Olesen et al. (2015) prove that technology $T^{\mathrm{F}}_{\mathrm{CRS}}$ is the set of all DMUs $(X, Y) \in \mathbb{R}^m_+ \times \mathbb{R}^s_+$ for which there exist vectors $\lambda, \sigma \in \mathbb{R}^n$ such that

$$\sum_{j\in J} \lambda_j \sigma_j Y^V_j \geq Y^V, \tag{3a}$$

$$\sum_{j\in J} \lambda_j \sigma_j X^V_j \leq X^V, \tag{3b}$$

$$\lambda_j \left(Y^R_j - Y^R \right) \geq \mathbf{0}, \quad \forall j \in J, \tag{3c}$$

$$\lambda_j \left(X^R_j - X^R \right) \leq \mathbf{0}, \quad \forall j \in J, \tag{3d}$$

$$\sum_{j\in J} \lambda_j = 1, \tag{3e}$$

$$X^R \leq \overline{X}^R, \tag{3f}$$

$$Y^R \leq \overline{Y}^R, \tag{3g}$$

$$\lambda, \sigma \geq \mathbf{0}. \tag{3h}$$

The meaning of the above conditions is clear. The inequalities (3a) and (3b) show that the multipliers σ_j, $j \in J$, are the scaling factors applied to the vectors of volume inputs and outputs X^V_j and Y^V_j of the observed DMUs. The resulting scaled vectors are subsequently combined in convex combinations using the components of vector λ as the weighting coefficients. As in the case of the R-VRS technology, conditions (3c) and (3d) mean that the observed scaled DMUs $(\sigma_j X^V_j, X^R_j, \sigma_j Y^V_J, X^R_j)$ whose volume inputs and outputs enter the convex combinations in inequalities (3a) and (3b) with a *positive* weight λ_j are not worse than the DMU (X, Y) on all ratio inputs and outputs.

Remark 2 To see the relationship of statement (3a)–(3h) to the standard statement of the CRS technology of Charnes et al. (1978), assume that there are no ratio inputs and outputs. In this case, all inequalities (3c), (3d), (3f) and (3g) are removed, and the resulting statement is an alternative equivalent statement of the standard CRS technology.[5] In contrast with the conventional statement of the CRS technology in which the single vector λ is used to describe all conical combinations of the observed DMUs, in the resulting statement (3a)–(3h), components of vector σ are first used to scale observed DMUs, and a separate vector λ is used to obtain convex combinations of the scaled DMUs.

For practical computations, Olesen et al. (2015) obtain two further partly linearized variants of the above statement. In particular, assuming that $Y^V \neq \mathbf{0}$, they prove that a DMU $(X, Y) = (X^V, X^R, Y^V, Y^R)$ is in technology $T^{\mathrm{F}}_{\mathrm{CRS}}$ if and only if DMU (X, Y) satisfies all conditions (2a)–(2h) with the exception of equality (2e), with some vector λ.[6] The assumption that $Y^V \neq \mathbf{0}$ should be satisfied in all applications of practical interest, in which case we can use simpler conditions (2a)–(2h) without equality (2e). However, in order to explore the structure of technology $T^{\mathrm{F}}_{\mathrm{CRS}}$ in the most general case, without any simplifying assumptions, below we rely on its statement (3a)–(3h).

4 The structure of the R-VRS technology

To simplify the main idea of the exposition, let us first consider the case in which the bounds (1) are not specified or are infinite and for this reason are not stated. (We remove this simplifying assumption afterwards.) In this case, the inequalities (2f) and (2g) are removed from the statement (2a)–(2h) of technology $T^{\mathrm{R}}_{\mathrm{VRS}}$.

Consider any non-empty subset J' of the set of observed DMUs, i.e., let $J' \subseteq J$, $J' \neq \emptyset$. The set J' may contain one or several observed DMUs.

Define the vector $X^{R^*}(J')$ of dimension $|\mathrm{I}^R|$ as the component-wise *maximum* of the vectors X^R_j, $j \in J'$. Similarly, define the vector $Y^{R^*}(J')$ of dimension $|\mathrm{O}^R|$ as the component-wise *minimum* of the vectors Y^R_j, $j \in J'$. Formally, we have:

$$\begin{aligned} X^{R^*}_i(J') &= \max_{j\in J'} \left\{ X^R_{ji} \right\}, \ \forall i \in \mathrm{I}^R, \\ Y^{R^*}_r(J') &= \min_{j\in J'} \left\{ Y^R_{jr} \right\}, \ \forall r \in \mathrm{O}^R. \end{aligned} \tag{4}$$

We now modify the observed DMUs $(X_j, Y_j) = (X^V_j, X^R_j, Y^V_j, Y^R_j)$, $j \in J'$, by replacing their subvectors of ratio inputs and outputs by the vectors $X^{R^*}(J')$

[5] In the described case with no ratio inputs and outputs, the standard statement of the CRS technology is obtained from (3a)–(3h) by the substitution $\hat{\lambda}_j = \lambda_j \sigma_j$, for all $j \in J$, in which case the normalizing equality (3e) becomes redundant and is also omitted.

[6] If we use statement (2a)–(2h) without condition (2e), the assumption $Y^V \neq \mathbf{0}$ implies $\lambda \neq \mathbf{0}$. Without the latter condition, statement (2a)–(2h) does not put any limits on the subvectors of ratio inputs and outputs X^R and Y^R of the DMU (X, Y), apart from the bounds (2f) and (2g). As a result, this allows DMUs (X, Y), for which $Y^V = \mathbf{0}$, that are not in technology $T^{\mathrm{F}}_{\mathrm{CRS}}$, for example, with the levels of ratio outputs higher than those of any observed DMUs.

and $Y^{R^*}(J')$, respectively. The modified DMUs become

$$(X'_j, Y'_j) = \left(X_j^V, X^{R^*}(J'), Y_j^V, Y^{R^*}(J')\right), \quad j \in J'. \tag{5}$$

Consider the standard VRS technology $T_{\mathrm{VRS}}(J')$ generated by the set of modified DMUs defined by (5). Although such technology does not differentiate between volume and ratio measures and treats all of them as volume measures, for clarity, we state the input and output conditions separately for the volume and ratio input and output subvectors. Because all modified DMUs in the set J' as stated by (5) have the same subvectors of ratio inputs and outputs, any convex combination of these DMUs will also have the subvectors of ratio inputs and outputs equal to the vectors $X^{R^*}(J')$ and $Y^{R^*}(J')$, respectively.

Technology $T_{\mathrm{VRS}}(J')$ is, therefore, the set of all DMUs $(X, Y) \in \mathbb{R}_+^m \times \mathbb{R}_+^s$ for which there exists a vector $\lambda \in \mathbb{R}^{|J'|}$ (whose dimension is equal to the number of observed DMUs in the set J' and whose components λ_j correspond to the observed DMUs (X_j, Y_j), $j \in J'$) such that

$$\sum_{j \in J'} \lambda_j Y_j^V \geq Y^V, \tag{6a}$$

$$Y^{R^*}(J') = Y^{R^*}(J') \sum_{j \in J'} \lambda_j = \sum_{j \in J'} \lambda_j Y^{R^*}(J') \geq Y^R, \tag{6b}$$

$$\sum_{j \in J'} \lambda_j X_j^V \leq X^V, \tag{6c}$$

$$X^{R^*}(J') = X^{R^*}(J') \sum_{j \in J'} \lambda_j = \sum_{j \in J'} \lambda_j X^{R^*}(J') \leq X^R, \tag{6d}$$

$$\sum_{j \in J'} \lambda_j = 1, \tag{6e}$$

$$\lambda \geq \mathbf{0}. \tag{6f}$$

Technology $T_{\mathrm{VRS}}(J')$ is defined for each of the $2^n - 1$ non-empty subsets $J' \subseteq J$. Our next main result shows that, if the bounds (1) are not specified, the R-VRS technology $T_{\mathrm{VRS}}^{\mathrm{R}}$ generated by the set of observed DMUs (X_j, Y_j), $j \in J$, is the union of the $2^n - 1$ *standard* VRS technologies $T_{\mathrm{VRS}}(J')$ generated by the modified DMUs (X'_j, Y'_j), $j \in J'$, where each VRS technology $T_{\mathrm{VRS}}(J')$ is stated by conditions (6a)–(6f).

Let $\mathcal{J}$ denote the set of all *non-empty* subsets of J, i.e., $\mathcal{J} = 2^J \setminus \{\varnothing\}$, where 2^J is the power set of J. Note that there are $2^n - 1$ elements of $\mathcal{J}$.

Theorem 1 *Suppose that the bounds (1) are not specified. Then*

$$T_{\mathrm{VRS}}^{\mathrm{R}} = \bigcup_{J' \in \mathcal{J}} T_{\mathrm{VRS}}(J'). \tag{7}$$

It is now easy to modify the statement of Theorem 1 for the general case in which the bounds (1) are specified. First, introduce the set $\mathcal{B}$ of all points in $\mathbb{R}^{m+s}$ that satisfy (1):

$$\mathcal{B} = \left\{(X, Y) = (X^V, X^R, Y^V, Y^R) \in \mathbb{R}^{m+s} | X^R \leq \overline{X}^R, Y^R \leq \overline{Y}^R\right\}. \tag{8}$$

We now have the following general result.

Corollary 1 *Let the bounds (1) be specified. Consider the R-VRS technology $T_{\mathrm{VRS}}^{\mathrm{R}}$ stated by conditions (2a)–(2h). Then*

$$T_{\mathrm{VRS}}^{\mathrm{R}} = \left(\bigcup_{J' \in \mathcal{J}} T_{\mathrm{VRS}}(J')\right) \cap \mathcal{B} = \bigcup_{J' \in \mathcal{J}} (T_{\mathrm{VRS}}(J') \cap \mathcal{B}).$$

We now note that each VRS technology $T_{\mathrm{VRS}}(J')$ and, therefore, technology $T_{\mathrm{VRS}}(J') \cap \mathcal{B}$ is a polyhedral set. Therefore, we have the following result:

Corollary 2 *Technology $T_{\mathrm{VRS}}^{\mathrm{R}}$ is the union of a finite number (more precisely, $2^n - 1$) of polyhedral (and therefore closed convex) sets.*

The following two examples illustrate Theorem 1.

Example 2 For illustration, we reuse the example given by Olesen et al. (2015). Figure 1 shows the R-VRS technology $T_{\mathrm{VRS}}^{\mathrm{R}}$ with a single volume input X^V, volume output Y^V and a ratio output Y^R generated by two observed DMUs A and B. This technology includes the DMUs located on and below the unbounded surfaces $PBKL$ and $HACG$. (Both surfaces are unbounded on the right and allow an unlimited increase of the input.)

We note that technology $T_{\mathrm{VRS}}^{\mathrm{R}}$ is not convex. In particular, it does not include convex combinations of the observed DMUs A and B located on the line segment

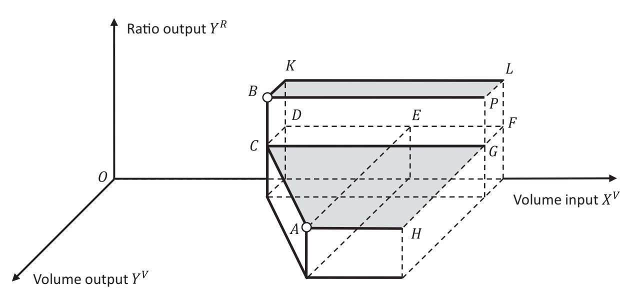

Fig. 1 The R-VRS technology in Example 2

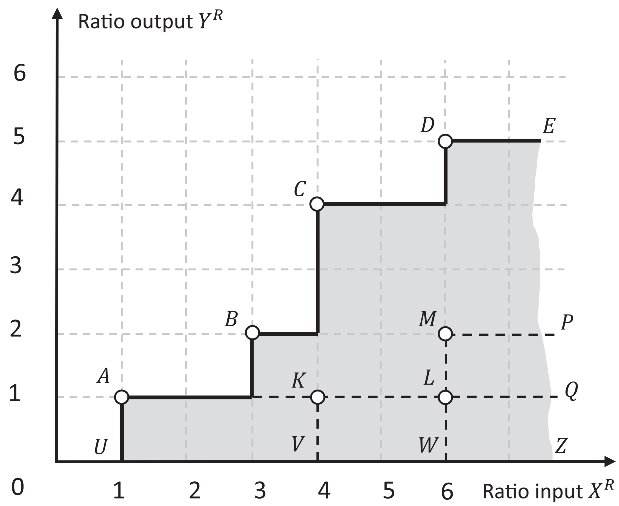

Fig. 2 The FDH technology in Example 3

joining them. However, according to Axiom 2 of free disposability, DMU C is included in $T^{\mathrm{R}}_{\mathrm{VRS}}$. Because the ratio output Y^R of DMUs A and C is equal, by Axiom 3 of selective convexity, these DMUs can be combined in convex combinations, and the line segment AC is included in technology $T^{\mathrm{R}}_{\mathrm{VRS}}$. We further note that, although technology $T^{\mathrm{R}}_{\mathrm{VRS}}$ is not convex, the section of this technology corresponding to any level of the ratio output Y^R is convex. For example, the section $HACDF$ is convex.

Let us now illustrate the statement of Theorem 1. The set of all observed DMUs is $J = \{A, B\}$. The set $\mathcal{J}$ of all non-empty subsets of J contains three elements:

$$\mathcal{J} = \{\{A\}, \{B\}, \{A, B\}\}. \tag{9}$$

For the single-element sets $J' \in \mathcal{J}$, i.e., for $J' = \{A\}$ and $J' = \{B\}$, the modified DMUs (X'_j, Y'_j) defined by (5) coincide with the observed DMUs A and B, respectively. We now have the following observations.

In the case $J' = \{A\}$, technology $T_{\mathrm{VRS}}(J')$ is the standard VRS technology generated by DMU A. It consists of all DMUs located on and below the surface $HAEF$. Similarly, in the case $J' = \{B\}$, $T_{\mathrm{VRS}}(J')$ is the standard VRS technology generated by DMU B. It consists of all DMUs located on and below the surface $PBKL$.

Now let $J' = \{A, B\}$. According to (5), we first modify the observed DMUs A and B by replacing their ratio output by its minimum across the observed DMUs. In our example, DMU A has the lowest ratio output and remains unchanged, and DMU B is changed to DMU C. Therefore, in this case, $T_{\mathrm{VRS}}(J')$ is the standard VRS technology generated by DMUs A and C. It consists of all DMUs located on and below the surface $HACDF$. (It is worth noting that, in this case, technology $T_{\mathrm{VRS}}(J')$ for $J' = \{A, B\}$ includes technology $T_{\mathrm{VRS}}(J')$ for $J' = \{A\}$ as a subset.)

In line with Theorem 1, the union of the three VRS technologies $T_{\mathrm{VRS}}(J')$, where J' is taken as $\{A\}$, $\{B\}$ and $\{A, B\}$, coincides with the R-VRS technology $T^{\mathrm{R}}_{\mathrm{VRS}}$.

Example 3 As a further illustration to Theorem 1, consider the R-VRS technology with a single ratio input X^R and a single ratio output Y^R generated by the observed DMUs A, B, C and D shown in Fig. 2. We assume that no bounds (1) are specified.

In this case, technology $T^{\mathrm{R}}_{\mathrm{VRS}}$ is stated by conditions (2a)–(2h) from which the inequalities (2a), (2b), (2f) and (2g) are removed. As noted by Olesen et al. (2015), because there are no volume measures and no bounds on the ratio measures, the R-VRS technology $T^{\mathrm{R}}_{\mathrm{VRS}}$ coincides with the free disposal hull (FDH) T_{FDH} of Deprins et al. (1984).

It is clear that T_{FDH} is the union of four trivial standard VRS technologies each generated by the single observed DMU A, B, C and D. (For example, the VRS technology generated by DMU A is the area $QAUZ$.)

Note that Theorem 1 remains true in this case as well. Indeed, the set $\mathcal{J}$ consists of $2^4 - 1 = 15$ non-empty subsets of observed DMUs, including the single element sets $\{A\}$, $\{B\}$, $\{C\}$ and $\{D\}$. Furthermore, for any subset $J' \subseteq J$ that consists of more than one observed DMU, the resulting VRS technology $T_{\mathrm{VRS}}(J')$ defined by conditions (6a)–(6f) is a subset of the VRS technology generated by any observed DMU in J'.

For example, consider $J' = \{A, C\}$. After modifying both DMUs according to (5), it is straightforward to verify that the VRS technology $T_{\mathrm{VRS}}(J')$ in this case is the set $QKVZ$. This technology is a subset of the VRS technologies generated by the single DMUs A and C.

Similarly, if $J' = \{B, D\}$, the VRS technology $T_{\mathrm{VRS}}(J')$ is the set $PMWZ$. If J' includes all observed DMUs, i.e., $J' = J = \{A, B, C, D\}$, the VRS technology $T_{\mathrm{VRS}}(J')$ is the set $QLWZ$. The latter is a subset of the VRS technology generated by any of the observed DMUs.

It is now clear that the union of all technologies $T_{\mathrm{VRS}}(J')$, $J' \in \mathcal{J}$, in (7) can be replaced by their union taken over the single element sets J' and omitting any other sets J' consisting of two or more observed DMUs. We note that the same treatment cannot be applied to any R-VRS technology. A counterexample is the R-VRS technology considered in Example 2.

5 The structure of the R-CRS technology with fixed ratio inputs and outputs

In this section we obtain a decomposition of the R-CRS technology $T^{\mathrm{F}}_{\mathrm{CRS}}$ with fixed ratio inputs and outputs into a finite union of polyhedral sets.

As in Section 4, let $\mathcal{J}$ be the set of all non-empty subsets of the set J. For each set $J' \in \mathcal{J}$, we first define the vectors $X^{R^*}(J')$ and $Y^{R^*}(J')$ according to (4). We subsequently define the modified observed DMUs (X'_j, Y'_j) as in (5), by replacing the vectors of ratio inputs and outputs of all observed DMUs (X_j, Y_j), $j \in J'$, by the same vectors $X^{R^*}(J')$ and $Y^{R^*}(J')$, respectively.

Similar to the case of the R-VRS technology considered in Section 4, we first assume that no bounds (1) on the ratio measures are specified. For each $J' \in \mathcal{J}$, define the *partial* cone extension $C(\mathrm{I}^V, \mathrm{O}^V, J')$ of technology $T_{\mathrm{VRS}}(J')$ taken with respect to its volume inputs and outputs only, while keeping the ratio measures unchanged.[7] This partial cone is defined as follows:[8]

$$\begin{aligned} C(\mathrm{I}^V, \mathrm{O}^V, J') \;=\; & \{(X^V, X^R, Y^V, Y^R) \in \mathbb{R}^{m+s} | \\ & \exists (\tilde{X}^V, \tilde{X}^R, \tilde{Y}^V, \tilde{Y}^R) \in T_{\mathrm{VRS}}(J'), \alpha \geq 0 : \\ & (X^V, Y^V) = \alpha(\tilde{X}^V, \tilde{Y}^V), (X^R, Y^R) = (\tilde{X}^R, \tilde{Y}^R)\}. \end{aligned} \tag{10}$$

It is straightforward to show that the partial cone $C(\mathrm{I}^V, \mathrm{O}^V, J')$ is generally not a closed set. Define technology $T_{\mathrm{PCone}}(J')$ as its closure (the subscript "PCone" in this notation stands for "partial cone"):

$$T_{\mathrm{PCone}}(J') = \mathrm{cl}\, C(\mathrm{I}^V, \mathrm{O}^V, J').$$

It can be shown that technology $T_{\mathrm{PCone}}(J')$ is the set of all DMUs $(X, Y) \in \mathbb{R}^m_+ \times \mathbb{R}^s_+$ for which there exists a vector $\lambda \in \mathbb{R}^{|J'|}$ whose components λ_j correspond to the observed DMUs (X_j, Y_j), $j \in J'$, such that

$$\sum_{j \in J'} \lambda_j Y_j^V \geq Y^V, \tag{11a}$$

$$\sum_{j \in J'} \lambda_j X_j^V \leq X^V, \tag{11b}$$

$$Y^{R^*}(J') \geq Y^R, \tag{11c}$$

$$X^{R^*}(J') \leq X^R, \tag{11d}$$

$$\lambda \geq \mathbf{0}. \tag{11e}$$

Let us compare the statement (11a)–(11e) of technology $T_{\mathrm{PCone}}(J')$ with the statement (6a)–(6f) of technology $T_{\mathrm{VRS}}(J')$. As discussed, the latter is the standard VRS technology generated by the modified set of observed DMUs (5). In contrast, the statement (11a)–(11e) of technology $T_{\mathrm{PCone}}(J')$ does not have a normalizing equality (6e) on the vector λ. Its components are used to scale and combine the volume inputs and outputs of the observed DMUs in conditions (11a) and (11b), resulting in their conical combinations. At the same time, the vectors of ratio inputs and outputs $X^{R^*}(J')$ and $Y^{R^*}(J')$ in conditions (11c) and (11d) are fixed and not scaled.

Any DMU $(X, Y) \in T_{\mathrm{VRS}}(J')$ satisfies conditions (6a)–(6f) with some vector λ'. Then DMU (X, Y) satisfies (11a)–(11e) with the same λ' and is in technology $T_{\mathrm{PCone}}(J')$. Therefore, $T_{\mathrm{VRS}}(J') \subset T_{\mathrm{PCone}}(J')$. The omission of the normalizing equality (6e) in the statement (11a)–(11e) defines $T_{\mathrm{PCone}}(J')$ as the closed partial cone extension of the VRS technology $T_{\mathrm{VRS}}(J')$.[9] In this closed partial cone, the scaling applies only to the volume measures, while keeping the vectors of ratio inputs and outputs fixed. Example 4 considered below provides an illustration to this observation.

The next result shows that the R-CRS technology $T^{\mathrm{F}}_{\mathrm{CRS}}$ generated by the set of observed DMUs (X_j, Y_j), $j \in J$, is the union of the $2^n - 1$ closed partial cone technologies $T_{\mathrm{PCone}}(J')$ generated by the modified DMUs (X'_j, Y'_j), $j \in J'$, and stated by conditions (11a)–(11e).

Theorem 2 *Suppose that the bounds (1) are not specified. Then*

$$T^{\mathrm{F}}_{\mathrm{CRS}} = \bigcup_{J' \in \mathcal{J}} T_{\mathrm{PCone}}(J').$$

Let us now consider the general case in which the bounds (1) are specified. Define the set $\mathcal{B}$ as in (8).

Corollary 3 *Let the bounds (1) be specified. Consider the R-CRS technology $T^{\mathrm{F}}_{\mathrm{CRS}}$ stated by conditions (3a)–(3h). Then*

$$T^{\mathrm{F}}_{\mathrm{CRS}} = \left(\bigcup_{J' \in \mathcal{J}} T_{\mathrm{PCone}}(J') \right) \cap \mathcal{B} = \bigcup_{J' \in \mathcal{J}} (T_{\mathrm{PCone}}(J') \cap \mathcal{B}). \tag{12}$$

The next result uses the fact that the sets $T_{\mathrm{PCone}}(J') \cap \mathcal{B}$ in (12) are polyhedral sets. This leads to the following statement.

[7] The specification of the sets I^V and O^V with respect to which the partial extension is obtained uniquely defines the specification of the fixed sets $\mathrm{I}^R = \mathrm{I} \setminus \mathrm{I}^V$ and $\mathrm{O}^R = \mathrm{O} \setminus \mathrm{O}^V$—see Section 2.

[8] If DMU $(\tilde{X}^V, \tilde{X}^R, \tilde{Y}^V, \tilde{Y}^R)$ in definition (10) is one of the modified observed DMUs (X'_j, Y'_j), $j = 1, \ldots, n$, defined by (5), or is their convex combination, then $\tilde{X}^R = X^{R^*}(J')$ and $\tilde{Y}^R = Y^{R^*}(J')$, where $X^{R^*}(J')$ and $Y^{R^*}(J')$ are defined by (4). Otherwise, DMU $(\tilde{X}^V, \tilde{X}^R, \tilde{Y}^V, \tilde{Y}^R)$ is dominated by DMUs in (5) or their convex combinations, and we generally have $\tilde{X}^R \geq X^{R^*}(J')$ and $\tilde{Y}^R \leq Y^{R^*}(J')$.

[9] The fact that $T_{\mathrm{PCone}}(J')$ is closed also follows from the fact that $T_{\mathrm{PCone}}(J')$ is a polyhedral set. The latter is established by Corollary 4.

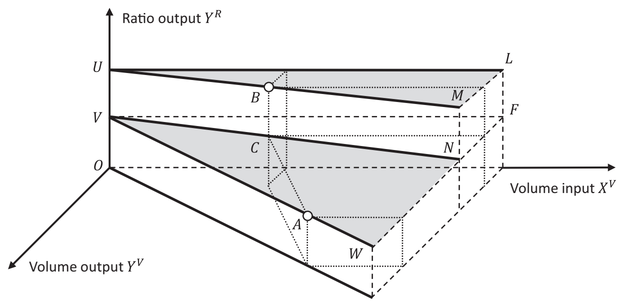

Fig. 3 The R-CRS technology in Example 4

Corollary 4 *Technology $T^{\mathrm{F}}_{\mathrm{CRS}}$ is the union of a finite number (more precisely, $2^n - 1$) of polyhedral (and therefore closed convex) sets.*

Example 4 Figure 3 shows technology $T^{\mathrm{F}}_{\mathrm{CRS}}$ generated by the two observed DMUs A and B as in Example 2. For reference, this figure also outlines the technology $T^{\mathrm{R}}_{\mathrm{VRS}}$ considered in Example 2. It is clear that $T^{\mathrm{F}}_{\mathrm{CRS}}$ is the partial (taken with respect to the volume measures only) closed cone extension of technology $T^{\mathrm{R}}_{\mathrm{VRS}}$.

As in Example 2, the set $\mathcal{J}$ consists of three elements as stated in (9). If $J' = \{A\}$, the set $T_{\mathrm{PCone}}(J')$ includes the points located on and below the surface WVF. If $J' = \{B\}$, $T_{\mathrm{PCone}}(J')$ consists of the points on and below the surface MUL. If $J' = \{A, B\}$, we first modify DMUs A and B according to (5). As in the same scenario in Example 2, this keeps DMU A unchanged and replaces DMU B by DMU C. The set $T_{\mathrm{PCone}}(J')$ generated by DMUs A and C includes the points located on and below the surface WVF, which in this case coincides with the set $T_{\mathrm{PCone}}(J')$ obtained with $J' = \{A\}$. In line with Theorem 2, technology $T^{\mathrm{F}}_{\mathrm{CRS}}$ is the union of technologies $T_{\mathrm{PCone}}(J')$ obtained, as described, for the three different sets J'.

6 Implications of the representation results

The geometric structure of the R-VRS and R-CRS technologies $T^{\mathrm{R}}_{\mathrm{VRS}}$ and $T^{\mathrm{F}}_{\mathrm{CRS}}$ may not be clear from their mathematical statements (2a)–(2h) and (3a)–(3h). Theorems 1 and 2 clarify this issue. As proved, both technologies, although being nonconvex, can be represented as the unions of a finite number of convex polyhedral technologies. In addition to providing clarity and intuition, these representations are useful in addressing various theoretical issues, e.g., those arising in frontier analysis of the R-VRS and R-CRS technologies.

It is worth emphasizing that the results obtained in our paper are not intended for their direct implementation in computational approaches. In particular, we are not suggesting the evaluation of efficiency of DMUs in the R-VRS technology by assessing their efficiency in each of the $2^n - 1$ VRS technologies $T_{\mathrm{VRS}}(J')$ used in the representation (7). For this purpose, much more efficient and practical linear and mixed integer linear programming methods were discussed in Olesen et al. (2017).

Below we discuss three examples demonstrating the usefulness of our results.

6.1 Properties of technologies and production frontiers

The representation of technologies $T^{\mathrm{R}}_{\mathrm{VRS}}$ and $T^{\mathrm{F}}_{\mathrm{CRS}}$ as finite unions of polyhedral sets could be useful in exploring their axiomatic properties and characterizing their efficient frontiers.

For example, from a theoretical perspective, it is important to verify that the R-VRS and R-CRS technologies are closed sets, which is one of the basic assumptions in production theory (Färe et al. 1985). Olesen et al. (2015) provide a rather complex bespoke proof of this (especially in the case of the R-CRS technology). The new Theorems 1 and 2 proved in this paper help us establish the closedness with less effort. Indeed, any polyhedral set is known to be a closed set. As shown, the R-VRS and R-CRS technologies $T^{\mathrm{R}}_{\mathrm{VRS}}$ and $T^{\mathrm{F}}_{\mathrm{CRS}}$ are finite unions of polyhedral sets and are therefore both closed.

There are large areas of theoretical research that rely on the clear structure of the technology. An example is the extensive literature exploring the structure of efficient frontiers of the standard VRS and CRS technologies and classification of their efficient points—see, e.g., Bougnol and Dulá (2009), Charnes et al. (1986, 1991), Fukuyama and Sekitani (2012), Krivonozhko et al. (2005, 2015), Mehdiloozad et al. (2017). A related field of research is concerned with the sensitivity and stability of efficiency classifications (see, e.g., a review in Cooper et al. (2004)) and stability of RTS characterizations (Podinovski & Bouzdine-Chameeva 2020, Seiford & Zhu 2005). The structure of the production technology is also important for exploring the effect of aggregation of DMUs into larger entities on the efficiency and scale efficiency measures (Briec et al. 2003, Färe and Grosskopf 1985, Zelenyuk 2015).

The decomposition of technologies $T^{\mathrm{R}}_{\mathrm{VRS}}$ and $T^{\mathrm{F}}_{\mathrm{CRS}}$ into finite unions of polyhedral sets obtained in this paper opens up further research avenues concerned with the structure of their efficient frontiers, sensitivity analysis of efficiency scores and RTS characterizations, and effects of aggregation on different notions of efficiency.

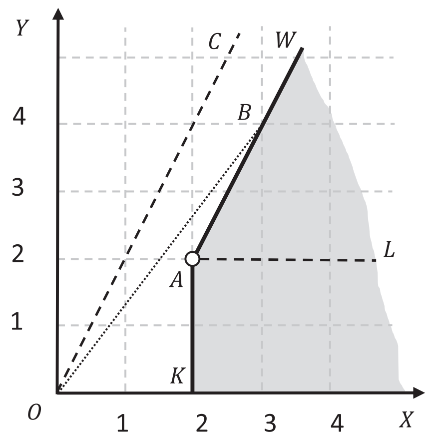

Fig. 4 Example of polyhedral technology in which there is no MPSS

6.2 The most productive scale size

Following Banker (1984), the most productive scale size (MPSS) of a DMU (X, Y) in the VRS technology T_{VRS} is defined by a DMU $(\alpha^* X, \beta^* Y)$ such that the scalars α^* and β^* maximize the ratio β/α under the condition that $(\alpha X, \beta Y) \in T_{\mathrm{VRS}}$. In the case of a single input and a single output, the MPSS simply maximizes the ratio of the output to the input.

It is well known that, in the standard VRS technology, the MPSS of any DMU (X, Y) exists (although it may not be unique). For example, the MPSS of DMU (X, Y) may be found by identifying its output radial projection in the corresponding CRS technology and subsequently scaling it back to the VRS technology, by dividing both the input and output vectors of the projected DMU by the sum of components of the optimal intensity vector λ in the output-oriented CRS model.

The notion of MPSS can be correctly defined and evaluated in many other technologies, for example, in FDH. However, in the general case of an arbitrary technology T (even assuming that T is a closed convex technology), DMU (X, Y) may have no MPSS, i.e., the maximum average productivity represented by MPSS may not be attained, although, by the definition of supremum, it can always be approximated. For an example of such situation, see Fig. 6 in Podinovski (2004a) and Fig. 6 in Podinovski (2004b).

The following example shows that MPSS may not exist even in a simple polyhedral technology.

Example 5 As a starting point, consider the VRS technology generated by a single DMU A with a single input $X = 2$ and single output $Y = 2$. This technology is shown in Fig. 4 as the area in the non-negative orthant spanned by KAL. Let us further assume the following trade-off between the input and output: an increase of the input by one unit is sufficient for the increase of the output by two units. Implementing this trade-off once, we move from DMU A to DMU B. Applying this trade-off in an arbitrary nonnegative proportion, we obtain the half-line AW. We also add the area in the nonnegative orthant below (or to the right) of AW by free disposability of the output and the input.

The resulting technology is the shaded area below the half-line AW. This technology is a polyhedral technology but it is not a standard VRS technology. (This technology is not a nondecreasing returns-to-scale technology either because its efficient boundary AW is steeper than the ray from the origin through point A.) Note that the specification of the above trade-off is equivalent to the weight restriction $2u \leq v$ in the dual multiplier models in which u and v are the output and input weights, respectively (Podinovski 2004c).

It is clear that there exists no MPSS for DMU A (and for any other DMU in the described technology) and that DMU A exhibits increasing returns to scale. Indeed, the average productivity Y/X at DMU A is equal to $2/2 = 1$. It gradually increases as we move away from A along the half-line AW. For example, the average productivity at DMU B is equal to 4/3. The average productivity then asymptotically tends to 2 which is the slope of the lines AW and OC. It is clear that this asymptotic value represents the supremum of the average productivity in the described technology, and that this supremum is not attained.

Let us use the results established in our paper to show that the MPSS of any DMU (X, Y) in the R-VRS technology $T^{\mathrm{R}}_{\mathrm{VRS}}$ is attained and therefore is correctly defined. For simplicity, we first assume that no bounds (1) on the ratio measures are specified. By Theorem 1, technology $T^{\mathrm{R}}_{\mathrm{VRS}}$ is the union of a finite number of VRS technologies $T_{\mathrm{VRS}}(J')$, where J' is an arbitrary nonempty subset of observed DMUs. Denote $\mathcal{J}_o$ the set of all subsets J' such that $(X, Y) \in T_{\mathrm{VRS}}(J')$. Because $(X, Y) \in T^{\mathrm{R}}_{\mathrm{VRS}}$, we have $\mathcal{J}_o \neq \varnothing$.

For every subset $J' \in \mathcal{J}_o$, the MPSS of DMU (X, Y) in the VRS technology $T_{\mathrm{VRS}}(J')$ exists and can be stated as $(\alpha' X, \beta' Y)$, with the corresponding average productivity β'/α'. Then the maximum ratio β'/α' taken across the finite number of VRS technologies $T_{\mathrm{VRS}}(J')$, $J' \in \mathcal{J}_o$, corresponds to the MPSS of DMU (X, Y) in technology $T^{\mathrm{R}}_{\mathrm{VRS}}$. This MPSS is therefore correctly defined and is attained.

The case in which the bounds (1) are specified is similar. By Corollary 1, technology $T^{\mathrm{R}}_{\mathrm{VRS}}$ is the union of a finite number of technologies $T_{\mathrm{VRS}}(J') \cap \mathcal{B}$. It is straightforward to prove that, in each such technology, the MPSS of DMU (X, Y) is attained (provided DMU (X, Y) belongs to it). The rest of the proof repeats the previous case and is omitted.

6.3 Marginal frontier characteristics

Theorems 1 and 2 show that we can correctly define and evaluate various marginal rates and scale characteristics on the production frontiers of technologies T^{R}_{VRS} and T^{F}_{CRS}. Such characteristics include, for example, (generally one-sided) marginal rates of transformation and substitution between different outputs and inputs, partial and full scale elasticities, and the local RTS characterization based on the latter. For *arbitrary* polyhedral technologies, a general linear programming approach to the evaluation of such characteristics is unproblematic and was developed by Podinovski et al. (2016).

To be specific, let us consider the case of the R-VRS technology T^{R}_{VRS}. (The case of R-CRS technology T^{F}_{CRS} is similar and is not discussed.) Note that all marginal characteristics and RTS are well-defined for convex technologies (Podinovski (2017)) but are generally undefined for arbitrary nonconvex technologies. Because technology T^{R}_{VRS} is not convex, a question arises whether all marginal characteristics can be correctly defined in this technology. Theorem 1 allows us to give a positive answer to this question.

Indeed, as proved, technology T^{R}_{VRS} is a finite union of polyhedral technologies $T_{VRS}(J')$ defined by (6a)–(6f). Let us consider two possible cases. Suppose that a DMU (X, Y) is efficient in the R-VRS technology and is located on the efficient boundary of a single polyhedral technology $T_{VRS}(J')$. An example of this is DMU B in the technology depicted in Fig. 1 which is located on the efficient boundary of technology $T_{VRS}(\{B\})$, i.e., in this case, we have $J' = \{B\}$. In this case, the required marginal characteristics at the DMU (X, Y) can be evaluated in the single polyhedral technology $T_{VRS}(J')$, for example, by the standard methods of Podinovski et al. (2016).

Now suppose that DMU (X, Y) is located on the efficient boundary of several polyhedral technologies $T_{VRS}(J')$ defined by different sets J'. An example of this is DMU A which, as shown in Example 2, is on the efficient boundary of two technologies $T_{VRS}(J')$, where $J' = \{A\}$ or $J' = \{A, B\}$. In this case, we can evaluate the required marginal characteristics in a way similar to their evaluation in nonconvex metatechnologies considered by Afsharian and Podinovski (2018). According to this approach, we first evaluate the required one-sided marginal value in all technologies $T_{VRS}(J')$ on whose boundary DMU (X, Y) is located and then, depending on the characteristic in question, select either the minimum or the maximum among these values.

It is now clear that we can (at least conceptually) extend the notions of various marginal rates and scale characteristics to the R-VRS technology T^{R}_{VRS} (and, in a similar way, to the R-CRS technology T^{F}_{CRS}). Given the large number of sets J' and the corresponding technologies $T_{VRS}(J')$, a direct practical implementation of the described procedure may be computationally challenging.

An exception is the case in which we are concerned with a marginal rate involving only volume measures. It is straightforward to show that this task is equivalent to the evaluation of the marginal rate in question in the VRS technology generated by the vectors of volume inputs and outputs only, by keeping all ratio measures fixed as in the DMU under the consideration. For example, the evaluation of the marginal rate of transformation between the volume input and volume output at DMU A in the R-VRS technology in Fig. 1 is equivalent to the evaluation of this marginal rate in the section *HACDF*, which is a VRS technology in the two volume dimensions.

Operationalizing the concept of RTS and developing practically acceptable computational approaches in the general case involving both volume and ratio measures (for example, by developing an efficient way of selecting a small number of relevant sets J') is left outside the scope of this paper for future research.

7 Conclusion

The R-VRS and R-CRS technologies developed by Olesen et al. (2015) are first defined axiomatically and then stated in an equivalent form by certain sets of mathematical conditions such as (2a)–(2h) and (3a)–(3h). These sets of conditions are useful for computational purposes and for establishing mathematical properties of the two technologies, but are less useful in helping us to think of the two technologies as geometric structures.

Theorems 1 and 2 establish an intuitively clear decomposition of the R-VRS and R-CRS technologies (the latter assuming the fixed nature of the ratio inputs and outputs) into the finite unions of polyhedral technologies. In addition to clarifying the structure of these technologies, the new results have various implications important for conceptual and theoretical exploration of their frontiers. In turn, these are important for the interpretation of the results of analysis involving applications of the R-VRS and R-CRS technologies.

For example, these results imply that any efficient DMU (X, Y) in the R-VRS technology is located on the boundary of a specially constructed VRS technology, one of those whose union becomes the R-VRS technology. In some rare cases the DMU (X, Y) may be located at the intersection of a finite number of efficient frontiers of such VRS technologies. A similar intuitive interpretation is true in the case of the R-CRS technology.

We outlined several areas in which the new results are either immediately useful or could facilitate additional research. First, the obtained representations of the two technologies as unions of polyhedral sets should be useful for exploring their properties. For example, the fact that both R-VRS and R-CRS technologies are closed sets is now obvious and follows directly from the decomposition of these technologies established in this paper.

Second, the new results suggest that the efficient frontiers of the R-VRS and R-CRS technologies could be explored using approaches similar to those already employed in the literature in the standard cases of the VRS and CRS production frontiers. This concerns various areas of research, including the classification of efficient DMUs, sensitivity and stability of efficiency measures and RTS characterizations, and effects of aggregation of DMUs into larger entities.

Third, we highlighted the fact that the notion of most productive scale size (MPSS) in an arbitrary production technology may be undefined. However, we used our main result stated by Theorem 1 to prove that the MPSS of any DMU in the R-VRS technology is defined correctly and always exists.

Fourth, we showed that the new representation results for the R-VRS and R-CRS technologies (the latter with the fixed type of ratio inputs and outputs) imply that we can correctly define various marginal characteristics of their frontiers.

This paper leaves as unresolved the possibility of representing the general R-CRS technology defined by Olesen et al. (2015) as a finite union of polyhedral sets. Such technology allows the specification of different types of ratio inputs and outputs and has a more complex structure than the technologies considered in this paper. Simple graphical examples of this general technology in different special cases considered by Olesen et al. (2015) suggest that its statement as the union of a finite number of polyhedral sets might also be possible. Whether this conjecture is true and could be proved formally remains an open question.

Compliance with ethical standards

Conflict of interest The authors declare no competing interests.

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8 Appendix: Proofs

Proof of Theorem 1 and Corollary 1. Denote $W = \bigcup_{J' \in \mathcal{J}} T_{\text{VRS}}(J')$. We need to prove that $T^{\text{R}}_{\text{VRS}} = W$. Consider any $(X, Y) = (X^V, X^R, Y^V, Y^R) \in T^{\text{R}}_{\text{VRS}}$. Then (X, Y) satisfies (2a)–(2h) with some vector λ', with the exception of conditions (2f) and (2g) which are omitted. Define $J' = \{j \in J | \lambda'_j > 0\}$. Because $\lambda'_j > 0$, $\forall j \in J'$, the inequalities (3c) and (3d) imply that $Y^R_j \geq Y^R$ and $X^R_j \leq X^R$, $\forall j \in J'$. Further define vectors $X^{R^*}(J')$ and $Y^{R^*}(J')$ as in (4) and observe that the inequalities (6b) and (6d) follow from (2c) and (2d). The inequalities (6a) and (6c) stated with λ' follow from (2a) and (2b), respectively. Therefore, $(X, Y) \in W$.

Conversely, consider any $(X, Y) = (X^V, X^R, Y^V, Y^R) \in W$. Then there exists a set $J' \subseteq J$ such that (X, Y) satisfies (6a)–(6f) with some vector $\lambda \in \mathbb{R}^{|J'|}$. Define vector $\lambda' \in \mathbb{R}^n$ such that $\lambda'_j = \lambda_j$, for all $j \in J'$, and $\lambda'_j = 0$, for all $j \in J \setminus J'$. Then conditions (6a)–(6f) imply that (X, Y) and λ' satisfy (2a)–(2h). Therefore, $(X, Y) \in T^{\text{R}}_{\text{VRS}}$.

For the proof of Corollary 1, we repeat the proof of Theorem 1 with the inequalities (2f) and (2g) included in (2a)–(2h) and additionally incorporated in (6a)–(6f).

Proof of Theorem 2 and Corollary 3. Denote $W = \bigcup_{J' \in \mathcal{J}} T_{\text{PCone}}(J')$. We need to prove that $T^{\text{F}}_{\text{CRS}} = W$. Consider any $(X, Y) = (X^V, X^R, Y^V, Y^R) \in T^{\text{F}}_{\text{CRS}}$. Then (X, Y) satisfies (3a)–(3h) with some vectors λ' and σ', with the exception of conditions (3f) and (3g) which are omitted. Define $J' = \{j \in J | \lambda'_j > 0\}$ and let $\hat{\lambda}_j = \lambda'_j \sigma'_j$, $\forall j \in J'$. Then (3a) and (3b) imply (11a) and (11b). Because $\lambda'_j > 0$, $\forall j \in J'$, the inequalities (2c) and (2d) imply (11c) and (11d). Therefore, $(X, Y) \in W$.

Conversely, consider any $(X, Y) = (X^V, X^R, Y^V, Y^R) \in W$. Then there exists a subset $J' \subseteq J$ such that (X, Y) satisfies (11a)–(11e) with some vector $\lambda^* \in \mathbb{R}^{|J'|}$. Two cases arise. First, let $Y^V \neq \mathbf{0}$. Then (11a) implies that $\lambda^* \neq \mathbf{0}$. Denote $\Lambda = \mathbf{1}^\top \lambda^* > 0$. Define vector $\lambda' \in \mathbb{R}^n$ such that $\lambda'_j = \lambda^*_j / \Lambda$, for all $j \in J'$, and $\lambda'_j = 0$, for all $j \in J \setminus J'$. Then (X, Y) satisfies (3a)–(3h) with the vectors λ' and $\sigma = \Lambda \mathbf{1}$.

Second, let $Y^V = \mathbf{0}$. Recall that $J' \neq \varnothing$. Define vector $\lambda' \in \mathbb{R}^n$ such that $\lambda'_j = 1/|J'|$, for all $j \in J'$, and $\lambda'_j = 0$, for all $j \in J \setminus J'$. Then $(X, Y) = (X^V, X^R, Y^V, Y^R)$ satisfies (3a)–(3h) with λ' and $\sigma = \mathbf{0}$. Therefore, $(X, Y) \in T^{\text{F}}_{\text{CRS}}$.

To prove Corollary 3, we repeat the proof of Theorem 2 with the inequalities (3f) and (3g) incorporated in (3a)–(3h) and additionally incorporated in (11a)–(11e).

Proof of Corollary 4. Technology $T_{\text{PCone}}(J')$ defined by conditions (11a)–(11e) is a special case of the general polyhedral technology introduced by Podinovski et al. (2016) and is, as shown in the latter paper, a polyhedral set. Technology $T_{\text{PCone}}(J') \cap \mathcal{B}$ is polyhedral as the intersection of two polyhedral sets.

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