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Elasticities of Substitution and Complementarity: A Synthesis

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Abstract

We derive the relationships between the net and gross elasticities of substitution and complementarity (i.e., the elasticities that refer either to the conditional or unconditional, direct or inverse demand system) in the general case of non-homothetic, variable-returns-to-scale technologies. We also show that the so-called Hicks Elasticity of Complementarity (Hicks, *Oxford economic Papers* 22, 289–296 (1970)) is dual to a full-fledged elasticity of *gross* input substitution that we call the Hotelling/Lau Elasticity of Substitution (Lau, *Production Economics: A Dual Approach to Theory and Applications.* Amsterdam: North-Holand (1978)). The former is, in fact, the proper elasticity of substitution in the case of the inverse, unconditional input demand. Our results should clarify some issues about the input substitutability classification.

JEL Classifications: D11, D24, D33

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1. Introduction

All economists are familiar with the concept of a demand function, and know that it can be defined conditionally (i.e., assuming that output or utility or some other quantity is somehow kept constant), or unconditionally (i.e., without presuming any "compensating variation"). It is also well known that demand systems can be *direct*, i.e., having quantities as a function of prices, or *inverse*, with prices as a function of quantities. Indeed, the inverse demand system, which played a historical role in the development of microeconomic theory (see the vast literature on the so-called integrability problem), is theoretically relevant to the analysis of the competitive equilibrium (see e.g., Seidman, 1989) and obviously of the monopolistic and oligopoly choices. Moreover, it is becoming an established tool of empirical economics (most applications are to agricultural and natural resources markets): see e.g., Barten and Bettendorf (1989), Eales (1994), Eales and Unnevehr (1994), Holt (2002) and the other references to the literature given in Park and Thurman (1999, p. 950) and Kim (2000, p. 246). However, when it comes to the properties of inputs (or commodities),[1] the previous four-way distinction gets somewhat confused, and it is sometimes unclear in the literature which concept is referred to.

In particular, according to standard terminology (see e.g., Mas-Colell et al., 1995, p. 70), *net* input substitution refers to changes along an isoquant, while *gross* substitution accounts also for the output change. That is, net substitution refers to a conditional ("compensated") demand system, while gross substitution uses unconditional ("uncompensated") demand. Moreover, Hicks (1956) made a fundamental step towards a clear classification by distinguishing between *p*-substitutes, according to the direct demand system, and *q*-substitutes, according to the inverse demand system. Now, it is well known that even inputs that are, say, *direct*, *net* substitutes, are not necessarily *direct*, *gross* substitutes, and conversely (see e.g., Chambers, 1988, Sec. 3.1). This notwithstanding, it is common to refer only to net substitutability, even when the distinction between direct and inverse measures is taken into account (see e.g., Kim, 2000).

One reason is perhaps that in the special but important case of constant returns to scale *direct* gross substitution is not well defined (this is so because in this case the profit-maximizing input vector is multiple or null or even undefined; i.e., it is not a continuous function of factor prices). Another is that in principle gross substitution can be characterized in terms of the conditional demand (see e.g., Bertoletti, 2001 in the case of the direct demand system). But more important seems the fact that the dual properties of the inverse demand systems in production theory, as far as we know, have not yet been fully established (see e.g., Anderson, 1980; Barten and Bettendorf, 1989; Kohli, 1985 for the relevant consumer demand theory). Actually, the relationships among direct and inverse and gross and net demand systems do not appear to be understood in the general case, i.e., when the technology may be non-homothetic and may also have variable returns to scale (again see Kim, 2000; Park and Thurman, 1999).

For example, consider the *elasticity of complementarity*, often so-called the Hicks Elasticity of Complementarity (HEC: see e.g., Kim, 2000). Hicks (1970) proposed it as a measure of *q*-substitutability dual to the well-known Allen/Uzawa Elasticity of Substitution (AUES: Allen and Hicks, 1934), a measure of *net p*-substitutability. Indeed, the two elasticities are intimately related, but as Sato and Koizumi (1973) showed for *k*-factor production functions, with $k \geq 3$ their duality relationship becomes rather awkward. In particular, the latter authors demonstrated, still assuming constant returns to scale (their analysis was extended to the general case by Syrquin and Hollender, 1982), that the HEC measures the cross-elasticity of the inverse conditional input demand, *with marginal cost held constant*. So characterized, the HEC does not clearly show its dual nature (i.e., its being a measure of *gross* complementarity), and in fact it has been convincingly criticized (Kim, 2000) for not being equivalent (except under constant returns to scale) to the so-called Antonelli Elasticity of Complementarity (AEC: Blackorby and Russell, 1981),[2] which is perfectly dual to the AUES. A second undesirable by-product of the original HEC characterization has been the creation of a somewhat inconsistent input classification. In fact, in a growing empirical literature (see references in Kim, 2000, p. 246) inputs are stratified as q-substitutes or q-complements either according to the sign of the HEC or to the sign of the AEC, and these signs need not to coincide.

In this paper, we show that, exploiting the duality between the direct production function and the profit function, the HEC can be characterized as a sound measure of *gross* q-complementarity, perfectly dual to a known elasticity of *gross* p-substitution that we call the Hotelling/Lau Elasticity of Substitution (HLES: see Lau, 1978, p. 197). Then, using this gross substitution approach[3] and by deriving some novel duality results concerning the inverse demand systems, we can establish the relationships between the net and gross elasticities of substitution and complementarity in the general case. These also make clear the obvious point that an input taxonomy should carefully distinguish between direct and inverse demand systems, *and* between gross and net substitution or complementarity. Thus, our results should be helpful in settling some issue in the long-standing questions about the proper input substitutability classification.

Finally, we would like to justify our concern with the AUES and related measures. We use them as a suitable tool to discuss input substitutability (they can also be used to investigate the functional features of the underlying technology: e.g., see Kim, 1997 on separability issues). But, while it is still common among practitioners to use the sign of those elasticities to stratify inputs (eventhough to some extent inconsistently, as argued above), we have to mention that Blackorby and Russell (1989) forcefully criticized the use of the AUES. These authors stated (p. 884) that: "In fact, the elasticity-of-substitution concept, as originally conceived by Hicks, has nothing to do with the substitute/complement taxonomy". In addition, they argued that, in contrast to the cross partial derivative of the demand function (which would serve the same purpose), the AUES is not a meaningful quantitative measure to assess input substitutability. However, our gross substitution perspective suggests that the AUES and the AEC play a role in the comparative statistics of the *unconditional* demand functions (see Sections 4 and 5), and provide interesting quantitative information for that purpose (to be interpreted jointly with the one provided by the relevant "size" elasticities of the cost and distance functions: also see Bertoletti, 2001). Accordingly, we believe that those elasticities deserve some attention and to be the subject of empirical investigation. Of course, to classify inputs one could directly refer to the sign of the cross second derivatives of the profit and of the production function, but it seems to us that they provide a (hopefully interesting) dual characterization of the gross substitution properties.

The paper is organized as follows: Section 2 illustrates the basic duality between the direct production function and the (normalized) profit function. Section 3 presents our fourfold input taxonomy, and shows the duality between the HEC and the HLES. Section 4 recalls some results concerning net and gross substitution effects in direct demand systems. Section 5 establishes the corresponding results concerning the inverse demand systems. Section 6 concludes.

2. Direct Versus Inverse Unconditional Demand System

Consider a (concave) production function $y = f(\boldsymbol{x})$ relating a vector of inputs $\boldsymbol{x}$ to the output y. Given a vector $\boldsymbol{w}$ of (positive) input price, and a (positive) output price p, the *normalized* profit function (see e.g., Chambers, 1988: Chapter 4) is

$$\pi(\underline{\boldsymbol{w}}) = \max_{\boldsymbol{x} \geq 0}(f(\boldsymbol{x}) - \underline{\boldsymbol{w}}\boldsymbol{x}), \tag{1}$$

where $\underline{\boldsymbol{w}} = \boldsymbol{w}/p$ is the vector of normalized input prices. To get well-defined unconditional demand functions, we assume that there are (at least locally) increasing marginal costs. The solution of the previous optimization problem is then given by the system $\boldsymbol{x}(\underline{\boldsymbol{w}})$ of the so-called unconditional (i.e., profit-maximizing) input demands, and satisfies the first-order conditions:

$$\underline{\boldsymbol{w}} = \boldsymbol{D}_{\boldsymbol{x}} f(\boldsymbol{x}), \tag{2}$$

where $\boldsymbol{D}_{\boldsymbol{x}} f(\boldsymbol{x})$ indicates the vector of the first derivatives of $f(\boldsymbol{x})$.

The demand system $\boldsymbol{x}(\underline{\boldsymbol{w}})$ has several well-known properties: in particular, it has a symmetric negative semi-definite Jacobian and satisfies the derivative property:

$$\boldsymbol{x}(\underline{\boldsymbol{w}}) = -\boldsymbol{D}_{\underline{\boldsymbol{w}}}\pi(\underline{\boldsymbol{w}}), \tag{3}$$

known as Hotelling's lemma. At the same time (2) also characterizes the system of the so-called *inverse* unconditional input demand functions, $\underline{\boldsymbol{w}}(\boldsymbol{x})$, which gives the normalized input prices $\underline{\boldsymbol{w}}$ of the input vector $\boldsymbol{x}$ (i.e., the relative input prices that would induce a competitive firm to adopt the input vector $\boldsymbol{x}$). Given the profit function (1), a basic duality result is given by the following way to recover the original production function (again, see Chambers, 1988, pp. 144–145)

$$f(\boldsymbol{x}) = \min_{\underline{\boldsymbol{w}} \geq 0}(\pi(\underline{\boldsymbol{w}}) + \underline{\boldsymbol{w}}\boldsymbol{x}). \tag{4}$$

Clearly, $\underline{\boldsymbol{w}}(\boldsymbol{x})$ solves (4) and satisfies (2), which also expresses its derivative property.

To complete the duality, note that of course (in matrix form):

$$\boldsymbol{D}_{\boldsymbol{x}}\underline{\boldsymbol{w}}(\boldsymbol{x}(\underline{\boldsymbol{w}})) = \boldsymbol{D}_{\underline{\boldsymbol{w}}}\boldsymbol{x}(\underline{\boldsymbol{w}})^{-1} = \boldsymbol{D}^2_{\boldsymbol{x}} f(\boldsymbol{x}(\underline{\boldsymbol{w}})), \tag{5}$$

where $\boldsymbol{D}^2_{\boldsymbol{x}}f(\boldsymbol{x})$ indicates the Hessian matrix of $f(\boldsymbol{x})$ and we make the "regularity" assumption[4] that it is invertible (of course this is *not* the case if there are constant returns of scale: see Section 3). As is indicated in Section 1 and discussed in next section, as far as *gross* substitutability is concerned inputs can be consistently classified according to the signs of the cross-derivatives of either of the two Jacobians in (5). In general, those signs need not coincide (matrix inversion is a rather involved operation). Note, however, that if all the off-diagonal elements of the Hessian matrix $\boldsymbol{D}^2_{\boldsymbol{x}} f(\boldsymbol{x}(\underline{\boldsymbol{w}}))$ are non-negative, a clear-cut conclusion concerning the substitutability properties of $\boldsymbol{x}(\underline{\boldsymbol{w}})$ follows. In fact, it is well known that in that case $\boldsymbol{D_w x}(\underline{\boldsymbol{w}})$ must be a non-positive matrix (see e.g., Takayama, 1985, Chapter 4, and in particular Theorem 4.D.3, p. 393). Thus, all inputs would be gross p-complements and, at the same time, gross q-complements (see next section). For the same reason, if the off-diagonal elements of $\boldsymbol{D_{\underline{w}} x}(\underline{\boldsymbol{w}})$ are all non-negative, $\boldsymbol{D_x}\ \underline{\boldsymbol{w}}(\boldsymbol{x}(\underline{\boldsymbol{w}}))$ must be non-positive, and all inputs are both gross p-substitutes and, at the same time, gross q-substitutes. While considering gross substitution, we are thus implying that such global properties are not *assumed* to hold.

3. Elasticities

Proper input classifications should carefully distinguish between the reference to direct and to inverse demand systems, and between gross and net substitution or complementarity. For example, in terms of the cost function, $c(y, \boldsymbol{w})$ (subscripts indicate partial derivatives), the AUES between inputs i and j, $\sigma_{ij}(y, \boldsymbol{w})$, can be written:

$$\sigma_{ij}(y, \boldsymbol{w}) = \frac{c_{ij}(y, \boldsymbol{w})\, c(y, \boldsymbol{w})}{C_{ij}(y, \boldsymbol{w})\, c_j(y, \boldsymbol{w})} = \frac{\tilde{\varepsilon}_{ij}(y, \boldsymbol{w})}{\theta_j(y, \boldsymbol{w})}, \tag{6}$$

where $\tilde{\varepsilon}_{ij}(y,\boldsymbol{w}) = \partial \ln \tilde{x}_i(y,\boldsymbol{w})/\partial \ln w_j$ is the cross-elasticity of the conditional (output-"compensated") demand of input i with respect to input price j,[5] and $\theta_j(y, \boldsymbol{w}) = w_j \tilde{x}_j(y, \boldsymbol{w})/c(y, \boldsymbol{w})$ is the cost share of input j. Because the AUES is defined in terms of the conditional factor demand, inputs should be stratified as *net* p-substitutes if and only if $\sigma_{ij}(y, \boldsymbol{w}) \geq 0$, and *net* p-complements otherwise.

Analogously, the AEC, $\rho_{ij}(y, \boldsymbol{x})$, is a measure of complementarity (see Blackorby and Russell, 1981, Section 4) which can be stated by making use of the distance function $d(y,\boldsymbol{x})$ (see e.g., Cornes, 1992 Sections 3.5 and 5.8; Deaton, 1979) as follows:

$$\rho_{ij}(y, \boldsymbol{x}) = \frac{d_{ij}(y, \boldsymbol{w}) d(y, \boldsymbol{x})}{d_i(y, \boldsymbol{x}) d_j(y, \boldsymbol{x})} = \frac{\tilde{\eta}_{ij}(y, \boldsymbol{x})}{\tau_j(y, \boldsymbol{x})}, \tag{7}$$

where $\tilde{\eta}_{ij}(y, \boldsymbol{x}) = \partial \ln a_i(y, \boldsymbol{x})/\partial \ln x_j$ is the cross-elasticity of the *inverse* conditional demand of input i, and $\tau_j(y, \boldsymbol{x}) = x_j a_j(y, \boldsymbol{x})/d(y, \boldsymbol{x})$ is the distance share of input j (it may be worth reminding the careful reader that $\boldsymbol{a}(y, \boldsymbol{x})$ gives the input price vector that would induce a cost-minimizing firm to adopt an input vector *proportional* to $\boldsymbol{x}$ when producing y, *normalized* by total cost; i.e., $c(y, \boldsymbol{a}) = 1$).[6] Accordingly, inputs i and j should be said to be *net* q-complements if and only if $\rho_{ij}(y, \boldsymbol{x}) \geq 0$, and *net* q-substitutes otherwise (see e.g. Kim, 1997).

Gross substitutability, as noted in Section 1, is somewhat less emphasized in the literature. Perhaps one reason is that the direct unconditional demand system $\boldsymbol{x}(\underline{\boldsymbol{w}})$ is not well defined in the theoretically important case of constant returns to scale[7] (but note that the *inverse* demand system *is* well defined even in that case).

However, an obvious elasticity of gross substitution is the following HLES:

$$\sigma_{ij}^{H}(\underline{\boldsymbol{w}}) = -\frac{\pi_{ij}(\underline{\boldsymbol{w}})\pi(\underline{\boldsymbol{w}})}{\pi_i(\underline{\boldsymbol{w}})\pi_j(\underline{\boldsymbol{w}})} = \frac{\varepsilon_{ij}(\underline{\boldsymbol{w}})}{\omega_j(\underline{\boldsymbol{w}})}, \tag{8}$$

where $\varepsilon_{ij}(\underline{\boldsymbol{w}}) = \partial \ln x_i(\underline{\boldsymbol{w}})/\partial \ln \underline{w}_j$ is the cross-elasticity of the unconditional input demand and $\omega_j(\underline{\boldsymbol{w}}) = \underline{w}_j x_j(\underline{\boldsymbol{w}})/\pi(\underline{\boldsymbol{w}})$ is the profit share of input j. The HLES was proposed by Lau (1978, p. 197): inputs i and j should be called either *gross p*-substitutes or *p*-complements according to the sign of $\sigma_{ij}^{H}(\underline{\boldsymbol{w}})$.

Finally, a measure of *gross* complementarity is defined by the following HEC (Hicks, 1970):

$$\rho_{ij}^{H}(\boldsymbol{x}) = \frac{f_{ij}(\boldsymbol{x})f(\boldsymbol{x})}{f_i(\boldsymbol{x})f_j(\boldsymbol{x})} = \frac{\eta_{ij}(\boldsymbol{x})}{\gamma_j(\boldsymbol{x})}, \tag{9}$$

where $\eta_{ij}(\boldsymbol{x}) = \partial \ln \underline{w}_i(\boldsymbol{x})/\partial \ln x_j$ is the cross-elasticity of the *inverse* unconditional input demand (see e.g., Kim, 1997, p. 1178) and $\gamma_j(\boldsymbol{x}) = x_j \underline{w}_j(\boldsymbol{x})/f(\boldsymbol{x})$ is the (normalized) output share of input j. The HEC is usually characterized in terms of the cost-minimization problem alone, by noting that the input price vector is implicitly changed so as to keep marginal cost constant (see Sato and Koizumi, 1973; Syrquin and Hollender, 1982). However, inputs should clearly be stratified either as *gross q*-complements or *q*-substitutes according to its sign (see Kim, 2000, p. 248).

The dual nature of the HLES and HEC follows immediately from (5): note that in matrix form $\boldsymbol{\varepsilon}(\underline{\boldsymbol{w}})^{-1} = \boldsymbol{\eta}(\boldsymbol{x}(\underline{\boldsymbol{w}}))$. Moreover, $\omega_j(\underline{\boldsymbol{w}})(1-\xi(\boldsymbol{x}(\underline{\boldsymbol{w}}))) = \gamma_j(\boldsymbol{x}(\underline{\boldsymbol{w}}))$, where $\xi(\boldsymbol{x}) = \partial \ln f(\lambda \boldsymbol{x})/\partial \ln \lambda|_{\lambda=1}$ is the elasticity of scale (a standard local measure of the returns to scale). Thus, with $\hat{\boldsymbol{\gamma}}(\boldsymbol{x})$ being the diagonal matrix whose generic (diagonal) element is $\gamma_j(\boldsymbol{x})$, we have that

$$\hat{\boldsymbol{\gamma}}(\boldsymbol{x})\boldsymbol{\sigma}^{H}(\underline{\boldsymbol{w}}(\boldsymbol{x}))\hat{\boldsymbol{\gamma}}(\boldsymbol{x})\boldsymbol{\rho}^{H}(\boldsymbol{x}) = (1-\xi(\boldsymbol{x}))\boldsymbol{I}, \tag{10}$$

where $\boldsymbol{I}$ is the identity matrix (note that all matrices in (10) are symmetric).[8] The duality between the HEC and HLES expressed in (10) should be compared with Syrquin and Hollender (1982), whose results found that the relationship between the HEC and AUES was not perfectly dual. Indeed, it would seem that the HLES/HEC dual relationship is the final piece Hicks (1970, p. 289) felt missing in the "Elasticity of Substitution Story" and thought he had found. In Sections 4 and 5 we establish the general relationships between the previous net and gross elasticities of substitution and complementarity.

4. Gross Versus Net Substitution

To illustrate the relationship between gross and net substitution, it can be shown that (see Appendix A and Bertoletti, 2001):

$$\sigma_{ij}^{H}(\underline{\boldsymbol{w}}) = (n(y,\boldsymbol{w})-1)\left[\sigma_{ij}(y,\boldsymbol{w}) - \frac{\tilde{\varepsilon}_{iy}(y,\boldsymbol{w})}{n(y,\boldsymbol{w})}\frac{\tilde{\varepsilon}_{jy}(y,\boldsymbol{w})}{\zeta(y,\boldsymbol{w})}\right], \tag{11}$$

where $\tilde{\varepsilon}_{iy}(y, \boldsymbol{w}) = \partial \ln \tilde{x}_i(y, \boldsymbol{w})/\partial \ln y$ is the output elasticity of the conditional demand of input i; $n(y, \boldsymbol{w}) = \partial \ln c(y, \boldsymbol{w})/\partial \ln y$ is the output elasticity of cost; and $\zeta(y, \boldsymbol{w}) = \partial \ln c_y(y, \boldsymbol{w})/\partial \ln y$ is the output elasticity of marginal cost.[9] These "size" elasticities are related to the scale properties of technology. In fact, as is well known, $n(y, \boldsymbol{w}) = \sum_j \theta_j(y, \boldsymbol{w})\tilde{\varepsilon}_{jy}(y, \boldsymbol{w})$ is equal to the reciprocal of the elasticity of scale (i.e., $n(y, \boldsymbol{w}) = \xi(\boldsymbol{x}(y, \boldsymbol{w}))^{-1}$; notice that under decreasing returns to scale $n(y, \boldsymbol{w}) > 1$) and it is sometimes called "the elasticity of size".[10] In addition, $\zeta(y, \boldsymbol{w}) = n(y, \boldsymbol{w}) - \partial \ln \xi(\tilde{\boldsymbol{x}}(y, \boldsymbol{w}))/\partial \ln y - 1$, where $\partial \ln \xi(\boldsymbol{x}(y,\boldsymbol{w}))/\partial \ln y$ is a straightforward measure of the change in the returns to scale *along the output expansion path* (see e.g., Hanoch, 1975).

It can be shown that $\tilde{\varepsilon}_{jy}(y, \boldsymbol{w}) = -\zeta(y, \boldsymbol{w}) \sum_i \varepsilon_{ji}(\underline{\boldsymbol{w}})$, i.e., the output elasticity of input j is given by the sum of the elements of the jth row of the elasticity matrix $\boldsymbol{\varepsilon}(\underline{\boldsymbol{w}})$, multiplied by the (negative) size elasticity of marginal cost. In matrix form: $\tilde{\boldsymbol{\varepsilon}}_y(y, \boldsymbol{w}) = -\zeta(y, \boldsymbol{w})\, \boldsymbol{\varepsilon}(\underline{\boldsymbol{w}})\boldsymbol{\iota}$, where $\tilde{\boldsymbol{\varepsilon}}_y(y, \boldsymbol{w})$ is the vector of the output elasticities of the direct conditional demand, and $\boldsymbol{\iota}' = [1, \ldots, 1]$ is the unit vector. It is also worth noticing the following duality result, which can be obtained by computing $\boldsymbol{\gamma}(\boldsymbol{x}(\underline{\boldsymbol{w}}))'\tilde{\boldsymbol{\varepsilon}}_y(y, \boldsymbol{w})$, where $\boldsymbol{\gamma}(\boldsymbol{x}(\underline{\boldsymbol{w}}))$ is the output share vector:[11]

$$-\frac{1}{\zeta(y, \boldsymbol{w})} = \boldsymbol{\gamma}(\boldsymbol{x}(\underline{\boldsymbol{w}}))'\boldsymbol{\varepsilon}(\underline{\boldsymbol{w}})\boldsymbol{\iota} = \frac{\underline{\boldsymbol{w}}'\boldsymbol{D}_{\underline{\boldsymbol{w}}}\boldsymbol{x}(\underline{\boldsymbol{w}})\underline{\boldsymbol{w}}}{y} = \boldsymbol{\iota}'\boldsymbol{\rho}^H(\boldsymbol{x}(\underline{\boldsymbol{w}}))^{-1}\boldsymbol{\iota}. \tag{12}$$

(the right-hand side of (12) follows from (10)).

From (11), inputs need not have the same p-classification according to $\sigma_{ij}^H(\underline{\boldsymbol{w}})$ and $\sigma_{ij}(y, \boldsymbol{w})$. However, as is well known, two inputs can revert from net substitutes to gross complements only if they are both normal or both inferior, while they may revert from net complements to gross substitutes only if one is normal and the other inferior (this is so because two inputs can change their substitutability status when account is taken of the variation in output only if the output effect does not strengthen the substitution effect). Also notice that, since $\sum_j \theta_j(y, \boldsymbol{w})\sigma_{ij}(y, \boldsymbol{w}) = 0$ (by homogeneity of degree zero with respect to $\boldsymbol{w}$ of the conditional demand), it follows that:

$$\sum_j \theta_j(y, \boldsymbol{w})\sigma_{ij}^H(\underline{\boldsymbol{w}}) = -\tilde{\varepsilon}_{iy}(y, \boldsymbol{w})\frac{n(y, \boldsymbol{w}) - 1}{\zeta(y, \boldsymbol{w})}. \tag{13}$$

A number of special cases of (11) emerge. Suppose for example that the output change per se generates no input bundle reallocation; i.e., that the isoclines are (locally) linear. This means that the technology is (locally) ray-homothetic. Since in this case $\tilde{\varepsilon}_{iy}(y, \boldsymbol{w}) = n(y, \boldsymbol{w}), i = 1, k$, no input plays a special role and thus (11) simplifies to:

$$\sigma_{ij}^H(\underline{\boldsymbol{w}}) = (n(y, \boldsymbol{w}) - 1)\left[\sigma_{ij}(y, \boldsymbol{w}) - \frac{n(y, \boldsymbol{w})}{\zeta(y, \boldsymbol{w})}\right]. \tag{11$'$}$$

Note that, in this case, inputs that are net p-complements will also be gross p-complements, but net p-substitutes may revert to gross p-complements (see e.g., Bertoletti, 2001). Suppose that, in addition, a movement along the output expansion path (which is in this case also a movement along a ray in the input space) does not change (locally) the degree of returns to scale, an assumption that as a global property characterizes the sub-class of ray-homogenous production functions. It must then be that $\zeta\,(y, \boldsymbol{w}) = n(y, \boldsymbol{w}) - 1$, and (11′) reduces to:

$$\sigma_{ij}^{H}(\underline{\boldsymbol{w}}) = [n(y, \boldsymbol{w}) - 1]\,\sigma_{ij}(y, \boldsymbol{w}) - n(y, \boldsymbol{w}). \tag{11''}$$

Finally, it is known that under weak "regularity assumptions" a ray-homothetic production function is also (simple) homothetic (see e.g., Färe, 1975). In this case the cost function is separable (i.e., $c(y, \boldsymbol{w}) = h(y)\mathrm{g}(\boldsymbol{w})$), and (11″) further specializes, since the AUES comes to depend only on the input prices, while the elasticity of size depends only on the output level: i.e., in (11″) we have $n(y)$ and $\sigma_{ij}(\boldsymbol{w})$. In the familiar sub-case of a homogeneous production function of degree r, we get $n = 1/r$; the HLES can thus be immediately computed from the AUES, and conversely:

$$\sigma_{ij}^{H}(\underline{\boldsymbol{w}}) = \frac{1}{r}\left[(1 - r)\sigma_{ij}(\boldsymbol{w}) - 1\right]. \tag{11'''}$$

5. Gross Versus Net Complementarity

It can be shown that (see Appendix B):

$$\rho_{ij}^{H}(\boldsymbol{x}) = \left[1 + \rho_{ij}(y, \boldsymbol{x}) + \mu_j(\boldsymbol{x})\right]\xi(\boldsymbol{x})^{-1} + \delta_i(y, \boldsymbol{x}), \tag{14}$$

where $\mu_j(\boldsymbol{x}) = \partial \ln \underline{w}_j(\lambda \boldsymbol{x})/\partial \ln \lambda|_{\lambda=1} = \sum_i \eta_{ji}(\boldsymbol{x})$ –which is sometimes called the "scale elasticity" of input j (see e.g., Anderson, 1980)— expresses the unconditional proportionate effect on $\underline{w}_j$ of a ray movement in the input space, and $\delta_i(y, \boldsymbol{x}) = \partial \ln a_i(y, \boldsymbol{x})/\partial \ln y$ is the output elasticity of the inverse conditional demand of input i. It is intuitive that $\delta_i(y, \boldsymbol{x})$ is closely related to the scale elasticity $\mu_i(\boldsymbol{x})$, since, with different normalization, both express the price effect of an output expansion along a ray in the input space. In fact (see Appendix B):

$$\delta_i(y, \boldsymbol{x}) = \xi(\boldsymbol{x})^{-1}(\mu_i(\boldsymbol{x}) - 1 - \underline{\mu}(\boldsymbol{x})), \tag{15}$$

where $\underline{\mu}(\boldsymbol{x}) = \sum_j \tau_j(y, \boldsymbol{x})\mu_j(\boldsymbol{x})$ is the "average" scale elasticity value (note that $\tau_i(y, \boldsymbol{x}) = \theta_i(y, \underline{\boldsymbol{w}}(\boldsymbol{x}))$). Also note that $\sum_i \tau_i(y, \boldsymbol{x})\delta_i(y, \boldsymbol{x}) = -1/\xi(\boldsymbol{x})$, and thus $\delta_i(y, \boldsymbol{x}) = -1/\xi(\boldsymbol{x})$ if the scale elasticities are all equal.[12]

According to (15), (14) can be rewritten as:

$$\rho_{ij}^{H}(\boldsymbol{x}) = (\rho_{ij}(y, \boldsymbol{x}) + \mu_i(\boldsymbol{x}) + \mu_j(\boldsymbol{x}) - \underline{\mu}(\boldsymbol{x}))\xi(\boldsymbol{x})^{-1}, \tag{16}$$

which gives the general relationship between the AEC and the HEC in terms of the inverse demand system. In fact, it can be easily proved that $\partial \ln d(y, \boldsymbol{x})/\partial \ln y = -1/\xi(\boldsymbol{x})$. Thus, the size elasticity of the distance function (evaluated at $y = f(\boldsymbol{x})$)

gives the (negative) elasticity of size. Moreover, by differentiating the identity $d_y(y, \boldsymbol{x}) \equiv -d(y, \boldsymbol{x})c_y(y, \boldsymbol{a}(y, \boldsymbol{x}))$,[13] after some computations we get that $\Lambda(y, \boldsymbol{x}) = \partial \ln d_y(y, \boldsymbol{x})/\partial \ln y = -(2 + \underline{\mu}(\boldsymbol{x}))/\xi(\boldsymbol{x})$, which provides a simple way to obtain $\underline{\mu}(\boldsymbol{x})$[14] and suggests that this quantity is related to the returns-to-scale *change* along a ray in the input space. By homogeneity of degree zero of the inverse conditional demand, it also follows immediately that:

$$\sum_j \tau_j(y, \boldsymbol{x})\rho_{ij}^H(\boldsymbol{x}) = \frac{\mu_i(\boldsymbol{x})}{\xi(\boldsymbol{x})}. \tag{17}$$

Note that (16) exhibits the expected interaction of a (net) substitution and an output effect, with the latter dependent on the "size" properties of the technology, summarized by the scale elasticities. Once again, inputs need not conserve the same q-classification when moving from the net to the gross measure. In particular, note that (in matrix form): $\boldsymbol{\mu}(\boldsymbol{x}) = \boldsymbol{\eta}(\boldsymbol{x})\boldsymbol{\iota}$, where $\boldsymbol{\mu}(\boldsymbol{x})$ is the vector of the scale elasticities. It follows that:

$$\underline{\mu}(\boldsymbol{x})\xi(\boldsymbol{x}) = \boldsymbol{\gamma}(\boldsymbol{x})'\boldsymbol{\eta}(\boldsymbol{x})\boldsymbol{\iota} = \frac{\boldsymbol{x}'\boldsymbol{D}_{\boldsymbol{x}}\underline{\boldsymbol{w}}(\boldsymbol{x})\boldsymbol{x}}{f(\boldsymbol{x})} = (1 - \xi(\boldsymbol{x}))\boldsymbol{\iota}'\boldsymbol{\sigma}^H(\boldsymbol{x})^{-1}\boldsymbol{\iota} \tag{18}$$

(notice the duality with (12)). By negative definiteness of $\boldsymbol{D}_{\boldsymbol{x}}^2 f(\boldsymbol{x})$ (18) shows that the scale elasticities are on average negative. Thus, net q-complements can revert to gross q-substitutes, but only if the sum of their scale elasticities is negative and sufficiently larger in absolute value than the average scale elasticity, $-\underline{\mu}(\boldsymbol{x})$; and net q-substitutes may revert to gross q-complements only if the sum of their scale elasticities is not negative and larger in absolute value than $-\mu(\boldsymbol{x})$. (18) also implies that $\underline{\mu}(\boldsymbol{x}) = \xi(\boldsymbol{x}) + \partial \ln \xi(\lambda \boldsymbol{x})/\partial \ln \lambda|_{\lambda=1} - 1$, where $\partial \ln \xi(\lambda \boldsymbol{x})/\partial \ln \lambda|_{\lambda=1}$ (sort of a "second-order elasticity of scale") is a straightforward measure of the change in the returns to scale *along a ray in the input space*.

Also note that in the system of inverse demand scale elasticities play a role similar to that of output elasticities in that of direct conditional demand. In particular, we have seen in Section 4 that the output elasticity of input j, $\tilde{\varepsilon}_{jy}(y, \boldsymbol{w})$, is given by the sum of the elements of the jth row of the elasticity matrix $\boldsymbol{\varepsilon}(\underline{\boldsymbol{w}})$, multiplied by $-\zeta(y, \boldsymbol{w})$. It follows from (5) that the scale elasticity of input i, $\mu_i(\boldsymbol{x}(\underline{\boldsymbol{w}}))$, is given by the sum of the elements of the ith row of the *inverse* of $\boldsymbol{\varepsilon}(\underline{\boldsymbol{w}})$. In matrix form: $-\boldsymbol{\varepsilon}(\underline{\boldsymbol{w}})^2\boldsymbol{\mu}(\boldsymbol{x}(\underline{\boldsymbol{w}})) = \tilde{\boldsymbol{\varepsilon}}_y(y, \boldsymbol{w})/\zeta(y, \boldsymbol{w})$.[15] However, the signs and sizes of the single scale elasticities are not, in general, related to those of the single output elasticities. For this reason, we believe that the temptation to class inputs according to the sign of their scale elasticities should be resisted, at least until these signs are shown to convey useful information (see e.g., Kohli, 1985, in the context of consumer demand).[16] Moreover, as suggested by (12) and (18), the dual relationship is actually between $\partial \ln f(\boldsymbol{x}(\underline{\boldsymbol{w}}))/\partial \ln \underline{w}_i = -\tilde{\varepsilon}_{iy}(y, \boldsymbol{w})/\zeta(y, \boldsymbol{w})$ (also see footnotes (9) and (11)) and $\partial \ln \underline{w}_i(\lambda \boldsymbol{x})/\partial \ln \lambda|_{\lambda=1} = \mu_i(\boldsymbol{x})$, and this is not, in general, given by simple scalar inversion.

A rather special case of (16) arises if, $i = 1, k$, either $\tilde{\varepsilon}_{iy}(y, \boldsymbol{w}) = n(y, \boldsymbol{w})$ or $\mu_i(\boldsymbol{x}) = \underline{\mu}(\boldsymbol{x})$, which are equivalent conditions. In fact, in that case $-n(y, \boldsymbol{w})/\zeta(y, \boldsymbol{w})$

is an eigenvalue of $\boldsymbol{\varepsilon}(\underline{\boldsymbol{w}})$ whose associated eigenvector is the unit vector $\boldsymbol{\iota}$, and then $\underline{\mu}(\boldsymbol{x}(\underline{\boldsymbol{w}})) = -\zeta(y, \boldsymbol{w})/n(y, \boldsymbol{w})$ must be the eigenvalue of $\boldsymbol{\eta}(\boldsymbol{x}(\underline{\boldsymbol{w}}))$ associated with the same eigenvector, and conversely. Thus, a fortiori, $\partial \ln f(\boldsymbol{x}(\underline{\boldsymbol{w}}))/\partial \ln \underline{w}_i = -\mu_i(\boldsymbol{x}(\underline{\boldsymbol{w}}))^{-1}$. Of course, this is so if the technology is (locally) ray-homothetic, in which case (16) simplifies to:

$$\rho_{ij}^{H}(\boldsymbol{x}) = \xi(\boldsymbol{x})^{-1}(\rho_{ij}(y, \boldsymbol{x}) + \underline{\mu}(\boldsymbol{x})). \tag{16$'$}$$

Note the nice duality between (11′) and (16′). In this case, inputs that are net q-substitutes will also be gross q-substitutes, but net q-complements may still revert to gross q-substitutes.

Similarly to what we have seen in Section 4, if in addition a movement along a ray in the input space (which is in this case also a movement along the output expansion path) does not change (locally) the degree of returns to scale (16′) reduces to:

$$\rho_{ij}^{H}(\boldsymbol{x}) = \xi(\boldsymbol{x})^{-1}(\rho_{ij}(y, \boldsymbol{x}) - 1) + 1, \tag{16$''$}$$

since in that case $\underline{\mu}(\boldsymbol{x}) = \xi(\boldsymbol{x}) - 1$. Finally, in the simple case of a homogeneous production function of degree $r \leq 1$, we get:

$$\rho_{ij}^{H}(\boldsymbol{x}) = \frac{\rho_{ij}(\boldsymbol{x}) - 1}{r} + 1. \tag{16$'''$}$$

Notice that (16‴) encompasses the particular case of constant returns to scale, for which the equality $\rho_{ij}^{H}(\boldsymbol{x}) = \rho_{ij}(\boldsymbol{x})$ holds (i.e., there is no output effect). However, other cases in which the HEC and the AEC are equal are clearly possible: for example, according to (16′) this happens under (local) ray-homotheticity if $\rho_{ij}(y, \boldsymbol{x}) = \underline{\mu}(\boldsymbol{x})/(\xi(\boldsymbol{x}) - 1)$. Note that, by (11′), in the same case the HLES is equal to the (negative)[17] AUES if $\sigma_{ij}(y, \boldsymbol{w}) = (n(y, \boldsymbol{w}) - 1)/\zeta(y, \boldsymbol{w})$. For example, for a Cobb–Douglas technology we get $\rho_{ij}^{H} = \rho_{ij} = \sigma_{ij} = 1 = -\sigma_{ij}^{H}$ (whenever the HLES is defined), $i \neq j$. Thus, all inputs are in such a case gross and net q-complements, net p-substitutes and gross p-complements. Finally, since Deaton (1979) has shown that the Slutsky matrix (i.e., the matrix whose generic element is given by $c_{ij}(y, \boldsymbol{w})$) is a generalized inverse of the Antonelli matrix (i.e., the matrix whose generic element is given by $d_{ij}(y, \boldsymbol{x})$) (16‴) implies that the same relationship exists under constant returns to scale between (in matrix form) $\hat{\boldsymbol{\gamma}}(\boldsymbol{x})\boldsymbol{\rho}^{H}(\boldsymbol{x})$ and $\hat{\boldsymbol{\gamma}}(\boldsymbol{x})\boldsymbol{\sigma}(y, \underline{\boldsymbol{w}}(\boldsymbol{x}))$,[18] which is perhaps where Hicks started more than seventy years ago.

6. Conclusion

There are at least four consistent ways to classify inputs according to their substitutability. They can be stratified according to the sign of the relevant cross-effects of either the direct or the inverse, the conditional or the unconditional demand functions. In this paper we have stressed the gross classification, based upon the unconditional demand. In particular, by deriving the familiar decomposition between a (net) substitution effect and an output effect, we have shown, in the general case of non-homotheticity and variable returns to scale, the relationships of the HLES and the HEC (two measures of gross substitution already used in the literature, although the former is less frequently referred to) with two well-known measures of net substitution, the dual AUES and AEC. Our approach, which uses the duality between the direct production function and the normalized profit function, also shows the perfect duality between the former two elasticities.

This paper suggests a way to reconcile some work on the elasticity of substitution by Allen and Hicks (1934), Hicks (1970), Sato and Koizumi (1973), Blackorby and Russell (1981), Syrquin and Hollender (1982) and, more recently, Kim (2000) and Bertoletti (2001). However, while our focus has been on elasticities of substitution or complementarity, this paper also provides a further exploration of the theoretical properties of the inverse demand systems in production theory. In particular, we have derived a number of (novel, as far as we know) findings concerning the scale elasticities, the output elasticity of the inverse conditional demand and the way the distance function can be used to account for the changes in returns to scale. As it turns out, not surprisingly an especially nice duality between direct and indirect demand systems exists under homotheticity.

It may also be worth noting that one reason a previous attempt (Kim, 2000) to asses input substitution failed to recognize the role of the HEC is because it was based too closely on some well-known analogy with consumer demand (i.e., it used the indirect production function: on this see also Park and Thurman, 1999). Indeed, as far as gross substitution/complementarity is concerned, there are significant differences between production and consumer theory, since in the latter there is no returns-to-scale analog (and in the former no "Giffen" input can exist).[19] It is hoped that the results of this paper will be useful in clarifying some issues in the debate on the proper input substitutability classification.

Appendix A

Differentiating the identity $x_i(\underline{\boldsymbol{w}}) \equiv \tilde{x}_i(f(\boldsymbol{x}(\underline{\boldsymbol{w}})), \underline{\boldsymbol{w}})$ gives:

$$x_{ij}(\underline{\boldsymbol{w}}) = \frac{\partial \tilde{x}_i(f(\boldsymbol{x}(\underline{\boldsymbol{w}})), \underline{\boldsymbol{w}})}{\partial y} \sum_i f_i(\boldsymbol{x}(\underline{\boldsymbol{w}})) \frac{\partial x_i(\underline{\boldsymbol{w}})}{\underline{w}_j} + \frac{\partial \tilde{x}_i(f(\boldsymbol{x}(\underline{\boldsymbol{w}})), \underline{\boldsymbol{w}})}{\partial w_j}. \tag{A.1}$$

By using the first-order condition $1 = c_y(f(\boldsymbol{x}(\underline{\boldsymbol{w}})), \underline{\boldsymbol{w}})$, it is easy to see that $\tilde{x}_{jy}(y, \boldsymbol{w}) = -(c_{yy}(y, \boldsymbol{w})/c_y(y, \boldsymbol{w})) \sum_i x_{ji}(\underline{\boldsymbol{w}})\underline{w}_i$, where $y = f(\boldsymbol{x}(\underline{\boldsymbol{w}}))$. Then (A.1) becomes

$$x_{ij}(\underline{\boldsymbol{w}}) = p\left[\tilde{x}_{ij}(y, \boldsymbol{w}) - \frac{\tilde{x}_{iy}(y, \boldsymbol{w})\tilde{x}_{jy}(y, \boldsymbol{w})}{c_{yy}(y, \boldsymbol{w})}\right], \tag{A.2}$$

which stresses the familiar decomposition between a (net) substitution and an output effect. By using $\omega_j(\underline{\boldsymbol{w}}) = \theta_j(y, \boldsymbol{w})/(n(y, \boldsymbol{w}) - 1)$ (11) immediately follows in elasticity terms.

Appendix B

Differentiating the identity $a_i(f(\boldsymbol{x}), \boldsymbol{x}) \equiv \underline{w}_i(\boldsymbol{x})/c(f(\boldsymbol{x}), \underline{\boldsymbol{w}}(\boldsymbol{x}))$ (which can also be written $a_i(f(\boldsymbol{x}), \boldsymbol{x}) \equiv \underline{w}_i(\boldsymbol{x})/(\underline{\boldsymbol{w}}(\boldsymbol{x})\boldsymbol{x})$ and directly derived by differentiating $y \equiv f(\boldsymbol{x}/d(y, \boldsymbol{x}))$[20] with respect to x_i and evaluating at $y = f(\boldsymbol{x})$) and using $c_y(f(\boldsymbol{x}), \underline{\boldsymbol{w}}(\boldsymbol{x})) = 1$ gives:

$$\begin{aligned} &a_{ij}(y, \boldsymbol{x}) + a_{iy}(y, \boldsymbol{x}) f_j(\boldsymbol{x}) \\ &\quad = \frac{\underline{w}_{ij}(\boldsymbol{x}) a_i(y, \boldsymbol{x})}{\underline{w}_i(\boldsymbol{x})} - \frac{a_i(y, \boldsymbol{x})(f_j(\boldsymbol{x}) + \sum_i \underline{w}_{ij}(\boldsymbol{x}) \tilde{x}_i(y, \underline{\boldsymbol{w}}(\boldsymbol{x})))}{c(y, \underline{\boldsymbol{w}}(\boldsymbol{x}))}, \end{aligned} \tag{B.1}$$

where $y = f(\boldsymbol{x})$. Since $d(f(\boldsymbol{x}), \boldsymbol{x}) = 1$, (14) follows.

In (15), $\delta_i(y, \boldsymbol{x})$ is computed by differentiating the identity $a_i(y, \boldsymbol{x}) \equiv \underline{w}_i(\lambda \boldsymbol{x})/c(f(\lambda \boldsymbol{x}), \underline{\boldsymbol{w}}(\lambda \boldsymbol{x}))$ with respect to y, where λ is implicitly defined by $y = f(\lambda \boldsymbol{x})$, and evaluating at $y = f(\boldsymbol{x})$ (i.e., at $\lambda = 1$).

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Notes

1. To classify goods (inputs) according to their substitutability is a popular exercise in empirical investigations, which also serves as a check on the economic implications of the various models: for a recent example see Holt (2002, Section 4.6). Obviously, it is also theoretically important (see e.g. Seidman, 1989).
2. Blackorby and Russell (1981, p. 152) gallantry attributed the AEC to Hicks (1970). However, even if the HEC and the AEC have the same value under the constant-returns-to-scale assumption (see Section 4) made in that paper, Hicks (1970) appears concerned with the former concept (see Sato and Koizumi, 1973).
3. Our approach is closely related to those of Kim (2000) and Bertoletti (2001). However, the latter does not deal with the inverse demand system, while the former uses the indirect production function (see e.g., Cornes, 1992, Section 5.1) to model "gross" complementarity, and thus fails to derive the general relationship between the HEC and the AEC.
4. That is, we assume that the Jacobian of $\boldsymbol{x}(\underline{\boldsymbol{w}})$ is negative *definite*, which implies that the production function is (locally) strictly concave.
5. It is well known that $\tilde{x}_i(y, \boldsymbol{w})$ is homogeneous of degree zero with respect to $\boldsymbol{w}$: the "size" (i.e., with respect to output) properties of this conditional demand are recalled in Section 4, where they are extensively exploited.
6. The distance function can be stated as follows: $d(y, \boldsymbol{x}) = \min_a\{\boldsymbol{a}x : c(y, \boldsymbol{a}) = 1\}$. $a_i(y, \boldsymbol{x})$ is then homogeneous of degree zero with respect to $\boldsymbol{x}$: its "size" properties at $y = f(\boldsymbol{x})$ are investigated in Section 5.

7. This has led some authors (e.g., Kim, 2000) to use the "unconditional" demand system that can be derived by the indirect production function (see e.g., Cornes, 1992, Section 5.1) and retains a closer formal analogy to the system of uncompensated (or Marshallian) demand in consumer theory. It turns out that our approach is in this case more fruitful.
8. It should be clear that expressions differing from (10) only for a scalar multiplication can be obtained by using instead of $\hat{\boldsymbol{\gamma}}$ either $\hat{\boldsymbol{\omega}}$ or $\hat{\boldsymbol{\theta}}$ or $\hat{\boldsymbol{\tau}}$ (where the latter diagonal matrices are analogously defined).
9. Thus ζ is a measure of the cost function curvature. It can be shown that $1/\zeta = \partial \ln y/\partial \ln p$ and $\tilde{\varepsilon}_{iy}/\zeta = (c_{iy}c)/(c_i c_y) = \partial \ln x_i/\partial \ln p$, where $y(p, \boldsymbol{w})$ and $x_i(p,\boldsymbol{w})$ are, respectively the (non-normalized) output supply function, and the (non-normalized) unconditional demand function of input i. Also note that $\zeta = n(c_{yy}c)/(c_y c_y)$ provides a simple way of computing it.
10. It is also easily shown that $\sum_j \gamma_j \tilde{\varepsilon}_{jy} = 1$ and $\sum_j \gamma_j = 1/n$.
11. It is easily seen that $\gamma_i \sum_j \varepsilon_{ij} = \partial \ln y/\partial \ln w_i$.
12. Notice that all computations are performed at $y = f(\boldsymbol{x})$.
13. This follows, by using the Envelope Theorem, from the first-order conditions that define the distance function (see footnotes 6 and 20). In particular, one can prove that the value of the distance function is equal to the relevant Lagrangean multiplier.
14. Λ is, by construction, a measure of the distance function curvature with respect to output. Note that $-\varepsilon\Lambda = \underline{\mu} + 2 = (d_{yy}d)/(d_y d_y)$.
15. Park and Thurman (1999) argue in the context of consumer demand that income elasticities and what they call "scale flexibilities" are fundamentally different measures, contending that the latter are equivalent to the scale elasticities considered in the literature. It appears that in production theory, as far as the inverse unconditional demand system is concerned, those elasticities are related just by matrix inversion.
16. Note that, instead of concentrating on the scale elasticities, we might focus on the output elasticities of the inverse conditional demand system: in fact, since $\mu_i = \varepsilon\delta_i + 1 + \underline{\mu} = \varepsilon\delta_i - 1 - \varepsilon\Lambda$ (16) can be rewritten as $\rho_{ij}^H = n[\rho_{ij} + 2 + \underline{\mu}] + \delta_i + \delta_j = n\rho_{ij} - \Lambda + \delta_i + \delta_j$.
17. The original Lau's definition of HLES (Lau, 1978, p. 197) was actually equal to $-\sigma_{ij}^H$: we have chosen to change the sign to conform to standard terminology concerning p-substitutes.
18. Again (see footnote 8), similar results apply when $\hat{\boldsymbol{\gamma}}$ is replaced either by $\hat{\boldsymbol{\tau}}$ or $\hat{\boldsymbol{\theta}}$ or $\hat{\boldsymbol{\omega}}$.
19. Since the "income effect", as stated by the Slutsky equation, is not necessarily symmetric (unless under homothetic preferences), in consumer theory gross substitutability does not appear well defined. An exploration of this issue is left for future work.
20. Alternatively, to confirm that $\boldsymbol{a}\,(f(\boldsymbol{x}),\boldsymbol{x}) \equiv \hat{\boldsymbol{w}}(\boldsymbol{x})/c(f(\boldsymbol{x}),\hat{\boldsymbol{w}}(\boldsymbol{x}))$, one might consider the FOCs which uniquely define (given concavity) its left-hand side (see footnote 6). They are: $\boldsymbol{x} - \beta\tilde{\boldsymbol{x}}(f(\boldsymbol{x}), \boldsymbol{a}) = \boldsymbol{0}$ and $\boldsymbol{a}\tilde{\boldsymbol{x}}(f(\boldsymbol{x}),\boldsymbol{a}) = 1$, where $\beta = 1$ is the relevant Lagrangean multiplier. It is obvious that the right-hand side of the previous expression does satisfy them. The economic intuition is also straightforward: once *normalized* in order to provide a unit cost, the ("inverse") unconditional (i.e., profit-maximizing), input relative (with respect to the output price p) price vector function $\hat{\boldsymbol{w}}(\boldsymbol{x})$ is then equivalent to the ("inverse") conditional (to the output level), input price vector function $\boldsymbol{a}(y, \boldsymbol{x})$, *if* $f(\boldsymbol{x}) = y$.

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