



# Estimation and efficiency evaluation of stochastic frontier models with interval dependent variables

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## Abstract

This paper considers the maximum likelihood estimation of a stochastic frontier production function with an interval outcome. We derive an analytical formula for calculating the likelihood function of interval stochastic frontier models. Monte Carlo experiments reveal that the finite sample performance of our method is promising even when the sample size is relatively moderate. We also provide an exact formula for evaluating technical efficiency with interval outcome and apply our method to measure information inefficiency in the labor market for newly graduated college students in Taiwan.

**Keywords** Stochastic frontier analysis · Interval dependent variable · Technical efficiency

JEL C13 · C24

## 1 Introduction

Consider a stochastic frontier model with the following matrix form:

$$y = X^T \beta + \varepsilon, \quad (1)$$

where  $y$  and  $\varepsilon$  are  $N \times 1$  vectors of observations on the dependent variable and the random disturbance, respectively;  $X$  is an  $N \times K$  matrix of observations on a constant term and  $K - 1$  regressors; and  $\beta$  is a  $K \times 1$  vector of

unknown regression coefficients to be estimated. The error specification is:

$$\varepsilon = v + Su, \quad (2)$$

where the elements of  $v$  are independent and identically distributed (iid) as  $N(0, \sigma_v^2)$ , and the elements of  $u$  are the absolute value of the variables that are iid as  $N(0, \sigma_u^2)$ . Here,  $S$  is a pre-specified number that equals  $-1$  for the production frontier or  $1$  for the cost frontier. All  $v$ 's and  $u$ 's are independent of each other and are also independent of  $X$ . We follow the reparameterization of Aigner et al. (1977, ALS hereafter):

$$\sigma^2 = \sigma_u^2 + \sigma_v^2, \quad \lambda = \sigma_u / \sigma_v. \quad (3)$$

ALS show that the log-likelihood function for the aforementioned stochastic frontier model is:

$$L = \frac{N}{2} \ln \left( \frac{2}{\pi} \right) - N \ln(\sigma) + \sum_{i=1}^N \ln \left[ \Phi \left( \frac{S\lambda}{\sigma} \varepsilon_i \right) \right] - \frac{1}{2\sigma^2} \sum_{i=1}^N \varepsilon_i^2. \quad (4)$$

Here,  $\varepsilon_i = y_i - x_i^T \beta$ ,  $x_i^T$  is the  $i$ th row of  $X$ , and  $\Phi(\cdot)$  is the standard normal cumulative distribution function (cdf). The maximum likelihood estimator (MLE) is obtained by maximizing Eq. (4) with respect to the parameters  $(\beta, \sigma_u, \sigma_v)^T$ .

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This paper extends the analysis of ALS to the scenario where the dependent variable  $y_i^*$  is an ordered interval variable located in  $J + 1$  regimes:

$$\begin{cases} y_i^* = x_i^\top \beta + \varepsilon_i, & i = 1, 2, \dots, N; \\ y_i = 0, & \text{if } y_i^* \leq w_0; \\ y_i = 1, & \text{if } w_0 < y_i^* \leq w_1; \\ \vdots & \\ y_i = J, & \text{if } y_i^* > w_{J-1}, \end{cases} \quad (5)$$

i.e., the cell limits  $w_0 < w_1 < w_2 < \dots < w_{J-1}$  are observable even though  $y_i^*$  is unobservable. For example, as regularly found in survey questionnaires, we are unable to observe the exact value of some variable, rather just the range of it. This kind of survey structure is ubiquitously prevalent in the literature.

This paper thus suggests an analytic and highly accurate formula for evaluating the likelihood function of the SFA model in Eq. (5) and proposes an exact formula to measure the technical efficiency of the model by extending the method in Jondrow et al. (1982, JLMS hereafter). Since neither a numerical integration nor simulation-based techniques are required in the two formulae proposed in this paper, the computational burden of our approach is lowered to the limit. Amsler et al. (2019, AST hereafter) also provide a detailed discussion of the importance of using highly accurate formula to evaluate the distribution function of  $\varepsilon$ . We focus on the case where the elements of  $u$  are the absolute value of the variables that are iid as  $N(0, \sigma_u^2)$ , because it is directly linked to the distribution assumption of ALS<sup>1</sup>. Interval stochastic frontier model can also be tackled with the Bayesian estimation strategy of Griffiths et al. (2014) given that  $u$  is an exponential distribution, but its implementation is not trivial. Moreover, the Monte Carlo experiments conducted herein show that the finite sample performance of our formulae is very promising.

We apply our method to the wage rates of newly graduated college students in Taiwan. The empirical results reveal the phenomenon of “school quality matters,” in which a worker graduating from a higher quality school has a higher potential wage. We also find that the undergraduate major affects the level of potential wage. Furthermore, the students of private universities are found to attain the highest efficiency score, implying private university students perform the best in their job market information search.

<sup>1</sup> Other distribution assumptions for  $u$  have also been discussed in the literature, including truncated normal distribution (Stevenson 1980), gamma distribution (Greene 1990), and Weibull distribution (Tsioulas 2007). Recently, Badade and Ramanathan (2020) propose a probabilistic frontier model using a logit model specification.

The remaining parts of this paper are arranged as follows. “Interval stochastic frontier model” presents the model and analytic formula of the likelihood function. “Interval efficiency estimate” provides an exact formula for measuring the technical efficiency when the conditional variable  $\varepsilon$  is only observed in a range. “Monte Carlo experiment” investigates the performance of our formulae via Monte Carlo experiments. “Empirical application” applies the proposed method to data on the wage rates of newly graduated college students in Taiwan. “Conclusion” concludes.

## 2 Interval stochastic frontier model

Defining  $f(\cdot)$  and  $F(\cdot)$  as the probability density and distribution functions of  $\varepsilon_i$ , respectively, the probability of  $y_i^*$  in each regime is:

$$\begin{cases} F_{i0} = F(w_0 - x_i^\top \beta), & \text{if } y_i^* \leq w_0; \\ F_{i1} = F(w_1 - x_i^\top \beta) - F(w_0 - x_i^\top \beta), & \text{if } w_0 < y_i^* \leq w_1; \\ \vdots & \\ F_{iJ} = 1 - F(w_{J-1} - x_i^\top \beta), & \text{if } y_i^* > w_{J-1}. \end{cases} \quad (6)$$

Consequently, the log-likelihood function for the model in Eq. (5) is:

$$\ln L = \sum_{i=1}^N \sum_{j=0}^J 1(y_i = j) \ln F_{ij}, \quad (7)$$

where  $1(\cdot)$  is the indicator function.

The major difficulty for the maximum likelihood estimation centers on computing the associated cdf functions in Eq. (7). For example, we have:

$$F(w_j - x_i^\top \beta) = \int_{-\infty}^{w_j - x_i^\top \beta} f(\varepsilon_i) d\varepsilon_i = \frac{2}{\sigma} \int_{-\infty}^{Q_{ij}} \left( \int_{-\infty}^{a\varepsilon_i} \phi(\xi) d\xi \right) \phi(b\varepsilon_i) d\varepsilon_i = \frac{2}{\sigma} I(Q_{ij}, a, b), \quad (8)$$

where  $j = 0, 1, \dots, J - 1$ ,  $Q_{ij} = w_j - x_i^\top \beta$ ,  $a = S \frac{\lambda}{\sigma}$ ,  $b = \frac{1}{\sigma}$ , and  $\phi(\cdot)$  stands for the standard normal density function. Provided that we have a good approximation formula for the value of  $I(Q_{ij}, a, b)$  in Eq. (8), we can accurately evaluate the likelihood function in Eq. (7). The first contribution of this paper is to derive such an approximation formula by extending the methodology of Tsay et al. (2013), who only deal with the cases  $S = 1$  in Eq. (8). Since the formula developed in this paper does not impose a constraint on the value of  $S$ , it can be used for both cost and production function scenarios encountered in interval stochastic frontier models. Before presenting  $I_{app}(Q_{ij}, a, b)$

in the following Proposition 1, we define an error function as:

$$Erf(z) = \frac{2}{\sqrt{\pi}} \int_0^z e^{-t^2} dt = 2 \int_0^{\sqrt{2}z} \phi(t) dt. \tag{9}$$

**Proposition 1**  $I(Q_{ij}, a, b)$  in Eq. (8) can be approximated by  $I_{app}(Q_{ij}, a, b)$ :

$$I_{app}(Q_{ij}, a, b) = \frac{\exp\left(\frac{a^2c_1^2}{4b^2-4a^2c_2}\right) \left[1 + Erf\left(\frac{\sqrt{a^2c_1}}{2\sqrt{b^2-a^2c_2}}\right)\right] S}{4\sqrt{b^2-a^2c_2}} + \frac{\sqrt{b^2-a^2c_2}}{2b^2} \frac{1-S}{2} - \frac{\exp\left(\frac{a^2c_1^2}{4b^2-4a^2c_2}\right) \left[Erf\left(\frac{-ac_1}{2\sqrt{b^2-a^2c_2}}\right) + Erf\left(\frac{-ac_1 + \sqrt{2}Q_{ij}(b^2-a^2c_2)\text{sign}(SQ_{ij})}{2\sqrt{b^2-a^2c_2}}\right)\right]}{4\sqrt{b^2-a^2c_2}} + \frac{Erf\left(\frac{bQ_{ij}}{\sqrt{2}}\right)}{2b} \frac{1 + \text{sign}(SQ_{ij})}{2}$$

where  $c_1 = -1.0950081470333$  and  $c_2 = -0.75651138383854$ .<sup>2</sup>

Proof of Proposition 1 is in the Appendix. AST and Amsler et al. (2020) address the issues of evaluating the cdf of the skew normal distribution or the composed error  $\varepsilon$  in Eq. (2). In the upper tail of the cdf of  $\varepsilon$ , the approximation formula of Tsay et al. (2013) and the formula APS-UT of APS works really well. In the lower tail, the approximation formula APS-LT of APS is recommended. We thus have the following proposition.

**Proposition 2**  $I(Q_{ij}, a, b)$  in Eq. (8) can also be approximated by the formula APS-LT of APS

$$I(Q_{ij}, a, b) \approx \begin{cases} 2\Phi\left(\frac{Q_{ij}}{\sigma}\right)\Phi\left(\frac{\lambda Q_{ij}}{\sigma}\right)\frac{1}{1+\lambda^2}, & \text{if } S = 1; \\ 1 - 2\Phi\left(\frac{-Q_{ij}}{\sigma}\right)\Phi\left(\frac{-\lambda Q_{ij}}{\sigma}\right)\frac{1}{1+\lambda^2}, & \text{if } S = -1; \end{cases}$$

when the value of  $Q_{ij}$  approaches the lower tail of the distribution.

For detailed simulations and discussions of Proposition 2, readers are referred to APS. With the combination of  $I_{app}(Q_{ij}, a, b)$  or APS-LT, the cdf of  $\varepsilon$  in Eq. (8) can be easily evaluated with a standard statistical package. This implies that the computation of the associated likelihood function in Eq. (7) is straightforward when using these formulae.

### 3 Interval efficiency estimate

This section addresses the important issue of ranking the relative efficiency level of different decision units given that the parameters in Eq. (5) are estimated. ALS defined the efficiency level of different units as  $\exp(Su_i)$ . Since  $u_i$  is unobservable, JLMS propose to estimate the efficiency level

<sup>2</sup> For discussion and evaluation of  $c_1$  and  $c_2$ , readers are referred to page 261 of Tsay et al. (2013).

of each unit for the case  $S = -1$  as:

$$\exp(-\hat{u}_i), \quad \hat{u}_i = E(u_i|\varepsilon_i) = S \frac{\sigma_u^2}{\sigma^2} \varepsilon_i + \frac{\sigma_u \sigma_v}{\sigma} \frac{\phi\left(\frac{\lambda \varepsilon_i}{\sigma}\right)}{\Phi\left(S\lambda \frac{\varepsilon_i}{\sigma}\right)}. \tag{10}$$

This result is widely used in the literature. Nevertheless, it only can be used when the dependent variable is continuous. The interval nature of the dependent variable in Eq. (5) requires a new formula to evaluate the conditional mean of  $u_i$  when  $\varepsilon_i$  is only observed in a range. In particular, for  $y_i = 0$  in Eq. (5), we observe the upper bound of  $\varepsilon_i$  as:

$$\hat{\varepsilon}_i \leq w_0 - x_i^\top \hat{\beta}, \tag{11}$$

where  $\hat{\beta}$  are the maximum likelihood estimates. For  $y_i \in \{1, \dots, J-1\}$ , we observe both the upper and the lower bound of  $\varepsilon_i$  as:

$$w_{j-1} - x_i^\top \hat{\beta} \leq \hat{\varepsilon}_i \leq w_j - x_i^\top \hat{\beta}, j = 1, \dots, J-1 \tag{12}$$

For  $y_i = J$ , we observe the lower bound of  $\varepsilon_i$  as:

$$\hat{\varepsilon}_i \geq w_{J-1} - x_i^\top \hat{\beta}. \tag{13}$$

Because we can only observe the boundary of  $\varepsilon_i$ , the conditional expectations cannot be directly evaluated with the method of JLMS. We thus fill the gap in the literature by providing an exact formula to fulfill our goal.

Without loss of generality, we assume the censoring interval of dependent variable is at  $(a, b)$ , and the conditional expectation denotes  $E(u_i|a < \varepsilon_i < b)$ . Defining  $[\mathbb{D}(\cdot)]_a^b = \mathbb{D}(b) - \mathbb{D}(a)$ , we state our theoretical finding in the following proposition.

**Proposition 3** Let  $\lambda^* = \sqrt{1 + \lambda^2}$ .  $E(u_i|a < \varepsilon_i < b)$  is expressed as:

$$\frac{S \frac{\sigma_u^2}{\sigma} \left\{ -\sigma [f(\varepsilon)]_a^b + \frac{2\lambda}{\sqrt{2\pi}\lambda^*} [\Phi\left(\frac{S\lambda^*}{\sigma} \varepsilon\right)]_a^b \right\} + \frac{\sigma_u \sigma_v}{\sqrt{2\pi}\lambda^* \sigma} \left[ erf\left(\frac{\lambda^*}{\sqrt{2}\sigma} \varepsilon\right) \right]_a^b}{[F(\varepsilon)]_a^b}$$

Derivations of Proposition 3 are in the Appendix. The formula in Proposition 3 is exact and can be easily computed with standard statistical packages, given that the denominator,  $[F(\varepsilon)]_a^b$ , can be evaluated with the results in Proposition 1 and Proposition 2. We also establish that the results in Proposition 3 are identical to those in JLMS when the censoring interval degenerates to 0 in the following corollary.

**Corollary 1** Let  $c$  and  $\xi$  be any finite constant such that:

$$\lim_{\xi \rightarrow 0} E(u_i|c < \varepsilon_i < c + \xi) \rightarrow E(u_i|\varepsilon_i = c),$$

where  $E(u_i|\varepsilon_i = c)$  is the formula in JLMS.

Proof of the above Corollary is also put in the Appendix.

### 4 Monte Carlo experiment

We consider two sets of Monte Carlo experiments in this section: (1) the finite sample performance of MLE when the data-generating processes (DGP) are the form in Eq. (5), and (2) the accuracy of the interval efficiency estimate in Proposition 3. All the programs are written in GAUSS.

#### 4.1 Finite sample performance of MLE

Following Olson et al. (1980, p. 76) and Tsay et al. (2013), we consider a set of experiments with a two-regressor model:

$$y_i^{*l} = \beta_0 + \beta_1 x_i + \varepsilon_i^l, \quad i = 1, 2, \dots, N, \quad l = 1, 2, \dots, 1000, \tag{14}$$

where  $l$  denotes the  $l$ th replication of the data, and the regressors  $x_i$  are only drawn once from  $N(0,1)$  and kept fixed throughout the simulation. Without loss of generality, we consider the case where  $J=3$ ; i.e., we divide the data into four regimes depending on whether  $y_i^{*l}$  is less than  $w_0$ , between  $w_0$  and  $w_1$ , between  $w_1$  and  $w_2$ , or greater than  $w_2$ . We consider three sets of  $(w_0, w_1, w_2, \beta_0, \beta_1, \sigma_u, \sigma_v)$  to check the robustness of the simulation results and four sample sizes,  $N = 100, 400, 1600,$  and  $6400$ . Since our model is directly linked to the model in ALS and  $y_i^{*l}$  is known to us, we also estimate Eq. (14) using the log-likelihood function in Eq. (4) for comparison purpose. The optimization algorithm used to implement MLE for both models is the quasi-Newton algorithm of Broyden, Fletcher, Goldfarb, and Shanno (BFGS) contained in the GAUSS MAXLIK library. The maximum number of iterations for each replication is 200.

The results in Tables 1 and 2 show the bias and mean squared errors (MSE) of the maximum likelihood estimates of ALS’s model and our model, respectively. For all the combinations of parameters presented in Tables 1 and 2, we find the bias is small and the MSE always decreases with an increasing sample size. The results confirm the promising performance of the proposed  $I_{app}$  approximation formula, and enhance our confidence in using  $I_{app}$  for the interval SFA models. We also find that the bias and MSE in Table 2 are larger than those in Table 1. This is consistent with the fact that our dependent variable is limited and thus contains less information than the continuous variable  $y_i^{*l}$ .

#### 4.2 Accuracy of the interval efficiency estimate

This subsection illustrates the accuracy of the interval efficiency estimator proposed in Proposition 3. Following Kumbhakar and Lothgren (1998), we obtain 100 million draws of  $u_i$  and  $v_i$ , which are generated from  $N^+(0, \sigma_u^2)$  and

**Table 1** Finite sample properties of MLE for ALS’s model in Eq. (4)

$N$	$\beta_0$	$\beta_1$	$\sigma_u$	$\sigma_v$
DGP 1: $\beta_0 = 1.28, \beta_1 = 1, \sigma_u = 1, \sigma_v = 0.5$				
<i>Bias</i>				
100	0.0518	-0.0053	0.0670	0.0098
400	0.0074	-0.0016	0.0121	0.0031
1600	0.0009	0.0000	0.0028	0.0007
6400	-0.0005	-0.0002	-0.0004	0.0006
<i>MSE</i>				
100	0.0621	0.0073	0.0908	0.0168
400	0.0088	0.0013	0.0127	0.0037
1600	0.0018	0.0003	0.0027	0.0009
6400	0.0004	0.0001	0.0006	0.0002
DGP 2: $\beta_0 = 1.37, \beta_1 = 1, \sigma_u = 1, \sigma_v = 0.67$				
<i>Bias</i>				
100	0.1001	-0.0060	0.1283	0.0116
400	0.0266	-0.0022	0.0368	0.0003
1600	0.0034	-0.0003	0.0063	0.0002
6400	-0.0002	-0.0002	0.0001	0.0005
<i>MSE</i>				
100	0.1331	0.0102	0.1965	0.0245
400	0.0286	0.0018	0.0426	0.0063
1600	0.0042	0.0004	0.0064	0.0014
6400	0.0010	0.0001	0.0014	0.0003
DGP 3: $\beta_0 = 1.56, \beta_1 = 1, \sigma_u = 1, \sigma_v = 1$				
<i>Bias</i>				
100	0.1519	-0.0074	0.1951	0.0377
400	0.1050	-0.0031	0.1365	-0.0012
1600	0.0293	-0.0011	0.0395	-0.0026
6400	0.0033	-0.0004	0.0047	0.0003
<i>MSE</i>				
100	0.2857	0.0175	0.4209	0.0402
400	0.1329	0.0031	0.2053	0.0119
1600	0.0316	0.0008	0.0498	0.0037
6400	0.0048	0.0002	0.0072	0.0009

All the results are based on 1000 replications. MSE denotes mean squared errors

$N(0, \sigma_v^2)$ , respectively. We control the variance of two error terms,  $\sigma_u$  and  $\sigma_v$ , and the variance ratio,  $\lambda$ , that reflects the contribution of the variance of  $u$  to the total variance of the error term  $\varepsilon$ . In particular, three different variance ratios  $\lambda = (0.5, 1, 2)$  and six censoring intervals of  $\varepsilon$ ,  $(a, b) = (-4, -2), (-2, -1), (-1, -0.5), (-0.5, 0), (0, 1),$  and  $(1, 3)$ , are employed.

Table 3 presents the simulation results. For all 18 combinations of parameters and censoring intervals, the largest difference between the values calculated by Proposition 3 and the values calculated by 100 million draws is 0.0006. The results again confirm the accuracy of Proposition 3.

**Table 2** Finite sample properties of MLE with the analytic formula in Proposition 1

$N$	$\beta_0$	$\beta_1$	$\sigma_u$	$\sigma_v$
DGP 1: $w_0 = 0, w_1 = 0.55, w_2 = 1.03$ $\beta_0 = 1.28, \beta_1 = 1, \sigma_u = 1, \sigma_v = 0.5$				
<i>Bias</i>				
100	0.1615	-0.0349	0.2015	0.0112
400	0.0464	-0.0085	0.0600	-0.0012
1600	0.0072	-0.0017	0.0104	-0.0006
6400	-0.0004	-0.0008	-0.0005	0.0012
<i>MSE</i>				
100	0.1842	0.0340	0.2969	0.0646
400	0.0508	0.0061	0.0843	0.0163
1600	0.0063	0.0016	0.0113	0.0037
6400	0.0013	0.0004	0.0023	0.0008
DGP 2: $w_0 = 0, w_1 = 0.62, w_2 = 1.2$ $\beta_0 = 1.37, \beta_1 = 1, \sigma_u = 1, \sigma_v = 0.67$				
<i>Bias</i>				
100	0.1829	-0.0297	0.2239	0.0371
400	0.0983	-0.0111	0.1268	-0.0052
1600	0.0303	-0.0028	0.0404	-0.0053
6400	0.0027	-0.0008	0.0036	0.0006
<i>MSE</i>				
100	0.2545	0.0337	0.4138	0.0842
400	0.1142	0.0066	0.1894	0.0227
1600	0.0258	0.0017	0.0446	0.0067
6400	0.0035	0.0004	0.0062	0.0015
DGP 3: $w_0 = 0, w_1 = 0.79, w_2 = 1.56$ $\beta_0 = 1.56, \beta_1 = 1, \sigma_u = 1, \sigma_v = 1$				
<i>Bias</i>				
100	0.1341	-0.0225	0.1505	0.1012
400	0.1578	-0.0076	0.1961	0.0195
1600	0.1294	-0.0015	0.1641	-0.0091
6400	0.0469	-0.0009	0.0595	-0.0058
<i>MSE</i>				
100	0.3681	0.0375	0.6079	0.1274
400	0.2560	0.0074	0.4229	0.0362
1600	0.1475	0.0020	0.2436	0.0139
6400	0.0479	0.0005	0.0789	0.0047

All the results are based on 1000 replications. MSE denotes mean squared errors

### 5 Empirical application

This paper applies the proposed method to the earning frontier analysis of newly graduated students in Taiwan. The data used in the empirical example reported below are drawn from a survey conducted by Peng (2005) on Taiwanese college graduates who graduated in 2003. The survey employed a proportionate stratified two-stage

**Table 3** Accuracy of interval efficiency estimate

Censoring interval	Exact	Simulation
DGP 1: $\sigma_v = 1.00, \sigma_u = 1.50, \lambda = 1.50$		
(-4, -2)	0.1461	0.1462
(-2, -1)	0.3001	0.2999
(-1, -0.5)	0.4089	0.4089
(-0.5, 0)	0.4789	0.4795
(0, 1)	0.5640	0.5636
(1, 3)	0.6642	0.6643
DGP 2: $\sigma_v = 1.00, \sigma_u = 1.00, \lambda = 1.00$		
(-4, -2)	0.2567	0.2568
(-2, -1)	0.3958	0.3956
(-1, -0.5)	0.4842	0.4844
(-0.5, 0)	0.5404	0.5409
(0, 1)	0.6091	0.6086
(1, 3)	0.6899	0.6905
DGP 3: $\sigma_v = 2.00, \sigma_u = 1.00, \lambda = 0.50$		
(-4, -2)	0.3809	0.3808
(-2, -1)	0.4340	0.4345
(-1, -0.5)	0.4619	0.4625
(-0.5, 0)	0.4804	0.4809
(0, 1)	0.5069	0.5064
(1, 3)	0.5500	0.5494

Exact is computed based on Proposition 3. Simulation is simulated from the Accept-Reject algorithm based on 100 million independent draws

random sampling framework in which schools and majors were used as the stratum. To serve the purpose of this study, the survey data are further screened to include only full-time workers, excluding part-time workers, self-employed, family workers, and graduates in military service or in prison. The above screening of the survey data yields a total of 3973 sample workers.

Following the literature of the human capital theory and the framework of Hofler and Polachek (1985), we develop a stochastic earning frontier function to empirically measure the labor market inefficiency of recent Taiwanese college graduates. Hofler and Polachek (1985) define the wage differential between the maximum potential wage that any given individual can attain and the wage the person actually earns to be the inefficiency that may be attributed to incomplete worker information. To accommodate the problem that the wage differential may include pure inefficiency as well as unobserved heterogeneity, previous studies have suggested to include demographic data in the earning function (Hofler and Polachek 1985; Polachek and Yoon 1996).

The dependent variable is the natural logarithm of a worker’s monthly wage (measured by New Taiwan Dollar).



It is located in four possible categories<sup>3</sup>: less than  $\ln(22800)$ , between  $\ln(22800)$  and  $\ln(28800)$ , between  $\ln(28800)$  and  $\ln(36300)$ , and more than  $\ln(36300)$ . The explanatory variables ( $X$ ) used in estimating the maximum potential wage cover a set of socio-demographic factors. More specifically, the explanatory variables are related to education or work experience, gender, work status, school type, and individual majors in college.

#### Education/Work experience:

Experience = 1 if a worker ever had a part-time job during undergraduate schooling; Experience = 0, otherwise. Since all subjects are recent graduates (with one year of work experience), they all have the same education level and number of working years. However, those who had a part-time work experience during college are expected to have more labor market information, which may lead to a better paying job.

#### Gender:

Male = 1 if male; and Male = 0, otherwise. Females are usually paid less due to sex discrimination in the job market.

#### Work Status:

Hours = Worker's weekly working hours. Workers with longer working hours are expected to earn more.

#### Knowledge Match:

Knowledge Match = 1 if a worker's undergraduate major matches the job requirement; Knowledge Match = 0, otherwise. The knowledge match is postulated to have a positive impact on the wage rate.

#### Public Sector:

Public Sector = 1 if working for the government; and Public Sector = 0, otherwise. Government employees in Taiwan on average are better paid than their private counterparts.

#### School Type:

Public University = 1 if a worker graduated from a public (comprehensive) university; and Public University = 0, otherwise.

Public Vocational College = 1 if a worker graduated from a public vocational college; and Public Vocational College = 0, otherwise.

Private Vocational College = 1 if a worker graduated from a private vocational college; and Private Vocational College = 0, otherwise.

Taiwan's higher education institutions can be classified by types into public/private or comprehensive/vocational. Among the four different types, Private University is used as the base for regression measurement. School type may reflect school quality. Workers who graduated from better quality schools are expected to earn more (Card and Krueger, 1992). In Taiwan, public schools are found to be of better quality than private schools, whereas comprehensive universities are on average better than vocational colleges in school quality (Fu, 2011).

#### Undergraduate Major:

Social Science = 1 if a worker graduated with a major in social science; Social Science = 0, otherwise.

Technology = 1 if a worker graduated with a major in engineering and technology; Technology = 0, otherwise.

The majors of undergraduates are classified into Social Science, Technology and Engineering, and Humanities. The Humanities category is used as the base for the regression analysis. In Taiwan's labor market, workers with a technology major are generally paid well, followed by Humanities and Social Science.

It is worth to note that our control variables did not include the variables of workers' seniority, i.e., age and experience of working, as our dataset does not provide the observations of these two variables. Nevertheless, this will not cause much loss of information, because the respondents of our dataset are newly graduated college students who have very little formal working experience, and they are about of the same age<sup>4</sup>.

Table 4 shows the mean statistics of variables used in the interval SFA model. Among all 3973 subjects of the four types of schools, 68.92% of them (38.54% from Private University and 30.38% from Private Vocational College) graduated from private schools. Public schools make up only 31.08% of the subjects, with 21.02% from Public University and 10.07% from Public Vocational College. Public University has a relatively high mean value in wage, knowledge match, and working in the public sector, as compared to the other three types of schools. The mean wages of public schools are averagely higher than those in private schools. We also find that the sample graduates from both Public and Private Vocational Schools have a higher percentage (46–55%) of obtaining a Technology and

<sup>3</sup> The original survey conducted by Peng (2005) has nine categories for worker's monthly wage: (1) less than 15,840, (2) between 15,840 and 22,800, (3) between 22,800 and 28,800, (4) between 28,800 and 36,300, (5) between 36,300 and 45,800, (6) between 45,800 and 57,800, (7) between 57,800 and 72,800, (8) between 72,800 and 83,900, and (9) more than 83,900. However, some categories contain only small amount of observations. For example, there are only 39 observations in the first category and only 213 observations in total in categories (6)–(9). To keep the number of observations in each category more balanced, we merge the first two categories, and we combine categories (5)–(9) into one.

<sup>4</sup> The efficiency estimates certainly depend on the specification of model used for empirical analysis and the distribution assumption imposed on composite error. Therefore, we treat our empirical results as a starting point of analyzing the market inefficiency of recent Taiwanese college graduates. More empirical works and data collection are useful to enhance our knowledge about this important issue.

**Table 4** Summary statistics (sample mean)

Variable	Total sample	Pub Uni	Pri Uni	Pub Col	Pri Col
Wage	31,824	37,381	32,076	32,099	27,570
Hours	45.38	44.46	45.26	46.71	45.72
Knowledge Match	0.44	0.66	0.39	0.44	0.36
Experience	0.86	0.90	0.88	0.80	0.83
Public Sector	0.19	0.55	0.12	0.11	0.06
Male	0.19	0.22	0.17	0.29	0.15
Social Science	0.43	0.25	0.53	0.35	0.47
Technology	0.36	0.25	0.29	0.55	0.46
Observations	3973	835	1531	400	1207

Average of wage is calculated based on mid-point of each category

**Table 5** Estimating log wage equation with the interval SFA model

Variable	Estimate
Public University	0.0955 (0.0199)***
Public Vocational College	-0.0252 (0.0131)*
Private Vocational College	-0.1150 (0.0089)***
Hours	0.0031 (0.0005)***
Knowledge Match	0.1124 (0.0078)***
Experience	0.0089 (0.0107)
Public Sector	0.1810 (0.0122)***
Male	0.1079 (0.0104)***
Social Science	-0.0676 (0.0109)***
Technology	0.0285 (0.0117)***
Constant	10.2775 (0.0283)***
$\sigma_u$	0.2252 (0.0176)***
$\sigma_v$	0.1619 (0.0085)***
Observation	3973
Log-Likelihood	-3971.86

The number in parenthesis denotes the standard error

The dependent variable,  $\ln(wage)$ , is located in four possible categories: less than  $\ln(22800)$ , between  $\ln(22800)$  and  $\ln(28800)$ , between  $\ln(28800)$  and  $\ln(36300)$ , and more than  $\ln(36300)$

\* $p < 0.10$ ; \*\*\* $p < 0.01$  (two-tailed tests)

Engineering degree than those from general comprehensive schools, whereas private schools are found to have a higher percentage of graduates who majored in Social Science than do public schools.

Table 5 shows that the sign of the estimates in the earning frontier function matches our expectations. Workers obtain a higher potential wage when working in the public sector, taking a job with a knowledge match, and having more working hours. The positive sign of Male indicates the existence of gender discrimination in Taiwan’s labor market. The school type variables show a positive sign for Public University and negative signs for both types of vocational schools, as compared to Private University.

School type variables are proxies for school quality, and it is widely recognized in Taiwan that the quality of public schools is on average higher than that of private schools, as are general comprehensive universities versus vocational colleges. Therefore, the empirical results indicate the phenomenon of "quality matters," in which a worker graduating from a higher quality school will have a higher potential wage. In our study, the sample graduates from public universities have the highest potential wage. Table 4 also shows that undergraduate major affects the level of potential wage. Those who majored in technology and engineering (social science) have a higher (lower) potential wage than those with a humanities major.

Given that the parameters are estimated, we now apply the results in Proposition 3 to measure the relative efficiency level of different units. The summary statistics and histograms of the resulting estimates are displayed in Table 6 and Figs. 1–4, respectively. From these efficiency estimates, we find that private university graduates achieve the highest average efficiency of 0.8406 as compared to the value of 0.8380 for public university graduates, but the difference is quite small. This implies that private universities try to help their graduates to get a better payoff so as to raise their ability of enrolling new students when facing the difficulty of attracting the best students in Taiwan. Interestingly, the lowest value of efficiency found is 0.5217, which is from graduates of a public university.

## 6 Conclusion

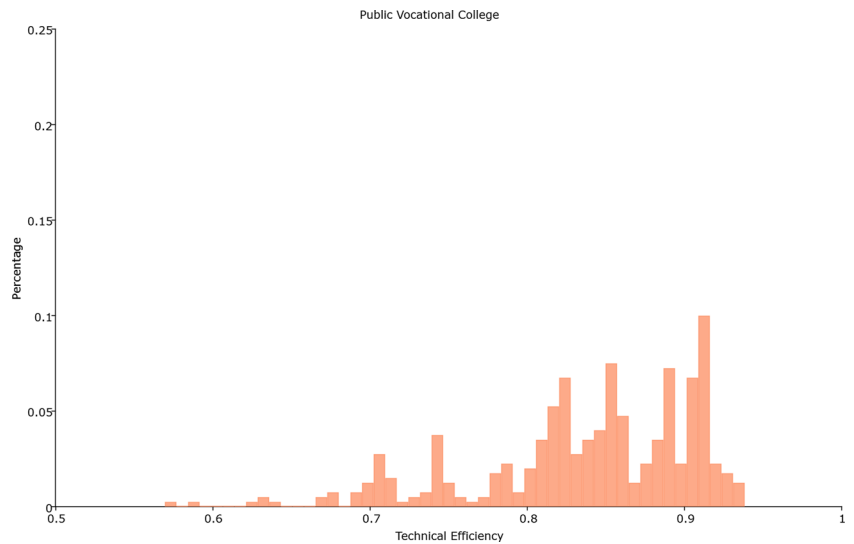
This paper extends the analysis of ALS to the scenario where the dependent variable is an ordered interval variable by recognizing that the interval coding data are commonly collected in surveys to individuals or decision units. In such surveys, the subjects are often asked to answer questions on their performance by choosing among some ordered interval coding values. We thus provide an analytic formula for evaluating the associated likelihood function and propose

**Table 6** Summary statistics of the efficiency estimates

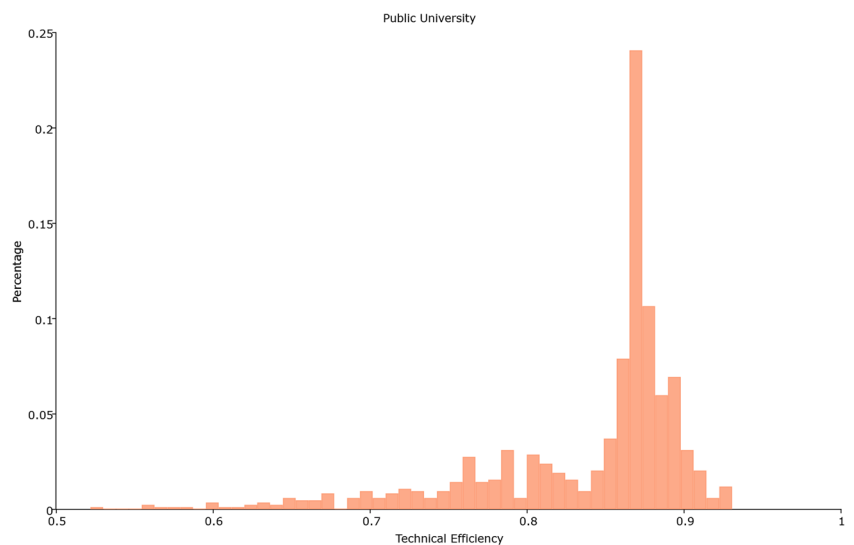
	Mean	SD	25th Pctl.	50th Pctl.	75th Pctl.
Public University	0.8380	0.0682	0.8098	0.8687	0.8773
Public Vocational College	0.8375	0.0696	0.8098	0.8516	0.8907
Private University	0.8406	0.0629	0.8062	0.8454	0.8884
Private Vocational College	0.8373	0.0659	0.7729	0.8505	0.8826

Mean and SD refer to the sample mean and standard deviation of the estimated relative efficiency, respectively, and 25th, 50th, and 75th Pctl. are sample percentiles of the estimated relative efficiency

**Fig. 1** The wage efficiency of public vocational college graduates



**Fig. 2** The wage efficiency of public university graduates



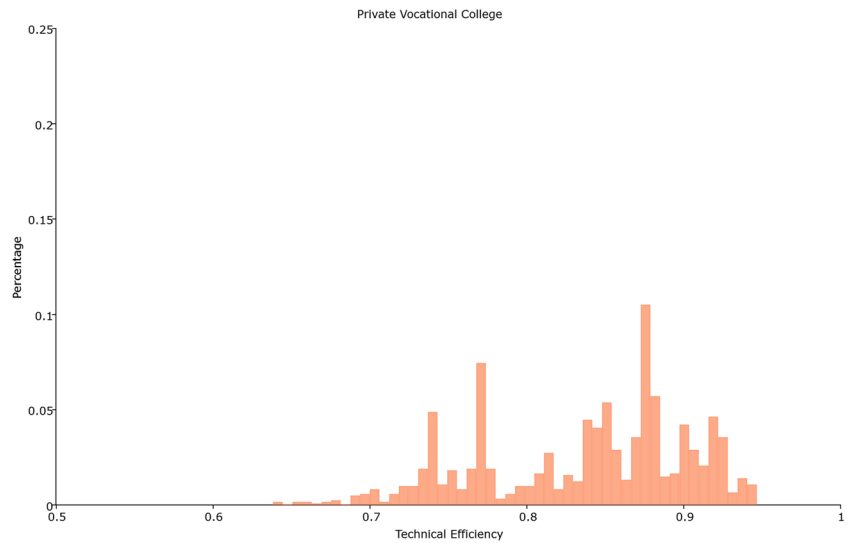
an exact formula to measure the technical efficiency by generalizing the method in JLMS. Our approaches do not depend on any numerical integration or simulation-based techniques and thus can be implemented straightforwardly.

The current paper illustrates an application of the proposed model for measuring labor market information inefficiency when the earning variable is measured by interval

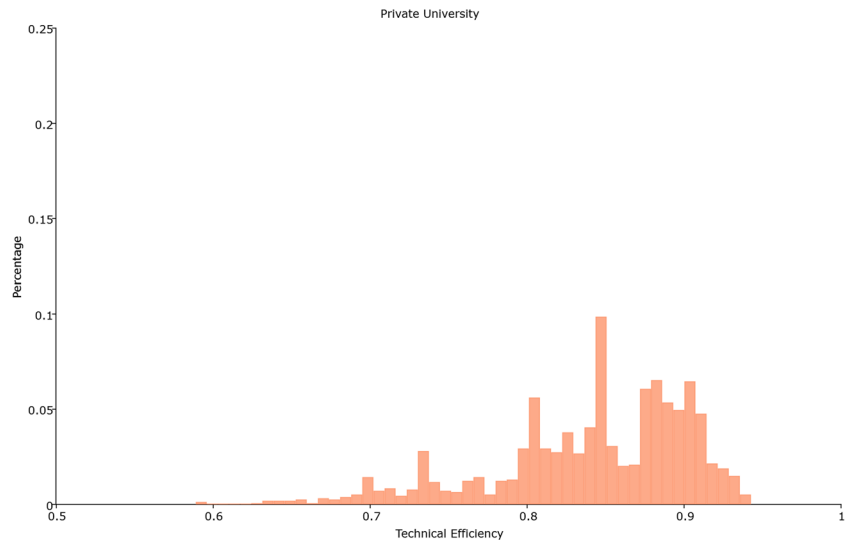
outcomes. Other potential applications are abundant, including survey studies that adopt contingency valuation methods to examine the values of hypothetical quality improvement of public goods or non-market goods (Carson 2012). In particular, we can use the willingness to pay (WTP) for quality improvement of environmental goods such as water, air, or any such amenity. Studies on WTP for



**Fig. 3** The wage efficiency of private vocational college graduates



**Fig. 4** The wage efficiency of private university graduates



hypothetical non-market goods such as national security, conservation of public parks or natural resources, and new medical treatments are also abundant in both the environment and health-related literature. Since WTP is often expressed in an interval form due to the contingency valuation survey design, our proposed model can therefore be used to measure the inefficiency of WTP reporting.

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**Compliance with ethical standards**

**Conflict of interest** The authors declare no competing interests.

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**7 Appendix**

For convenience of exposition, we suppress subindex throughout the Appendix, and two equations from (Abramowitz and Stegun, 1970, Eqs. 7.11 and 7.4.32) are given:

$$Erf(z) = \frac{2}{\sqrt{\pi}} \int_0^z \exp(-t^2) dt = 2 \int_0^{\sqrt{2}z} \phi(t) dt,$$

$$\int \exp-(kx^2 + 2mx + n) dx = \frac{1}{2} \sqrt{\frac{\pi}{k}} \exp\left(\frac{m^2 - kn}{k}\right) Erf\left(\sqrt{k}x + \frac{m}{\sqrt{k}}\right) + C, \quad k \neq 0,$$

where *C* denotes a finite constant.

## 7.1 Derivation of Proposition 1

Given that  $(Q, a, b) \in \mathbb{R}$ ,  $b > 0$ ,  $\text{Erf}(-x) = -\text{Erf}(x)$ , and define  $\varepsilon = \sqrt{2}v$ , we have:

$$\begin{aligned} I(Q, a, b) &= \sqrt{2} \int_{-\infty}^{\frac{Q}{\sqrt{2}}} \left( \int_{-\infty}^{\sqrt{2}av} \phi(\zeta) d\zeta \right) \phi(\sqrt{2}bv) dv \\ &= \frac{\sqrt{2}}{2} \int_{-\infty}^{\frac{Q}{\sqrt{2}}} (1 + \text{Erf}(av)) \phi(\sqrt{2}bv) dv \end{aligned}$$

Tsay et al. (2013) show that  $\text{Erf}(x)$  can be well approximated by a function,  $g(x) = 1 - \exp(c_1x + c_2x^2)$  for  $x \geq 0$ , where  $c_1$  and  $c_2$  are discussed in “Interval stochastic frontier model.” We divide the derivation into four cases:  $(Q \geq 0, a \geq 0)$ ,  $(Q \leq 0, a \geq 0)$ ,  $(Q \geq 0, a \leq 0)$ , and  $(Q \leq 0, a \leq 0)$ .

**Case 1.**  $(Q \geq 0, a \geq 0)$ :

$$\begin{aligned} I(Q, a, b) &= \frac{\sqrt{2}}{2} \int_{-\infty}^{\frac{Q}{\sqrt{2}}} (1 + \text{Erf}(av)) \phi(\sqrt{2}bv) dv \\ &= \frac{\sqrt{2}}{2} \int_{-\infty}^0 (1 + \text{Erf}(av)) \phi(\sqrt{2}bv) dv \\ &\quad + \frac{\sqrt{2}}{2} \int_0^{\frac{Q}{\sqrt{2}}} (1 + \text{Erf}(av)) \phi(\sqrt{2}bv) dv \\ &= \frac{\sqrt{2}}{2} \int_0^{\infty} (1 - \text{Erf}(av)) \phi(\sqrt{2}bv) dv \\ &\quad + \frac{\sqrt{2}}{2} \int_0^{\frac{Q}{\sqrt{2}}} (1 + \text{Erf}(av)) \phi(\sqrt{2}bv) dv \\ &\approx \frac{\sqrt{2}}{2} \int_0^{\infty} \exp(c_1av + c_2a^2v^2) \frac{1}{\sqrt{2\pi}} \exp(-b^2v^2) dv \\ &\quad + \sqrt{2} \int_0^{\frac{Q}{\sqrt{2}}} \frac{1}{\sqrt{2\pi}} \exp(-b^2v^2) dv \\ &\quad - \frac{\sqrt{2}}{2} \int_0^{\frac{Q}{\sqrt{2}}} \exp(c_1av + c_2a^2v^2) \frac{1}{\sqrt{2\pi}} \exp(-b^2v^2) dv \\ &= \frac{1}{2\sqrt{\pi}} \int_0^{\infty} \exp(-(b^2 - a^2c_2)v^2 + ac_1v) dv \\ &\quad + \frac{1}{\sqrt{\pi}} \int_0^{\frac{Q}{\sqrt{2}}} \exp(-b^2v^2) dv \\ &\quad - \frac{1}{2\sqrt{\pi}} \int_0^{\frac{Q}{\sqrt{2}}} \exp(-(b^2 - a^2c_2)v^2 + ac_1v) dv. \end{aligned}$$

When we use (7.4.32) of Abramowitz and Stegun (1970),  $I(Q, a, b)$  in this case can be approximated by:

$$I(Q, a, b) \approx \frac{\exp\left(\frac{a^2c_1^2}{4b^2-4a^2c_2}\right) \left[1 - \text{Erf}\left(\frac{-ac_1 + \sqrt{2}Q(b^2 - a^2c_2)}{2\sqrt{b^2 - a^2c_2}}\right)\right]}{4\sqrt{b^2 - a^2c_2}} + \frac{1}{2b} \text{Erf}\left(\frac{bQ}{\sqrt{2}}\right).$$

**Case 2.**  $(Q \leq 0, a \geq 0)$ :

$$\begin{aligned} I(Q, a, b) &= \frac{\sqrt{2}}{2} \int_{-\infty}^{\frac{Q}{\sqrt{2}}} (1 + \text{Erf}(av)) \phi(\sqrt{2}bv) dv \\ &= \frac{\sqrt{2}}{2} \int_{-\infty}^{\infty} (1 - \text{Erf}(av)) \phi(\sqrt{2}bv) dv \\ &\approx \frac{\sqrt{2}}{2} \int_{-\infty}^{\infty} \exp(c_1av + c_2a^2v^2) \frac{1}{\sqrt{2\pi}} \exp(-b^2v^2) dv \\ &= \frac{1}{2\sqrt{\pi}} \int_{-\infty}^{\infty} \exp(-(b^2 - a^2c_2)v^2 + ac_1v) dv. \end{aligned}$$

When we use (7.4.32) of Abramowitz and Stegun (1970),  $I(Q, a, b)$  in this case can be approximated by:

$$I(Q, a, b) \approx \frac{\exp\left(\frac{a^2c_1^2}{4b^2-4a^2c_2}\right) \left[1 - \text{Erf}\left(\frac{-ac_1 - \sqrt{2}Q(b^2 - a^2c_2)}{2\sqrt{b^2 - a^2c_2}}\right)\right]}{4\sqrt{b^2 - a^2c_2}}.$$

**Case 3.**  $(Q \geq 0, a \leq 0)$ :

$$\begin{aligned} I(Q, a, b) &= \frac{\sqrt{2}}{2} \int_{-\infty}^{\frac{Q}{\sqrt{2}}} (1 + \text{Erf}(av)) \phi(\sqrt{2}bv) dv \\ &= \frac{\sqrt{2}}{2} \int_{-\infty}^0 (1 + \text{Erf}(av)) \phi(\sqrt{2}bv) dv \\ &\quad + \frac{\sqrt{2}}{2} \int_0^{\frac{Q}{\sqrt{2}}} (1 + \text{Erf}(av)) \phi(\sqrt{2}bv) dv \\ &= \frac{\sqrt{2}}{2} \int_{-\infty}^0 (1 + \text{Erf}(av)) \phi(\sqrt{2}bv) dv \\ &\quad + \frac{\sqrt{2}}{2} \int_{-\frac{Q}{\sqrt{2}}}^0 (1 - \text{Erf}(av)) \phi(\sqrt{2}bv) dv \\ &\approx -\frac{\sqrt{2}}{2} \int_{-\infty}^0 \exp(c_1av + c_2a^2v^2) \frac{1}{\sqrt{2\pi}} \exp(-b^2v^2) dv \\ &\quad + \sqrt{2} \int_{-\infty}^0 \frac{1}{\sqrt{2\pi}} \exp(-b^2v^2) dv \\ &\quad + \frac{\sqrt{2}}{2} \int_{-\frac{Q}{\sqrt{2}}}^0 \exp(c_1av + c_2a^2v^2) \frac{1}{\sqrt{2\pi}} \exp(-b^2v^2) dv \\ &= -\frac{1}{2\sqrt{\pi}} \int_{-\infty}^0 \exp(-(b^2 - a^2c_2)v^2 + ac_1v) dv \\ &\quad + \frac{1}{\sqrt{\pi}} \int_{-\infty}^0 \exp(-b^2v^2) dv \\ &\quad + \frac{1}{2\sqrt{\pi}} \int_{-\frac{Q}{\sqrt{2}}}^0 \exp(-(b^2 - a^2c_2)v^2 + ac_1v) dv. \end{aligned}$$

When we use (7.4.32) of Abramowitz and Stegun (1970),  $I(Q, a, b)$  in this case can be approximated by:

$$\begin{aligned} I(Q, a, b) &\approx -\frac{\exp\left(\frac{a^2c_1^2}{4b^2-4a^2c_2}\right) \left[1 + \text{Erf}\left(\frac{\sqrt{a^2c_1}}{2\sqrt{b^2 - a^2c_2}}\right)\right]}{4\sqrt{b^2 - a^2c_2}} + \frac{1}{2b} \\ &\quad - \frac{\exp\left(\frac{a^2c_1^2}{4b^2-4a^2c_2}\right) \left[\text{Erf}\left(\frac{\sqrt{a^2c_1}}{2\sqrt{b^2 - a^2c_2}}\right) + \text{Erf}\left(\frac{-ac_1 - \sqrt{2}Q(b^2 - a^2c_2)}{2\sqrt{b^2 - a^2c_2}}\right)\right]}{4\sqrt{b^2 - a^2c_2}}. \end{aligned}$$

**Case 4.**  $(Q \leq 0, a \leq 0)$ :

$$\begin{aligned} I(Q, a, b) &= \frac{\sqrt{2}}{2} \int_{-\infty}^{\frac{Q}{\sqrt{2}}} (1 + \text{Erf}(av)) \phi(\sqrt{2}bv) dv \\ &\approx -\frac{\sqrt{2}}{2} \int_{-\infty}^{\frac{Q}{\sqrt{2}}} \exp(c_1av + c_2a^2v^2) \frac{1}{\sqrt{2\pi}} \exp(-b^2v^2) dv \\ &\quad + \sqrt{2} \int_{-\infty}^{\frac{Q}{\sqrt{2}}} \frac{1}{\sqrt{2\pi}} \exp(-b^2v^2) dv \\ &= -\frac{1}{2\sqrt{\pi}} \int_{-\infty}^{\frac{Q}{\sqrt{2}}} \exp(-(b^2 - a^2c_2)v^2 + ac_1v) dv \\ &\quad + \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\frac{Q}{\sqrt{2}}} \exp(-b^2v^2) dv. \end{aligned}$$

When we use (7.4.32) of Abramowitz and Stegun (1970),  $I(Q, a, b)$  in this case can be approximated by:

$$I(Q, a, b) \approx -\frac{\exp\left(\frac{a^2 c_1^2}{4b^2 - 4a^2 c_2}\right) \left[1 + \operatorname{Erf}\left(\frac{\sqrt{a^2 c_1}}{2\sqrt{b^2 - a^2 c_2}}\right)\right]}{4\sqrt{b^2 - a^2 c_2}} + \frac{1}{2b} - \frac{\exp\left(\frac{a^2 c_1^2}{4b^2 - 4a^2 c_2}\right) \left[\operatorname{Erf}\left(\frac{\sqrt{a^2 c_1}}{2\sqrt{b^2 - a^2 c_2}}\right) + \operatorname{Erf}\left(\frac{-ac_1 + \sqrt{2Q(b^2 - a^2 c_2)}}{2\sqrt{b^2 - a^2 c_2}}\right)\right]}{4\sqrt{b^2 - a^2 c_2}} + \frac{1}{2b} \operatorname{Erf}\left(\frac{bQ}{\sqrt{2}}\right).$$

Combining the results in Cases 1–4, we obtain the result in Proposition 1.

### 7.2 Derivation of Proposition 3

$$\begin{aligned} E(u|a < \varepsilon < b) &= E(E(u|\varepsilon)|a < \varepsilon < b) \\ &= \frac{1}{[F(\varepsilon)]_a^b} \int_a^b E(u|\varepsilon) f(\varepsilon) d\varepsilon \\ &= \frac{1}{[F(\varepsilon)]_a^b} \int_a^b \left( S \frac{\sigma_u^2}{\sigma^2} \varepsilon + \frac{\sigma_u \sigma_v}{\sigma} \frac{\phi\left(\frac{\lambda \varepsilon}{\sigma}\right)}{\Phi\left(\frac{\lambda \varepsilon}{\sigma}\right)} \right) \left( \frac{2}{\sigma} \phi\left(\frac{\varepsilon}{\sigma}\right) \Phi\left(S \lambda \frac{\varepsilon}{\sigma}\right) \right) d\varepsilon \\ &= \frac{1}{[F(\varepsilon)]_a^b} \left[ \int_a^b S \frac{\sigma_u^2}{\sigma^2} \varepsilon \frac{2}{\sigma} \phi\left(\frac{\varepsilon}{\sigma}\right) \Phi\left(S \lambda \frac{\varepsilon}{\sigma}\right) d\varepsilon + \int_a^b \frac{2\sigma_u \sigma_v}{\sigma^2} \phi\left(\frac{\varepsilon}{\sigma}\right) \phi\left(\lambda \frac{\varepsilon}{\sigma}\right) d\varepsilon \right] \\ &= E_1 + E_2. \end{aligned}$$

We first rewrite  $E_1$  as:

$$\begin{aligned} E_1 &= \frac{1}{[F(\varepsilon)]_a^b} S \frac{\sigma_u^2}{\sigma^2} \int_a^b \varepsilon f(\varepsilon) d\varepsilon \\ &= S \frac{\sigma_u^2}{\sigma^2} E(\varepsilon|a < \varepsilon < b). \end{aligned}$$

As we can see, the integral part in  $E_1$  is a censored conditional expectation of  $f(\varepsilon)$  given that  $\varepsilon \in (a, b)$ . Using Proposition 1 in Flecher et al. (2009), we can evaluate  $E_1$  as:

$$E_1 = S \frac{\sigma_u^2}{\sigma^2} \left\{ -\sigma \frac{[f(\varepsilon)]_a^b}{[F(\varepsilon)]_a^b} + \frac{2\lambda}{\sqrt{2\pi}\lambda^*} \frac{[\Phi\left(\frac{\lambda^* \varepsilon}{\sigma}\right)]_a^b}{[F(\varepsilon)]_a^b} \right\},$$

where  $\lambda^* = \sqrt{1 + \lambda^2}$ .

We can also recast  $E_2$  as:

$$\begin{aligned} E_2 &= \frac{2\sigma_u \sigma_v}{\sigma^2} \frac{1}{[F(\varepsilon)]_a^b} \int_a^b \phi\left(\frac{\varepsilon}{\sigma}\right) \phi\left(\lambda \frac{\varepsilon}{\sigma}\right) d\varepsilon \\ &= \frac{2\sigma_u \sigma_v}{\sigma^2} \frac{1}{[F(\varepsilon)]_a^b} \int_a^b \frac{1}{2\pi} \exp\left[-\frac{1}{2}\left(\frac{\varepsilon}{\sigma}\right)^2\right] \exp\left[-\frac{\lambda^2}{2}\left(\frac{\varepsilon}{\sigma}\right)^2\right] d\varepsilon \\ &= \frac{\sigma_u \sigma_v}{\pi \sigma^2} \frac{1}{[F(\varepsilon)]_a^b} \int_a^b \exp\left(-\frac{\lambda^{*2} \varepsilon^2}{2\sigma^2}\right) d\varepsilon. \end{aligned}$$

Using (7.4.32) of Abramowitz and Stegun (1970), we can evaluate  $E_2$  as:

$$\begin{aligned} E_2 &= \frac{\sigma_u \sigma_v}{\pi \sigma^2} \frac{1}{[F(\varepsilon)]_a^b} \left[ \frac{1}{2} \sqrt{\frac{\pi}{\lambda^{*2}}} \operatorname{erf}\left(\frac{\lambda^*}{\sqrt{2}\sigma} \varepsilon\right) \right]_a^b \\ &= \frac{\sigma_u \sigma_v}{\sqrt{2\pi}\sigma\lambda^*} \left[ \frac{\operatorname{erf}\left(\frac{\lambda^*}{\sqrt{2}\sigma} \varepsilon\right)}{[F(\varepsilon)]_a^b} \right]_a^b. \end{aligned}$$

With the above results, we now obtain the JLMS interval efficiency estimate.

### 7.3 Proof of Corollary 1

We express the limit of Proposition 3 as:

$$\begin{aligned} \lim_{\delta \rightarrow 0} E(u|c < \varepsilon < c + \delta) &= \lim_{\delta \rightarrow 0} S \frac{\sigma_u^2}{\sigma^2} \left\{ -\sigma \frac{[f(\varepsilon)]_c^{c+\delta}}{[F(\varepsilon)]_c^{c+\delta}} + \frac{2\lambda}{\sqrt{2\pi}\lambda^*} \frac{[\Phi\left(\frac{\lambda^* \varepsilon}{\sigma}\right)]_c^{c+\delta}}{[F(\varepsilon)]_c^{c+\delta}} \right\} \\ &\quad + \lim_{\delta \rightarrow 0} \frac{\frac{\sigma_u \sigma_v}{\sqrt{2\pi}\sigma\lambda^*} \left[ \operatorname{erf}\left(\frac{\lambda^*}{\sqrt{2}\sigma} \varepsilon\right) \right]_c^{c+\delta}}{[F(\varepsilon)]_c^{c+\delta}} \\ &= \lim_{\delta \rightarrow 0} S \frac{\sigma_u^2}{\sigma^2} \frac{-\frac{2\lambda}{\sigma} \phi\left(\frac{c+\delta}{\sigma}\right) \phi\left(\lambda \frac{c+\delta}{\sigma}\right) - \frac{2\lambda}{\sigma} \phi\left(\frac{c+\delta}{\sigma}\right) \Phi\left(\lambda \frac{c+\delta}{\sigma}\right)}{f(c+\delta)} \\ &\quad + \lim_{\delta \rightarrow 0} S \frac{\sigma_u^2}{\sigma^2} \frac{2S\lambda}{\sqrt{2\pi}} \frac{\phi\left(\lambda \frac{c+\delta}{\sigma}\right)}{f(c+\delta)} \\ &\quad + \lim_{\delta \rightarrow 0} \frac{2}{\sqrt{2\pi}} \frac{\sigma_u \sigma_v}{\sigma^2} \frac{\phi\left(\lambda \frac{c+\delta}{\sigma}\right)}{f(c+\delta)} \\ &= S \frac{\sigma_u^2}{\sigma^2} c - \frac{2\lambda}{\sigma} \phi\left(\frac{c}{\sigma}\right) \phi\left(\lambda \frac{c}{\sigma}\right) - \frac{2\lambda}{\sigma} \phi\left(\lambda \frac{c}{\sigma}\right) \Phi\left(\lambda \frac{c}{\sigma}\right) \\ &\quad + \frac{2}{\sqrt{2\pi}} \frac{\sigma_u \sigma_v}{\sigma^2} \frac{\phi\left(\lambda \frac{c}{\sigma}\right)}{f(c)} \\ &= S \frac{\sigma_u^2}{\sigma^2} c + \frac{\sigma_u \sigma_v}{\sigma} \frac{\phi\left(\lambda \frac{c}{\sigma}\right)}{\Phi\left(\lambda \frac{c}{\sigma}\right)}, \end{aligned}$$

where the second equality is based on L’hopital Rule, and the fourth equality follows from the observation that  $\phi\left(\frac{c+\delta}{\sigma}\right) \phi\left(\lambda \frac{c+\delta}{\sigma}\right) = \frac{1}{\sqrt{2\pi}} \phi\left(\lambda^* \frac{c+\delta}{\sigma}\right)$ .

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