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RESEARCH

Spectral Asymptotics of the Cauchy Operator and its Product with Bergman's Projection on a Doubly Connected Domain

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Abstract
We found the exact asymptotics of the singular numbers for the Cauchy transform and its product with Bergman's projection over the space $L^2(\Omega)$, where Ω is a doubly-connected domain in the complex plane.

Keywords Cauchy operator · Singular numbers · Doubly-connected domain

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1 Introduction

Let Ω be a doubly-connected domain of the complex plane $\mathbb{C}$. Throughout the paper the quantity $\frac{1}{\rho}$, $0 < \rho < 1$, will be the fixed modulus of the doubly-connected domain Ω. In other words, the domain Ω can be conformally mapped onto a circular ring $A' = \{z : \rho < |z| < 1\}$.

By $dA(z) = dxdy$ we denote the ordinary Lebesgue measure in $\mathbb{C}$. Denote by $L^2(\Omega)$ the space of all complex-valued functions on Ω for which the norm

$$\|f\|_{L^2(\Omega)} = \left(\int_\Omega |f(z)|^2 dA(z)\right)^{1/2}$$

is finite.

Specially, $L^2_a(\Omega)$ denotes a closed subspace of analytic functions in $L^2(\Omega)$ known as the Bergman space and the orthogonal projection $P_\Omega : L^2(\Omega) \to L^2_a(\Omega)$ which appears in this setting is known as the Bergman projection.

The Cauchy integral operator

$$C : L^2(\Omega) \to L^2(\Omega)$$

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is defined in the following manner

$$Cf(z) = -\frac{1}{\pi}\int_{\Omega}\frac{f(\xi)}{\xi - z}dA(\xi).$$

It is well known that the Cauchy operator is bounded on $L^2(\Omega)$. Moreover, in the case when Ω is the unit disc $\mathbb{D} = \{z : |z| < 1\}$ it was shown in [1] that

$$\|C\|_{L^2(\mathbb{D})\to L^2(\mathbb{D})} = 2j_0^{-1},$$

where j_0 is the smallest positive zero of the Bessel function

$$J_0(z) = \sum_{k=0}^{\infty}\frac{(-1)^k}{(k!)^2}\left(\frac{z}{2}\right)^{2k}.$$

For the general domain $\Omega \subset \mathbb{C}$ with smooth boundary it was proved that

$$\|C\|_{L^2(\Omega)\to L^2(\Omega)} \geq \frac{2}{\sqrt{\lambda_1}},$$

where λ_1 is the smallest eigenvalue of the boundary value problem

$$-\triangle u = \lambda u, \, u|_{\partial\Omega} = 0.$$

At some places in the paper we will use also the notion of the Cauchy transform which refers to the integral operator defined on a closed curve. Namely, for a bounded domain Ω with C^∞ smooth boundary $\partial\Omega$, Cauchy's transform $\tilde{C} : L^2(\partial\Omega) \to L^2(\partial\Omega)$ is defined as

$$\tilde{C}(f)(z) = \frac{1}{2\pi i}\int_{\partial\Omega}\frac{f(\xi)}{\xi - z}d\xi.$$

The Cauchy transform maps $C^\infty(\partial\Omega)$ into the space of all holomorphic functions on Ω that are in $C^\infty(\overline{\Omega})$. We refer to [4] for a comprehensive study related to the Cauchy transform. Also, for the various L^p-norm estimation of the Cauchy transform in the unit disc we refer to [11] and [15].

Let us emphasize that the domain Ω we consider is bounded by analytic curves. In this case the doubly connected domain Ω can be mapped univalently onto the circular annulus A' by some function F which is analytic in $\bar{\Omega}$, and whose inverse function φ is analytic in the closed annulus A'. Moreover, the circular annulus onto which the domain Ω is mapped is unique up to the linear transform. For more details on this topic we refer to [17] and [13].

1.1 Singular Numbers of Compact Operators

Let us denote by S_∞ the space of all compact operators on a Hilbert space H and let $T \in S_\infty$. The eigenvalues of a nonnegative operator $(T^*T)^{1/2}$ managed in decreasing order are called singular numbers (values) of the operator T.

For the compact operator T let us denote by $\mathcal{N}_t(T)$ the number of singular numbers of T which exceed the positive number t :

$$\mathcal{N}_t(T) = \sum_{s_n(T)\geq t} 1.$$

We refer to [12] for a study on properties of singular numbers and many nontrivial inequalities among them.

For instance, throughout the paper we will use the following expected inequality

$$s_n(TT_1) \le \|T_1\| s_n(T), \tag{1.1}$$

where T_1 is bounded operator on H and $T \in S_\infty$.

The following theorem (see [6], pp.78) plays an important role in proving the main results of this paper and we state it in a slightly adjusted version. Firstly, for the kernel $K(z,\xi)$ defined in a bounded domain Ω we say that it is (real-)analytic uniformly with respect to z, if for every $\xi_0 \in \Omega$ there is a neighbourhood where the following expansion holds

$$K(z,\xi) = \sum_{|\alpha|=0}^{\infty} c_\alpha(z,\xi_0)(\xi-\xi_0)^\alpha \tag{1.2}$$

and there are constants M and r which don't depend from $\xi_0 \in \Omega$, $z \in \Omega$, such that

$$|c_\alpha(z,\xi_0)| \le M r^{-|\alpha|}. \tag{1.3}$$

The operator $T : L^2(\Omega) \to L^2(\Omega)$ induced by $K(z,\xi)$ is, as usual, defined by

$$Tf(z) = \int_\Omega K(z,\xi) f(\xi) dA(\xi).$$

Theorem 1.1 *Let $\Omega \subset \mathbb{C}$ be a bounded subset and let the kernel $K(z,w)$ satisfy the conditions* Eqs. 1.2 and 1.3. *Then*

$$s_n(T) \le CMe^{-\beta n^{1/2}}|\Omega|, \;\; C = C(\Omega, r), \beta = \beta(\Omega, r) > 0,$$

where $|\Omega|$ stands for the measure area of Ω.

In [8] the reader may find the following useful result.

Lemma 1.2 *Let T be a compact operator such that for any $\epsilon > 0$ there exists a decomposition*

$$T = T'_\epsilon + T''_\epsilon,$$

where $T'_\epsilon, T''_\epsilon$ are compact operators such that

(1) the limit

$$\lim_{t\to 0^+} t^{\frac{1}{\alpha}} \mathcal{N}_t(T'_\epsilon) = C(T'_\epsilon), \;\; (\alpha > 0)$$

exists, and $C(T'_\epsilon)$ is a bounded function in a neighbourhood of $\epsilon = 0$,

(2)

$$\limsup_{n\to+\infty} n^\alpha s_n(T''_\epsilon) \le \epsilon.$$

Then the following limits exist:

$$\lim_{\epsilon\to 0^+} C(T'_\epsilon) = C(T) \;\; and \;\; \lim_{t\to 0^+} t^{\frac{1}{\alpha}} \mathcal{N}_t(T) = C(T).$$

1.2 Spectral Properties of Cauchy Operator

Since the Cauchy operator is a compact operator on $L^2(\Omega)$, where Ω is a bounded domain, the asymptotic behavior of its singular number was the subject of numerous researches.

We refer to [2] where was determined the asymptotic behaviour of singular numbers for the Cauchy operator in the case when $\Omega = \mathbb{D}$.

The asymptotic behaviour of singular numbers for the Cauchy operator and its product with harmonic Bergman's projection was considered in [10, 21] and [22].

The exact asymptotic behavior of $s_n(C)$ for a bounded domain with a piecewise smooth boundary was established in [8]. Namely,

$$s_n(C) \sim \sqrt{\frac{|\Omega|}{\pi n}},$$

where $a_n \sim b_n$ means that $\lim_{n\to\infty} \frac{a_n}{b_n} = 1$. Also for the best possible estimate of the second term in the spectral asymptotic of Cauchy's operator on a bounded domain we refer to the additional paper [7].

In [9] M. Dostanić considered the spectral asymptotic of the Cauchy integral operator and its product with Bergman's projection P on a bounded simply connected domain Ω with analytic boundary. In fact, the author explained the phenomenon of the "acceleration" of the decrease of singular numbers for the Cauchy operator when multiplied by Bergman's projection. Moreover, a certain dependence between the spectral asymptotics and the length of the boundary was established. We want to point out the main result of the mentioned article (given in a shorter version) which reads as follows

Theorem 1.3 *Let Ω be a simply connected domain in $\mathbb{C}$ with analytic boundary. Then*

$$\lim_{n\to\infty} n s_n(PC) = \frac{|\partial\Omega|}{2\pi}, \tag{1.4}$$

where $|\partial\Omega|$ denotes the length of the boundary Ω.

The main purpose of this paper is to extend the above result to the context of a doubly connected domain Ω. Here Ω is domain with analytic boundary $\partial\Omega = \Gamma_1 \cup \Gamma_{\frac{1}{\rho}}$, where $\Gamma_1 = \{\varphi(e^{i\theta}) : \theta \in [0, 2\pi)\}$ and $\Gamma_{\frac{1}{\rho}} = \{\varphi(\frac{e^{i\theta}}{\rho}) : \theta \in [0, 2\pi)\}$. In this setting we showed

Theorem 1.4 *Let $\Omega \subset \mathbb{C}$ be a doubly connected domain with the modulus $\frac{1}{\rho}$, $0 < \rho < 1$ and analytic boundary. Then*

$$\lim_{n\to\infty} n s_n(P_\Omega C) = \frac{|\Gamma_1|}{2\pi} + \frac{\rho^3 |\Gamma_{\frac{1}{\rho}}|}{2\pi}. \tag{1.5}$$

Remark 1.5 The result of the above theorem is dedicated to the case of domain Ω which is conformally isomorphic to the annulus $\{z : 1 < |z| < \frac{1}{\rho}\}$ or $\{z : \rho < |z| < 1\}$. We may notice that the presence of a "hole" in the domain Ω ensures another summand in Eq. 1.5 in comparison to Eq. 1.4. In the limiting case when Ω is a simply connected domain the formula Eq. 1.5 reduces to the formula Eq. 1.4 when $\rho \to 0$.

Together with this introduction, the paper contains two more sections. In Sect. 2 we compute the exact formula for the kernel $P_\Omega C$. In Sect. 3 we determine the spectral asymptotics for certain special operators appearing in the formula of $P_\Omega C$.

2 The Kernel of Operator $P_\Omega C$

2.1 The Bergman Kernel of Annulus

As it was stated the Bergman space $L^2_a(A)$ is a closed Hilbert subspace of $L^2(A)$ space and the orthogonal projection $P : L^2(A) \to L^2_a(A)$ which arises in this case is an integral operator whose acting is determined by the reproducing Bergman kernel $K_A(z, w)$ in the following way

$$Pf(z) = \int_A K_A(z, w) f(w) dA(w).$$

In [5] the Bergman kernel was calculated for a circular annulus $A' = \{z : \rho < |z| < 1\}$,

$$K_{A'}(z, w) = \frac{1}{\pi z \bar{w}} \left(\mathcal{P}(\ln (z\bar{w})) + \frac{\eta_1}{i\pi} - \frac{1}{2 \ln \rho} \right),$$

where $\mathcal{P}$ is the Weierstrass function with the periods $\omega_1 = \pi i, \omega_2 = \ln \rho$, and η_1 is the half-increment of the Weierstrass ζ-function related to the period ω_1. We refer interested reader to [14] for another presentation of Bergman kernel in terms of a Poincaré series for every circular multiply connected domain in the plane.

At this point, we want to underlie that in the rest of the paper we will consider the annulus $A = \{z | 1 < |z| < \frac{1}{\rho}\}$ (conformally isomorphic to A') instead of A'.

For our purpose we will present here a brief outline of calculating the kernel K_A given in [16]. Let us recall one of the basic results from [16] which relies on existence of complete orthonormal base $\{\phi_j(z)\}_{j=-\infty}^{\infty}$ in $L_a^2(A)$.

Proposition 2.1 *Let K be a compact set in A. Then the series*

$$\sum_{j=-\infty}^{\infty} \phi_j(z) \overline{\phi_j(w)}$$

uniformly converges to the Bergman kernel $K_A(z, w)$ on K.

The functions $\phi_j = z^j$, where $j = ... - 2, -1, 0, 1, 2...$ form a complete orthogonal system in A.

Further,

$$\|\phi_j\|^2_{L^2(A)} = \begin{cases} \frac{\pi}{j+1} \left(\frac{1}{\rho^{2j+2}} - 1 \right), & j \neq -1, \\ 2\pi \ln (\frac{1}{\rho}), & j = -1. \end{cases}$$

According to Proposition 2.1 we have

$$\begin{aligned} K_A(z, w) &= \sum_{j=-\infty, j\neq -1}^{\infty} \frac{j+1}{\pi(\frac{1}{\rho^{2j+2}} - 1)} z^j \bar{w}^j + \frac{1}{2\pi \ln (\frac{1}{\rho})} z^{-1} \bar{w}^{-1} \\ &= \sum_{j \le -2} \frac{(j+1)\rho^{2j+2}}{\pi(1-\rho^{2j+2})} z^j \bar{w}^j + \frac{1}{2\pi \ln (\frac{1}{\rho})} z^{-1} \bar{w}^{-1} \\ &+ \sum_{j \ge 0} \frac{(j+1)\rho^{2j+2}}{\pi(1-\rho^{2j+2})} z^j \bar{w}^j = I_1 + I_2 + I_3. \end{aligned} \tag{2.1}$$

On the other hand,

$$\begin{aligned} I_1(z, w) &= - \sum_{j=-\infty}^{-2} \frac{j+1}{\pi} z^j \bar{w}^j + \sum_{j=-\infty}^{-2} \frac{j+1}{\pi} \frac{z^j \bar{w}^j}{1-\rho^{2j+2}} \\ &= \frac{1}{\pi(1-z\bar{w})^2} + \sum_{j=-\infty}^{-2} \frac{j+1}{\pi} \frac{z^j \bar{w}^j}{1-\rho^{2j+2}}, \end{aligned} \tag{2.2}$$

and

$$I_2(z,w) = \frac{1}{2\pi \ln\left(\frac{1}{\rho}\right)} z^{-1}\bar{w}^{-1},$$

$$\begin{aligned} I_3(z,w) &= \sum_{j=0}^{\infty} \frac{(j+1)\rho^{2j+2}}{\pi} z^j \bar{w}^j + \sum_{j\ge 0}\left(\frac{(j+1)\rho^{2j+2}}{\pi(1-\rho^{2j+2})} - \frac{(j+1)\rho^{2j+2}}{\pi}\right) z^j \bar{w}^j \\ &= \frac{\rho^2}{\pi(1-\rho^2 z\bar{w})^2} + \sum_{j\ge 0}\left(\frac{(j+1)\rho^{4j+4}}{\pi(1-\rho^{2j+2})}\right) z^j \bar{w}^j. \end{aligned} \tag{2.3}$$

It is worth to mention that the previous calculations imply that the kernel $K_A(z,w)$ can be viewed as sum of the Bergman kernel for the discs of radius 1 and the Bergman kernel for the disc of radius $\frac{1}{\rho}$, and certain series which converge absolutely and uniformly with all their derivatives in A.

2.2 Computation the Kernel of Operator $P_\Omega C$

At the beginning of this subsection let us recall one important transformation formula related to the reproducing kernels of domains ([19], pp.184).

Theorem 2.2 *Suppose $f : \Omega_1 \to \Omega_2$ is a biholomorphic map between boundened domains in $\mathbb{C}$. Then*

$$K_{\Omega_1}(z,w) = f'(z) K_{\Omega_2}(f(z), f(w)) \overline{f'(w)}.$$

Keeping in mind that F is a conformal mapping from Ω onto the annulus A ($\varphi = F^{-1}$) we can deduce the formula for the reproducing kernel of the domain Ω, denoted by $K_\Omega(z,w)$, which now is given by

$$K_\Omega(z,w) = F'(z) K_A(F(z), F(w)) \overline{F'(w)}.$$

More explicitly,

$$K_\Omega(z,w) = \sum_{i=1}^{4} G_i(z,w), \tag{2.4}$$

where

$$\begin{aligned} G_1(z,w) &= \frac{1}{2\pi \ln\left(\frac{1}{\rho}\right)} \frac{F'(z)\overline{F'(w)}}{F(z)\overline{F(w)}}, \\ G_2(z,w) &= \frac{F'(z)\overline{F'(w)}}{\pi(1-F(z)\overline{F(w)})^2} + \frac{\rho^2 F'(z)\overline{F'(w)}}{\pi(1-\rho^2 F(z)\overline{F(w)})^2}, \\ G_3(z,w) &= F'(z)\overline{F'(w)} \sum_{j\ge 0}\left(\frac{(j+1)\rho^{4j+4}}{\pi(1-\rho^{2j+2})}\right)(F(z)\overline{F(w)})^j, \\ G_4(z,w) &= F'(z)\overline{F'(w)} \sum_{j=-\infty}^{-2} \frac{j+1}{\pi} \frac{1}{1-\rho^{2j+2}} (F(z)\overline{F(w)})^j. \end{aligned} \tag{2.5}$$

On the other hand,

$$P_\Omega C f(z) = \int_\Omega H_\Omega(z,\xi) f(\xi) dA(\xi),$$

where $H_\Omega(z,\xi) = -\frac{1}{\pi}\int_\Omega \frac{K_\Omega(z,w)}{\xi - w} dA(w)$.

The exact formula for the kernel $H_\Omega(z,\xi)$ will be calculated in Lemma 2.3. In order to write the final formulas in a more concise way, let us denote by

$$S_i(z,\xi) = -\frac{1}{\pi}\int_\Omega \frac{G_i(z,w)}{\xi - w} dA(w), \tag{2.6}$$

for $i = 1, 2, 3, 4$, and

$$\Phi_\rho(z,w) = \frac{\rho^2 w}{\rho^2 w - z} - \frac{w}{w - \rho^2 z}, z, w \in A,$$

$$\Phi_\rho^+(z,w) = \frac{\rho^2 w}{\rho^2 w - z} + \frac{w}{w - z}, z, w \in A,$$

$$\Xi_\rho(z,w) = \frac{1}{\bar{w}z - 1} + \frac{1}{\rho^2 \bar{w} z - 1}, z, w \in A,$$

$$\Psi_\rho(z,w) = \sum_{j=0}^{\infty} \frac{\rho^{2j+2} z^j (1 - (\rho w)^{2j+2})}{(1 - \rho^{2j+2}) w^{j+1}}, z, w \in A,$$

$$\Theta_\rho(z,w) = \sum_{j=1}^{\infty} \frac{\rho^{2j}(\rho^{-2j} - |w|^{-2j}) w^j}{(\rho^{2j} - 1) z^{j+1}}, z, w \in A.$$

Moreover, the Cauchy transform $\tilde{\mathcal{C}} : L^2(\Omega) \to L^2(\Omega)$, which will be used further is defined by

$$\tilde{\mathcal{C}}(f)(\xi) = \frac{1}{2\pi i}\int_{\Gamma_1} \frac{f(\zeta)}{\zeta - \xi} d\zeta,$$

$\Gamma_1 = \{\varphi(e^{i\theta}) : 0 \le \theta < 2\pi\}$.

Lemma 2.3 $H_\Omega(z,\xi) = \sum_{i=1}^{5} \tilde{S}_i(z,\xi)$. *Here,*

$$\tilde{S}_1(z,\xi) = \frac{2F'(z)}{\pi F(z)}\left(1 - \frac{\ln|F(\xi)|}{\ln\left(\frac{1}{\rho}\right)}\right),$$

$$\tilde{S}_2(z,\xi) = \frac{F'(z)}{\pi F(z)}\tilde{\mathcal{C}}(\Phi_\rho(F(z), F(\cdot))(\xi),$$

$$\tilde{S}_3(z,\xi) = \frac{F'(z)}{\pi F(z)}\left(\Xi_\rho(F(z), F(\xi)) + \Phi_\rho^+(F(\xi), F(z))\right) + D(z,\xi),$$

$$\tilde{S}_4(z,\xi) = \frac{F'(z)}{\pi}\left(\rho^2\tilde{\mathcal{C}}(\frac{1}{F(\cdot) - \rho^2 F(z)})(\xi) + \Psi_\rho(F(z), F(\xi))\right),$$

$$\tilde{S}_5(z,\xi) = \frac{F'(z)}{\pi}\left(\frac{\rho^2}{F(z)}\tilde{\mathcal{C}}(\frac{F(\cdot)}{F(z) - \rho^2 F(\cdot)})(\xi) + \Theta_\rho(F(z), F(\xi))\right),$$

where

$$D(z,\xi) = \frac{1}{\pi}\frac{1}{z - \xi}, z, \xi \in \Omega.$$

Proof Let us note that for any $\xi \in \Omega$ and $\varphi(r_0 e^{i\theta_0}) = \xi$ the closed curves $\{\varphi(re^{i\theta}) : \theta \in [0, 2\pi)\}$, $r > r_0$ and $\{\varphi(e^{i\theta}) : \theta \in [0, 2\pi)\}$ enclose the area in Ω which contains ξ. Let us denote by $\Gamma_r = \{\varphi(re^{i\theta}) | 0 \le \theta < 2\pi\}$ the closed curve in Ω for the fixed radius $r \in [1, \frac{1}{\rho}]$.

After the change of variable $w = \varphi(\omega)$ and by applying the Cauchy's integral formula for the function $S_1(z, \xi)$ we get

$$\begin{aligned}
&S_1(z,\xi)\\
&= \frac{\frac{F'(z)}{F(z)}}{\pi^2 \ln(\frac{1}{\rho})} \int_\Omega \frac{\overline{F'(w)}}{(w-\xi)\overline{F(w)}} dA(w) = \frac{\frac{F'(z)}{F(z)}}{\pi^2 \ln(\frac{1}{\rho})} \int_A \frac{\varphi'(\omega)}{(\varphi(\omega)-\xi)\overline{\omega}} dA(\omega)\\
&= \frac{\frac{F'(z)}{F(z)}}{\pi^2 \ln(\frac{1}{\rho})} \int_1^{\frac{1}{\rho}} dr \int_0^{2\pi} \frac{\varphi'(re^{i\theta})e^{i\theta}}{\varphi(re^{i\theta})-\xi} d\theta = \frac{\frac{F'(z)}{F(z)}}{i\pi^2 \ln(\frac{1}{\rho})} \int_1^{\frac{1}{\rho}} \frac{dr}{r} \int_{\Gamma_r} \frac{d\zeta}{\zeta-\xi}\\
&= \frac{\frac{F'(z)}{F(z)}}{i\pi^2 \ln(\frac{1}{\rho})} \int_{|F(\xi)|}^{\frac{1}{\rho}} \frac{dr}{r} \int_{\Gamma_r} \frac{d\zeta}{\zeta-\xi} = \frac{2F'(z)}{\pi F(z)} \left(1 - \frac{\ln|F(\xi)|}{\ln\left(\frac{1}{\rho}\right)}\right).
\end{aligned}$$

Note that in this case $\tilde{S}_1 = S_1$. In a similar manner we obtain

$$\begin{aligned}
S_2(z,\xi) =& \frac{F'(z)}{\pi^2} \int_A \frac{\varphi'(\omega) dA(\omega)}{(\varphi(\omega)-\xi)(1-F(z)\overline{\omega})^2}\\
&+ \frac{\rho^2 F'(z)}{\pi^2} \int_A \frac{\varphi'(\omega) dA(\omega)}{(\varphi(\omega)-\xi)(1-\rho^2 F(z)\overline{\omega})^2}.
\end{aligned} \tag{2.7}$$

Let us calculate the first summand in Eq. 2.7.

$$\begin{aligned}
&\frac{F'(z)}{\pi^2} \int_A \frac{\varphi'(\omega) dA(\omega)}{(\varphi(\omega)-\xi)(1-F(z)\overline{\omega})^2}\\
&= \frac{F'(z)}{\pi^2} \int_1^{\frac{1}{\rho}} r dr \int_0^{2\pi} \frac{\varphi'(re^{i\theta}) d\theta}{(\varphi(re^{i\theta})-\xi)(1-F(z)re^{-i\theta})^2}\\
&= \frac{F'(z)}{i\pi^2} \int_1^{\frac{1}{\rho}} \frac{1}{r^3} dr \int_{|\zeta|=r} \frac{\varphi'(\zeta)\zeta}{(\varphi(\zeta)-\xi)(\frac{\zeta}{r^2}-F(z))^2} d\zeta\\
&= I_1(z,\xi) + I_2(z,\xi),
\end{aligned}$$

where

$$I_1(z,\xi) = \frac{F'(z)}{i\pi^2} \int_1^{|F(\xi)|} \frac{1}{r^3} dr \int_{|\zeta|=r} \frac{\varphi'(\zeta)\zeta}{(\varphi(\zeta)-\xi)(\frac{\zeta}{r^2}-F(z))^2} d\zeta,$$

$$I_2(z,\xi) = \frac{F'(z)}{i\pi^2} \int_{|F(\xi)|}^{\frac{1}{\rho}} \frac{1}{r^3} dr \int_{|\zeta|=r} \frac{\varphi'(\zeta)\zeta}{(\varphi(\zeta)-\xi)(\frac{\zeta}{r^2}-F(z))^2} d\zeta.$$

Using the Cauchy formula for multiply connected domains we get the following formulas

$$\begin{aligned}
I_1(z,\xi) &= \frac{F'(z)}{i\pi^2} \int_1^{|F(\xi)|} r dr \int_{|\zeta|=1} \frac{\varphi'(\zeta)\zeta}{(\varphi(\zeta)-\xi)(\zeta-r^2F(z))^2} d\zeta\\
&= \frac{F'(z)}{i\pi^2} \int_{|\zeta|=1} \frac{\varphi'(\zeta)\zeta}{\varphi(\zeta)-\xi} d\zeta \int_1^{|F(\xi)|} \frac{r}{(\zeta-r^2F(z))^2} dr\\
&= \frac{F'(z)}{2\pi^2 i F(z)} \int_{\Gamma_1} \frac{F(\zeta)}{\zeta-\xi} \left(\frac{1}{F(\zeta)-|F(\xi)|^2F(z)} - \frac{1}{F(\zeta)-F(z)}\right) d\zeta,
\end{aligned}$$

and

$$\begin{aligned}
I_2(z,\xi) &= \frac{F'(z)}{i\pi^2}\int_{|F(\xi)|}^{\frac{1}{\rho}} r dr \int_{|\zeta|=1} \frac{\varphi'(\zeta)\zeta}{(\varphi(\zeta)-\xi)(\zeta - r^2F(z))^2} d\zeta \\
&+ \frac{F'(z)}{i\pi^2}\int_{|F(\xi)|}^{\frac{1}{\rho}} r dr \int_{\gamma_r} \frac{F(\zeta)d\zeta}{(\zeta-\xi)(F(\zeta)-r^2F(z))^2}, \\
&= \frac{F'(z)}{2\pi^2 i F(z)}\int_{\Gamma_1} \frac{F(\zeta)}{\zeta-\xi}\left(\frac{\rho^2}{\rho^2F(\zeta)-F(z)} - \frac{1}{F(\zeta)-|F(\xi)|^2F(z)}\right)d\zeta \\
&+ \frac{F'(z)}{\pi F(z)}\left(\frac{\rho^2F(\xi)}{\rho^2F(\xi)-F(z)} - \frac{1}{1-F(z)\overline{F(\xi)}}\right),
\end{aligned}$$

where γ_r is a closed analytic contour positively oriented in $\Gamma_1^r = \{z \in \Omega : d(z,\Gamma_1) < r\}$ which encloses the point ξ.

In the following computations we use the Cauchy-Green formula for a multiply connected domain.

$$\begin{aligned}
&\int_\Omega \frac{\rho^2F'(z)\overline{F'(w)}dA(w)}{\pi^2(w-\xi)(1-\rho^2F(z)\overline{F(w)})^2} \\
&= \frac{F'(z)}{\pi^2F(z)}\int_\Omega \frac{\partial}{\partial\bar{w}}\left(\frac{1}{1-\rho^2F(z)\overline{F(w)}}\right)\frac{dA(w)}{w-\xi} \\
&= -\frac{F'(z)}{\pi F(z)(1-\rho^2F(z)\overline{F(\xi)})} + \frac{F'(z)}{2\pi^2 iF(z)}\int_{\Gamma_{\frac{1}{\rho}}} \frac{1}{1-\rho^2F(z)\overline{F(w)}}\frac{dw}{w-\xi} \\
&- \frac{F'(z)}{2\pi^2 iF(z)}\int_{\Gamma_1} \frac{1}{1-\rho^2F(z)\overline{F(w)}}\frac{dw}{w-\xi}.
\end{aligned}$$

Note that the fact that the mapping $F:\Omega\to A$ extends analytically to $\overline{\Omega}$ such that F is a homeomorphism on the boundaries ($F:\partial\Omega\to\partial A$) implies

$$\frac{1}{2\pi i}\int_{\Gamma_{\frac{1}{\rho}}} \frac{1}{1-\rho^2F(z)\overline{F(w)}}\frac{dw}{w-\xi} = \frac{1}{2\pi i}\int_{\Gamma_{\frac{1}{\rho}}} \frac{F(w)}{F(w)-F(z)}\frac{dw}{w-\xi}. \tag{2.8}$$

Applying Cauchy's Theorem for a multiply connected domain to the last integral in Eq. 2.8 one obtains

$$\begin{aligned}
&\frac{1}{2\pi i}\int_{\Gamma_{\frac{1}{\rho}}} \frac{F(w)}{F(w)-F(z)}\frac{dw}{w-\xi} \\
&= \frac{F(\xi)}{F(\xi)-F(z)} + \frac{F(z)}{F'(z)}\frac{1}{z-\xi} + \frac{1}{2\pi i}\int_{\Gamma_1} \frac{F(w)}{F(w)-F(z)}\frac{dw}{w-\xi}.
\end{aligned} \tag{2.9}$$

Collecting the computations from Eqs. 2.8 and 2.9 we get

$$\begin{aligned}
&\int_\Omega \frac{\rho^2F'(z)\overline{F'(w)}dA(w)}{\pi^2(w-\xi)(1-\rho^2F(z)\overline{F(w)})^2} \\
&= \frac{F'(z)}{2\pi^2 iF(z)}\int_{\Gamma_1} \frac{F(\zeta)}{\zeta-\xi}\left(\frac{1}{F(\zeta)-F(z)} - \frac{1}{F(\zeta)-\rho^2F(z)}\right)d\zeta \\
&+ \frac{F'(z)}{\pi F(z)}\left(\frac{F(\xi)}{F(\xi)-F(z)} - \frac{1}{1-\rho^2F(z)\overline{F(\xi)}}\right) + \frac{1}{\pi}\frac{1}{z-\xi}.
\end{aligned}$$

Further according to Eq. 2.6 we have

$$
\begin{aligned}
&S_3(z,\xi)\\
&=\frac{F'(z)}{\pi^2}\sum_{j\geq 0}\frac{(j+1)(F(z))^j\rho^{4j+4}}{1-\rho^{2j+2}}\int_\Omega\frac{(\overline{F'(w)})(F(w))^j}{w-\xi}dA(w)\\
&=\frac{F'(z)}{i\pi^2}\sum_{j\geq 0}\frac{(j+1)(F(z))^j\rho^{4j+4}}{1-\rho^{2j+2}}\int_1^{\frac{1}{\rho}}r^{2j+1}\left(\int_{|\zeta|=r}\frac{\varphi'(\zeta)d\zeta}{\zeta^{j+1}(\varphi(\zeta)-\xi)}\right)dr\\
&=\frac{F'(z)}{i\pi^2}\sum_{j\geq 0}\frac{(j+1)(F(z))^j\rho^{4j+4}}{1-\rho^{2j+2}}\int_1^{\frac{1}{\rho}}r^{2j+1}\left(\int_{|\zeta|=1}\frac{\varphi'(\zeta)d\zeta}{\zeta^{j+1}(\varphi(\zeta)-\xi)}\right)dr\\
&+\frac{F'(z)}{i\pi^2}\sum_{j\geq 0}\frac{(j+1)(F(z))^j\rho^{4j+4}}{1-\rho^{2j+2}}\int_{|F(\xi)|}^{\frac{1}{\rho}}r^{2j+1}\left(\int_{\gamma_r}\frac{\varphi'(\zeta)d\zeta}{\zeta^{j+1}(\varphi(\zeta)-\xi)}\right)dr,
\end{aligned}
$$

where the curve γ_r encloses the point $F(\xi)$ within the region $\{\zeta|1<|\zeta|<r\}$. Therefore we obtain

$$
\begin{aligned}
S_3(z,\xi)=&\frac{\rho^2F'(z)}{2\pi^2 i}\int_{|\zeta|=1}\frac{\varphi'(\zeta)d\zeta}{(\zeta-\rho^2F(z))(\varphi(\zeta)-\xi)}\\
&+\frac{F'(z)}{\pi}\sum_{j\geq 0}\frac{\rho^{2j+2}(F(z))^j(1-(\rho|F(\xi)|)^{2j+2})}{(1-\rho^{2j+2})(F(\xi))^{j+1}},
\end{aligned}
$$

and

$$
\begin{aligned}
S_4(z,\xi)=&\frac{\rho^2F'(z)}{2\pi^2 iF(z)}\int_{|\zeta|=1}\frac{\varphi'(\zeta)\zeta}{(F(z)-\rho^2\zeta)(\varphi(\zeta)-\xi)}d\zeta\\
&+\frac{F'(z)}{\pi}\sum_{j=1}^{\infty}\frac{\rho^{2j}(\rho^{2j}-|F(\xi)|^{-2j})(F(\xi))^j}{(\rho^{2j}-1)(F(z))^{j+1}}.
\end{aligned}
\tag{2.10}
$$

□

In terms of the last lemma, we have

$$P_\Omega Cf(z)=\sum_{i=1}^{5}M_if(z),$$

where $M_i:L^2(\Omega)\to L^2(\Omega)$,

$$M_if(z)=\int_\Omega\tilde{S}_i(z,\xi)f(\xi)dA(\xi),\ \ i\in\{1,2,3,4,5\}. \tag{2.11}$$

3 Proof of the Main Result

Lemma 3.1 *Let $\{M_i\}_{i\geq 1}$ be the operators defined in* Eq. 2.11.
The following relation holds

$$\lim_{n\to+\infty}ns_n(M_1+M_2+M_4+M_5)=0.$$

Proof First of all, it is clear that the operator M_1 is one-dimensional finite rank operator. Thus for $n > 1$ we have the following inequality (see [12])

$$s_{n+1}(M_2 + M_4 + M_5) \leq s_n(M_1 + M_2 + M_4 + M_5) \leq s_{n-1}(M_2 + M_4 + M_5).$$

In the sequel we will consider the singular numbers for the operators $\{s_n(M_i)\}_{n\geq 1}$, $i \in \{2, 4, 5\}$.

We give the proof concerning the operators M_5. The proof for the operators M_4 and M_2 is analogous.

Let us note that

$$M_5 f(z) = \sum_{i=1}^{3} M_5^i f(z),\ f \in L^2(\Omega),$$

where

$$M_5^i f(z) = \int_\Omega S_5^i(z, \xi) f(\xi) dA(\xi),$$

and

$$S_5^1(z, \xi) = \frac{\rho^2 F'(z)}{2\pi^2 i F(z)} \int_{|\zeta|=1} \frac{\varphi'(\zeta)\zeta}{(F(z) - \rho^2\zeta)(\varphi(\zeta) - \xi)} d\zeta,$$

$$S_5^2(z, \xi) = \frac{F'(z)}{\pi} \sum_{j=1}^{\infty} \frac{\rho^{4j}(F(\xi))^j}{(\rho^{2j} - 1)(F(z))^{j+1}},$$

$$S_5^3(z, \xi) = \frac{F'(z)}{\pi} \sum_{j=1}^{\infty} \frac{\rho^{2j}(\overline{F(\xi)})^{-j}}{(\rho^{2j} - 1)(F(z))^{j+1}}.$$

Using the linear isometry $V : L^2(\Omega) \to L^2(A)$, given by $Vf = (f \circ \varphi)\varphi'$, we have that $VM_5^2 = \tilde{M}_2 V$, where $\tilde{M}_2 : L^2(A) \to L^2(A)$,

$$\tilde{M}_2 f(z) = \sum_{j=1}^{\infty} \frac{\rho^{4j}}{\pi(\rho^{2j} - 1)} \int_A \frac{\xi^j \overline{\varphi'(\xi)} f(\xi)}{z^{j+1}} dA(\xi),$$

and therefore $s_n(M_5^2) = s_n(\tilde{M}_2)$.

On the other hand, $\tilde{M}_2 = M^2 Q$, where $Qf(z) = \overline{\varphi'(z)} f(z)$, and

$$M^2 f(z) = \sum_{j=1}^{\infty} \frac{\rho^{4j}}{\pi(\rho^{2j} - 1)} \int_A \frac{\xi^j f(\xi)}{z^{j+1}} dA(\xi).$$

Taking into account that the family $\{\psi_j, \overline{\psi}_j\}_{j=-\infty}^{\infty}$

$$\psi_j(z) = \|\phi_j\|_{L^2(A)}^{-1} \frac{1}{z^j},\ z \in A$$

presents an orthonormal system in $L^2(A)$, the operator M^2 admits the following Schmidt expansion

$$M^2 = \sum_{j=1}^{\infty} \frac{\rho^{4j} c_j(\rho)}{(\rho^{2j} - 1)} \psi_{-j-1} \langle \cdot, \overline{\psi}_j \rangle,$$

$c_j(\rho) = \left(\frac{(1-\rho^{2j+2})(1-\rho^{2j})}{j(j+1)\rho^{2j+2}} \right)^{\frac{1}{2}}$.

Therefore,

$$s_n(M^2) \sim \frac{\rho^{3n-1}}{n}, n \to +\infty.$$

Since, $s_n(\tilde{M}_2) \le \|\varphi'\|_\infty s_n(M^2), n \in \mathbb{N}$ (the inequality Eq. 1.1) we have that

$$\lim_{n\to+\infty} n^\alpha s_n(M_5^2) = 0, \alpha > 1.$$

In a similar fashion it can be proved that $s_n(M_5^3) \sim \frac{\rho^{n-1}}{n}, n \to +\infty$ and consequently

$$\lim_{n\to+\infty} n^\alpha s_n(M_5^3) = 0, \alpha > 1.$$

In order to examine the behavior of singular numbers $s_n(M_5^1)$ we have to apply a different type of arguments.

Firstly, after change of variable the kernel $S_1(z,\xi)$ can be written by the following formula

$$S_1(z,\xi) = \frac{\rho^2 F'(z)}{2\pi^2 i F(z)} \int_{\Gamma_1} \frac{d\zeta}{(F(z) - \rho^2 F(\zeta))(\zeta - \xi)}.$$

Let us recall that for any $z \in \Omega$,

$$\begin{aligned} &\frac{1}{F(z) - \rho^2 F(\xi)} \\ &= \frac{1}{2\pi i} \int_{\partial\Omega} \frac{d\zeta}{(F(z) - \rho^2 F(\zeta))(\zeta - \xi)} \\ &= \frac{1}{2\pi i} \int_{\Gamma_{\frac{1}{\rho}}} \frac{d\zeta}{(F(z) - \rho^2 F(\zeta))(\zeta - \xi)} - \frac{1}{2\pi i} \int_{\Gamma_1} \frac{d\zeta}{(F(z) - \rho^2 F(\zeta))(\zeta - \xi)}. \end{aligned} \tag{3.1}$$

In what follows we consider the integral operator $T' : L^2(\Omega) \to L^2(\Omega)$

$$T' f(z) = \int_\Omega \frac{f(\xi)}{F(z) - \rho^2 F(\xi)} dA(\xi)$$

and we prove that

$$s_n(T') = o(n^{-1}), n \to +\infty. \tag{3.2}$$

First of all, note that the operator $B : L^2(A) \to L^2(A)$ defined by the formula

$$Bf(z) = \int_A \frac{f(\xi)}{z - \rho^2 \xi} dA(\xi)$$

admits the sequel Schmidt expansion

$$B = \sum_{j=1}^{\infty} \rho^{2(j-1)} c_j(\rho) \psi_{-j-1} \left\langle \cdot, \overline{\psi}_j \right\rangle.$$

Therefore $s_n(B) \sim \frac{\rho^{n-3}}{n}$, $n \to +\infty$. Since $VT' = Q'BQ''$, where $Q' : L^2(A) \to L^2(A)$, $Q'' : L^2(A) \to L^2(A)$, $Q' f(z) = \varphi'(z) f(z)$ and $Q'' f(z) = |\varphi'(z)|^2 f(z)$ are the multiplication operators, we get the relation $s_n(T') = o(n^{-1})$, $n \to +\infty$.

It is not hard to check that the kernel $S_1(z,\xi)$ is an analytic function with respect to ξ in Ω which satisfies the conditions from Theorem 1.1. Namely, if we denote by $\delta = d(\Gamma_1, \Gamma_{\frac{1}{\rho}})$, then for any $\xi_0 \in \Omega$ such that $d(\xi_0, \Gamma_1) \ge \frac{\delta}{2}$ the last summand in Eq. 3.1

(the kernel of $S_1(z,\xi)$) can be directly expanded in some neighbourhood of ξ_0 satisfying the conditions from Theorem 1.1. If this is not the case, i.e. $d(\xi_0,\Gamma_1)\leq\frac{\delta}{2}$ then the same procedure can be conducted for the first summand in the Eq. 3.1 (the kernel $\tilde{K}(z,\xi)=\frac{1}{2\pi i}\int_{\Gamma_{\frac{1}{\rho}}}\frac{d\zeta}{(F(z)-\rho^2F(\zeta))(\zeta-\xi)}$).

Taking into account the relation Eq. 3.2 and according to Theorem 1.1 we have

$$\lim_{n\to+\infty} ns_n(M_5^1)=0.$$

According to Theorem 2.3 in [12], we have that

$$\lim_{n\to+\infty} ns_n(M_5)=0,$$

and consequently

$$\lim_{n\to+\infty} ns_n(M_2+M_4+M_5)=0.$$

□

Before proving Lemma 3.3 we state a slightly modified form of Lemma 3 from [9].

Lemma 3.2 *Let* $K=\{z|0<\Re z<a, 0<\Im z<a\}$ *with* $a>0$ *and let* $T:L^2(K)\to L^2(K)$ *be an integral operator defined as*

$$Tf(z)=\int_K\frac{f(\xi)}{z+\xi}dA(\xi),\ f\in L^2(K).$$

Then

$$s_n(T)=o(n^{-1}), n\to+\infty. \tag{3.3}$$

Here we should point out the fact that the kernel of the operator T has a singularity only in the point $\xi=z=0$. The proof of the lemma follows by applying the results from the paper [20] or the relation Eq. 3.3 can be also derived from the kernel condition established in the paper [18].

Lemma 3.3 *Let* $K_a=\{z|0<\Re z<a, 0<\Im z<a\}$, *where* $1<a<2\pi$, *then*

$$s_n\left(\int_{K_a}\left(\frac{c_0}{z+\bar{\xi}}+\frac{c_1}{z+\bar{\xi}-2a}\right)dA(\xi)\right)\sim\frac{a|c_1|}{2n},\ \text{as } n\to+\infty,$$

$$s_{-n}\left(\int_{K_a}\left(\frac{c_0}{z+\bar{\xi}}+\frac{c_1}{z+\bar{\xi}-2a}\right)dA(\xi)\right)\sim\frac{a|c_0|}{2n},\ \text{as } n\to+\infty,$$

where c_0 *and* c_1 *are fixed complex numbers.*

Proof Let us first consider the annulus $A_\pi=\{z:1<|z|<e^{2\pi}\}$, and the operator $T:L^2(A_\pi)\to L^2(A_\pi)$ which is defined by the sequel formula

$$Tf(z)=\int_{A_\pi}f(\xi)\left(\frac{c_0}{z\bar{\xi}(z\bar{\xi}-1)}+\frac{c_1}{z\bar{\xi}(\rho^2z\bar{\xi}-1)}\right)dA(\xi),$$

where $\rho=e^{-2\pi}$.

Using the Laurent expansion of the function $\frac{c_0}{z\bar{\xi}(z\bar{\xi}-1)}+\frac{c_1}{z\bar{\xi}(\rho^2 z\bar{\xi}-1)}$ in A_π, we get the sequel expansion for the operator T

$$\begin{aligned} Tf(z) = & -2c_1\pi \ln(\frac{1}{\rho})(f,\phi_{-1})\phi_{-1} + \sum_{n=1}^{\infty}\frac{\pi c_0}{n}(1-\rho^{2n})(f,\phi_{-n-1})\phi_{-n-1} \\ & -\sum_{n=1}^{\infty}\frac{\pi c_1}{n}\left(1-\rho^{2n}\right)(f,\phi_{n-1})\phi_{n-1}. \end{aligned} \tag{3.4}$$

From the previous representation we may conclude that

$$s_n(T) \sim \frac{\pi|c_1|}{n}, \text{ as } n \to +\infty$$

and

$$s_{-n}(T) \sim \frac{\pi|c_0|}{n}, \text{ as } n \to +\infty.$$

Let $S : L^2(A_\pi) \to L^2(K_{2\pi})$ be the linear isometry given by

$$Sf(z) = e^z f(e^z),$$

where $K_{2\pi} = \{z|0 < \Re z < 2\pi, 0 < \Im z < 2\pi\}$ and $T_1 : L^2(K_{2\pi}) \to L^2(K_{2\pi})$ as follows

$$T_1 f(z) = \int_{K_{2\pi}} \left(\frac{c_0}{e^{z+\bar{\xi}}-1} + \frac{c_1}{\rho^2 e^{z+\bar{\xi}}-1}\right) f(\xi)dA(\xi).$$

Then $ST = T_1 S$. From the last equality we conclude that $s_n(T_1) = s_n(T)$, $n \in \mathbb{Z}$.

Further, the function $m(u)$ defined as

$$m(u) = \frac{c_0}{e^u-1} - \frac{c_0}{u} - \phi_1(u) + \frac{c_1}{\rho^2 e^u - 1} - \frac{c_1}{u-4\pi} - \phi_2(u)$$

is analytic in the disc $|u| < 6\pi$, where

$$\phi_1(u) = \frac{c_0}{u-2\pi i} + \frac{c_0}{u+2\pi i} + \frac{c_0}{u-4\pi i} + \frac{c_0}{u+4\pi i},$$

and

$$\phi_2(u) = \frac{c_1}{u-4\pi+2\pi i} + \frac{c_1}{u-4\pi-2\pi i}.$$

Note that

$$\frac{c_0}{e^{z+\bar{\xi}}-1} + \frac{c_1}{\rho^2 e^{z+\bar{\xi}}-1} = m(z+\bar{\xi}) + \frac{c_0}{z+\bar{\xi}} + \frac{c_1}{z+\bar{\xi}-4\pi} + \phi_1(z+\bar{\xi}) + \phi_2(z+\bar{\xi}).$$

Further, Theorem 1.1 and Lemma 3.3 imply

$$\lim_{n\to+\infty} n^\alpha s_n\left(\int_{K_{2\pi}} m(z+\bar{\xi})dA(\xi)\right) = 0, \alpha \geq 1,$$

$$\lim_{n\to+\infty} ns_n\left(\int_{K_{2\pi}} (\phi_1(z+\bar{\xi}) + \phi_2(z+\bar{\xi}))dA(\xi)\right) = 0,$$

respectively.

Theorem 2.3 from [12] concludes the proof of this lemma when $a = 2\pi$.

In case we observe K_a, where $1 < a < 2\pi$, let us note that the mapping $V : L^2(K_a) \to L^2(K_{2\pi})$ defined by

$$Vf(z) = \frac{a}{2\pi} f(\frac{a}{2\pi}z)$$

is a linear isometry. Then, it is not hard do check that $T_{2\pi} V = \frac{2\pi}{a} V T_a$, where $T_a : L^2(K_a) \to L^2(K_a)$,

$$T_a f(z) = \int_{K_a} \left(\frac{c_0}{z+\bar{\xi}} + \frac{c_1}{z+\bar{\xi}-2a} \right) f(\xi) dA(\xi),$$

and the result follows by the appropriate substitution. □

Corollary 3.4

$$s_n \left(\int_{K_a^h} \left(\frac{c_0}{z+\bar{\xi}} + \frac{c_1}{z+\bar{\xi}-2a} \right) dA(\xi) \right) \sim \frac{a|c_1|}{2n}, \ \ as \ n \to +\infty,$$

$$s_{-n} \left(\int_{K_a^h} \left(\frac{c_0}{z+\bar{\xi}} + \frac{c_1}{z+\bar{\xi}-2a} \right) dA(\xi) \right) \sim \frac{a|c_0|}{2n}, \ \ as \ n \to +\infty,$$

where c_0 and c_1 are fixed complex numbers, $K_a^h = \{z + ih : z \in K_a\}$, and $h \geq 0$ is an arbitrary positive number.

Proof The proof follows directly from the relation $T_a S = S T_a^h$, where $Sf(z) = f(z+ih)$ is linear isometry from $L^2(K_a^h)$ onto $L^2(K_a)$ and

$$T_a^h f(z) = \int_{K_a^h} \left(\frac{c_0}{z+\bar{\xi}} + \frac{c_1}{z+\bar{\xi}-2a} \right) f(\xi) dA(\xi).$$

□

Remark 3.5 In the sequence of singular numbers $\{s_n(T_a)\}_{n\in\mathbb{Z}}$ in Lemma 3.3 and Corollary 3.4, occur two particular subsequences $\{s_n(T_a)\}_{n\geq 0}$ and $\{s_{-n}(T_a)\}_{n>0}$. In fact, this ambiguity can be explained by the presence of two types of singularities in the kernel of the operator T_a, the vertical sides $\{z : \Re z = 0, 0 \leq \Im z \leq a\}$ and $\{z : \Re z = a, 0 \leq \Im z \leq a\}$.

Lemma 3.6

$$s_n \left(\int_{K_a^h} \left(\frac{c_0}{z+\bar{\xi}} + \frac{c_1}{z+\bar{\xi}-2H} \right) dA(\xi) \right) \sim \frac{a|c_0|}{2n}, \ \ as \ n \to +\infty,$$

$$s_{-n} \left(\int_{K_a'} \left(\frac{c_0}{z+\bar{\xi}} + \frac{c_1}{z+\bar{\xi}-2H} \right) dA(\xi) \right) \sim \frac{a|c_0|}{2n}, \ \ as \ n \to +\infty,$$

where $0 < a < \frac{H}{3}$ *and* $K_a^{'} = \{z : H - a < \Re z < H, h < \Im z < a + h\}$, $h > 0$.

The proof of Lemma 3.6 can be easily established by using the main results from [20] along with Lemma 3.6 (see also Lemma 4 from [9]) and the K.Fan theorem (Theorem 2.3 from [12]).

Let

$$K_i^1 = \{z : \frac{2\pi}{N}(i-1) < \arg z < \frac{2\pi}{N} i, 1 < |z| < e^{\frac{2\pi}{N}}\}$$

and

$$K_i^2 = \{z : \frac{2\pi}{N}(i-1) < \arg z < \frac{2\pi}{N} i, e^{\ln \frac{1}{\rho} - \frac{2\pi}{N}} < |z| < e^{\ln \frac{1}{\rho}}\},$$

$i = 1, 2, ..., N$. By K_{1i} and K_{2i}, $i = 1, 2, ..., N$ we denote the corresponding sets defined as

$$K_{1i} = \{z : 0 < \Re z < \frac{2\pi}{N}, \frac{2\pi}{N}(i-1) < \Im z < \frac{2\pi}{N} i\},$$

$$K_{2i} = \{z : \ln\frac{1}{\rho} - \frac{2\pi}{N} < \Re z < \ln\frac{1}{\rho}, \frac{2\pi}{N}(i-1) < \Im z < \frac{2\pi}{N}i\}.$$

Here we suppose that the integer N is large enough so that the inequality $\frac{2\pi}{N} < \ln\frac{1}{\rho} - \frac{2\pi}{N}$ holds.

In correspondence to the sets K_i^1 and K_i^2, we define the operators T_i^1 and T_i^2 in the following way

$$T_i^1 f(z) = \int_{K_i^1} \left(\frac{c_0}{z\bar{\xi}(z\bar{\xi}-1)} + \frac{c_1}{z\bar{\xi}(\rho^2 z\bar{\xi}-1)} \right) f(\xi) dA(\xi),\ f \in L^2(K_i^1)$$

and

$$T_i^2 f(z) = \int_{K_i^2} \left(\frac{c_0}{z\bar{\xi}(z\bar{\xi}-1)} + \frac{c_1}{z\bar{\xi}(\rho^2 z\bar{\xi}-1)} \right) f(\xi) dA(\xi),\ f \in L^2(K_i^2),$$

$i = 1, 2, ..., N$.

Lemma 3.7 *The following asymptotic formulas hold*

$$s_n(T_i^1) \sim \frac{|c_0|\pi}{nN}, n \to +\infty,$$

and

$$s_n(T_i^2) \sim \frac{|c_1|\pi}{nN}, n \to +\infty.$$

Proof Using again the linear isometry $S_i^j f(z) = e^z f(e^z)$,

$$S_i^j : L^2(K_i^j) \to L^2(K_{ji})$$

we get that $S_i^j T_i^j = \tilde{T}_i^j S_i^j$, $j = 1, 2$. Here,

$$\tilde{T}_i^j f(z) = \int_{K_{ji}} \left(\frac{c_0}{e^{z+\bar{\xi}}-1} + \frac{c_1}{\rho^2 e^{z+\bar{\xi}}-1} \right) f(\xi) dA(\xi),$$

$j = 1, 2$ and $i = 1, 2, ..., N$. On the other hand, the function

$$(z, \xi) \to \frac{c_0}{e^{z+\bar{\xi}}-1} + \frac{c_1}{\rho^2 e^{z+\bar{\xi}}-1} - \frac{c_0}{z+\bar{\xi}} - \frac{c_1}{z+\bar{\xi}-2\ln\frac{1}{\rho}},$$

is a real analytic function in $K_{ji} \times K_{ji}$. We are now in position to apply Birman-Solomjak theorem (see [12], pp.78), which we stated in a modified form as Theorem 1.1, and we get

$$\lim_{n\to\infty} n s_n(\tilde{T}_i^j - T_{ij}) = 0,\ j = 1, 2,\ \ i = 1, ..., N,$$

where $T_{ij} : L^2(K_{ij}) \to L^2(K_{ij})$,

$$T_{ij} f(z) = \int_{K_{ij}} \left(\frac{c_0}{z+\bar{\xi}} + \frac{c_1}{z+\bar{\xi}-2\ln\frac{1}{\rho}} \right) f(\xi) dA(\xi).$$

According to Lemma 3.6 and Theorem 2.3 from [12] we have that

$$s_n(T_i^1) \sim \frac{|c_0|\pi}{nN},\ \ s_n(T_i^2) \sim \frac{|c_1|\pi}{nN},\ \ n \to \infty,$$

for $i = 1, ..., N$. □

The proof of Lemma 3.8 follows the main line of argumentation given in Lemma 7 in [9].

Lemma 3.8 *Let $\psi \in C(\bar{A})$ be a complex function and the operator $T : L^2(A) \to L^2(A)$ is defined by*

$$Tf(z) = \frac{1}{\pi}\int_A \left(\frac{1}{z\bar{\xi}(z\bar{\xi}-1)} + \frac{1}{z\bar{\xi}(\rho^2 z\bar{\xi}-1)}\right)\psi(\xi)f(\xi)dA(\xi).$$

Then

$$s_n(T) \sim \frac{1}{2\pi n}\int_0^{2\pi} |\psi(e^{it})|dt + \frac{\rho}{2\pi n}\int_0^{2\pi}\left|\psi\left(\frac{e^{it}}{\rho}\right)\right| dt, n \to +\infty.$$

Proof Let us introduce the domain $K_0 = \{z|e^{\frac{2\pi}{N}} < |z| < e^{\ln\frac{1}{\rho}-\frac{2\pi}{N}}\}$ and the complex numbers $\xi_j = e^{i\theta_i}$, where $\frac{2\pi}{N}(i-1) \le \theta_i \le \frac{2\pi i}{N}$, for $i = 1, 2, ..., N$.

Specially, for the operators

$$T_i^1 f(z) = \int_{K_i^1}\left(\frac{\psi(\xi_i)}{z\bar{\xi}(z\bar{\xi}-1)} + \frac{\psi(\frac{\xi_i}{\rho})}{z\bar{\xi}(\rho^2 z\bar{\xi}-1)}\right) f(\xi)dA(\xi),$$

$$T_i^2 f(z) = \int_{K_i^2}\left(\frac{\psi(\xi_i)}{z\bar{\xi}(z\bar{\xi}-1)} + \frac{\psi(\frac{\xi_i}{\rho})}{z\bar{\xi}(\rho^2 z\bar{\xi}-1)}\right) f(\xi)dA(\xi),$$

where $T_i^j : L^2(K_i^j) \to L^2(K_i^j)$, $j = 1, 2$.

From Lemma 3.7 we have that

$$s_n(T_i^1) \sim \frac{|\psi(\xi_i)|}{nN}, n \to +\infty \tag{3.5}$$

and

$$s_n(T_i^2) \sim \frac{|\psi(\frac{\xi_i}{\rho})|}{nN}, n \to +\infty. \tag{3.6}$$

At this point, we introduce a family of orthoprojectors $\{P_i^j\}_{i,j=1}^{N,2}$, on $L^2(A)$ defined as

$$P_i^j f(z) = \chi_{K_i^j}(z)f(z)$$

and

$$P_0 f(z) = \chi_{K_0}(z)f(z),$$

where χ_K is a characteristic function of the set K.

Then,

$$T = P_0 T P_0 + \sum_{i\neq 0} P_0 T P_i^j + \sum_{i\neq 0} P_i^j T P_0 + \sum_{j'\neq j \vee i\neq l} P_i^{j'} T P_l^j + \sum_{j=1}^{2}\sum_{i=1}^{N} P_i^j T P_i^j.$$

On the other hand Theorem 1.1 implies

$$\lim_{n\to+\infty} ns_n(P_0 T P_0) = 0, \lim_{n\to+\infty} ns_n(\sum_{i\neq 0} P_i^j T P_0) = 0,$$

and

$$\lim_{n\to+\infty} ns_n(\sum_{j'\neq j\vee i\neq l} P_i^{j'} T P_l^j) = 0, \lim_{n\to+\infty} ns_n(\sum_{i\neq 0} P_0 T P_i^j) = 0.$$

The proof of the last relations reduces to Lemma 3.2 (see also [20]). Appealing to Theorem 2.3 from [12] we obtain

$$\lim_{n\to+\infty} ns_n(P_0TP_0+\sum_{i\neq 0}P_0TP_i^j+\sum_{i\neq 0}P_i^jTP_0+\sum_{j'\neq j\vee i\neq l}P_i^{j'}TP_l^j)=0.$$

Further,

$$\sum_{j=1}^{2}\sum_{i=1}^{N}P_i^jTP_i^j=L_N+P_N,$$

$$L_Nf(z)$$
$$=\sum_{j=1}^{2}\sum_{i=1}^{N}\chi_{K_i^j}(z)\int_A\left(\frac{\psi(\xi)-\psi(\xi_i)}{z\bar{\xi}(z\bar{\xi}-1)}+\frac{\psi(\xi)-\psi(\frac{\xi_i}{\rho})}{z\bar{\xi}(\rho^2z\bar{\xi}-1)}\right)\chi_{K_i^j}(\xi)f(\xi)dA(\xi)$$

and

$$P_Nf(z)$$
$$=\sum_{j=1}^{2}\sum_{i=1}^{N}\chi_{K_i^j}(z)\int_A\left(\frac{\psi(\xi_i)}{z\bar{\xi}(z\bar{\xi}-1)}+\frac{\psi(\frac{\xi_i}{\rho})}{z\bar{\xi}(\rho^2z\bar{\xi}-1)}\right)\chi_{K_i^j}(\xi)f(\xi)dA(\xi).$$

For $\epsilon>0$ we may take N sufficiently large, such that $|\psi(\xi)-\psi(\xi_i)|<\frac{\epsilon}{2}$, $\xi\in K_i^1$ and $|\psi(\xi)-\psi(\frac{\xi_i}{\rho})|<\frac{\epsilon}{2}$, $\xi\in K_i^2$.

According to Lemma 3.7 we get

$$\begin{aligned}&s_n\left(\chi_{K_i^j}(z)\int_A\left(\frac{\psi(\xi)-\psi(\xi_i)}{z\bar{\xi}(z\bar{\xi}-1)}+\frac{\psi(\xi)-\psi(\frac{\xi_i}{\rho})}{z\bar{\xi}(\rho^2z\bar{\xi}-1)}\right)\chi_{K_i^j}(\xi)dA(\xi)\right)\\&=s_n\left(\int_{K_i^j}\left(\frac{\psi(\xi)-\psi(\xi_i)}{z\bar{\xi}(z\bar{\xi}-1)}+\frac{\psi(\xi)-\psi(\frac{\xi_i}{\rho})}{z\bar{\xi}(\rho^2z\bar{\xi}-1)}\right)dA(\xi)\right)\\&\leq c\frac{\epsilon}{2nN},\end{aligned}\tag{3.7}$$

where the constant c is independent from ϵ, n, N. In the last relation we used the inequality Eq. 1.1.

On the other hand, L_N is a direct sum of operators $\tilde{T}_i^j: L^2(K_i^j)\to L^2(K_i^j)$

$$\tilde{T}_i^jf(z)=\int_{K_i^j}\left(\frac{\psi(\xi)-\psi(\xi_i)}{z\bar{\xi}(z\bar{\xi}-1)}+\frac{\psi(\xi)-\psi(\frac{\xi_i}{\rho})}{z\bar{\xi}(\rho^2z\bar{\xi}-1)}\right)dA(\xi),$$

$j=1,2$ and $i=1,2,...,N$. Thus,

$$s_n(L_N)\leq\sum_{j=1}^{2}\sum_{i=1}^{N}s_n(P_{K_i^j}TP_{K_i^j})\leq c\frac{\epsilon}{n},\tag{3.8}$$

i.e, $s_n(L_N)<c\frac{\epsilon}{n}$.

Therefore we have that

$$\limsup_{n\to+\infty} ns_n\left(L_N + P_0TP_0 + \sum_{i\neq l} P_i^{j'}TP_l^j\right) \leq C\epsilon,$$

where C is a new constant independent from ϵ, n, N.

From Eqs. 3.5 and 3.6, and the fact that from the orthogonality of operators T_i^1 and T_i^2 we have

$$\mathcal{N}_t(T_i^1 + T_i^2) = \mathcal{N}_t(T_i^1) + \mathcal{N}_t(T_i^2),$$

for $i = 1, 2, ..., N$, we get

$$\mathcal{N}_t(T_i^1 + T_i^2) \sim \frac{|\psi(\xi_i)|}{tN} + \frac{|\psi(\frac{\xi_i}{\rho})|}{tN}, t \to 0^+.$$

Therefore,

$$\lim_{t\to 0^+} t\mathcal{N}_t(P_N) = \sum_{i=1}^{N} \frac{|\psi(\xi_i)|}{N} + \sum_{i=1}^{N} \frac{|\psi(\frac{\xi_i}{\rho})|}{N}.$$

Now, from Lemma 1.2 we have that

$$\begin{aligned}\lim_{t\to 0^+} t\mathcal{N}_t(T) &= \frac{1}{2\pi}\lim_{N\to+\infty}\frac{2\pi}{N}|\psi(\xi_i)| + \frac{\rho}{2\pi}\lim_{N\to+\infty}\sum_{i=1}^{N}\frac{2\pi}{N\rho}|\psi(\frac{\xi_i}{\rho})| \\ &= \frac{1}{2\pi}\int_0^{2\pi}|\psi(e^{i\theta})|d\theta + \frac{\rho}{2\pi}\int_0^{2\pi}\left|\psi\left(\frac{e^{i\theta}}{\rho}\right)\right|d\theta.\end{aligned} \tag{3.9}$$

Finally, taking that $t = s_n(T)$ in Eq. 3.9 we obtain the desired result. □

3.1 Proof of Theorem 1.4

At the beginning let us first notice that for the fixed $z \in \Omega$ the function

$$H_z(\xi) = \frac{1}{\pi}\frac{1}{z-\xi} + \frac{F'(z)}{\pi F(z)}\frac{F(\xi)}{F(\xi)-F(z)}$$

is analytic in Ω for $\xi \neq z$, while

$$\lim_{\xi\to z} H_z(\xi) = -\frac{1}{2\pi}\frac{F''(z)}{F'(z)} + \frac{1}{\pi}\frac{F'(z)}{F(z)},$$

which implies that $H_z(\xi)$ can be analytically continued in z. Again, appealing to Theorem 1.1 we have that

$$\lim_{n\to+\infty} ns_n\left(\frac{1}{\pi}\frac{1}{z-\xi} + \frac{F'(z)}{\pi F(z)}\frac{F(\xi)}{F(\xi)-F(z)}\right) = 0.$$

The operator induced with the kernel $\frac{F'(z)}{\pi F(z)}\frac{\rho^2 F(\xi)}{\rho^2 F(\xi)-F(z)}$ is unitarily equivalent with the operator

$$T^1 f(z) = \frac{1}{\pi z}\int_A \frac{\rho^2\xi\overline{\varphi'(\xi)}}{\rho^2\xi - z} f(\xi)dA(\xi),\ f \in L^2(A).$$

Thus,

$$\lim_{n\to+\infty} ns_n(T^1) \leq 2\|\varphi'\|_\infty \lim_{n\to+\infty}\rho^n = 0.$$

Further, the operator induced with the kernel

$$\frac{F'(z)}{\pi F(z)}\left(\frac{1}{F(z)\overline{F(\xi)}-1}+\frac{1}{\rho^2 F(z)\overline{F(\xi)}-1}\right)$$

is unitarily equivalent with the operator

$$\tilde{T}f(z)=\frac{1}{\pi}\int_A\left(\frac{\overline{\varphi'(\xi)}}{z(z\bar{\xi}-1)}+\frac{\overline{\varphi'(\xi)}}{z(\rho^2 z\bar{\xi}-1)}\right)f(\xi)dA(\xi),\quad f\in L^2(A).$$

Lemma 3.8 implies that

$$\begin{aligned} &s_n\left(\frac{F'(z)}{\pi F(z)}\left(\frac{1}{F(z)\overline{F(\xi)}-1}+\frac{1}{\rho^2 F(z)\overline{F(\xi)}-1}\right)\right)\\ &\sim\frac{1}{2\pi n}\int_0^{2\pi}|\varphi'(e^{it})|dt+\frac{\rho^2}{2\pi n}\int_0^{2\pi}\left|\varphi'\left(\frac{e^{it}}{\rho}\right)\right|dt, \end{aligned}\tag{3.10}$$

as $n\to+\infty$.

Finally, by applying Theorem 2.3 from [12] we conclude the proof.

Author Contributions There is only one author.

Data Availability The data that support the findings of this study are available from the corresponding author on request.

Declarations

Competing interests The authors declare no competing interests.

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