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Stochastic Generalized Porous Media Equations Over σ-finite Measure Spaces with Non-continuous Diffusivity Function

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Abstract

In this paper, we prove that stochastic porous media equations over σ-finite measure spaces $(E, \mathcal{B}, \mu)$, driven by time-dependent multiplicative noise, with the Laplacian replaced by a self-adjoint transient Dirichlet operator L and the diffusivity function given by a maximal monotone multi-valued function Ψ of polynomial growth, have a unique solution. This generalizes previous results in that we work on general measurable state spaces, allow non-continuous monotone functions Ψ, for which, no further assumptions (as e.g. coercivity) are needed, but only that their multi-valued extensions are maximal monotone and of at most polynomial growth. Furthermore, an $L^p(\mu)$-Itô formula in expectation is proved, which is not only crucial for the proof of our main result, but also of independent interest. The result in particular applies to fast diffusion stochastic porous media equations (in particular self-organized criticality models) and cases where E is a manifold or a fractal, and to non-local operators L, as e.g. $L = -f(-\Delta)$, where f is a Bernstein function.

Keywords Wiener process · Porous media equation · Dirichlet form · Maximal monotone graph · Yosida approximation · L^p-Itô formula in expectation

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1 Introduction

The purpose of this paper is to solve multi-valued stochastic porous media equations (SPMEs) on $(E, \mathcal{B}, \mu)$ of the following type:

$$\begin{cases} dX(t) - L\Psi(X(t))dt \ni B(t, X(t))dW(t), \text{ in } [0, T] \times E, \\ X(0) = x \text{ on } E \ (x \in \mathscr{F}_e^*), \end{cases} \tag{1.1}$$

where $(E, \mathcal{B})$ is a standard measurable space (see [31]) with a σ-finite measure μ and $T \in (0, \infty)$ is fixed. $(L, D(L))$ is the generator of a symmetric strongly continuous contraction sub-Markovian semigroup on $L^2(\mu)$, which additionally is assumed to be the generator of transient Dirichlet form, $\mathscr{F}_e^*$ is the dual space of the corresponding extended transient Dirichlet space $\mathscr{F}_e$ (cf. Section 2.1 below). $\Psi(\cdot) : \mathbb{R} \to 2^{\mathbb{R}}$ denotes a maximal monotone graph with polynomial growth (cf. **(H1)** in Section 3 below, Ψ is called diffusivity function, see [41]). B is a Hilbert-Schmidt operator-valued map fulfilling certain Lipschitz and growth conditions (cf. **(H2)** and **(H3)** in Section 3 below). W is a cylindrical Wiener process in $L^2(\mu)$ (cf. [26, Proposition 2.5.2]) defined on a probability space $(\Omega, \mathscr{F}, \mathbb{P})$ with normal filtration $(\mathscr{F}_t)_{t \geq 0}$. Explicit assumptions and more explanations will be given in Section 3.

At least since [12], there has been a lot of papers concerning stochastic porous media equations, e.g., about strong solutions [4–6, 8, 9, 13, 24], stochastic variational inequalities [17, 29] or finite time extinction of solutions [6, 8, 9, 16] to

$$dX(t) - \Delta\Psi(X(t))dt = B(t, X(t))dW(t) \text{ on } \mathcal{O}, \tag{1.2}$$

with maximal monotone (multi-valued) diffusivity Ψ, where $\mathcal{O}$ is an open and bounded subset of the Euclidean space $\mathbb{R}^d$ (see also [7, 26] and references therein). In the classical deterministic case, i.e., $B \equiv 0$ and $\Psi(r) = |r|^{m-1}r$, $r \in \mathbb{R}$, $m \geq 1$ (see [1]), for $x = $ a probability density on $\mathcal{O}$, its solution $X(t)$, $t \geq 0$, describes the time evolution for the density of a substance in a porous medium. Heuristically (because Ψ is in general not even assumed to be (single-valued) continuous and $\xi \mapsto X(t)(\xi)$ is not C^2) applying the chain rule we have

$$\Delta\Psi(X(t)) = \Psi'(X(t))\Delta X(t) + \Psi''(X(t))|\nabla X(t)|^2. \tag{1.3}$$

This shows that $\Psi'(X(t))$ is the (solution dependent) diffusion coefficient of the equation. This explains the name "diffusivity (function)" for Ψ and why Ψ must be assumed to be increasing. If Ψ is strictly increasing, which corresponds to $\Psi' > 0$ on $\mathbb{R}$ in Eq. 1.3, this would mean that we have local strict ellipticity in Eq. 1.3, hence we would be in the nondegenerate case. We stress, however, that in this paper we do not assume this, so the degenerate case is covered. Equation 1.3 also reveals why it is important to include multivalued diffusivities, because it implies that we can cover non-continuous Ψ (See Example 6.1 below). This means that its generalized derivative Ψ' (in the sense of Schwartz distributions) would be a weighted Dirac measure δ_{r_0} at a point of discontinuity r_0 of Ψ. So, if we consider the time evolution $t \mapsto X(t)(\xi)$ of the density of the substance at a point $\xi \in E$ and if it "hits" such a discontinuity point $r_0 \in \mathbb{R}$ of Ψ, the diffusion coefficient $\Psi'(X(t)(\xi))$ would jump to $+\infty$, describing a "very large" diffusion of "the system" at that moment, which is an interesting case of high relevance, e.g. in physics. This is also the reason why the solutions to Eq. 1.2 are sometimes called singular diffusions. A prominent example is the so-called self-organized criticality (SOC) model developed by Bak, Tang and Wiesenfeld [2], which can also be used to model the dynamics of phase-transition (including melting and solidification processes) as well as for the description of a large class of other diffusion problems. It remains to "justify"

the type of noise in Eq. 1.2. For a general explanation of this we refer [26, Page:3, Section 1.2], which explains that the noise must have this term under some reasonable assumptions.

One interesting direction of research is to replace $-\Delta$ by a nonlocal pseudodifferential operator (e.g., $(-\Delta)^\alpha$, $\alpha \in (0, 1]$, cf. [39] or more generally $f(-\Delta)$, where f is a Bernstein function, see [37]) in Eq. 1.2, or even more generally by a self-adjoint operator L satisfying certain properties, as e.g., being the generator of a transient Dirichlet form. But there exist many interesting operators on $\mathbb{R}^d$ which are not self-adjoint on $L^2(\mathbb{R}^d, dx)$ (where dx denotes the Lebesgue measure), but on $L^2(\mathbb{R}^d, \mu)$ for some other measure μ replacing dx, as for instance the Friedrichs extension of the operator $L_0 = \Delta + 2\frac{\nabla\rho}{\rho} \cdot \nabla$ on $L^2(\mathbb{R}^d, \rho^2 dx)$ [36], where $\rho \in H^1(\mathbb{R}^d)$ and H^1 is the usual Sobolev space. It is also interesting to change $\mathbb{R}^d$ (the "state space") and replace it by a smooth or even not smooth Riemannian manifold as e.g., a fractal, or allow infinite dimensional state spaces, e.g., the Wiener space. Furthermore, it is very desirable to do all this for multivalued diffusivity functions Ψ, as explained above. So, all this motivates the study of Eq. 1.1, in such a general form.

There are few results about Eq. 1.1 on general measure spaces. One paper known to the authors about Eq. 1.1 with Ψ being a multi-valued graph is [18, Example 7.3], in which the existence and uniqueness of (limit) solutions and the ergodicity for Eq. 1.1 are proved. However, in [18], $(E, \mathcal{B}, \mu)$ is assumed to be a finite measure space and L to have compact resolvent, and Ψ is of linear growth, which simplifies the situation substantially. Furthermore, in [18] the extended transient Dirichlet space $\mathscr{F}_e$ is the same as the Dirichlet space $F_{1,2}$, which results in a Gelfand triple $L^2(\mu) \subset \mathscr{F}_e^*(= F_{1,2}^*) \subset (L^2(\mu))^*$. The proof of the well posedness theorem in [18] heavily relies on this Gelfand triple. We work on general σ-finite measure spaces with no further conditions on L, so it turns out that we must construct solutions in the smaller state space $\mathscr{F}_e^*$. Also the idea of the proof in [18] is based on a viscosity approximation, while our proof is based on the Yosida approximation. We would also like to mention another paper, where Eq. 1.2 was studied with multivalued Ψ and with Lévy noise replacing the Wiener noise in Eq. 1.2, namely in Section 6 of [27]. However, this is done only on $(\mathcal{O}, \mathcal{B}(\mathcal{O}), dx)$ and $\mathcal{O}$ is assumed to have finite Lebesgue measure.

The present work was also motivated by papers [9, 34, 36]. In [9] existence and uniqueness of solutions for Eq. 1.2 with linear multiplicative noise was proved for multivalued Ψ with $E := \mathcal{O} = \mathbb{R}^d$, $d \geq 3$, and one of our aims here is to generalize this result to Eq. 1.1 on general measure spaces and include more examples of L, where e.g. it is the generator of a transient Dirichlet form not only on $\mathbb{R}^d$, but also on e.g. a manifold or a fractal (cf. Examples 6.3-6.5 in Section 6). In [34], by constructing a suitable Gelfand triple with $\mathscr{F}_e^*$ as the pivôt space and using the variational framework [26, 33], the first named author of the present paper and his collaborators proved existence and uniqueness for the following stochastic generalized porous media equation in the state space $\mathscr{F}_e^*$:

$$dX(t) = (L\Psi(t, X(t)) + \Phi(t, X(t)))dt + B(t, X(t))dW(t), \tag{1.4}$$

where L is as above and Ψ is continuous, single-valued and maximal monotone, additionally satisfying a number of other somewhat restrictive conditions (see [34, Page: 135, condition (A1)], in particular, $(\Psi 2)$ and $(\Psi 3)$, where the appearing Young function N is assumed to be Δ_2-regular), which, in particular, imply that $\Psi(r) \to \infty$ as $r \to \infty$. In the case where $\Phi \equiv 0$ and Ψ is a Lipschitz increasing function which is independent of t and ω, the well-posedness of strong solutions to Eq. 1.4 in $F_{1,2}^*$ was proved in [36]. In contrast to that, in the present paper, Ψ is not assumed to be either Lipschitz or single-valued. In particular, we can cover the SOC model which is not included in [34] or [36]. A second aim of our paper is to generalize the results in [34, 36] to multi-valued diffusivities Ψ, being just maximal monotone and of at

most polynomial growth, with no further assumptions such as e.g. that $\lim_{r \to \infty} \Psi(r) = \infty$, which allows our framework to apply to the SOC model (cf. Example 6.1 in Section 6). Our method is completely different from [34], but it is a generalization of that in [9]. Because of our much more general situation, the methods and techniques available for investigating SPMEs on $\mathbb{R}^d$ are insufficient for covering the case of general measure spaces and cannot be extended to general measure spaces in a straightforward way. Let us now describe our method. Though the overall strategy is borrowed from [9], some severe obstacles had to be overcome in our case, one of which was to find a proper version of Itô's formula for the L^p-norm of solutions for processes taking values in $L^p(E, \mathcal{B}, \mu)$ with no further assumption on $(E, \mathcal{B}, \mu)$, so that our results apply to general state spaces E, as those mentioned above.

As a first step, we consider Yosida approximating equations to Eq. 1.1 of the following form with initial value $X_\lambda(0) \in \mathscr{F}_e^*$:

$$dX_\lambda(t) - L(\Psi_\lambda(X_\lambda(t)) + \lambda X_\lambda(t))dt = B(t, X_\lambda(t))dW(t), \quad t \in (0, T). \qquad (1.5)$$

Here $\lambda > 0$ and

$$\Psi_\lambda(x) = \frac{1}{\lambda}\big(x - (1 + \lambda\Psi)^{-1}(x)\big) \in \Psi\big((1 + \lambda\Psi)^{-1}(x)\big)$$

is the Yosida approximation of Ψ, which is Lipschitz continuous. One of the main points in this paper is first to prove the well posedness of Eq. 1.5 in $\mathscr{F}_e^*$. To this end, we consider the following approximating equations to Eq. 1.5

$$dX_\lambda^\nu(t) + (\nu - L)(\Psi_\lambda(X_\lambda^\nu(t)) + \lambda X_\lambda^\nu(t))dt = B(t, X_\lambda^\nu(t))dW(t), \quad t \in (0, T), 0 < \nu \leq 1, (1.6)$$

with initial value $X_\lambda^\nu(0) \in F_{1,2}^*$ (which contains $\mathscr{F}_e^*$, see Section 2.1 below). This approach allows us to estimate $\|X_\lambda^\nu\|_{F_{1,2,\nu_0}^*}$, $0 < \nu_0 \leq 1$, where $\|\cdot\|_{F_{1,2,\nu_0}^*}$ is an equivalent norm on $F_{1,2}^*$ (cf. Section 2.1 below). The key point here is to relate $\|\cdot\|_{F_{1,2,\nu_0}^*}$ with $\|\cdot\|_{\mathscr{F}_e^*}$ to prove the convergence of X_λ^ν to X_λ in $\mathscr{F}_e^*$ as $\nu \to 0$ (see Proposition 2.1 and the last part of the proof of Theorem 3.2).

To prove the convergence of solutions to Eq. 1.6 as $\nu \to 0$ to those of Eq. 1.5 and in turn that solutions to Eq. 1.5 as $\lambda \to 0$ converge to those of Eq. 1.1, since Ψ has a growth of at most order $m \geq 1$ (see **(H1)** in Section 3), one has to control the $\|\cdot\|_{L^{2m}}$ norm of the solutions uniformly in the approximation parameters. To obatin such bounds (see Eqs. 4.24 and 3.9) we need to apply Itô's formula to $|X_\lambda^\nu(t)|_{2m}^{2m}$. But this is not possible directly since X_λ^ν is not (right) continuous in $L^{2m}(\mu)$. Therefore, we consider the following approximating equation to Eq. 1.6:

$$dX_\lambda^{\nu,\varepsilon}(t) + A_\lambda^{\nu,\varepsilon}(X_\lambda^{\nu,\varepsilon}(t))dt = B(t, X_\lambda^{\nu,\varepsilon}(t))dW(t), \text{ in } (0, T) \times E, \qquad (1.7)$$

where $\varepsilon > 0$ and

$$A_\lambda^{\nu,\varepsilon}(x) = \frac{1}{\varepsilon}\big(x - (I + \varepsilon A_\lambda^\nu)^{-1}(x)\big),$$

is the Yosida approximation of the operator $A_\lambda^\nu(x) := (\nu - L)(\Psi_\lambda(x) + \lambda I(x))$ and I denotes the identity map on the respective space (see more details in Section 4 below). In [9, Section 4] the authors can directly use the L^p-Itô formula ($p \geq 2$) proved in [25] for the case $E = \mathbb{R}^d$, which is not possible here. Also one cannot expect to prove such a formula on arbitrary σ-finite measure spaces without further assumptions. The reason is that in [25], approximations by convolution with smooth functions were crucial, which heavily depend on the linear structure of $\mathbb{R}^d$. To overcome this difficulty, we prove an $L^p(\mu)$-Itô formula ($p \geq 2$) in expectation (see Subsection A.2) to get the crucial a priori $L^{2m}(\mu)$-estimates first

for the solution to Eq. 1.7 uniformly in the approximation parameters ε, ν, λ, subsequently, by letting $\varepsilon \to 0$ and then $\nu \to 0$ we get a solution to Eq. 1.5 satisfying Eq. 3.9 (-Eq. 3.12) (see Theorem 3.2 below), which then allows us finally to take $\lambda \to 0$ to obtain well-posedness of Eq. 1.1 (see Theorem 3.1). So, the L^p-Itô formula in expectation (see Theorem A.1 below), which we think is also of its own interest, is the main new tool that leads to the desired generalization of [9]. Apart from other smaller obstacles that had to be overcome in comparison to [9] (see e.g. Proposition A.1 in the Appendix), there is one other novel point we would like to mention here. Since our aim was to be able to treat non-local operators (such as pseudo-differential operators) L in Eq. 1.1 in our approach, we had to include non-local extended Dirichlet forms in our framework. We identified the crucial condition on the square field operators of the non-local Dirichlet forms needed to implement our approach. This is condition **(H4)(ii)** which is fulfilled for an abundance of (local and) non-local Dirichlet forms (see the examples in Section 6), in particular for such with generator $L = -f(-\Delta)$, where $f : [0, \infty) \to \mathbb{R}$ is a Bernstein function (in the sense of [37]). Such examples for the operator L together with discontinuous diffusivities in Eq. 1.1 are completely new.

This paper is organized as follows. In Section 2, we introduce some notations and recall some known results for preparation. In addition, we prove some necessary technical auxiliary results, which will be used to construct the solutions to Eq. 1.1 in $\mathscr{F}_e^*$. In Section 3, we present our assumptions and the two main results for Eqs. 1.1 and 1.5. A detailed proof of the existence result for Eq. 1.5 will be given in Section 4, while the existence and uniqueness result for Eq. 1.1 will be given in Section 5. A number of examples that are covered by our framework will be presented in Section 6, including local (nonlocal) operators L on manifold or fractals. In order to make the main structure of the proofs more transparent, we shift the proofs of some estimates to Appendix A.1. In addition, we present a detailed proof of the mentioned $L^p(\mu)$-Itô formula ($p \geq 2$) in expectation in Appendix A.2.

2 Notations and Preliminaries

2.1 Dirichlet Spaces

Let $(E, \mathcal{B}, \mu)$ be a σ-finite measure space, which we fix in the entire paper. We assume that $(E, \mathcal{B})$ is a standard measurable space (i.e., σ-isomorphic to a Polish space, see [31]). This assumption is used in the proof of the $L^p(\mu)$-Itô formula ($p \geq 2$) in expectation, but also in the proof of Lemma 4.1 below, where we apply [35, Lemma 5.1], in which this assumption on $(E, \mathcal{B})$ was crucially used. Let $(P_t)_{t \geq 0}$ be a strongly continuous, symmetric, sub-Markovian contraction semigroup on $L^2(\mu)$. Let $(L, D(L))$ be its infinitesimal generator (see e.g. [15, 28]), which is a negative definite self-adjoint operator on $L^2(\mu)$. We use $\langle \cdot, \cdot \rangle_2$ and $| \cdot |_2$ for the inner product and the norm in $L^2(\mu)$ respectively. More generally, we set $\langle f, g \rangle_2 := \mu(fg) := \int fg \, d\mu$ for any two measurable functions f, g such that $fg \in L^1(\mu)$. For the rest of this paper we fix $(P_t)_{t \geq 0}$ with generator $(L, D(L))$ on $L^2(\mu)$ with $(E, \mathcal{B}, \mu)$ as above.

Consider the Γ-transform $V_r (r > 0)$ of $(P_t)_{t \geq 0}$

$$V_r u = \Gamma(\frac{r}{2})^{-1} \int_0^\infty s^{\frac{r}{2}-1} e^{-s} P_s u \, ds, \; r > 0, \; u \in L^2(\mu).$$

From [14], we can define the Bessel-potential space $(F_{1,2}, \| \cdot \|_{F_{1,2}})$ by

$$F_{1,2} := V_1(L^2(\mu)), \text{ with norm } \|u\|_{F_{1,2}} = |f|_2, \text{ for } u = V_1 f, \; f \in L^2(\mu).$$

Consequently,

$$V_1 = (1 - L)^{-\frac{1}{2}}, \quad \text{so that} \quad F_{1,2} = D\big((1 - L)^{\frac{1}{2}}\big) \quad \text{and} \quad \|u\|_{F_{1,2}} = |(1 - L)^{\frac{1}{2}} u|_2.$$

The dual space of $F_{1,2}$ is denoted by $F_{1,2}^*$ and $F_{1,2}^* = D((1 - L)^{-\frac{1}{2}})$, it is equipped with the norms

$$\|\eta\|_{F_{1,2,\nu}^*} := \langle \eta, (\nu - L)^{-1} \eta \rangle_2^{\frac{1}{2}}, \quad \eta \in F_{1,2}^*, \ 0 < \nu < \infty.$$

Denote the duality between $F_{1,2}^*$ and $F_{1,2}$ by $_{F_{1,2}^*}\langle \cdot, \cdot \rangle_{F_{1,2}}$.

Consider the Dirichlet form $(\mathscr{E}, D(\mathscr{E}))$ on $L^2(\mu)$ associated with $(L, D(L))$, i.e.,

$$D(\mathscr{E}) := F_{1,2}, \quad \text{and}$$
$$\mathscr{E}(u, v) := \mu(\sqrt{-L}u\sqrt{-L}v), \quad u, v \in F_{1,2}.$$

Let $D(\mathscr{E})$ be equipped with the inner product $\mathscr{E}_1 := \mathscr{E} + \langle \cdot, \cdot \rangle_2$.

If $(\mathscr{E}, D(\mathscr{E}))$ is a transient Dirichlet space, that is, there exists $g \in L^1(\mu) \cap L^\infty(\mu), g > 0$, such that $\mathscr{F}_e \subset L^1(g \cdot \mu)$ continuously, let $(\mathscr{E}, \mathscr{F}_e)$ be the corresponding extended Dirichlet space (see [15]), which is the completion of $F_{1,2}$, with respect to the norm

$$\| \cdot \|_{\mathscr{F}_e} := \mathscr{E}(\cdot, \cdot)^{\frac{1}{2}}.$$

Then $F_{1,2} = \mathscr{F}_e \cap L^2(\mu)$. Let $\mathscr{F}_e^*$ be its dual space with inner product $\langle \cdot, \cdot \rangle_{\mathscr{F}_e^*}$ and corresponding norm $\| \cdot \|_{\mathscr{F}_e^*}$, which is induced by the Riesz map $\mathscr{F}_e \ni u \mapsto \mathscr{E}(\cdot, u) \in \mathscr{F}_e^*$. Denote the duality between $\mathscr{F}_e^*$ and $\mathscr{F}_e$ by $_{\mathscr{F}_e^*}\langle \cdot, \cdot \rangle_{\mathscr{F}_e}$. Both $\mathscr{F}_e$ and $\mathscr{F}_e^*$ are Hilbert spaces. For more background knowledge on Dirichlet forms, we refer to [15, 28]. From now on we assume:

(L.1) The symmetric Dirichlet form $(\mathscr{E}, D(\mathscr{E}))$ associated with $(L, D(L))$ is transient.

Consider the inner product $\mathscr{E}_\nu := \mathscr{E} + \nu \langle \cdot, \cdot \rangle_2, \ \nu \in (0, \infty)$, on $F_{1,2}$, i.e.,

$$\|v\|_{F_{1,2,\nu}}^2 := \mathscr{E}(v, v) + \nu \int |v|^2 d\mu = \|v\|_{\mathscr{F}_e}^2 + \nu \int |v|^2 d\mu, \quad \text{for } v \in F_{1,2}, \tag{2.1}$$

and

$$\|l\|_{F_{1,2,\nu}^*} := {}_{F_{1,2}^*}\langle l, (\nu - L)^{-1} l \rangle_{F_{1,2}}^{\frac{1}{2}} := \sup_{\substack{v \in F_{1,2} \\ \|v\|_{F_{1,2,\nu}} \le 1}} l(v), \ l \in F_{1,2}^*,$$

$$\|l\|_{\mathscr{F}_e^*} := \sup_{\substack{v \in \mathscr{F}_e \\ \|v\|_{\mathscr{F}_e} \le 1}} l(v), \ l \in \mathscr{F}_e^*.$$

Since $F_{1,2} \subset \mathscr{F}_e$ continuously and densely, we have

$$\mathscr{F}_e^* \subset F_{1,2}^* \quad \text{continuously and densely.}$$

Proposition 2.1 *Let $l \in \mathscr{F}_e^*$. Then $\nu \mapsto \|l\|_{F_{1,2,\nu}^*}$ is decreasing,*

$$\lim_{\nu \to 0} \|l\|_{F_{1,2,\nu}^*} = \sup_{\nu > 0} \|l\|_{F_{1,2,\nu}^*} = \|l\|_{\mathscr{F}_e^*}, \tag{2.2}$$

$$\|l\|_{F_{1,2}^*} \le \|l\|_{F_{1,2,\nu}^*} \le \frac{1}{\sqrt{\nu}} \|l\|_{F_{1,2}^*}, \quad \forall 0 < \nu \le 1. \tag{2.3}$$

Proof Firstly, note that for all $l \in F_{1,2}^*$ and $0 < \nu' \leq \nu < \infty$, we have

$$\|l\|_{F_{1,2,\nu}^*} =: \sup_{\substack{v \in F_{1,2} \\ \|v\|_{F_{1,2,\nu}} \leq 1}} l(v) \leq \sup_{\substack{v \in F_{1,2} \\ \|v\|_{F_{1,2,\nu'}} \leq 1}} l(v) = \|l\|_{F_{1,2,\nu'}^*},$$

i.e., $\forall\, l \in F_{1,2}^*$, $\|l\|_{F_{1,2,\nu}^*}$ is decreasing in ν. In particular, the first equality in Eq. 2.2 and the first inequality in Eq. 2.3 hold.

Let $l \in \mathscr{F}_e^*$. Since $\mathscr{F}_e^* \subset F_{1,2}^*$ continuously and densely, we have $l \in F_{1,2}^*$ and

$$\|l\|_{F_{1,2,\nu}^*} = \sup_{\substack{v \in F_{1,2} \\ \|v\|_{F_{1,2,\nu}} \leq 1}} l(v) \leq \sup_{\substack{v \in \mathscr{F}_e \\ \|v\|_{\mathscr{F}_e} \leq 1}} l(v) = \|l\|_{\mathscr{F}_e^*}.$$

Hence $\forall l \in \mathscr{F}_e^*$,

$$\lim_{\nu \to 0} \|l\|_{F_{1,2,\nu}^*} = \sup_{\nu > 0} \|l\|_{F_{1,2,\nu}^*} \leq \|l\|_{\mathscr{F}_e^*}. \tag{2.4}$$

To prove the converse inequality of Eq. 2.4, fix $l \in \mathscr{F}_e^*$ and let $\epsilon, \ \delta \in (0, 1)$. Then there exists $v_\epsilon \in F_{1,2}$ with $\|v_\epsilon\|_{\mathscr{F}_e} = 1$ and

$$l(v_\epsilon) \geq \|l\|_{\mathscr{F}_e^*} - \epsilon.$$

Let $\nu_0 := \frac{\delta^2}{1+|v_\epsilon|_2^2}$. From Eq. 2.1, we see that

$$\|v_\epsilon\|_{F_{1,2,\nu_0}} = \sqrt{\|v_\epsilon\|_{\mathscr{F}_e}^2 + \nu_0 |v_\epsilon|_2^2} \leq \sqrt{1 + \delta^2} \leq 1 + \delta,$$

so for $\bar{v}_\epsilon := \frac{v_\epsilon}{1+\delta}$, we have

$$\|\bar{v}_\epsilon\|_{F_{1,2,\nu_0}} \leq 1.$$

Consequently,

$$\lim_{\nu \to 0} \|l\|_{F_{1,2,\nu}^*} = \sup_{\nu > 0} \|l\|_{F_{1,2,\nu}^*}$$

$$\geq \|l\|_{F_{1,2,\nu_0}^*} \geq l(\bar{v}_\epsilon) = \frac{1}{1+\delta} l(v_\epsilon) \geq \frac{1}{1+\delta} (\|l\|_{\mathscr{F}_e^*} - \epsilon),$$

letting $\delta \to 0$, $\epsilon \to 0$, yields the desired converse inequality. Hence Eq. 2.2 is proved.

It remains to prove the second inequality in Eq. 2.3. Note that

$$\|v\|_{F_{1,2,\nu}}^2 = \mathscr{E}(v, v) + \nu \langle v, v \rangle_2 \quad \text{and} \quad \|v\|_{F_{1,2}}^2 = \mathscr{E}(v, v) + \langle v, v \rangle_2,$$

hence for $0 < \nu \leq 1$,

$$\|l\|_{F_{1,2}^*} = \sup_{\|v\|_{F_{1,2}} \leq 1} l(v) = \sqrt{\nu} \sup_{\sqrt{\nu}\|v\|_{F_{1,2}} \leq 1} l(v) \geq \sqrt{\nu} \sup_{\|v\|_{F_{1,2,\nu}} \leq 1} l(v) = \sqrt{\nu}\|l\|_{F_{1,2,\nu}^*}.$$

2.2 Gelfand Triples

Let H be a separable Hilbert space with inner product $\langle \cdot, \cdot \rangle_H$ and let H^* be its dual space. Let V be a reflexive Banach space, such that $V \subset H$ continuously and densely. Then for its dual space V^* it follows that $H^* \subset V^*$ continuously and densely. Identifying H and H^* via the Riesz isomorphism we have that

$$V \subset H \subset V^*$$

continuously and densely. Let $_{V^*}\langle\cdot,\cdot\rangle_V$ denote the dualization between V^* and V (i.e. $_{V^*}\langle z, v\rangle_V := z(v)$ for $z \in V^*$, $v \in V$). Then it follows that

$$_{V^*}\langle z, v\rangle_V = \langle z, v\rangle_H, \text{ for all } z \in H, \ v \in V. \tag{2.5}$$

(V, H, V^*) is called a Gelfand triple.

A Gelfand triple with $V := L^2(\mu)$, $H := F_{1,2}^*$ was constructed in [36], the Riesz map which identifies $F_{1,2}$ and $F_{1,2}^*$ is $(1 - L)^{-1} : F_{1,2}^* \to F_{1,2}$. We need the following two lemmas.

Lemma 2.1 *([36, Lemma 2.1.]) The map $(1 - L) : F_{1,2} \to F_{1,2}^*$ is an isometric isomorphism. In particular,*

$$\big\langle (1 - L)u, \ (1 - L)v \big\rangle_{F_{1,2}^*} = \langle u, \ v\rangle_{F_{1,2}} \text{ for all } u, v \in F_{1,2}. \tag{2.6}$$

Furthermore, $(1 - L)^{-1} : F_{1,2}^* \to F_{1,2}$ is the Riesz isomorphism for $F_{1,2}^*$, i.e., for every $u \in F_{1,2}^*$,

$$\langle u, \ \cdot\rangle_{F_{1,2}^*} =_{F_{1,2}}\langle (1 - L)^{-1}u, \ \cdot\rangle_{F_{1,2}^*}. \tag{2.7}$$

Lemma 2.2 *([34, Lemma 3.3(i)]) The map $\bar{L} : \mathscr{F}_e \to \mathscr{F}_e^*$ defined by*

$$\bar{L}v := -\mathscr{E}(v, \cdot), \ v \in \mathscr{F}_e$$

(i.e. the Riesz isomorphism of $\mathscr{F}_e$ and $\mathscr{F}_e^*$ multiplied by (-1)) is the unique continuous linear extension of the map

$$D(L) \ni v \mapsto \mu(Lv\cdot) \in \mathscr{F}_e^*.$$

If there is no danger of confusion, we write L instead of $\bar{L}$ below. For a Hilbert space $\mathbb{H}$, throughout the paper, let $L^2([0, T] \times \Omega; \mathbb{H})$ denote the space of all $\mathbb{H}$-valued square-integrable functions on $[0, T] \times \Omega$, $C([0, T]; \mathbb{H})$ the space of all continuous $\mathbb{H}$-valued functions on $[0, T]$, and $L^\infty([0, T]; \mathbb{H})$ the space of all $\mathbb{H}$-valued uniformly bounded measurable functions on $[0, T]$. For two Hilbert spaces H_1 and H_2, the space of Hilbert-Schmidt operators from H_1 to H_2 is denoted by $L_2(H_1, H_2)$. For simplicity, the positive constants $C, C_i, i = 1, ..., 5$, used in this paper may change from line to line. We would like to refer to [26, 33] for more background information and results on SPDEs, [1, 41] on PMEs and [7] on SPMEs.

3 Assumptions and Main Results

Let $K := L^1(\mu) \cap L^\infty(\mu) \cap \mathscr{F}_e^*$. In addition to condition **(L.1)** above, we study Eq. 1.1 under the following assumptions.

(H1) $\Psi(\cdot) : \mathbb{R} \to 2^{\mathbb{R}}$ is a maximal monotone graph such that $0 \in \Psi(0)$ and there exist constants $C \in (0, \infty)$ and $m \in [1, \infty)$ such that

$$|\Psi(r)| = \inf\{|\eta|; \eta \in \Psi(r)\} \leq C(|r|^m + 1_{\{\mu(E)<\infty\}}), \ \forall r \in \mathbb{R}. \tag{3.1}$$

(H2) $B : [0, T] \times K \times \Omega \to L_2(L^2(\mu), L^2(\mu))$ is progressively measurable (i.e. for any $t \in [0, T]$, this mapping restricted to $[0, t] \times K \times \Omega$ is measurable w.r.t. $\mathscr{B}([0, t]) \times \mathscr{B}(K) \times \mathscr{F}_t$, where $\mathscr{B}(\cdot)$ is the Borel σ-field for a topological space). Furthermore, $B(t, u)$ satisfies the following conditions:

(i) There exists a constant $C_1 \in [0, \infty)$ such that for all $\nu \in (0, 1]$,

$$\|B(\cdot, u) - B(\cdot, v)\|_{L_2(L^2(\mu), F_{1,2,\nu}^*)} \leq C_1 \|u - v\|_{F_{1,2,\nu}^*}, \quad \forall u, v \in K \text{ on } [0, T] \times \Omega.$$

(ii) There exists a constant $C_2 \in (0, \infty)$ such that for all $\nu \in (0, 1]$,

$$\|B(\cdot, u)\|_{L_2(L^2(\mu), F_{1,2,\nu}^*)} \leq C_2 \|u\|_{F_{1,2,\nu}^*}, \quad \forall u \in K \text{ on } [0, T] \times \Omega.$$

(H3)(i) There exists a constant $C_3 \in (0, \infty)$ satisfying

$$\|B(\cdot, u)\|_{L_2(L^2(\mu), L^2(\mu))} \leq C_3 |u|_2, \quad \forall u \in K \text{ on } [0, T] \times \Omega.$$

(ii) Let m be as in Eq. 3.1, there exist an orthonormal basis $\{e_k\}_{k \geq 1}$ of $L^2(\mu)$ and a constant $C_4 \in (0, \infty)$ satisfying

$$\int_E \Big(\sum_{k=1}^\infty |B(\cdot, u) e_k|^2 \Big)^m d\mu \leq C_4 |u|_{2m}^{2m}, \quad \forall u \in K \text{ on } [0, T] \times \Omega.$$

(H4) There exists a symmetric, positive, bilinear mapping $\Gamma : F_{1,2} \times F_{1,2} \to L^1(\mu)$ satisfying:
(i)

$$\mathscr{E}(u, u) = \int \frac{1}{2} \Gamma(u, u) d\mu, \quad \forall u \in F_{1,2}.$$

(ii) There exists a constant $C_5 \in (0, \infty)$ such that

$$\Gamma(\varphi(u), \varphi(u)) \leq C_5 \Gamma(u, \varphi(u)), \quad \forall u \in F_{1,2},$$

for every non-decreasing Lipschitz function $\varphi : \mathbb{R} \to \mathbb{R}$ with $\varphi(0) = 0$.

Remark 3.1 *(i) Eq. 2.2 and (H2)(i) imply that for all $u, v \in K$,*

$$\|B(\cdot, u) - B(\cdot, v)\|_{L_2(L^2(\mu), \mathscr{F}_e^*)}^2 \leq C_1 \|u - v\|_{\mathscr{F}_e^*}^2 \quad on \ [0, T] \times \Omega. \tag{3.2}$$

(ii) We emphasize that (H4)(ii) is automatically fulfilled, if $(\mathcal{E}, D(\mathcal{E}))$ is a local Dirichlet form.

(iii) By (L.1) there exists $g \in L^1(\mu) \cap L^\infty(\mu)$, $g > 0$, μ-a.e., such that $\mathscr{F}_e \subset L^1(g \cdot \mu)$ continuously and it was proved in [34] (see the last part of the proof of Proposition 3.1 in [34]) that the linear space

$$\mathscr{G} := \{h \cdot g \,|\, h \in L^\infty(\mu)\}$$

is dense in $\mathscr{F}_e^*$. Furthermore, obviously $\mathscr{G} \subset L^1(\mu) \cap L^\infty(\mu)$. Hence it follows that K (defined in (H2)) is dense in $\mathscr{F}_e^*$, and hence in $F_{1,2,\nu_0}^*$ for every $\nu_0 \in (0, 1]$. Here $F_{1,2,\nu_0}^*$ denotes the space $F_{1,2}^*$ equipped with the norm $\| \cdot \|_{F_{1,2,\nu_0}^*}$. Therefore, by (H2)(i) the map

$$K \ni u \longrightarrow B(t, u) \in L_2(L^2(\mu), F_{1,2,\nu_0}^*)$$

can be extended uniquely to a Lipschitz continuous map on all of $F_{1,2,\nu_0}^*$. Furthermore, (H2)(ii) trivially also holds for this extension, as well as Eq. 3.2. We shall use this extension below without further notice.

(iv) From [34, Proposition 3.1], we also know that $\mathscr{G}$ is dense in $L^p(\mu)$ $(1 < p < \infty)$, since $\mathscr{G} \subset L^1(\mu) \cap L^\infty(\mu)$, $\mathscr{G} \subset \mathcal{F}_e^*$, $L^1(\mu) \cap L^\infty(\mu)$ is dense in $L^p(\mu)$, hence K is dense in $L^p(\mu)$, so (H3)(i) is also true for all $u \in L^2(\mu)$ and (H3)(ii) is true for all $u \in L^{2m}(\mu)$. Note that if $m = 1$, then (H3)(i) and (H3)(ii) are the same.

Definition 3.1 *Let $x \in \mathscr{F}_e^*$. An $\mathscr{F}_e^*$-valued adapted process $X = X(t)$ is called strong solution to Eq. 1.1 if there exists $q \in [2, \infty)$ such that the following conditions hold:*

$$X \text{ is } \mathscr{F}_e^* - valued \; continuous \; on \; [0, T], \; \mathbb{P} - a.s.;$$
$$X \in L^q(\Omega \times (0, T) \times E);$$

there is $\eta \in L^{\frac{q}{m}}(\Omega \times (0, T) \times E)$ such that

$$\eta \in \Psi(X), \; dt \otimes \mathbb{P} \otimes d\mu - a.e. \; on \; \Omega \times (0, T) \times E;$$

and $\mathbb{P}$-a.s.,

$$\int_0^{\cdot} \eta(s)ds \in C([0, T]; \mathscr{F}_e), \tag{3.3}$$

$$X(t) = x + L \int_0^t \eta(s)ds + \int_0^t B(s, X(s))dW(s) \text{ holds in } \mathscr{F}_e^*, \; \forall t \in [0, T].$$

Theorem 3.1 below is the main existence and uniqueness result for Eq. 1.1.

Theorem 3.1 *Assume that (L.1), (H1)-(H4) are satisfied and let m be as in Eq. 3.1. Let $x \in L^2(\mu) \cap L^{2m}(\mu) \cap \mathscr{F}_e^*$. Then there is a unique strong solution X to Eq. 1.1 such that*

$$X \in L^2(\Omega; C([0, T]; \mathscr{F}_e^*)) \cap L^\infty([0, T]; (L^2 \cap L^{2m})(\Omega \times E)).$$

Remark 3.2 *Uniqueness means that if X_1 and X_2 are two strong solutions to Eq. 1.1 in the sense of Definition 3.1 with the same initial value $x \in L^2(\mu) \cap L^{2m}(\mu) \cap \mathscr{F}_e^*$, then $X_1 = X_2$, $\forall t \in [0, T]$, $\mathbb{P}$-a.s..*

Theorem 3.1 will be proved in Section 5. The proof is based on an approximating equation of Eq. 1.1. More precisely, in Section 4 we shall establish the existence of solutions for the following Yosida approximating equation of Eq. 1.1

$$\begin{cases} dX_\lambda - L(\Psi_\lambda(X_\lambda) + \lambda X_\lambda)dt = B(t, X_\lambda)dW(t), \; t \in [0, T], \\ X_\lambda(0) = x \text{ on } E. \end{cases} \tag{3.4}$$

Here $\lambda > 0$ and

$$\Psi_\lambda(x) = \frac{1}{\lambda}(x - (1 + \lambda\Psi)^{-1}(x)) \in \Psi((1 + \lambda\Psi)^{-1}(x))$$

is the Yosida approximation of Ψ. We recall that (see [3, page:38, Proposition 2.2]) Ψ_λ is single-valued, monotone and for all $r \in \mathbb{R}$

$$|\Psi_\lambda(r)| \le |\Psi(r)|. \tag{3.5}$$

We have the following result for Eq. 3.4.

Theorem 3.2 *Assume that (L.1), (H1)-(H4) are satisfied and let m be as in Eq. 3.1. Let $\lambda \in (0, 1]$, and $x \in L^2(\mu) \cap L^{2m}(\mu) \cap \mathscr{F}_e^*$. Then Eq. 3.4 has a strong solution*

$$X_\lambda \in L^2(\Omega; C([0, T]; \mathscr{F}_e^*)) \cap L^\infty([0, T]; (L^2 \cap L^{2m})(\Omega \times E)), \tag{3.6}$$

satisfying

$$\int_0^{\cdot} \Psi_\lambda(X_\lambda(s)) + \lambda X_\lambda(s)ds \in C([0, T]; \mathscr{F}_e), \; \mathbb{P}\text{-a.s.,} \tag{3.7}$$

and $\mathbb{P}$-*a.s.,*

$$X_\lambda(t) = x + L \int_0^t \Psi_\lambda(X_\lambda(s)) + \lambda X_\lambda(s)ds + \int_0^t B(s, X_\lambda(s))dW(s) \text{ holds in } \mathscr{F}_e^*, \ \forall t \in [0, T].$$
(3.8)

Moreover, there exists $C \in (0, \infty)$ *such that for all* $\lambda, \ \lambda' \in (0, 1], \ t \in [0, T]$,

$$\mathbb{E}|X_\lambda(t)|_{2m}^{2m} \leq C|x|_{2m}^{2m},$$
(3.9)

$$\mathbb{E} \int_0^T \int_E |\Psi_\lambda(X_\lambda(t))|^2 d\mu dt \leq C(|x|_{2m}^{2m} + \mu(E) \cdot 1_{\{\mu(E) < \infty\}}),$$
(3.10)

$$\mathbb{E}\left[\sup_{0 \leq t \leq T} \|X_\lambda(t)\|_{\mathscr{F}_e^*}^2 \right] \leq C\left(\|x\|_{\mathscr{F}_e^*}^2 + |x|_{2m}^{2m} + |x|_2^2 + \mu(E) \cdot 1_{\{\mu(E) < \infty\}} \right),$$
(3.11)

$$\mathbb{E}\left[\sup_{0 \leq t \leq T} \|X_\lambda(t) - X_{\lambda'}(t)\|_{\mathscr{F}_e^*}^2 \right] \leq C(\lambda + \lambda')(|x|_2^2 + |x|_{2m}^{2m} + \mu(E) \cdot 1_{\{\mu(E) < \infty\}}).$$
(3.12)

4 Proof of Theorem 3.2

Proof For each fixed constant $\lambda \in (0, 1]$, firstly we consider the following approximating equation for Eq. 3.4

$$\begin{cases} dX_\lambda^\nu(t) + (\nu - L)\big(\Psi_\lambda(X_\lambda^\nu(t)) + \lambda X_\lambda^\nu(t)\big)dt = B(t, X_\lambda^\nu(t))dW(t), \text{ in } (0, T) \times E, \\ X_\lambda^\nu(0) = x \in L^2(\mu) \cap L^{2m}(\mu), \end{cases}$$
(4.1)

where $\nu \in (0, 1]$. Since $\Psi_\lambda + \lambda I$ is Lipschitz (where here and below I denotes the identity map on the respective space), by the proof of Lemma 3.1, particularly Claim 3.1 in [36], we know that Eq. 4.1 has a unique $(\mathscr{F}_t)_{t \geq 0}$-adapted solution in the sense that $X_\lambda^\nu \in L^2([0, T] \times \Omega; L^2(\mu)) \cap L^2(\Omega; C([0, T]; F_{1,2}^*))$,

$$X_\lambda^\nu(t) + (\nu - L) \int_0^T \Psi_\lambda(X_\lambda^\nu(s)) + \lambda X_\lambda^\nu(s)ds = x + \int_0^T B(s, X_\lambda^\nu(s))dW(s), \text{ holds in } F_{1,2}^*, \ \mathbb{P} - a.s.,$$

and there exists a positive constant $C \in (0, \infty)$, such that for all $\lambda, \nu \in (0, 1]$,

$$\mathbb{E}\left[\sup_{t \in [0, T]} |X_\lambda^\nu(t)|_2^2 \right] \leq e^{CT}|x|_2^2.$$
(4.2)

To prove that Eqs. 3.6-3.11 hold with X_λ^ν replacing X_λ, with a constant C independent of ν and λ, we consider the following approximating equation for Eq. 4.1.

$$\begin{cases} dX_\lambda^{\nu,\varepsilon}(t) + A_\lambda^{\nu,\varepsilon}(X_\lambda^{\nu,\varepsilon}(t))dt = B(t, X_\lambda^{\nu,\varepsilon}(t))dW(t), \text{ in } (0, T) \times E, \\ X_\lambda^{\nu,\varepsilon}(0) = x \in L^2(\mu) \cap L^{2m}(\mu), \end{cases}$$
(4.3)

where $A_\lambda^{\nu,\varepsilon} : F_{1,2}^* \to F_{1,2}^*$, defined by

$$A_\lambda^{\nu,\varepsilon}(x) = \frac{1}{\varepsilon}\big(x - (I + \varepsilon A_\lambda^\nu)^{-1}(x)\big), \ x \in F_{1,2}^*, \ \varepsilon > 0,$$

is the Yosida approximation of the operator $A_\lambda^\nu(x) := (\nu - L)(\Psi_\lambda(x) + \lambda I(x))$ on $F_{1,2}^*$, $x \in D(A_\lambda^\nu) := F_{1,2}$. Clearly, $I + \varepsilon A_\lambda^\nu : F_{1,2} \to F_{1,2}^*$ is a bijection, since so is $\Psi_\lambda + \lambda I : F_{1,2} \to F_{1,2}^*$. Furthermore, since by Eq. 4.9 below, A_λ^ν with domain $F_{1,2}$ is monotone on

$F_{1,2,\nu}^*$, it follows that A_λ^ν is maximal monotone on $F_{1,2,\nu}^*$. Fix $x \in F_{1,2}^*$ and set $y := J_\varepsilon(x) := (I + \varepsilon A_\lambda^\nu)^{-1}x \in F_{1,2}$, i.e., $(I + \varepsilon A_\lambda^\nu)(y) = x$, equivalently,

$$y + \varepsilon(\nu - L)(\Psi_\lambda + \lambda I)(y) = x. \tag{4.4}$$

In particular, $(\Psi_\lambda + \lambda I)(y) \in D(L)$, if $x \in L^2(\mu)$.

Before giving the well-posedness result for Eq. 4.3, we need some preparations.

Lemma 4.1 *Set*

$$J_\varepsilon(x) := (I + \varepsilon A_\lambda^\nu)^{-1}x, \ \forall x \in F_{1,2}^*.$$

For all $\varepsilon \in (0, 1]$, we have

$$\|J_\varepsilon(x) - J_\varepsilon(\widetilde{x})\|_{F_{1,2,\nu}^*} \leq \|x - \widetilde{x}\|_{F_{1,2,\nu}^*}, \forall x, \widetilde{x} \in F_{1,2}^*. \tag{4.5}$$

$$|J_\varepsilon(x) - J_\varepsilon(\widetilde{x})|_2 \leq \frac{1}{\sqrt{\nu\varepsilon\lambda}}|x - \widetilde{x}|_2, \ \forall x, \widetilde{x} \in L^2(\mu). \tag{4.6}$$

$$|J_\varepsilon(x)|_p \leq |x|_p, \ \ \forall x \in L^p(\mu) \cap L^2(\mu), \ 2 \leq p < \infty. \tag{4.7}$$

Proof Firstly, let us prove Eq. 4.5. For x, $\tilde{x} \in F_{1,2}^*$, set $y := J_\varepsilon(x)$ and $\widetilde{y} := J_\varepsilon(\widetilde{x})$, we have

$$y - \tilde{y} + \varepsilon A_\lambda^\nu(y) - \varepsilon A_\lambda^\nu(\tilde{y}) = x - \tilde{x}.$$

Taking the scalar product of $y - \tilde{y}$ with both sides in $(F_{1,2}^*, \|\cdot\|_{F_{1,2,\nu}^*})$, we get

$$\langle y - \tilde{y}, y - \tilde{y}\rangle_{F_{1,2,\nu}^*} + \varepsilon\langle A_\lambda^\nu(y) - A_\lambda^\nu(\tilde{y}), y - \tilde{y}\rangle_{F_{1,2,\nu}^*} = \langle x - \tilde{x}, y - \tilde{y}\rangle_{F_{1,2,\nu}^*}. \tag{4.8}$$

For the second term in the left hand-side of Eq. 4.8, by Eq. 2.7, we know

$$\begin{aligned}
&\langle A_\lambda^\nu(y) - A_\lambda^\nu(\widetilde{y}), y - \widetilde{y}\rangle_{F_{1,2,\nu}^*} \\
&= \left\langle (\nu - L)((\Psi_\lambda + \lambda I)(y) - (\Psi_\lambda + \lambda I)(\widetilde{y})), y - \widetilde{y}\right\rangle_{F_{1,2,\nu}^*} \\
&= {}_{F_{1,2}}\left\langle (\Psi_\lambda + \lambda I)(y) - (\Psi_\lambda + \lambda I)(\widetilde{y}), y - \widetilde{y}\right\rangle_{F_{1,2}^*} \\
&= \left\langle (\Psi_\lambda + \lambda I)(y) - (\Psi_\lambda + \lambda I)(\widetilde{y}), y - \widetilde{y}\right\rangle_2 \geq 0,
\end{aligned} \tag{4.9}$$

since $y - \tilde{y} \in F_{1,2} \subset L^2(\mu)$.

Equations 4.8 and 4.9 imply

$$\|y - \tilde{y}\|_{F_{1,2,\nu}^*}^2 \leq \|x - \tilde{x}\|_{F_{1,2,\nu}^*} \cdot \|y - \tilde{y}\|_{F_{1,2,\nu}^*},$$

from which Eq. 4.5 follows.

Secondly, to prove the Lipschitz continuity of J_ε in $L^2(\mu)$, we take $x, \tilde{x} \in L^2(\mu)$ and apply

$${}_{F_{1,2}^*}\left\langle \cdot, \ (\Psi_\lambda + \lambda I)(y) - (\Psi_\lambda + \lambda I)(\tilde{y})\right\rangle_{F_{1,2}}$$

to both sides of Eq. 4.8. Then

$$\begin{aligned}
&{}_{F_{1,2}^*}\left\langle y - \tilde{y}, (\Psi_\lambda + \lambda I)(y) - (\Psi_\lambda + \lambda I)(\tilde{y})\right\rangle_{F_{1,2}} \\
&+ {}_{F_{1,2}^*}\left\langle \varepsilon A_\lambda^\nu(y) - \varepsilon A_\lambda^\nu(\tilde{y}), (\Psi_\lambda + \lambda I)(y) - (\Psi_\lambda + \lambda I)(\tilde{y})\right\rangle_{F_{1,2}} \\
&= {}_{F_{1,2}^*}\left\langle x - \tilde{x}, (\Psi_\lambda + \lambda I)(y) - (\Psi_\lambda + \lambda I)(\tilde{y})\right\rangle_{F_{1,2}}.
\end{aligned} \tag{4.10}$$

For the second term in the left hand-side of Eq. 4.10, by Eqs. 2.5-2.7 (under the Gelfand triple $F_{1,2} \subset L^2(\mu) \subset F_{1,2}^*$), we obtain

$$\begin{aligned}
&{}_{F_{1,2}^*}\langle \varepsilon A_\lambda^\nu(y) - \varepsilon A_\lambda^\nu(\widetilde{y}), (\Psi_\lambda + \lambda I)(y) - (\Psi_\lambda + \lambda I)(\widetilde{y})\rangle_{F_{1,2}} \\
&= {}_{F_{1,2}^*}\langle (1-L)(\varepsilon(\Psi_\lambda + \lambda I)(y) - \varepsilon(\Psi_\lambda + \lambda I)(\widetilde{y})), (\Psi_\lambda + \lambda I)(y) - (\Psi_\lambda + \lambda I)(\widetilde{y})\rangle_{F_{1,2}} \\
&\quad + {}_{F_{1,2}^*}\langle \varepsilon(\nu - 1)((\Psi_\lambda + \lambda I)(y) - (\Psi_\lambda + \lambda)(\widetilde{y})), (\Psi_\lambda + \lambda I)(y) - (\Psi_\lambda + \lambda I)(\widetilde{y})\rangle_{F_{1,2}} \\
&= \varepsilon\|(\Psi_\lambda + \lambda I)(y) - (\Psi_\lambda + \lambda I)(\widetilde{y})\|_{F_{1,2}}^2 + \varepsilon(\nu - 1)|(\Psi_\lambda + \lambda I)(y) - (\Psi_\lambda + \lambda I)(\widetilde{y})|_2^2 \\
&\geq \nu\varepsilon|(\Psi_\lambda + \lambda I)(y) - (\Psi_\lambda + \lambda I)(\widetilde{y})|_2^2.
\end{aligned} \tag{4.11}$$

For the first term in the left hand-side of Eq. 4.10, since Ψ_λ is monotone, by Eq. 2.5 (under the Gelfand triple $F_{1,2} \subset L^2(\mu) \subset F_{1,2}^*$), we know

$$\begin{aligned}
&{}_{F_{1,2}^*}\langle y - \widetilde{y}, (\Psi_\lambda + \lambda I)(y) - (\Psi_\lambda + \lambda I)(\widetilde{y})\rangle_{F_{1,2}} \\
&= \langle y - \widetilde{y}, (\Psi_\lambda + \lambda I)(y) - (\Psi_\lambda + \lambda I)(\widetilde{y})\rangle_2 \\
&\geq \lambda|y - \widetilde{y}|_2^2.
\end{aligned} \tag{4.12}$$

Similarly, since $x, \widetilde{x} \in L^2(\mu)$, by Eq. 2.5, we have

$$\begin{aligned}
&{}_{F_{1,2}^*}\langle x - \widetilde{x}, (\Psi_\lambda + \lambda I)(y) - (\Psi_\lambda + \lambda I)(\widetilde{y})\rangle_{F_{1,2}} \\
&= \langle x - \widetilde{x}, (\Psi_\lambda + \lambda I)(y) - (\Psi_\lambda + \lambda I)(\widetilde{y})\rangle_2.
\end{aligned} \tag{4.13}$$

Taking Eqs. 4.11, 4.12, and 4.13 into Eq. 4.10, by Young's inequality, we obtain

$$\begin{aligned}
&\lambda|y - \widetilde{y}|_2^2 + \nu\varepsilon\big|(\Psi_\lambda + \lambda I)(y) - (\Psi_\lambda + \lambda I)(\widetilde{y})\big|_2^2 \\
&\leq |x - \widetilde{x}|_2 \cdot \big|(\Psi_\lambda + \lambda I)(y) - (\Psi_\lambda + \lambda I)(\widetilde{y})\big|_2 \\
&\leq \frac{1}{\nu\varepsilon}|x - \widetilde{x}|_2^2 + \nu\varepsilon\big|(\Psi_\lambda + \lambda I)(y) - (\Psi_\lambda + \lambda I)(\widetilde{y})\big|_2^2,
\end{aligned}$$

and therefore

$$|y - \widetilde{y}|_2^2 \leq \frac{1}{\nu\varepsilon\lambda}|x - \widetilde{x}|_2^2,$$

which yields Eq. 4.6 as claimed.

Now, let us prove Eq. 4.7. Let $x \in L^2(\mu) \cap L^p(\mu)$, $p \geq 2$. Since the function $h(r) := r|r|^{p-2}(1 + k|r|^{p-2})^{-1}$ is Lipschitz, and $h(0) = 0$, we have $h(y) \in F_{1,2}$, because $y \in F_{1,2}$. Hence applying ${}_{F_{1,2}^*}\langle \cdot, \; y|y|^{p-2}(1 + k|y|^{p-2})^{-1}\rangle_{F_{1,2}}$, $k > 0$, to both sides of Eq. 4.4, we obtain

$$\begin{aligned}
&{}_{F_{1,2}^*}\langle y, \frac{y|y|^{p-2}}{1 + k|y|^{p-2}}\rangle_{F_{1,2}} + {}_{F_{1,2}^*}\langle \varepsilon(\nu - L)(\Psi_\lambda(y) + \lambda y), \frac{y|y|^{p-2}}{1 + k|y|^{p-2}}\rangle_{F_{1,2}} \\
&= {}_{F_{1,2}^*}\langle x, \frac{y|y|^{p-2}}{1 + k|y|^{p-2}}\rangle_{F_{1,2}}.
\end{aligned} \tag{4.14}$$

Under the Gelfand triple $F_{1,2} \subset L^2(\mu) \subset F_{1,2}^*$, by Eqs. 2.5, 4.14 yields

$$\begin{aligned}
&\langle y, \frac{y|y|^{p-2}}{1 + k|y|^{p-2}}\rangle_2 + {}_{F_{1,2}^*}\langle \varepsilon(\nu - L)(\Psi_\lambda(y) + \lambda y), \frac{y|y|^{p-2}}{1 + k|y|^{p-2}}\rangle_{F_{1,2}} \\
&= \langle x, \frac{y|y|^{p-2}}{1 + k|y|^{p-2}}\rangle_2.
\end{aligned} \tag{4.15}$$

For the second term in the left hand-side of Eq. 4.15, since $x \in L^2(\mu)$, $y \in F_{1,2} \subset L^2(\mu)$, from Eq. 4.4 we deduce that

$$(\nu - L)(\Psi_\lambda(y) + \lambda y) \in L^2(\mu).$$

Then by Eq. 2.5, we know

$$_{F_{1,2}^*}\langle \varepsilon(\nu - L)(\Psi_\lambda(y) + \lambda y), \frac{y|y|^{p-2}}{1 + k|y|^{p-2}}\rangle_{F_{1,2}} = \langle \varepsilon(\nu - L)(\Psi_\lambda(y) + \lambda y), \frac{y|y|^{p-2}}{1 + k|y|^{p-2}}\rangle_2.$$

To estimate the term above, notice that for all Lipschitz and increasing function $g : \mathbb{R} \to \mathbb{R}$ with $g(0) = 0$, we have

$$\int_E (\nu - L)(\Psi_\lambda(y) + \lambda y) \cdot g(y)d\mu \geq 0,$$

because on one hand, Ψ_λ is Lipschitz and monotone with $\Psi_\lambda(0) = 0$, then obviously,

$$\int_E \nu(\Psi_\lambda(y) + \lambda y) \cdot g(y)d\mu \geq 0.$$

On the other hand, we can prove the following term, i.e.,

$$\begin{aligned}
&\langle(-L)(\Psi_\lambda(y) + \lambda y), g(y)\rangle \\
&= \mathscr{E}(\Psi_\lambda(y) + \lambda y, g(y)) \\
&= \lim_{\varepsilon \to 0} \mathscr{E}^{(\varepsilon)}(\Psi_\lambda(y) + \lambda y, g(y)),
\end{aligned} \tag{4.16}$$

is non-negative. Indeed, by [35, Lemma 5.1], with κ being the kernel corresponding to $P := (I - \varepsilon L)^{-1}$, we know, setting $f := \Psi_\lambda + \lambda I$,

$$\begin{aligned}
\mathscr{E}^{(\varepsilon)}(f(y), g(y)) :&= \frac{1}{\varepsilon}\langle f(y), (I - (I - \varepsilon L)^{-1})g(y)\rangle_2 \\
&= \frac{1}{2\varepsilon}\int_E \int_E ((f(y(\tilde{\xi}))) - f(y(\xi))) \cdot (g(y(\tilde{\xi})) - g(y(\xi)))\kappa(\xi, d\tilde{\xi})\mu(d\xi) \\
&\quad + \frac{1}{\varepsilon}\int_E (1 - P1(\xi))f(y(\xi))g(y(\xi))\mu(d\xi),
\end{aligned}$$

since f, g are monotone with $f(0) = g(0) = 0$ and $P1 \leq 1$, we deduce that

$$\mathscr{E}^{(\varepsilon)}(f(y), g(y)) \geq 0,$$

which implies that Eq. 4.16 is non-negative. As a short remark, the assumption that $(E, \mathcal{B})$ is a standard measurable space is needed in [35, Lemma 5.1] to ensure the existence of the kernel κ above.

Thus,

$$\int_E \frac{|y|^p}{1 + k|y|^{p-2}}d\mu \leq \int_E \frac{xy|y|^{p-2}}{1 + k|y|^{p-2}}d\mu.$$

Letting $k \to 0$ and by Hölder's inequality, we obtain

$$|y|_p^p \leq \int_E xy|y|^{p-2}d\mu \leq |x|_p|y|_p^{p-1}.$$

Hence, since $y = J_\varepsilon(x)$,

$$|J_\varepsilon(x)|_p \leq |x|_p.$$

As shown in Lemma 4.1, J_ε is Lipschitz in both $L^2(\mu)$ and $F_{1,2}^*$. Since $A_\lambda^{\nu,\varepsilon} = \frac{1}{\varepsilon}(I - J_\varepsilon)$, $A_\lambda^{\nu,\varepsilon}$ is also Lipschitz in $L^2(\mu)$ and $F_{1,2}^*$. If $x \in F_{1,2}^*$, Eq. 4.3 has a unique adapted solution $X_\lambda^{\nu,\varepsilon} \in L^2(\Omega; C([0, T]; F_{1,2}^*))$ and by Itô's formula (see e.g. [26, Theorem 4.2.5]) we have

$$\mathbb{E}\|X_\lambda^{\nu,\varepsilon}(t)\|_{F_{1,2,\nu}^*}^2 + 2\mathbb{E}\int_0^t \left\langle A_\lambda^{\nu,\varepsilon}(X_\lambda^{\nu,\varepsilon}(s)), X_\lambda^{\nu,\varepsilon}(s)\right\rangle_{F_{1,2,\nu}^*} ds$$

$$= \|x\|_{F_{1,2,\nu}^*}^2 + \mathbb{E}\int_0^t \|B(s, X_\lambda^{\nu,\varepsilon}(s))\|_{L_2(L^2(\mu), F_{1,2,\nu}^*)}^2 ds, \quad t \in [0, T],$$

which, by virtue of **(H2)(ii)** and the fact that the second term on the left hand-side is non-negative by Eq. 4.5, yields

$$\mathbb{E}\|X_\lambda^{\nu,\varepsilon}(t)\|_{F_{1,2,\nu}^*}^2 \leq e^{C_2 T}\|x\|_{F_{1,2,\nu}^*}^2, \quad \forall \varepsilon, \lambda, \nu \in (0, 1], \ t \in [0, T], \ x \in F_{1,2}^*.$$

Similarly, if $x \in L^2(\mu)$, we know that $X_\lambda^{\nu,\varepsilon} \in L^2(\Omega; C([0, T]; L^2(\mu)))$ and again by Itô's formula we obtain

$$\mathbb{E}|X_\lambda^{\nu,\varepsilon}(t)|_2^2 + 2\mathbb{E}\int_0^t \left\langle A_\lambda^{\nu,\varepsilon}(X_\lambda^{\nu,\varepsilon}(s)), X_\lambda^{\nu,\varepsilon}(s)\right\rangle_2 ds$$

$$= |x|_2^2 + \mathbb{E}\int_0^t \|B(s, X_\lambda^{\nu,\varepsilon}(s))\|_{L_2(L^2(\mu), L^2(\mu))}^2 ds,$$

which, by virtue of **(H3)(i)** and the fact that the second summand on the left hand-side is nonnegative by Eq. 4.7 applied to $p = 2$, yields

$$\mathbb{E}|X_\lambda^{\nu,\varepsilon}(t)|_2^2 \leq e^{C_3 T}|x|_2^2, \quad \forall \varepsilon, \lambda, \nu \in (0, 1], \ t \in [0, T], \ x \in L^2(\mu). \tag{4.17}$$

Lemma 4.2 *For $x \in L^2(\mu) \cap L^{2m}(\mu)$, we have that*

$$X_\lambda^{\nu,\varepsilon} \in L^\infty\big([0, T]; L^{2m}(\Omega; L^{2m}(\mu))\big), \quad \forall \varepsilon, \lambda, \nu \in (0, 1].$$

Proof For $\alpha, R > 0$, consider the set

$$\mathcal{K}_R = \left\{X \in L^2\big([0, T]; C([0, T]; L^2(\mu))\big), e^{-2m\alpha t}\mathbb{E}|X(t)|_{2m}^{2m} \leq R^{2m}, \ t \in [0, T]\right\}.$$

Since, by Eq. 4.3, $X_\lambda^{\nu,\varepsilon}$ is a fixed point of the map

$$F : X \mapsto e^{-\frac{\bullet}{\varepsilon}}x + \frac{1}{\varepsilon}\int_0^\bullet e^{-\frac{\bullet-s}{\varepsilon}} J_\varepsilon(X(s))ds + \int_0^\bullet e^{-\frac{\bullet-s}{\varepsilon}} B(s, X(s))dW(s),$$

obtained by iteration in $L^2\big(\Omega; C([0, T]; L^2(\mu))\big)$, it suffices to show that F leaves the set $\mathcal{K}_R$ invariant for $\alpha, R > 0$ large enough. By Eq. 4.7 we have that for $X \in \mathcal{K}_R, t \geq 0$

$$\left[e^{-2m\alpha t}\mathbb{E}\left|e^{-\frac{t}{\varepsilon}}x + \frac{1}{\varepsilon}\int_0^t e^{-\frac{t-s}{\varepsilon}} J_\varepsilon(X(s))ds\right|_{2m}^{2m}\right]^{\frac{1}{2m}}$$

$$\leq e^{-\alpha t}e^{-\frac{t}{\varepsilon}}|x|_{2m} + e^{-\alpha t}\left[\mathbb{E}\left(\int_0^t \frac{1}{\varepsilon}e^{-\frac{t-s}{\varepsilon}}|X(s)|_{2m}ds\right)^{2m}\right]^{\frac{1}{2m}}$$

$$\leq e^{-(\alpha+\frac{1}{\varepsilon})t}|x|_{2m} + e^{-\alpha t}\int_0^t \frac{1}{\varepsilon}e^{-\frac{t-s}{\varepsilon}}\left(\mathbb{E}|X(s)|_{2m}^{2m}\right)^{\frac{1}{2m}}ds$$

$$\leq e^{-(\alpha+\frac{1}{\varepsilon})t}|x|_{2m} + \frac{R}{1 + \alpha\varepsilon}. \tag{4.18}$$

Set

$$Y(t) = \int_0^t e^{-\frac{t-s}{\varepsilon}} B(s, X(s)) dW(s), \ t \geq 0.$$

Then Y is a solution to the following SDE on $L^2(\mu)$:

$$\begin{cases} dY(t) + \dfrac{1}{\varepsilon} Y(t) dt = B(t, X(t)) dW(t), \ t \geq 0, \\ Y(0) = 0, \end{cases}$$

equivalently,

$$d\left(e^{\frac{t}{\varepsilon}} Y(t)\right) = e^{\frac{t}{\varepsilon}} B(t, X(t)) dW(t), \ t \geq 0, \ Y(0) = 0.$$

By Hypothesis **(H3)(ii)**, we may apply Itô's formula in expectation from Theorem A.1 in the Appendix with $u(t)$ replaced by $e^{\frac{t}{\varepsilon}} Y(t)$. Then by Hölder's and Young's inequality and **(H3)(ii)**, we obtain for $t \in [0, T]$

$$\mathbb{E} \left| e^{\frac{t}{\varepsilon}} Y(t) \right|_{2m}^{2m}$$

$$= m(2m-1)\mathbb{E} \int_0^t \int_E \left| e^{\frac{s}{\varepsilon}} Y(s) \right|^{2m-2} \cdot \sum_{k=1}^{\infty} \left| e^{\frac{s}{\varepsilon}} B(s, X(s)) e_k \right|^2 d\mu ds$$

$$\leq m(2m-1)\mathbb{E} \int_0^t \left(\int_E |e^{\frac{s}{\varepsilon}} Y(s)|^{2m-2 \cdot \frac{m}{m-1}} d\mu \right)^{\frac{m-1}{m}} \cdot \left(\int_E \left(\sum_{k=1}^{\infty} |e^{\frac{s}{\varepsilon}} B(s, X(s)) e_k|^2 \right)^m d\mu \right)^{\frac{1}{m}} ds$$

$$= m(2m-1)\mathbb{E} \int_0^t |e^{\frac{s}{\varepsilon}} Y(s)|_{2m}^{2m-2} \cdot \left(\int_E \left(\sum_{k=1}^{\infty} |e^{\frac{s}{\varepsilon}} B(s, X(s)) e_k|^2 \right)^m d\mu \right)^{\frac{1}{m}} ds$$

$$\leq m(2m-1)\mathbb{E} \int_0^t \frac{\left(|e^{\frac{s}{\varepsilon}} Y(s)|_{2m}^{2m-2} \right)^{\frac{m}{m-1}}}{\frac{m}{m-1}} + \frac{\left(\int_E \left(\sum_{k=1}^{\infty} |e^{\frac{s}{\varepsilon}} B(s, X(s)) e_k|^2 \right)^m d\mu \right)^{\frac{1}{m} \cdot m}}{m} ds$$

$$= (m-1)(2m-1)\mathbb{E} \int_0^t |e^{\frac{s}{\varepsilon}} Y(s)|_{2m}^{2m} ds + (2m-1)\mathbb{E} \int_0^t \int_E \left(\sum_{k=1}^{\infty} |e^{\frac{s}{\varepsilon}} B(s, X(s)) e_k|^2 \right)^m ds$$

$$\leq (m-1)(2m-1)\mathbb{E} \int_0^t |e^{\frac{s}{\varepsilon}} Y(s)|_{2m}^{2m} ds + C_4(2m-1)\mathbb{E} \int_0^t |e^{\frac{s}{\varepsilon}} X(s)|_{2m}^{2m} ds,$$

and therefore, by Gronwall's lemma, we obtain

$$\mathbb{E} \left| e^{\frac{t}{\varepsilon}} Y(t) \right|_{2m}^{2m} \leq C_4(2m-1)e^{(m-1)(2m-1)T} \int_0^t \mathbb{E} |e^{\frac{s}{\varepsilon}} X(s)|_{2m}^{2m} ds$$

$$\leq C_4(2m-1)e^{(m-1)(2m-1)T} \int_0^t R^{2m} e^{(\frac{2m}{\varepsilon} + 2m\alpha)s} ds$$

$$\leq \frac{\varepsilon R^{2m} C_{T,m}}{2m(1+\varepsilon\alpha)} e^{\frac{2mt(1+\varepsilon\alpha)}{\varepsilon}},$$

which yields

$$e^{-2m\alpha t} \mathbb{E} |Y(t)|_{2m}^{2m} \leq \frac{\varepsilon R^{2m} C_{T,m}}{(1+\varepsilon\alpha)}, \ \forall t \in [0, T]. \tag{4.19}$$

Then, by formulas Eqs. 4.18, 4.19, we infer that for α large enough and $R \geq 2|x|_{2m}$, the map F leaves $\mathcal{K}_R$ invariant as claimed.

Lemma 4.3 *For $x \in L^2(\mu) \cap L^{2m}(\mu)$, there exists $C \in (0, \infty)$ such that*

$$\sup_{t \in [0,T]} \mathbb{E}|X_\lambda^{\nu,\varepsilon}(t)|_{2m}^{2m} \leq C|x|_{2m}^{2m}, \quad \forall \varepsilon, \lambda, \nu \in (0, 1]. \tag{4.20}$$

Proof Applying the Itô formula in expectation to $\mathbb{E}|X_\lambda^{\nu,\varepsilon}(t)|_{2m}^{2m}$ from Theorem A.1 in the Appendix, we obtain

$$\mathbb{E}|X_\lambda^{\nu,\varepsilon}(t)|_{2m}^{2m} = |x|_{2m}^{2m} - 2m\mathbb{E}\int_0^t \int_E A_\lambda^{\nu,\varepsilon}(X_\lambda^{\nu,\varepsilon}(s))X_\lambda^{\nu,\varepsilon}(s)|X_\lambda^{\nu,\varepsilon}(s)|^{2m-2}d\mu ds$$

$$+ m(2m-1)\mathbb{E}\int_0^t \int_E |X_\lambda^{\nu,\varepsilon}(s)|^{2m-2} \cdot \sum_{k=1}^\infty |B(s, X_\lambda^{\nu,\varepsilon}(s))e_k|^2 d\mu ds. \tag{4.21}$$

Recall that $A_\lambda^{\nu,\varepsilon}(X_\lambda^{\nu,\varepsilon}(s)) = \frac{1}{\varepsilon}(X_\lambda^{\nu,\varepsilon}(s) - J_\varepsilon(X_\lambda^{\nu,\varepsilon}(s)))$, so we have

$$\int_E A_\lambda^{\nu,\varepsilon}(X_\lambda^{\nu,\varepsilon}(s))X_\lambda^{\nu,\varepsilon}(s)|X_\lambda^{\nu,\varepsilon}(s)|^{2m-2}d\mu$$

$$= \frac{1}{\varepsilon}\int_E |X_\lambda^{\nu,\varepsilon}(s)|^{2m}d\mu - \frac{1}{\varepsilon}\int_E J_\varepsilon(X_\lambda^{\nu,\varepsilon}(s))X_\lambda^{\nu,\varepsilon}(s)|X_\lambda^{\nu,\varepsilon}(s)|^{2m-2}d\mu. \tag{4.22}$$

By Hölder's inequality and Eq. 4.7, we conclude

$$\frac{1}{\varepsilon}\int_E |X_\lambda^{\nu,\varepsilon}(s)|^{2m}d\mu - \frac{1}{\varepsilon}\int_E J_\varepsilon(X_\lambda^{\nu,\varepsilon}(s))X_\lambda^{\nu,\varepsilon}(s)|X_\lambda^{\nu,\varepsilon}(s)|^{2m-2}d\mu$$

$$\geq \frac{1}{\varepsilon}\int_E |X_\lambda^{\nu,\varepsilon}(s)|^{2m}d\mu - \frac{1}{\varepsilon}\left[|J_\varepsilon(X_\lambda^{\nu,\varepsilon}(s))|_{2m} \cdot |X_\lambda^{\nu,\varepsilon}(s)|_{2m}^{2m-1}\right]$$

$$\geq \frac{1}{\varepsilon}\int_E |X_\lambda^{\nu,\varepsilon}(s)|^{2m}d\mu - \frac{1}{\varepsilon}|X_\lambda^{\nu,\varepsilon}(s)|_{2m} \cdot |X_\lambda^{\nu,\varepsilon}(s)|_{2m}^{2m-1}$$

$$= 0. \tag{4.23}$$

By Eqs. 4.21-4.23 and using a similar argument as in Eq. 4.19, we get

$$\mathbb{E}|X_\lambda^{\nu,\varepsilon}(t)|_{2m}^{2m} \leq |x|_{2m}^{2m} + (2m-1)\mathbb{E}\int_0^t (m-1)|X_\lambda^{\nu,\varepsilon}(s)|_{2m}^{2m} + C_4|X_\lambda^{\nu,\varepsilon}(s)|_{2m}^{2m}ds$$

$$= |x|_{2m}^{2m} + (2m-1)(m-1+C_4)\mathbb{E}\int_0^t |X_\lambda^{\nu,\varepsilon}(s)|_{2m}^{2m}ds.$$

As a result, by Gronwall's lemma, there exists $C \in (0, \infty)$ such that

$$esssup_{t \in [0,T]}\mathbb{E}|X_\lambda^{\nu,\varepsilon}(t)|_{2m}^{2m} \leq C|x|_{2m}^{2m}, \quad \forall \varepsilon, \lambda, \nu \in (0, 1].$$

Since $t \mapsto |X_\lambda^{\nu,\varepsilon}(t)|_{2m}$ is lower semi-continuous and hence so is $t \mapsto \mathbb{E}|X_\lambda^{\nu,\varepsilon}(t)|_{2m}^{2m}$, Eq. 4.20 follows.

Lemma 4.4 *For $x \in L^2(\mu) \cap L^{2m}(\mu)$ and $X_\lambda^{\nu,\varepsilon}$ as above. Then as $\varepsilon \longrightarrow 0$, we have*

$$X_\lambda^{\nu,\varepsilon} \longrightarrow X_\lambda^\nu \text{ strongly in } L^2(\Omega; C([0, T]; F_{1,2}^*)),$$

where X_λ^ν is the solution to Eq. 4.1. Furthermore, there exists $C \in (0, \infty)$ such that

$$\sup_{t \in [0,T]} \mathbb{E}|X_\lambda^\nu(t)|_{2m}^{2m} \leq C|x|_{2m}^{2m}, \quad \forall \lambda, \nu \in (0, 1]. \tag{4.24}$$

Proof We prove the lemma in two steps, which are given as two claims.

Claim 4.1 *For each* $x \in L^2(\mu)$, *the sequence* $\{X_\lambda^{\nu,\varepsilon}\}$, $\varepsilon \in (0,1]$, *is Cauchy in* $L^2(\Omega; C([0,T]; F_{1,2}^*))$.

Proof Let $\varepsilon, \eta > 0$. Applying the Itô formula ([26, Theorem 4.2.5] with $V := L^2(\mu)$, $H := F_{1,2,\nu}^*, \alpha = 2, X_0 = x$) to $\|X_\lambda^{\nu,\varepsilon} - X_\lambda^{\nu,\eta}\|_{F_{1,2,\nu}^*}^2$, we have

$$d\|X_\lambda^{\nu,\varepsilon}(t) - X_\lambda^{\nu,\eta}(t)\|_{F_{1,2,\nu}^*}^2$$
$$+ 2\langle (\nu - L)\big((\Psi_\lambda + \lambda I)(J_\varepsilon(X_\lambda^{\nu,\varepsilon}(t))) - (\Psi_\lambda + \lambda)(J_\eta(X_\lambda^{\nu,\eta}(t)))\big), X_\lambda^{\nu,\varepsilon}(t) - X_\lambda^{\nu,\eta}(t)\rangle_{F_{1,2,\nu}^*} dt$$
$$= 2\langle X_\lambda^{\nu,\varepsilon}(t) - X_\lambda^{\nu,\eta}(t), \big(B(t, X_\lambda^{\nu,\varepsilon}(t)) - B(t, X_\lambda^{\nu,\eta}(t))\big)dW(t)\rangle_{F_{1,2,\nu}^*}$$
$$+ \big\|B(t, X_\lambda^{\nu,\varepsilon}(t)) - B(t, X_\lambda^{\nu,\eta}(t))\big\|_{L_2(L^2(\mu), F_{1,2,\nu}^*)}^2 dt. \qquad (4.25)$$

The second term in the left hand-side of the above equality, by Eqs. 4.4, 2.7, and 2.5, is equal to

$$2\langle (\Psi_\lambda + \lambda I)(J_\varepsilon(X_\lambda^{\nu,\varepsilon}(t))) - (\Psi_\lambda + \lambda I)(J_\eta(X_\lambda^{\nu,\eta}(t))), J_\varepsilon(X_\lambda^{\nu,\varepsilon}(t)) - J_\eta(X_\lambda^{\nu,\eta}(t))\rangle_2 dt$$
$$+ 2\langle (\nu - L)\big((\Psi_\lambda + \lambda I)(J_\varepsilon(X_\lambda^{\nu,\varepsilon}(s)))\big) - (\nu - L)\big((\Psi_\lambda + \lambda I)(J_\eta(X_\lambda^{\nu,\eta}(s)))\big),$$
$$\varepsilon(\nu - L)\big((\Psi_\lambda + \lambda I)(J_\varepsilon(X_\lambda^{\nu,\varepsilon}(s)))\big) - \eta(\nu - L)\big((\Psi_\lambda + \lambda I)(J_\eta(X_\lambda^{\nu,\eta}(s)))\big)\rangle_{F_{1,2,\nu}^*} dt. \quad (4.26)$$

Taking Eq. 4.26 into Eq. 4.25, then taking expectation of both sides, we obtain for all $t \in [0,T]$

$$\mathbb{E} \sup_{r \in [0,t]} \|X_\lambda^{\nu,\varepsilon}(r) - X_\lambda^{\nu,\eta}(r)\|_{F_{1,2,\nu}^*}^2$$
$$- 2\mathbb{E}\Big[\sup_{r \in [0,t]} \Big|\int_0^r \langle X_\lambda^{\nu,\varepsilon}(s) - X_\lambda^{\nu,\eta}(s), (B(s, X_\lambda^{\nu,\varepsilon}(s)) - B(s, X_\lambda^{\nu,\eta}(s)))dW(s)\rangle_{F_{1,2,\nu}^*}\Big|\Big]$$
$$+ 2\mathbb{E}\int_0^t \langle (\Psi_\lambda + \lambda I)\big(J_\varepsilon(X_\lambda^{\nu,\varepsilon}(s))\big)$$
$$- (\Psi_\lambda + \lambda I)\big(J_\eta(X_\lambda^{\nu,\eta}(s))\big), J_\varepsilon(X_\lambda^{\nu,\varepsilon}(s)) - J_\eta(X_\lambda^{\nu,\eta}(s))\rangle_2 ds$$
$$\leq 2\mathbb{E}\int_0^T \big|\langle (\nu - L)\big((\Psi_\lambda + \lambda I)(J_\varepsilon(X_\lambda^{\nu,\varepsilon}(s)))\big) - (\nu - L)\big((\Psi_\lambda + \lambda I)(J_\eta(X_\lambda^{\nu,\eta}(s)))\big),$$
$$\varepsilon(\nu - L)\big((\Psi_\lambda + \lambda I)(J_\varepsilon(X_\lambda^{\nu,\varepsilon}(s)))\big) - \eta(\nu - L)\big((\Psi_\lambda + \lambda I)(J_\eta(X_\lambda^{\nu,\eta}(s)))\big)\rangle_{F_{1,2,\nu}^*}\big| ds$$
$$+ \mathbb{E}\int_0^t \big\|B(s, X_\lambda^{\nu,\varepsilon}(s)) - B(s, X_\lambda^{\nu,\eta}(s))\big\|_{L_2(L^2(\mu), F_{1,2,\nu}^*)}^2 ds$$
$$\leq 3(\varepsilon + \eta)\mathbb{E}\int_0^T \|(\nu - L)(\Psi_\lambda + \lambda I)(J_\varepsilon(X_\lambda^{\nu,\varepsilon}(s)))\|_{F_{1,2,\nu}^*}^2$$
$$+ \|(\nu - L)(\Psi_\lambda + \lambda I)(J_\eta(X_\lambda^{\nu,\eta}(s)))\|_{F_{1,2,\nu}^*}^2 ds$$
$$+ \mathbb{E}\int_0^t \big\|B(s, X_\lambda^{\nu,\varepsilon}(s)) - B(s, X_\lambda^{\nu,\eta}(s))\big\|_{L_2(L^2(\mu), F_{1,2,\nu}^*)}^2 ds$$
$$\leq 3(\varepsilon + \eta)(\frac{1}{\lambda} + \lambda + C_5)e^{C_3 T}|x|_2^2 + \mathbb{E}\int_0^t C_1\|X_\lambda^{\nu,\varepsilon}(s) - X_\lambda^{\nu,\eta}(s)\|_{F_{1,2,\nu}^*}^2 ds, \qquad (4.27)$$

where we used Proposition A.1 (see Appendix) and **(H2)(i)** in the last inequality. For the

second term in the left hand-side of Eq. 4.27, by using the Burkholder-Davis-Gundy (BDG) inequality for $p = 1$, we obtain for all $t \in [0, T]$,

$$\mathbb{E}\left[\sup_{r \in [0,t]} \left| \int_0^r \left\langle X_\lambda^{\nu,\varepsilon}(s) - X_\lambda^{\nu,\eta}(s), (B(s, X_\lambda^{\nu,\varepsilon}(s)) - B(s, X_\lambda^{\nu,\eta}(s)))dW(s) \right\rangle_{F_{1,2,\nu}^*} \right| \right]$$

$$\leq \mathbb{E}\left[\int_0^t \|X_\lambda^{\nu,\varepsilon}(s) - X_\lambda^{\nu,\eta}(s)\|_{F_{1,2,\nu}^*}^2 \cdot C_1 \|X_\lambda^{\nu,\varepsilon}(s) - X_\lambda^{\nu,\eta}(s)\|_{F_{1,2,\nu}^*}^2 ds \right]^{\frac{1}{2}}$$

$$\leq \mathbb{E}\left[\sup_{r \in [0,t]} \|X_\lambda^{\nu,\varepsilon}(r) - X_\lambda^{\nu,\eta}(r)\|_{F_{1,2,\nu}^*}^2 \cdot C_1 \int_0^t \|X_\lambda^{\nu,\varepsilon}(s) - X_\lambda^{\nu,\eta}(s)\|_{F_{1,2,\nu}^*}^2 ds \right]^{\frac{1}{2}}$$

$$\leq \frac{1}{4}\mathbb{E} \sup_{r \in [0,t]} \|X_\lambda^{\nu,\varepsilon}(r) - X_\lambda^{\nu,\eta}(r)\|_{F_{1,2,\nu}^*}^2 + C_1\mathbb{E} \int_0^t \|X_\lambda^{\nu,\varepsilon}(s) - X_\lambda^{\nu,\eta}(s)\|_{F_{1,2,\nu}^*}^2 ds. \quad (4.28)$$

Substituting Eq. 4.28 into Eq. 4.27, we obtain

$$\frac{1}{2}\mathbb{E} \sup_{r \in [0,t]} \|X_\lambda^{\nu,\varepsilon}(r) - X_\lambda^{\nu,\eta}(r)\|_{F_{1,2,\nu}^*}^2$$

$$+2\mathbb{E} \int_0^t \left\langle (\Psi_\lambda + \lambda I)\left(J_\varepsilon(X_\lambda^{\nu,\varepsilon}(s))\right) - (\Psi_\lambda + \lambda I)\left(J_\eta(X_\lambda^{\nu,\eta}(s))\right), J_\varepsilon(X_\lambda^{\nu,\varepsilon}(s)) - J_\eta(X_\lambda^{\nu,\eta}(s)) \right\rangle_2 ds$$

$$\leq 3(\varepsilon + \eta)(\frac{1}{\lambda} + \lambda + C_5)e^{C_3 T}|x|_2^2 + 3C_1\mathbb{E} \int_0^t \sup_{r \in [0,s]} \|X_\lambda^{\nu,\varepsilon}(r) - X_\lambda^{\nu,\eta}(r)\|_{F_{1,2,\nu}^*}^2 ds.$$

By Gronwall's lemma, we obtain

$$\mathbb{E} \sup_{t \in [0,T]} \|X_\lambda^{\nu,\varepsilon}(t) - X_\lambda^{\nu,\eta}(t)\|_{F_{1,2}^*}^2$$

$$+4\mathbb{E} \int_0^t \left\langle (\Psi_\lambda + \lambda I)\left(J_\varepsilon(X_\lambda^{\nu,\varepsilon}(s))\right) - (\Psi_\lambda + \lambda I)\left(J_\eta(X_\lambda^{\nu,\eta}(s))\right), J_\varepsilon(X_\lambda^{\nu,\varepsilon}(s)) - J_\eta(X_\lambda^{\nu,\eta}(s)) \right\rangle_2 ds$$

$$\leq 6(\varepsilon + \eta)(\frac{1}{\lambda} + \lambda + C)e^{(6C_1 + C_3)T}|x|_2^2. \quad (4.29)$$

Since by the monotonicity of Ψ_λ the second term on the left-hand-side of inequality Eq. 4.29 is nonnegative, letting $\varepsilon, \eta \to 0$, we see that $\{X_\lambda^{\nu,\varepsilon}\}$ is Cauchy in $L^2(\Omega; C([0, T]; F_{1,2}^*))$.

From Claim 4.1, we know there exists $\widetilde{X} \in L^2(\Omega; C([0, T]; F_{1,2}^*))$ such that

$$\lim_{\varepsilon \to 0} X_\lambda^{\nu,\varepsilon} = \widetilde{X} \text{ in } L^2(\Omega; C([0, T]; F_{1,2}^*)), \quad (4.30)$$

Claim 4.2 $\widetilde{X} = X_\lambda^\nu$.

Proof We have

$$\lim_{\varepsilon \to 0} \int_0^\cdot B(s, X_\lambda^{\nu,\varepsilon}(s))dW(s) = \int_0^\cdot B(s, \widetilde{X}(s))dW(s) \text{ in } L^2(\Omega; C([0, T]; F_{1,2}^*)), (4.31)$$

since by the BDG inequality for $p = 1$ and **(H2)(i)**, we have

$$\mathbb{E} \sup_{r \in [0,T]} \left\| \int_0^r (B(s, X_\lambda^{\nu,\varepsilon}(s)) - B(s, \widetilde{X}(s)))dW(s) \right\|_{F_{1,2,\nu}^*}^2$$

$$\leq C\mathbb{E} \int_0^T \|B(s, X_\lambda^{\nu,\varepsilon}(s)) - B(s, \widetilde{X}(s))\|_{L_2(L^2(\mu), F_{1,2,\nu}^*)}^2 ds$$

$$\leq CT\mathbb{E} \sup_{s \in [0,T]} \|X_\lambda^{\nu,\varepsilon}(s) - \widetilde{X}(s)\|_{F_{1,2,\nu}^*}^2.$$

Next we show that $(\Psi_\lambda + \lambda)(\widetilde{X}) \in L^2((0, T); L^2(\Omega; F_{1,2}))$ and that Eq. 4.1 is satisfied. From Eq. 4.17 we know that $\{X_\lambda^{\nu,\varepsilon}\}$ is bounded in $L^2((0, T) \times \Omega \times E)$ and therefore along a subsequence, again denoted by $\{\varepsilon\}$, we have

$$\lim_{\varepsilon \to 0} X_\lambda^{\nu,\varepsilon} = \widetilde{X} \text{ weakly in } L^2((0, T) \times \Omega \times E). \tag{4.32}$$

From Eqs. 4.4 and A.4, we know

$$\mathbb{E} \int_0^T \|X_\lambda^{\nu,\varepsilon}(s) - J_\varepsilon(X_\lambda^{\nu,\varepsilon}(s))\|_{F_{1,2,\nu}^*}^2 ds$$

$$= \varepsilon^2 \mathbb{E} \int_0^T \|(\nu - L)(\Psi_\lambda(J_\varepsilon(X_\lambda^{\nu,\varepsilon}(s))) + \lambda J_\varepsilon(X_\lambda^{\nu,\varepsilon}(s)))\|_{F_{1,2,\nu}^*}^2 ds$$

$$\leq \frac{\varepsilon}{2}(\frac{1}{\lambda} + \lambda + C_5)e^{C_3 T}|x|_2^2, \tag{4.33}$$

which yields,

$$\lim_{\varepsilon \to 0} J_\varepsilon(X_\lambda^{\nu,\varepsilon}) = \widetilde{X} \text{ in } L^2((0, T); L^2(\Omega; F_{1,2}^*)).$$

Recall from Eq. 4.7 that

$$|J_\varepsilon(X_\lambda^{\nu,\varepsilon}(t))|_2 \leq |X_\lambda^{\nu,\varepsilon}(t)|_2, \quad \forall t \in [0, T].$$

Therefore, we infer by Eqs. 4.32 and 4.33 that

$$\lim_{\varepsilon \to 0} J_\varepsilon(X_\lambda^{\nu,\varepsilon}) = \widetilde{X}, \text{ weakly in } L^2((0, T) \times \Omega \times E). \tag{4.34}$$

By the monotonicity of Ψ_λ, it follows from Eq. 4.29 that $J_\varepsilon(X_\lambda^{\nu,\varepsilon})$, $\varepsilon \in (0, 1]$, is Cauchy in $L^2((0, T) \times \Omega \times E)$, so the convergence in Eq. 4.34 is strong and thus

$$\lim_{\varepsilon \to 0} (\Psi_\lambda + \lambda I)(J_\varepsilon(X_\lambda^{\nu,\varepsilon})) = (\Psi_\lambda + \lambda I)(\widetilde{X}) \text{ in } L^2((0, T) \times \Omega \times E),$$

since $\Psi_\lambda + \lambda I$ is Lipschitz.

From Eq. A.4, we know that $(\nu - L)(\Psi_\lambda + \lambda I)(J_\varepsilon(X_\lambda^{\nu,\varepsilon}))$, $\varepsilon \in (0, 1]$, is bounded in $L^2([0, T] \times \Omega; F_{1,2}^*)$, so $(\Psi_\lambda + \lambda I)(J_\varepsilon(X_\lambda^{\nu,\varepsilon}))$ is bounded in $L^2([0, T]; L^2(\Omega, F_{1,2}))$. Hence there exists a subsequence, again denoted by $\{\varepsilon\}$ such that

$$\lim_{\varepsilon \to 0} (\Psi_\lambda + \lambda I)(J_\varepsilon(X_\lambda^{\nu,\varepsilon})) = (\Psi_\lambda + \lambda I)(\widetilde{X}) \text{ weakly in } L^2([0, T] \times \Omega; F_{1,2}). \tag{4.35}$$

It is then easy to see that also

$$\lim_{\varepsilon \to 0} \int_0^\cdot (\Psi_\lambda + \lambda I)(J_\varepsilon(X_\lambda^{\nu,\varepsilon}(s)))ds = \int_0^\cdot (\Psi_\lambda + \lambda I)(\widetilde{X}(s))ds$$

weakly in $L^2([0, T] \times \Omega; F_{1,2})$, and thus

$$\lim_{\varepsilon \to 0} (\nu - L) \int_0^\cdot (\Psi_\lambda + \lambda I) J_\varepsilon(X_\lambda^{\nu,\varepsilon}(s))ds = (\nu - L) \int_0^\cdot (\Psi_\lambda + \lambda I)(\widetilde{X}(s))ds$$

weakly in $L^2([0, T] \times \Omega; F_{1,2}^*)$.

Consequently, taking into account Eqs. 4.30, 4.31, as $\varepsilon \to 0$, we can pass to the weak limit in $L^2([0, T] \times \Omega; F_{1,2}^*)$ in the equation

$$X_\lambda^{\nu,\varepsilon}(t) = x + (\nu - L) \int_0^t (\Psi_\lambda + \lambda I)(J_\varepsilon(X_\lambda^{\nu,\varepsilon}(s)))ds + \int_0^t B(s, X_\lambda^{\nu,\varepsilon}(s))dW(s),$$

and since each term is a $\mathbb{P}$-a.s. continuous path in $F_{1,2}^*$, we conclude that $\widetilde{X}$ is a strong solution to Eq. 4.1 in the sense of the definition in Lemma 3.1 of [36]. Furthermore, by the uniqueness part of [36, Lemma 3.1], it follows that $X_\lambda^\nu = \widetilde{X}$ a.e. in $[0, T] \times \Omega \times E$.

It remains to prove Eq. 4.24. By Eqs. 4.17 and 4.20, we know that for fixed ν and λ, there exists a subsequence denoted as $\{X_\lambda^{\nu, \varepsilon_n}\}$, $\varepsilon_n \in (0, 1]$ and an element $Y_\lambda^\nu \in L^\infty([0, T]; (L^2 \cap L^{2m})(\Omega \times E))$ such that as $\varepsilon_n \to 0$,

$$X_\lambda^{\nu, \varepsilon_n} \longrightarrow Y_\lambda^\nu \text{ weakly in } L^\infty([0, T]; (L^2 \cap L^{2m})(\Omega \times E)),$$

hence also in $L^2([0, T] \times \Omega; L^2(\mu))$ and in $L^2([0, T] \times \Omega; F_{1,2}^*)$, but $\{X_\lambda^{\nu, \varepsilon_n}\}$ is Cauchy in $L^2(\Omega; C([0, T]; F_{1,2}^*))$ by Claim 4.1, therefore we have $Y_\lambda^\nu = X_\lambda^\nu$ a.e. in $[0, T] \times \Omega \times E$. Note that the above weak convergence also holds in $L^\infty([0, T]; L^{2m}(\Omega \times E))$, hence by the (sequentially) weakly lower-semicontinuity of norms, letting $\varepsilon \to 0$ in Eq. 4.20, we can get Eq. 4.24. This completes the proof of Lemma 4.4.

Remark 4.1 *By Lemma 4.4 we know that*

$$X_\lambda^\nu(t) = x + (\nu - L) \int_0^t \left(\Psi_\lambda(X_\lambda^\nu(s)) + \lambda X_\lambda^\nu(s) \right) ds + \int_0^t B(s, X_\lambda^\nu(s)) dW(s), \ t \in [0, T].$$

But, since $X_\lambda^\nu = \widetilde{X}$, by Eq. 4.35 we may interchange $(\nu - L)$ with the integral w.r.t. ds.

Let us now continue to prove Theorem 3.2. Choose $0 < \nu \le \nu_0 \le 1$, rewrite Eq. 4.1 as

$$dX_\lambda^\nu(t) + (\nu_0 - L)(\Psi_\lambda(X_\lambda^\nu(t)) + \lambda X_\lambda^\nu(t)) dt$$
$$= (\nu_0 - \nu)(\Psi_\lambda(X_\lambda^\nu(t)) + \lambda X_\lambda^\nu(t)) dt + B(t, X_\lambda^\nu(t)) dW(t).$$

Now by Remark 4.1 we may apply Itô's formula ([26, Theorem 4.2.5] with $V := L^2(\mu)$, $H := F_{1,2,\nu_0}^*$) to $\|X_\lambda^\nu - X_\lambda^{\nu'}\|_{F_{1,2,\nu_0}^*}^2$, $\nu, \nu' \in (0, \nu_0]$, to obtain for all $t \in [0, T]$, $\lambda \in (0, 1)$,

$$\|X_\lambda^\nu(t) - X_\lambda^{\nu'}(t)\|_{F_{1,2,\nu_0}^*}^2$$

$$+2 \int_0^t \int_E \left(\Psi_\lambda(X_\lambda^\nu(s)) + \lambda X_\lambda^\nu(s) - \Psi_\lambda(X_\lambda^{\nu'}(s)) - \lambda X_\lambda^{\nu'}(s) \right) \cdot (X_\lambda^\nu(s) - X_\lambda^{\nu'}(s)) d\mu ds$$

$$= 2 \int_0^t \left\langle \nu_0 \left(\Psi_\lambda(X_\lambda^\nu(s)) - \Psi_\lambda(X_\lambda^{\nu'}(s)) + \lambda X_\lambda^\nu(s) - \lambda X_\lambda^{\nu'}(s) \right), X_\lambda^\nu(s) - X_\lambda^{\nu'}(s) \right\rangle_{F_{1,2,\nu_0}^*} ds$$

$$+2\nu' \int_0^t \left\langle \Psi_\lambda(X_\lambda^{\nu'}(s)) + \lambda X_\lambda^{\nu'}(s), X_\lambda^\nu(s) - X_\lambda^{\nu'}(s) \right\rangle_{F_{1,2,\nu_0}^*} ds$$

$$-2\nu \int_0^t \left\langle \Psi_\lambda(X_\lambda^\nu(s)) + \lambda X_\lambda^\nu(s), X_\lambda^\nu(s) - X_\lambda^{\nu'}(s) \right\rangle_{F_{1,2,\nu_0}^*} ds$$

$$+ \int_0^t \|B(s, X_\lambda^\nu(s)) - B(s, X_\lambda^{\nu'}(s))\|_{L_2(L^2(\mu), F_{1,2,\nu_0}^*)}^2 ds$$

$$+2 \int_0^t \left\langle X_\lambda^\nu(s) - X_\lambda^{\nu'}(s), \left(B(s, X_\lambda^\nu(s)) - B(s, X_\lambda^{\nu'}(s)) \right) dW(s) \right\rangle_{F_{1,2,\nu_0}^*}. \tag{4.36}$$

Since Ψ_λ is $\frac{1}{\lambda}$-Lipschitz, we have $\Psi_\lambda(r) - \Psi_\lambda(r')(r - r') \ge \lambda |\Psi_\lambda(r) - \Psi_\lambda(r')|^2, \forall r, r' \in \mathbb{R}$, then

$$2 \int_0^t \int_E \left(\Psi_\lambda(X_\lambda^\nu(s)) + \lambda X_\lambda^\nu(s) - \Psi_\lambda(X_\lambda^{\nu'}(s)) - \lambda X_\lambda^{\nu'}(s) \right) \cdot (X_\lambda^\nu(s) - X_\lambda^{\nu'}(s)) ds$$

$$\ge 2\lambda \int_0^t |X_\lambda^\nu(s) - X_\lambda^{\nu'}(s)|_2^2 ds + 2\lambda \int_0^t |\Psi_\lambda(X_\lambda^\nu(s)) - \Psi_\lambda(X_\lambda^{\nu'}(s))|_2^2 ds. \tag{4.37}$$

Since $L^2(\mu) \subset F_{1,2}^*$ continuously, by Eq. 2.3 we have $\|u\|_{F_{1,2,v_0}^*} \leq \frac{1}{\sqrt{v_0}} |u|_2, \forall u \in L^2(\mu)$, by Minkowski's inequality and Young's inequality, we get for all $t \in [0, T]$

$$2 \int_0^t \left\langle v_0 \big(\Psi_\lambda(X_\lambda^v(s)) - \Psi_\lambda(X_\lambda^{v'}(s)) + \lambda X_\lambda^v(s) - \lambda X_\lambda^{v'}(s) \big), X_\lambda^v(s) - X_\lambda^{v'}(s) \right\rangle_{F_{1,2,v_0}^*} ds$$

$$\leq 2v_0 \int_0^t \|\Psi_\lambda(X_\lambda^v(s)) - \Psi_\lambda(X_\lambda^{v'}(s))\|_{F_{1,2,v_0}^*} \cdot \|X_\lambda^v(s) - X_\lambda^{v'}(s)\|_{F_{1,2,v_0}^*} ds$$

$$+ 2v_0 \lambda \int_0^t \|X_\lambda^v(s) - X_\lambda^{v'}(s)\|_{F_{1,2,v_0}^*}^2 ds$$

$$\leq 2\lambda \int_0^t |\Psi_\lambda(X_\lambda^v(s)) - \Psi_\lambda(X_\lambda^{v'}(s))|_2^2 ds + \frac{v_0}{2\lambda} \int_0^t \|X_\lambda^v(s) - X_\lambda^{v'}(s)\|_{F_{1,2,v_0}^*}^2 ds$$

$$+ 2v_0 \lambda \int_0^t \|X_\lambda^v(s) - X_\lambda^{v'}(s)\|_{F_{1,2,v_0}^*}^2 ds. \tag{4.38}$$

Using similar arguments as above and the fact that, by **(H1)** and Eq. 3.5, $|\Psi_\lambda(r)| \leq C(|r|^m + 1_{\{\mu(E)<\infty\}}), \forall r \in \mathbb{R}$ with C independent of λ, we have for all $t \in [0, T]$

$$2v' \int_0^t \left\langle \Psi_\lambda(X_\lambda^{v'}(s)) + \lambda X_\lambda^{v'}(s), X_\lambda^v(s) - X_\lambda^{v'}(s) \right\rangle_{F_{1,2,v_0}^*} ds$$

$$- 2v \int_0^t \left\langle \Psi_\lambda(X_\lambda^v(s)) + \lambda X_\lambda^v(s), X_\lambda^v(s) - X_\lambda^{v'}(s) \right\rangle_{F_{1,2,v_0}^*} ds$$

$$\leq \frac{2v'}{v_0} \int_0^t |\Psi_\lambda(X_\lambda^{v'}(s))|_2^2 + \lambda |X_\lambda^{v'}(s)|_2^2 ds + \int_0^t \|X_\lambda^v(s) - X_\lambda^{v'}(s)\|_{F_{1,2,v_0}^*}^2 ds$$

$$+ \frac{2v}{v_0} \int_0^t |\Psi_\lambda(X_\lambda^v(s))|_2^2 + \lambda |X_\lambda^v(s)|_2^2 ds + \int_0^t \|X_\lambda^v(s) - X_\lambda^{v'}(s)\|_{F_{1,2,v_0}^*}^2 ds$$

$$\leq \frac{2v'}{v_0} \int_0^t C\big(|X_\lambda^{v'}(s)|_{2m}^{2m} + \mu(E) \cdot 1_{\{\mu(E)<\infty\}}\big) + \lambda |X_\lambda^{v'}(s)|_2^2 ds$$

$$+ \frac{2v}{v_0} \int_0^t C\big(|X_\lambda^v(s)|_{2m}^{2m} + \mu(E) \cdot 1_{\{\mu(E)<\infty\}}\big) + \lambda |X_\lambda^v(s)|_2^2 ds$$

$$+ 2 \int_0^t \|X_\lambda^v(s) - X_\lambda^{v'}(s)\|_{F_{1,2,v_0}^*}^2 ds$$

$$\leq \frac{2(v+v')(C+\lambda)}{v_0} \int_0^t |X_\lambda^{v'}(s)|_{2m}^{2m} + |X_\lambda^v(s)|_{2m}^{2m} + |X_\lambda^{v'}(s)|_2^2 + |X_\lambda^v(s)|_2^2 + \mu(E) \cdot 1_{\{\mu(E)<\infty\}} ds$$

$$+ 2 \int_0^t \|X_\lambda^v(s) - X_\lambda^{v'}(s)\|_{F_{1,2,v_0}^*}^2 ds. \tag{4.39}$$

Taking expectation to both sides of Eq. 4.36, by the BDG inequality for $p = 1$, taking **(H2)(i)**, Eqs. 4.37-4.39, 4.24 into account, we obtain for all $t \in [0, T]$

$$\frac{1}{2} \mathbb{E} \Big[\sup_{s \in [0,t]} \|X_\lambda^v(s) - X_\lambda^{v'}(s)\|_{F_{1,2,v_0}^*}^2 \Big] + 2\lambda \mathbb{E} \int_0^t |X_\lambda^v(s) - X_\lambda^{v'}(s)|_2^2 ds$$

$$\leq \frac{(C+\lambda)(v+v')C_T}{v_0} \big(|x|_{2m}^{2m} + |x|_2^2 + \mu(E) \cdot 1_{\{\mu(E)<\infty\}}\big)$$

$$+ \Big(\frac{v_0}{2\lambda} + 2v_0\lambda + C\Big) \mathbb{E} \int_0^t \sup_{r \in [0,s]} \|X_\lambda^v(r) - X_\lambda^{v'}(r)\|_{F_{1,2,v_0}^*}^2 ds.$$

Hence by Gronwall's lemma, we have for some $C_{T,\lambda} \in (0, \infty)$

$$\mathbb{E}\Big[\sup_{s \in [0,T]} \|X_\lambda^\nu(s) - X_\lambda^{\nu'}(s)\|_{F_{1,2,\nu_0}^*}^2 \Big] + 2\lambda \mathbb{E} \int_0^T |X_\lambda^\nu(s) - X_\lambda^{\nu'}(s)|_2^2 ds$$

$$\leq \frac{C_{T,\lambda}(\nu + \nu')}{\nu_0} \big(|x|_{2m}^{2m} + |x|_2^2 + \mu(E) \cdot 1_{\{\mu(E) < \infty\}} \big), \quad \forall \nu, \nu' \in (0, \nu_0]. \tag{4.40}$$

Hence there exists an $(\mathscr{F}_t)_{t \geq 0}$-adapted process

$$X_\lambda \in L^2(\Omega; C([0, T]; F_{1,2}^*)) \cap L^2((0, T) \times \Omega \times E), \tag{4.41}$$

such that

$$\lim_{\nu \to 0} \Big\{ \mathbb{E}\Big[\sup_{s \in [0,T]} \|X_\lambda^\nu(s) - X_\lambda(s)\|_{F_{1,2,\nu_0}^*}^2 \Big] + 2\lambda \mathbb{E} \int_0^T |X_\lambda^\nu(s) - X_\lambda(s)|_2^2 ds \Big\} = 0. \tag{4.42}$$

Consequently, by **(H2)(i)** we can pass to the limit with $\nu \to 0$ in Eq. 4.1 to obtain

$$X_\lambda(t) = x - \lim_{\nu \to 0} (\nu - L) \int_0^t \Psi_\lambda(X_\lambda^\nu(s)) + \lambda X_\lambda^\nu(s) ds$$

$$+ \int_0^t B(s, X_\lambda(s)) dW(s), \quad t \in [0, T], \tag{4.43}$$

where the limit exists in $L^2(\Omega; C([0, T]; F_{1,2}^*))$. Furthermore, it follows by Eq. 4.42, since Ψ_λ is Lipschitz, that

$$\lim_{\nu \to 0} \int_0^\cdot \Psi_\lambda(X_\lambda^\nu(s)) + \lambda X_\lambda^\nu(s) ds = \int_0^\cdot \Psi_\lambda(X_\lambda(s)) + \lambda X_\lambda(s) ds, \tag{4.44}$$

in $L^2(\Omega; C([0, T]; L^2(\mu)))$, hence in $L^2(\Omega; C([0, T]; F_{1,2}^*))$. Writing $\nu - L = (1 - L) + (\nu - 1)I$, Eqs. 4.43 and 4.44 imply that the convergence in Eq. 4.44 holds even in $L^2(\Omega; C([0, T]; F_{1,2}))$ and that the second term on the right hand-side of Eq. 4.43 is equal to

$$L \int_0^t \Psi_\lambda(X_\lambda(s)) + \lambda X_\lambda(s) ds,$$

which together with Eq. 4.41 show that X_λ is a solution of Eq. 3.8 in the sense of Definition 3.1 in [36] with state space $F_{1,2}^*$.

Now let us prove that, since $x \in \mathscr{F}_e^* (\subset F_{1,2}^*)$, which so far we have not used, that X_λ is indeed a solution of Eq. 3.8 on the smaller state space $\mathscr{F}_e^*$ and that Eqs. 3.9-3.12 hold. Note that Eq. 3.7 trivially holds, since the convergence in Eq. 4.44 is in $L^2(\Omega; C([0, T]; F_{1,2}))$ and since $F_{1,2} \subset \mathscr{F}_e$ continuously.

To prove Eq. 3.9, we observe that by Eq. 4.42 it follows that as $\nu \to 0$, $X_\lambda^\nu \to X_\lambda$ in $dt \otimes \mathbb{P} \otimes d\mu$-measure. Hence we have by Fatou's lemma and Eq. 4.24 for all $\varphi \in L^1([0, T]; \mathbb{R})$

$$\int_0^T |\varphi(t)| \mathbb{E} |X_\lambda(t)|_{2m}^{2m} dt \leq \liminf_{\nu \to 0} \int_0^T |\varphi(t)| \mathbb{E} |X_\lambda^\nu(t)|_{2m}^{2m} dt$$

$$\leq |\varphi|_{L^1([0,T];\mathbb{R})} C |x|_{2m}^{2m},$$

which implies Eq. 3.9. Now Eq. 3.10 follows by **(H1)**.

To prove Eq. 3.11, applying Itô's formula ([26, Theorem 4.2.5] with $V := L^2(\mu)$, $H := F_{1,2,\nu_0}^*$) to $\|X_\lambda^\nu(t)\|_{F_{1,2,\nu_0}^*}^2$, for all $t \in [0, T]$, we have

$$
\|X_\lambda^\nu(t)\|_{F_{1,2,\nu_0}^*}^2 + 2\int_0^t \int_E \left(\Psi_\lambda(X_\lambda^\nu(s)) + \lambda X_\lambda^\nu(s)\right) \cdot X_\lambda^\nu(s) d\mu ds
$$

$$
= \|x\|_{F_{1,2,\nu_0}^*}^2 + 2\int_0^t \left\langle (\nu_0 - \nu)\left(\Psi_\lambda(X_\lambda^\nu(s)) + \lambda X_\lambda^\nu(s)\right), X_\lambda^\nu(s)\right\rangle_{F_{1,2,\nu_0}^*} ds
$$

$$
+ 2\int_0^t \left\langle X_\lambda^\nu(s), B(s, X_\lambda^\nu(s)) dW(s)\right\rangle_{F_{1,2,\nu_0}^*} + \int_0^t \|B(s, X_\lambda^\nu(s))\|_{L_2(L^2(\mu), F_{1,2,\nu_0}^*)} \tag{4.45}
$$

Since Ψ_λ is monotone,

$$
2\int_0^t \int_E \left(\Psi_\lambda(X_\lambda^\nu(s)) + \lambda X_\lambda^\nu(s)\right) \cdot X_\lambda^\nu(s) d\mu ds \geq 2\lambda \int_0^t |X_\lambda^\nu(s)|_2^2 ds. \tag{4.46}
$$

Since $\|u\|_{F_{1,2,\nu_0}^*} \leq \frac{1}{\sqrt{\nu_0}}|u|_2$, $\forall u \in L^2(\mu)$, by Minkowski's inequality, Young's inequality and $|\Psi_\lambda(r)| \leq C(|r|^m + 1_{\{\mu(E)<\infty\}})$, $\forall r \in \mathbb{R}$ with C independent of λ, we get for all $t \in [0, T]$, $\lambda \in (0, 1]$, $\nu \in (0, \nu_0]$ with $\nu_0 \in (0, 1]$ that

$$
2\int_0^t \left\langle (\nu_0 - \nu)\left(\Psi_\lambda(X_\lambda^\nu(s)) + \lambda X_\lambda^\nu(s)\right), X_\lambda^\nu(s)\right\rangle_{F_{1,2,\nu_0}^*} ds
$$

$$
\leq 2\int_0^t \frac{\nu_0 - \nu}{\sqrt{\nu_0}} |\Psi_\lambda(X_\lambda^\nu(s)) + \lambda X_\lambda^\nu(s)|_2 \cdot \|X_\lambda^\nu(s)\|_{F_{1,2,\nu_0}^*} ds
$$

$$
\leq 2\int_0^t \sqrt{\nu_0}\left(|\Psi_\lambda(X_\lambda^\nu(s))|_2 + \lambda |X_\lambda^\nu(s)|_2\right) \cdot \|X_\lambda^\nu(s)\|_{F_{1,2,\nu_0}^*} ds
$$

$$
\leq 2\int_0^t |\Psi_\lambda(X_\lambda^\nu(s))|_2^2 + |X_\lambda^\nu(s)|_2^2 ds + \int_0^t \|X_\lambda^\nu(s)\|_{F_{1,2,\nu_0}^*}^2 ds
$$

$$
\leq C\int_0^t |X_\lambda^\nu(s)|_{2m}^{2m} + |X_\lambda^\nu(s)|_2^2 + \mu(E) \cdot 1_{\{\mu(E)<\infty\}} ds + \int_0^t \|X_\lambda^\nu(s)\|_{F_{1,2,\nu_0}^*}^2 ds. \tag{4.47}
$$

Taking expectation to both sides of Eq. 4.45, and taking Eqs. 4.46 and 4.47 into Eq. 4.45, by exactly the same arguments as in the proof of Eq. 4.40, except for using **(H2)(ii)** instead of **(H2)(i)**, we obtain

$$
\mathbb{E}\left[\sup_{s\in[0,T]} \|X_\lambda^\nu(s)\|_{F_{1,2,\nu_0}^*}^2\right] + \lambda \mathbb{E}\int_0^T |X_\lambda^\nu(s)|_2^2 ds
$$

$$
\leq C_T\left(\|x\|_{F_{1,2,\nu_0}^*}^2 + |x|_{2m}^{2m} + |x|_2^2 + \mu(E) \cdot 1_{\{\mu(E)<\infty\}}\right), \quad \forall \lambda \in (0, 1], \ \nu \in (0, \nu_0].
$$

Hence we get by Fatou's lemma

$$
\mathbb{E}\left[\sup_{t\in[0,T]} \|X_\lambda(t)\|_{F_{1,2,\nu_0}^*}^2\right] + \lambda \mathbb{E}\int_0^T |X_\lambda(s)|_2^2 ds
$$

$$
\leq C_T\left(\|x\|_{F_{1,2,\nu_0}^*}^2 + |x|_{2m}^{2m} + |x|_2^2 + \mu(E) \cdot 1_{\{\mu(E)<\infty\}}\right), \quad \forall \lambda \in (0, 1]. \tag{4.48}
$$

Letting $\nu_0 \to 0$ and taking Eq. 2.2 into account, we get

$$
\mathbb{E}\left[\sup_{t\in[0,T]} \|X_\lambda(t)\|_{\mathscr{F}_e^*}^2\right] + \lambda \mathbb{E}\int_0^T |X_\lambda(s)|_2^2 ds
$$

$$
\leq C_T\left(\|x\|_{\mathscr{F}_e^*}^2 + |x|_{2m}^{2m} + |x|_2^2 + \mu(E) \cdot 1_{\{\mu(E)<\infty\}}\right), \quad \forall \lambda \in (0, 1],
$$

hence Eq. 3.11 follows.

Now let us prove that X_λ is a solution to Eq. 3.4 with state space $\mathscr{F}_e^*$. By Eq. 3.7 and Lemma 2.2, we have

$$L \int_0^\cdot \Psi_\lambda(X_\lambda(s)) + \lambda X_\lambda(s)ds \in L^2(\Omega; C([0, T]; \mathscr{F}_e^*)).$$

Furthermore, letting $\nu \to 0$ in **(H2)(ii)**, we conclude from Eq. 4.48 that the stochastic integral in Eq. 3.8 is in $L^2(\Omega; C([0, T]; \mathscr{F}_e^*))$ as well. Since $x \in \mathscr{F}_e^*$, Eq. 3.8 (which holds in $F_{1,2}^*$) implies that $X_\lambda \in L^2(\Omega; C([0, T]; \mathscr{F}_e^*))$. So, altogether this implies that X_λ is a strong solution of Eq. 3.4 with state space $\mathscr{F}_e^*$ in the sense of Eqs. 3.6-3.8.

Now finally we prove Eq. 3.12. Firstly, we have

$$d(X_\lambda^\nu(t) - X_{\lambda'}^\nu(t)) + (\nu_0 - L)\big(\Psi_\lambda(X_\lambda^\nu(t)) - \Psi_{\lambda'}(X_{\lambda'}^\nu(t)) + \lambda X_\lambda^\nu(t) - \lambda' X_{\lambda'}^\nu(t)\big)dt$$
$$+(\nu - \nu_0)\big(\Psi_\lambda(X_\lambda^\nu(t)) - \Psi_{\lambda'}(X_{\lambda'}^\nu(t)) + \lambda X_\lambda^\nu(t) - \lambda' X_{\lambda'}^\nu(t)\big)dt$$
$$= \big(B(t, X_\lambda^\nu(t)) - B(t, X_{\lambda'}^\nu(t))\big)dW(t).$$

By Remark 4.1 we may apply Itô's formula ([26, Theorem 4.2.5] with $V := L^2(\mu)$, $H := F_{1,2,\nu_0}^*$) to $\frac{1}{2}\|X_\lambda^\nu - X_{\lambda'}^\nu\|_{F_{1,2,\nu_0}^*}^2$, to obtain for $\nu \in (0, \nu_0]$, $t \in [0, T]$,

$$\frac{1}{2}\|X_\lambda^\nu(t) - X_{\lambda'}^\nu(t)\|_{F_{1,2,\nu_0}^*}^2$$
$$+ \int_0^t \int_E \big(\Psi_\lambda(X_\lambda^\nu(s)) + \lambda X_\lambda^\nu(s) - \Psi_{\lambda'}(X_{\lambda'}^\nu(s)) - \lambda' X_{\lambda'}^\nu(s)\big) \cdot (X_\lambda^\nu(s) - X_{\lambda'}^\nu(s))d\mu ds$$
$$+(\nu-\nu_0)\int_0^t \big\langle\Psi_\lambda(X_\lambda^\nu(s)) + \lambda X_\lambda^\nu(s) - \Psi_{\lambda'}(X_{\lambda'}^\nu(s)) - \lambda' X_{\lambda'}^\nu(s), X_\lambda^\nu(s) - X_{\lambda'}^\nu(s)\big\rangle_{F_{1,2,\nu_0}^*} ds$$
$$= \frac{1}{2}\int_0^t \big\|B(s, X_\lambda^\nu(s)) - B(s, X_{\lambda'}^\nu(s))\big\|_{L_2(L^2(\mu), F_{1,2,\nu_0}^*)}^2 ds$$
$$+ \int_0^t \big\langle X_\lambda^\nu(s) - X_{\lambda'}^\nu(s), (B(s, X_\lambda^\nu(s)) - B(s, X_{\lambda'}^\nu(s)))dW(s)\big\rangle_{F_{1,2,\nu_0}^*}. \tag{4.49}$$

Since $r = \lambda\Psi_\lambda(r) + (I + \lambda\Psi)^{-1}(r)$, for all $r \in \mathbb{R}$, we have for all $r' \in \mathbb{R}$

$$(\Psi_\lambda(r) - \Psi_{\lambda'}(r'))(r - r') = \big[\Psi_\lambda(r) - \Psi_{\lambda'}(r')\big] \cdot \big[(I + \lambda\Psi)^{-1}(r) - (I + \lambda'\Psi)^{-1}(r')\big]$$
$$+\big[\Psi_\lambda(r) - \Psi_{\lambda'}(r')\big] \cdot \big[\lambda\Psi_\lambda(r) - \lambda'\Psi_{\lambda'}(r')\big]. \tag{4.50}$$

Note that the first summand in the right hand-side is nonnegative since Ψ is maximal monotone and since $\Psi_\lambda(r) \in \Psi((I + \lambda\Psi)^{-1}(r))$ (see [3, page:61]). Plugging Eq. 4.50 into Eq. 4.49, and using that $\|\cdot\|_{F_{1,2,\nu_0}^*} \le \frac{1}{\sqrt{\nu_0}}|\cdot|_2$ and **(H2)(i)**, we obtain for $\nu \in (0, \nu_0]$, $t \in [0, T]$

$$\frac{1}{2}\|X_\lambda^\nu(t) - X_{\lambda'}^\nu(t)\|_{F_{1,2,\nu_0}^*}^2$$
$$+ \int_0^t \int_E \big(\Psi_\lambda(X_\lambda^\nu(s)) - \Psi_{\lambda'}(X_{\lambda'}^\nu(s))\big) \cdot \big(\lambda\Psi_\lambda(X_\lambda^\nu(s)) - \lambda'\Psi_{\lambda'}(X_{\lambda'}^\nu(s))\big)d\mu ds$$
$$+ \int_0^t \int_E \big(\lambda X_\lambda^\nu(s) - \lambda' X_{\lambda'}^\nu(s)\big) \cdot \big(X_\lambda^\nu(s) - X_{\lambda'}^\nu(s)\big)d\mu ds$$
$$\le \frac{(\nu_0 - \nu)}{\sqrt{\nu_0}} \int_0^t \big|\Psi_\lambda(X_\lambda^\nu(s)) + \lambda X_\lambda^\nu(s) - \Psi_{\lambda'}(X_{\lambda'}^\nu(s)) - \lambda' X_{\lambda'}^\nu(s)\big|_2 \cdot \big\|X_\lambda^\nu(s) - X_{\lambda'}^\nu(s)\big\|_{F_{1,2,\nu_0}^*} ds$$

$$+ \frac{C_1}{2} \int_0^t \| X_\lambda^\nu(s) - X_{\lambda'}^\nu(s) \|^2_{F^*_{1,2,\nu_0}} \, ds$$

$$+ \int_0^t \left\langle X_\lambda^\nu(s) - X_{\lambda'}^\nu(s), (B(s, X_\lambda^\nu(s)) - B(s, X_{\lambda'}^\nu(s))) dW(s) \right\rangle_{F^*_{1,2,\nu_0}}.$$

By the BDG inequality for $p = 1$ we get for all $\lambda, \lambda' \in (0, 1], t \in [0, T]$

$$\frac{1}{4} \mathbb{E}\left[\sup_{s \in [0,t]} \| X_\lambda^\nu(s) - X_{\lambda'}^\nu(s) \|^2_{F^*_{1,2,\nu_0}} \right]$$

$$\leq C(\lambda + \lambda' + \nu_0)\mathbb{E} \int_0^t \left(|\Psi_\lambda(X_\lambda^\nu(s))|_2^2 + |\Psi_{\lambda'}(X_{\lambda'}^\nu(s))|_2^2 + |X_\lambda^\nu(s)|_2^2 + |X_{\lambda'}^\nu(s)|_2^2 \right) ds$$

$$+ C\mathbb{E} \int_0^t \sup_{r \in [0,s]} \| X_\lambda^\nu(r) - X_{\lambda'}^\nu(r) \|^2_{F^*_{1,2,\nu_0}} \, ds.$$

Hence by **(H1)**, Eq. 4.24 and Gronwall's lemma, there exists $C_T \in (0, \infty)$ independent of ν_0, such that for all $\nu \in (0, \nu_0], \lambda, \lambda' \in (0, 1]$,

$$\mathbb{E}\left[\sup_{t \in [0,T]} \| X_\lambda^\nu(t) - X_{\lambda'}^\nu(t) \|^2_{F^*_{1,2,\nu_0}} \right]$$

$$\leq C_T(\lambda + \lambda' + \nu_0)(|x|_2^2 + |x|_{2m}^{2m} + \mu(E) \cdot 1_{\{\mu(E) < \infty\}}).$$

Then letting $\nu \to 0$, we obtain

$$\mathbb{E}\left[\sup_{t \in [0,T]} \| X_\lambda(t) - X_{\lambda'}(t) \|^2_{F^*_{1,2,\nu_0}} \right]$$

$$\leq C_T(\lambda + \lambda' + \nu_0)(|x|_2^2 + |x|_{2m}^{2m} + \mu(E) \cdot 1_{\{\mu(E) < \infty\}}), \tag{4.51}$$

so by letting $\nu_0 \to 0$ in Eq. 4.51 and taking into account Eq. 2.2 we obtain Eq. 3.12. Consequently, Theorem 3.2 is proved.

5 Proof of Theorem 3.1

After all our preparations, to deduce that the solution X_λ, $\lambda \in (0, 1]$, of Eq. 3.4 as $\lambda \to 0$ converges to the unique solution of Eq. 1.1 is now in principle quite standard (at least for experts on stochastic porous media equations), maybe except for proving Eq. 3.3. Since, however, there is no proof in the literature that covers our general case, we give a complete presentation of the arguments in this section.

Proof Let X_λ be as in Theorem 3.2. Then it follows by Theorem 3.2 that there exists a process $X \in L^2(\Omega; C([0, T]; \mathscr{F}_e^*))$ such that, as $\lambda \to 0$,

$$X_\lambda \to X \quad \text{strongly in } L^2(\Omega; C([0, T]; \mathscr{F}_e^*)),$$

$$X_\lambda \to X \quad \text{weak-star in } L^\infty([0, T]; (L^2 \cap L^{2m})(\Omega \times E)) \subset L^\infty([0, T]; L^{m+1}(\Omega \times E)),$$

$$\lambda X_\lambda \to 0 \quad \text{strongly in } L^2([0, T] \times \Omega \times E), \tag{5.1}$$

$$\Psi_\lambda(X_\lambda) \to \eta \quad \text{weakly in } L^{\frac{2}{m}}([0, T] \times \Omega \times E) \cap L^2([0, T] \times \Omega \times E) \subset L^{\frac{m+1}{m}}([0, T] \times \Omega \times E).$$

Note that $\Psi_\lambda(X_\lambda) \to \eta$ weakly in $L^2([0, T] \times \Omega \times E)$ since Eq. 3.10, also in $L^{\frac{2}{m}}([0, T] \times \Omega \times E)$ since by Eqs. 3.5, 3.1, 4.17 and the weakly lower-semicontinuity of norms one has

$$\mathbb{E} \int_0^T \int_E |\Psi_\lambda(X_\lambda(t))|^{\frac{2}{m}} d\mu dt$$

$$\leq \mathbb{E} \int_0^T \int_E |\Psi(X_\lambda(t))|^{\frac{2}{m}} d\mu dt$$

$$\leq \mathbb{E} \int_0^T \int_E C^{\frac{2}{m}}(|X_\lambda(t)|^m + 1_{\{\mu(E)<\infty\}})^{\frac{2}{m}} d\mu dt$$

$$\leq \mathbb{E} \int_0^T \int_E C^{\frac{2}{m}} \cdot 2(|X_\lambda(t)|^2 + 1_{\{\mu(E)<\infty\}}) d\mu dt$$

$$\leq C_{m,T,C_3}(|x|_2^2 + 1_{\{\mu(E)<\infty\}}),$$

where C_{m,T,C_3} is dependent on m, T and C_3 in Eq. 4.17, but independent of λ. Therefore, $\Psi_\lambda(X_\lambda) \to \eta$ weakly in $L^{\frac{m+1}{m}}([0, T] \times \Omega \times E)$ by interpolation.

By Eq. 5.1 and **(H2)(i)** we may take the limit $\lambda \to 0$ in Eq. 3.8 in $L^2(\Omega; C([0, T]; \mathscr{F}_e^*))$ to obtain that

$$X = x + \lim_{\lambda \to 0} L \int_0^{\cdot} \Psi_\lambda(X_\lambda(s)) + \lambda X_\lambda(s)ds + \int_0^{\cdot} B(s, X(s))dW(s), \qquad (5.2)$$

where we have used that by Eq. 2.2 we may take the limit $\nu_0 \to 0$ in **(H2)(i)**. By Lemma 2.2 we conclude that

$$\lim_{\lambda \to 0} \int_0^{\cdot} \Psi_\lambda(X_\lambda(s)) + \lambda X_\lambda(s)ds \qquad (5.3)$$

exists in $L^2(\Omega; C([0, T]; \mathscr{F}_e))$, hence by **(L.1)** in $L^2(\Omega; C([0, T]; L^1(g \cdot \mu)))$ for some $g \in L^1(\mu) \cap L^\infty(\mu)$, $g > 0$. Hence the limit in Eq. 5.3 coincides with the limit in $dt \otimes d\mathbb{P} \otimes d\mu$-measure. Therefore, $\int_0^{\cdot} \eta(s)ds \in L^2(\Omega; C([0, T]; \mathscr{F}_e))$ and Eq. 5.2 implies

$$X(t) = x + L \int_0^t \eta(s)ds + \int_0^t B(s, X(s))dW(s), \quad t \in [0, T]. \qquad (5.4)$$

Hence $X(t)$, $t \in [0, T]$, is a solution to Eq. 1.1 in the sense of Definition 3.1 if we can show that

$$\eta \in \Psi(X), \quad dt \otimes \mathbb{P} \otimes \mu - a.e..$$

For this it suffices to show that

$$\lim_{\lambda \to 0} \sup \mathbb{E} \int_0^T \int_E \Psi_\lambda(X_\lambda)X_\lambda d\mu dt \leq \mathbb{E} \int_0^T \int_E \eta X d\mu dt. \qquad (5.5)$$

Indeed, since $\Psi_\lambda = \partial j_\lambda$, i.e. Ψ_λ is the subdifferential of j_λ (cf. [3, page:7]), where j_λ is as in Eq. 5.9, we have for all $\lambda \in (0, 1]$

$$\mathbb{E} \int_0^T \int_E \Psi_\lambda(X_\lambda)(X_\lambda - Z)d\mu dt \geq \mathbb{E} \int_0^T \int_E j_\lambda(X_\lambda) - j_\lambda(Z)d\mu dt, \qquad (5.6)$$

for all $Z \in L^{m+1}((0, T) \times \Omega \times E)$, since $X_\lambda \in (L^2 \cap L^{2m})((0, T) \times \Omega \times E) \subset L^{m+1}((0, T) \times \Omega \times E)$, $\Psi_\lambda(X_\lambda) \in (L^{\frac{2}{m}} \cap L^2)((0, T) \times \Omega \times E) \subset L^{\frac{m+1}{m}}((0, T) \times \Omega \times E)$.

Let $\Psi^0 : \mathbb{R} \to \mathbb{R}$ be defined by (see [3, page: 54]),

$$\Psi^0(r) = \{y \in \Psi(r); |y| = \inf\{|z|; z \in \Psi(r)\}\}, \quad \forall r \in \mathbb{R},$$

Ψ^0 is monotonically increasing. Define the integral (see [3, page:54])

$$j(r) := \int_0^r \Psi^0(s)ds, \quad \forall r \in \mathbb{R},$$

the function j is continuous on $(0, r)$ and convex on $\mathbb{R}$ satisfying $\Psi = \partial j$

$$0 \leq j(r) \leq r\Psi^0(r), \quad \forall r \in \mathbb{R}, \tag{5.7}$$

and from [3, page:41, Proposition 2.3]

$$\lim_{\lambda \to 0} \Psi_\lambda(r) = \Psi^0(r), \quad \forall r \in \mathbb{R}. \tag{5.8}$$

Recall from [3, page:48, Theorem 2.9] that $\Psi_\lambda = \partial j_\lambda$ with

$$j_\lambda(r) := \inf\left\{\frac{|r - \bar{r}|^2}{2\lambda} + j(\bar{r}); \bar{r} \in \mathbb{R}\right\}, \quad \forall r \in \mathbb{R}. \tag{5.9}$$

Moreover,

$$j_\lambda \geq 0,$$
$$\lim_{\lambda \to 0} j_\lambda(r) = j(r), \forall r \in \mathbb{R}, \tag{5.10}$$
$$j_\lambda(r) \leq j(r), \forall r \in \mathbb{R}. \tag{5.11}$$

Consequently, for all $Z \in L^{m+1}((0, T) \times \Omega \times E)$

$$\limsup_{\lambda \to 0} \mathbb{E}\int_0^T \int_E j_\lambda(X_\lambda) - j_\lambda(Z)d\mu dt \geq \mathbb{E}\int_0^T \int_E j(X) - j(Z)d\mu dt. \tag{5.12}$$

Indeed, by Eqs. 5.7, 5.11 and **(H1)**

$$|j_\lambda(z)| \leq C(|z|^{m+1} + |z| \cdot 1_{\{\mu(E)<\infty\}}),$$

hence by Lebesgue's dominated convergence theorem and Eq. 5.10

$$\lim_{\lambda \to 0} \mathbb{E}\int_0^T \int_E j_\lambda(Z)d\mu dt = \mathbb{E}\int_0^T \int_E j(Z)d\mu dt.$$

Furthermore, by Eqs. 3.5, 5.8, **(H1)** and because $X \in L^{2m}((0, T) \times \Omega \times E)$ we have as $\lambda \to 0$, $\Psi_\lambda(X) \to \Psi^0(X)$ in $L^2((0, T) \times \Omega \times E)$, hence

$$\limsup_{\lambda \to 0} \mathbb{E}\int_0^T \int_E j_\lambda(X_\lambda) - j(X)d\mu dt$$
$$= \limsup_{\lambda \to 0} \mathbb{E}\int_0^T \int_E j_\lambda(X_\lambda) - j_\lambda(X)d\mu dt$$
$$\geq \limsup_{\lambda \to 0} \mathbb{E}\int_0^T \int_E \Psi_\lambda(X)(X_\lambda - X)d\mu dt$$
$$= 0.$$

Hence by Eqs. 5.5, 5.6, and 5.12, we have $\forall Z \in L^{m+1}((0, T) \times \Omega \times E)$,

$$\mathbb{E}\int_0^T \int_E \eta(X - Z)d\mu dt \geq \mathbb{E}\int_0^T \int_E \left(j(X) - j(Z)\right)d\mu dt.$$

This yields

$$\mathbb{E}\int_0^T \int_E \eta(X-Z)d\mu dt \geq \mathbb{E}\int_0^T \int_E \zeta(X-Z)d\mu dt, \tag{5.13}$$

for all $Z \in L^{m+1}((0,T) \times \Omega \times E)$ and $\zeta \in L^{\frac{m+1}{m}}((0,T) \times \Omega \times E)$ such that $\zeta \in \Psi(Z)$ a.e. on $(0,T) \times \Omega \times E$.

By virtue of assumption **(H1)**, Ψ is maximal monotone in $L^{m+1}((0,T) \times \Omega \times E) \times L^{\frac{m+1}{m}}((0,T) \times \Omega \times E)$, so by [3, page:34, Theorem 2.2] one knows that the equation

$$J(Z) + \Psi(Z) \ni J(X) + \eta, \tag{5.14}$$

where $J(Z) = |Z|^{m-1}Z$, has a unique solution $Z \in L^{m+1}((0,T) \times \Omega \times E)$.

Now if in Eq. 5.13, we take Z to be the solution of Eq. 5.14 and $\zeta := J(X) - J(Z) + \eta$, we obtain

$$\mathbb{E}\int_0^T \int_E (J(X) - J(Z))(X-Z)d\mu dt \leq 0,$$

i.e.,

$$\mathbb{E}\int_0^T \int_E (|X|^{m-1}X - |Z|^{m-1}Z)(X-Z)d\mu dt \leq 0.$$

Since $J : r \to |r|^{m-1}r$ is strictly increasing, it follows that

$$\mathbb{E}\int_0^T \int_E ((|X|^{m-1}X - |Z|^{m-1}Z)(X-Z)d\mu dt = 0.$$

Hence $X = Z$ a.e. on $(0,T) \times \Omega \times E$, and thus by Eq. 5.14, we have $\eta \in \Psi(X)$, $\mathbb{P}\otimes dt \otimes \mu$, $a.e.$.

It remains to prove Eq. 5.5. In order to apply the Itô formula from [26, Theorem 4.2.5] to the equations Eqs. 3.4 and 5.4, we need to use an appropriate Gelfand triple. Recall a special case of [34, Proposition 3.1] that

$$\mathscr{F}_e \cap L^{\frac{m+1}{m}}(\mu) \text{ is dense both in } \mathscr{F}_e \text{ and } L^{\frac{m+1}{m}}(\mu). \tag{5.15}$$

Define

$$V := \{u \in L^{m+1}(\mu) | \exists \, C \in (0, \infty) \text{ such that } |\mu(uv)| \leq C\|v\|_{\mathscr{F}_e}, \, \forall \, v \in \mathscr{F}_e \cap L^{\frac{m+1}{m}}(\mu)\}.$$

By Eq. 5.15, V is a subspace of $\mathscr{F}_e^*$ and can be symbolically written as $V = L^{m+1}(\mu) \cap \mathscr{F}_e^*$. We note that V is reflexive, since $L^{m+1}(\mu)$ and $\mathscr{F}_e^*$ are reflexive, hence so is $L^{m+1}(\mu) \times \mathscr{F}_e^*$. But

$$V \ni u \mapsto (u, \mu(u \cdot)) \in L^{m+1} \times \mathscr{F}_e^*$$

is a homeomorphic isomorphism, mapping V onto a closed subspace of $L^{m+1} \times \mathscr{F}_e^*$, which is reflexive.

Recall special cases of [34, Proposition 3.1] and [34, Lemma 3.3(ii)] that

$$V \text{ is dense both in } \mathscr{F}_e^* \text{ and } L^{m+1}(\mu), \tag{5.16}$$

and for the map $L := \overline{L} : \mathscr{F}_e \to \mathscr{F}_e^*$ defined in Lemma 2.2 we have for all $v \in \mathscr{F}_e \cap L^{\frac{m+1}{m}}(\mu)$, $u \in V$,

$$\langle Lv, u \rangle_{\mathscr{F}_e^*} = -\mu(vu).$$

Now we set $H := \mathscr{F}_e^*$ and consider the Gelfand triple

$$V \subset H \subset V^*.$$

Consider the operator

$$L : \mathscr{F}_e \cap L^{\frac{m+1}{m}}(\mu) \to \mathscr{F}_e^* \subset V^*,$$

as V^*-valued, i.e., $Lv = -\mu(v \cdot) \in V^*$. Then by Eq. 5.16, L is continuous w.r.t. the norm $|\cdot|_{\frac{m+1}{m}}$ on $\mathscr{F}_e \cap L^{\frac{m+1}{m}}(\mu)$, hence by Eq. 5.15 has a unique continuous linear extension

$$\widetilde{L} : L^{\frac{m+1}{m}}(\mu) \to V^*,$$

such that

$$_{V^*}\langle \widetilde{L}v, u \rangle_V = \mu(vu), \quad \forall v \in L^{\frac{m+1}{m}}(\mu), u \in V. \tag{5.17}$$

Now we consider Eq. 5.4 in the large space V^*. Then, since $\eta \in L^{\frac{m+1}{m}}([0,T] \times \Omega \times E)$,

$$L \int_0^t \eta(s)ds = \widetilde{L} \int_0^t \eta(s)ds = \int_0^t \widetilde{L}\eta(s)ds \in V^*,$$

so

$$X(t) = x + \int_0^t \widetilde{L}\eta(s)ds + \int_0^t B(s, X(s))dW(s), t \in [0,T],$$

hence by the Itô formula from [26, Theorem 4.2.5] with the Gelfand triple $V = L^{m+1}(\mu) \cap \mathscr{F}_e^* \subset \mathscr{F}_e^* \subset V^*$, we have

$$\frac{1}{2}\mathbb{E}\|X(t)\|_{\mathscr{F}_e^*}^2 + \mathbb{E}\int_0^t \int_E \eta \cdot X(s)d\mu ds$$
$$= \frac{1}{2}\|x\|_{\mathscr{F}_e^*}^2 + \frac{1}{2}\mathbb{E}\int_0^t \|B(s, X(s))\|_{L_2(L^2(\mu), \mathscr{F}_e^*)}^2 ds. \tag{5.18}$$

By similar arguments (a special case with $m = 1$) we get for Eq. 3.4

$$X_\lambda(t) = x + \int_0^t \widetilde{L}(\Psi_\lambda(X_\lambda(s)) + \lambda X_\lambda(s))ds + \int_0^t B(s, X_\lambda(s))dW(s), t \in [0,T],$$

hence

$$\frac{1}{2}\mathbb{E}\|X_\lambda(t)\|_{\mathscr{F}_e^*}^2 - \mathbb{E}\int_0^t {}_{V^*}\langle \widetilde{L}(\Psi_\lambda(X_\lambda(s)) + \lambda X_\lambda(s)), X_\lambda(s) \rangle_V ds$$
$$= \frac{1}{2}\|x\|_{\mathscr{F}_e^*}^2 + \frac{1}{2}\mathbb{E}\int_0^t \|B(s, X_\lambda(s))\|_{L_2(L^2(\mu), \mathscr{F}_e^*)}^2 ds. \tag{5.19}$$

By Eq. 5.17, we know that

$$- {}_{V^*}\langle \widetilde{L}(\Psi_\lambda(X_\lambda(s)) + \lambda X_\lambda(s)), X_\lambda(s) \rangle_V = \int_E (\Psi_\lambda(X_\lambda(s)) + \lambda X_\lambda(s)) \cdot X_\lambda(s)d\mu \tag{5.20}$$

Letting $\lambda \to 0$ in Eq. 5.19 after plugging in Eq. 5.20, using Eq. 5.2 and comparing with Eq. 5.18, we obtain Eq. 5.5 (even with "=" replacing "≤").

(**Uniqueness**) Suppose X_1, X_2 are two strong solutions to Eq. 1.1. We have with $\widetilde{L}$ that

$$\begin{cases} d(X_1 - X_2) - \widetilde{L}(\eta_1 - \eta_2)dt = (B(t, X_1) - B(t, X_2))dW(t), \text{ in } [0,T] \times E, \\ X_1 - X_2 = 0 \text{ on } E, \end{cases}$$

where $\eta_i \in \Psi(X_i)$, $i = 1, 2$, a.e. on $\Omega \times (0, T) \times E$.

As above we may apply Itô's formula from [26, Theorem 4.2.5] with the same Gelfand triple as used in Eq. 5.18 to get

$$\frac{1}{2}d\|X_1 - X_2\|_{\mathscr{F}_e^*}^2 - {}_{V^*}\langle \widetilde{L}(\eta_1 - \eta_2), X_1 - X_2\rangle_V dt$$
$$= \frac{1}{2}\|B(t, X_1) - B(t, X_2)\|_{L_2(L^2(\mu), \mathscr{F}_e^*)}^2 dt$$
$$+ \langle X_1 - X_2, (B(s, X_1) - B(s, X_2))dW_t\rangle_{\mathscr{F}_e^*}. \tag{5.21}$$

Since Ψ is monotone, by Eq. 5.17 we have

$$\mathbb{E}\int_0^T {}_{V^*}\langle -\widetilde{L}(\eta_1 - \eta_2), X_1 - X_2\rangle_V dt$$
$$= \mathbb{E}\int_0^T \int_E (\eta_1 - \eta_2) \cdot (X_1 - X_2)d\mu dt \geq 0. \tag{5.22}$$

Therefore, integrating Eq. 5.21 from 0 to t and taking expectation, by Eq. 5.22 and Remark 3.1 (i), we obtain

$$\mathbb{E}\|X_1 - X_2\|_{\mathscr{F}_e^*}^2 \leq C_1 \int_0^t \mathbb{E}\|X_1 - X_2\|_{\mathscr{F}_e^*}^2 ds, \quad \forall t \in [0, T],$$

and by Gronwall's inequality we get $X_1 = X_2$, $\mathbb{P}$−a.s.. Thus Theorem 3.1 is proved.

6 Applications

Example 6.1 (Example for Ψ) Recall a known fact from [3, page:54]: if $\tilde{\Psi} : \mathbb{R} \to \mathbb{R}$ is an increasing function and if $\{r_i | i \in \mathbb{N}\} \subset \mathbb{R}$ is the set of all $r \in \mathbb{R}$ for which $\tilde{\Psi}$ is discontinuous in r, then one gets a maximal monotone multivalued function $\Psi : \mathbb{R} \to 2^{\mathbb{R}}$ by filling the gaps, i.e., define

$$\Psi(r) := \begin{cases} \tilde{\Psi}(r), & \text{for } r \notin \{r_i | i \in \mathbb{N}\}, \\ [\tilde{\Psi}(r_j - 0), \tilde{\Psi}(r_j + 0)], & \text{else}. \end{cases}$$

A concrete example of maximal monotone multivalued Ψ satisfying **(H1)** could be constructed as following: for $m \in [1, \infty)$, define

$$\Psi(r) := \begin{cases} |r + 2|^{m-1}(r + 2) - 2, & \text{if } r \in (-\infty, -2], \\ -2, & \text{if } r \in (-2, -1), \\ [-2, -1], & \text{if } r = -1, \\ |r|^{m-1}r, & \text{if } r \in (-1, 1), \\ [1, 2], & \text{if } r = 1, \\ 2, & \text{if } r \in (1, 2), \\ |r - 2|^{m-1}(r - 2) + 2, & \text{if } r \in [2, \infty). \end{cases}$$

Hence our result covers non-continuous nonlinearities Ψ, as is indicated in the title of the paper.

Another typical example for Ψ satisfying **(H1)** is

$$\Psi(r) := |r|^\gamma \text{sign}(r), \quad \forall r \in \mathbb{R}, \quad \gamma \in [0, 1],$$

where sign : $\mathbb{R} \to 2^{\mathbb{R}}$ is defined as

$$\text{sign}(r) := \begin{cases} 1, & \text{if } r > 0, \\ [-1, 1], & \text{if } r = 0, \\ -1, & \text{if } r < 0, \end{cases}$$

if $L = \Delta$ on an open bounded domain of $\mathbb{R}^d$ ($d \geq 3$), then Eq. 1.1 corresponds to the fast diffusion equation (FDE) perturbed by multiplicative noise and the special case $\gamma = 0$ is related to the SOC model (see [2]) mentioned in the introduction. Applications of the FDE have been proposed in different areas. The equation appears in plasma physics. One behavior of the solution to FDE is it decays to zero in finite time. The equation in dimensions $d \geq 3$ with exponent $\gamma = \frac{d-2}{d+2}$ has an important application in geometry (the Yamabe flow) (see [40, Section 7.5]), while the Okuda-Dawson law corresponds to $\gamma = \frac{1}{2}$ (see [30]), for more mathematical studies of the (stochastic) FDE we refer to [17, 20, 34] and references therein.

Example 6.2 (Example for B) For the convenience of the reader, here we recall that there is a standard large class of example for B already described in [35, page: 3914, Remark 2.9(iii)], which satisfy **(H2)** and **(H3)**. Let $N \in \mathbb{N} \cup \{+\infty\}$ and $e_k \in L^2(\mu) \cap L^\infty(\mu)$, $1 \leq k < N$, be an orthonormal system in $L^2(\mu)$ such that for every $1 \leq k < N$ there exists $\xi_k \in (0, \infty)$ such that for all $\nu \in (0, \infty)$

$$|_{F_{1,2}^*}\langle x, e_k u \rangle_{F_{1,2}}| \leq \xi_k \|x\|_{F_{1,2,\nu}^*} \|u\|_{F_{1,2,\nu}}, \quad \forall u \in F_{1,2}, \ x \in F_{1,2}^*.$$

The above assumption means that each e_k is a multiplier in $(F_{1,2}^*, \| \cdot \|_{F_{1,2,\nu}^*})$ with norm independent of $\nu > 0$. Choose $\mu_k \in (0, \infty)$ such that

$$\sum_{k=1}^{\infty} \mu_k^2 (\xi_k^2 + |e_k|_\infty^2) < \infty, \tag{6.1}$$

and define for $x \in F_{1,2}^*$, $B(x) \in L_2(L^2(\mu), F_{1,2}^*)$ by

$$B(x)h := \sum_{k=1}^{\infty} \mu_k \langle e_k, h \rangle x \cdot e_k, \quad h \in L^2(\mu). \tag{6.2}$$

By extending $\{e_k | k \in \mathbb{N}\}$ to an orthonormal basis of $L^2(\mu)$, we can get **(H2)(ii)** as in [35, page:3915], **(H2)(i)** follows from the linearity of $B(x)$. From Eqs. 6.1 and 6.2, **(H3)(i)** and **(H3)(ii)** are obvious.

Example 6.3 (Example for local $\mathscr{E}$) Suppose $(\mathscr{E}, F_{1,2})$ is a local transient Dirichlet form with generator L such that it admits a carré du champ Γ ([11, Definition 4.1.2]), which is a unique positive symmetric and continuous bilinear map from $F_{1,2} \times F_{1,2}$ into $L^1(\mu)$ such that

$$\mathscr{E}(uw, v) + \mathscr{E}(vw, u) - \mathscr{E}(w, uv) = \int w \Gamma(u, v) d\mu, \ \forall u, v, w \in F_{1,2}.$$

From [11, Proposition 6.1.1], we know that then

$$\mathscr{E}(u, v) = \frac{1}{2} \int \Gamma(u, v) d\mu, \quad u, v \in F_{1,2}, \tag{6.3}$$

which implies **(H4)(i)**.

By [11, Corollary 7.1.2], we know that for every Lipschitz function $\varphi : \mathbb{R} \to \mathbb{R}$ which satisfies $\varphi(0) = 0$,

$$\Gamma(\varphi(u), v) = \varphi'(u)\Gamma(u, v), \quad \forall\, u, v \in F_{1,2},$$

where φ' is any version of the derivatives (defined Lebesgue-a.e.) of φ. Furthermore, if φ is nondecreasing, then

$$\Gamma(u, \varphi(u)) = \varphi'(u)\Gamma(u, u) \geq 0, \quad \forall\, u \in F_{1,2},$$

and

$$\Gamma(\varphi(u), \varphi(u)) = \varphi'(u)\Gamma(u, \varphi(u)) \leq esssup_{r \in \mathbb{R}}\, \varphi'(r)\Gamma(u, \varphi(u)), \quad \forall\, u \in F_{1,2},$$

which implies **(H4)(ii)**.

There is an abundance of examples of such Dirichlet forms on very general state spaces E, as e.g. finite or infinite dimensional manifolds. A typical example is the Ornstein- Uhlenbeck operator on the Wiener space ([11, page:83, Proposition 2.3.2]), for more details and examples we refer e.g. to [11, 15, 28] and also [21]. In the following, we briefly cite an example of manifold-valued transient Dirichlet form from [32].

Let $\widetilde{M}$ be a C^∞ manifold and let μ be a measure with positive C^∞ density with respect to the Lebesgue measure on each local chart. If $(\Xi_i; 1 \leq i \leq q)$ are C^∞ vector fields, we can consider the operator

$$\Gamma(u, v) = \sum_{i=1}^{q}(\Xi_i u)(\Xi_i v)$$

and the form $\widetilde{\mathscr{E}}$ associated by Eq. 6.3. We can close this form and obtain a Dirichlet space $\widetilde{\mathbb{D}}$. If now M is an open subset of $\widetilde{M}$, one can let $(\mathscr{E}, \mathbb{D})$ be the part of $(\widetilde{\mathscr{E}}, \widetilde{\mathbb{D}})$ on M; this is the subset of functions, a quasicontinuous modification of which is zero on $\widetilde{M}\backslash M$. If $\widetilde{\mathscr{E}}$ is irreducible and $\widetilde{M}\backslash M$ has positive capacity, then $\mathscr{E}$ is transient. For more details and explanations, we refer to [32, page:57, Example 1] (see also [32, page:57, Example 2] for the example on Riemannian manifold).

Example 6.4 (Example for nonlocal $\mathscr{E}$) As is well-known, under quite general assumptions according to the Beurling-Deny representation formula a Dirichlet can be written as the sum of a local Dirichlet form $\mathscr{E}^{(1)}$ (i.e. if it has a square field operator, it satisfies Eq. 6.4) and a non-local part $\mathscr{E}^{(2)}$ (see [15, Section 3.2] or [22] for details). A typical form of $\mathscr{E}^{(2)}$ is as follows

$$\mathscr{E}^{(2)}(u, v) = \int_{E}\int_{E}(u(x) - u(y))(v(x) - v(y))J(x, dy)\mu(dx), \quad u, v \in D(\mathscr{E}^{(2)}),$$

where J is a kernel on $(E, \mathcal{B})$ and μ is a σ-finite measure on $(E, \mathcal{B})$. Therefore,

$$\mathscr{E}^{(2)}(u, v) = \frac{1}{2}\int_{E}\Gamma(u, v)d\mu, \quad u, v \in D(\mathscr{E}^{(2)}),$$

where for $x \in E$

$$\Gamma(u, v)(x) = 2\int_{E}(u(x) - u(y))(v(x) - v(y))J(x, dy).$$

Clearly, Γ does not satisfy Eq. 6.4, but it satisfies our condition **(H4)(ii)**. Indeed, for every non-decreasing Lipschitz function $\varphi : \mathbb{R} \to \mathbb{R}$ with $\varphi(0) = 0$ and $u \in D(\mathscr{E}^{(2)})$ we have

$$
\begin{aligned}
\Gamma(\varphi(u), \varphi(u))(x) &= 2 \int_E \left(\varphi(u(x)) - \varphi(u(y)) \right)^2 J(x, dy) \\
&\leq 2 Lip\varphi \Big(\int_E 1_{\{u(x) \geq u(y)\}}(u(x) - u(y))\big(\varphi(u(x)) - \varphi(u(y))\big) J(x, dy) \\
&\quad + \int_E 1_{\{u(x) < u(y)\}}(u(y) - u(x))\big(\varphi(u(y)) - \varphi(u(x))\big) J(x, dy) \Big) \\
&= 2 Lip\varphi \Gamma(u, \varphi(u)).
\end{aligned}
$$

There are plenty of examples of such Dirichlet forms, not only when the corresponding state space $E := \mathbb{R}^d$, but also when E is a general state space, as e.g. a fractal. Concrete Dirichlet forms on $\mathbb{R}^d$ will be given in Example 6.5, while in this section, we recall an example from [19] with E being a Sierpiński graph.

Let $c : E \times E \to [0, \infty)$ be conductance satisfying

$$
\begin{aligned}
c(x, y) &= c(y, x), \\
\pi(x) &= \sum_{y \in E} c(x, y) \in (0, +\infty), \\
c(x, y) &> 0 \Leftrightarrow x \sim y,
\end{aligned}
$$

for all $x, y \in E$. Let $\mu : E \to (0, \infty)$ be a positive function given by

$$
\mu(x) = \Big(\frac{c}{3\lambda}\Big)^{|x|}, \quad x \in E,
$$

where $c \in (0, \lambda) \subseteq (0, 1)$. Then μ can be regarded as a finite measure on E. Set a symmetric form on $L^2(E; \mu)$ by

$$
\begin{cases}
\mathscr{E}(u, v) = \frac{1}{2} \sum_{x, y \in E} c(x, y)(u(x) - u(y))(v(x) - v(y)), \\
D(\mathscr{E}) = \text{the } (\mathscr{E})_1 - \text{closure of } C_0(E),
\end{cases}
$$

where $C_0(E)$ is the set of all functions with finite support. Then, $(\mathscr{E}, D(\mathscr{E}))$ is a regular Dirichlet form on $L^2(E; \mu)$ and it is transient. For the detailed proofs and more information about Sierpiński gasket, we refer to [19] and references therein.

Example 6.5 (Nonlocal $\mathscr{E}$ on $\mathbb{R}^d$) Based on [15, page: 31, Example 1.4.1], [15, page:48, Example 1.5.2] and Example 6.4, one can give a large class of examples of transient nonlocal Dirichlet forms on $\mathbb{R}^d$ satisfying assumptions **(H4)**. Here we only briefly present the main framework. Let $\{\nu_t, t > 0\}$ on $\mathbb{R}^d$ be a continuous symmetric convolution semigroup of probability measures, i.e.,

$$
\begin{aligned}
\nu_t * \nu_s &= \nu_{t+s}, \quad t, s > 0, \\
\nu_t(A) &= \nu_t(-A), \quad A \in \mathcal{B}(\mathbb{R}^d), \\
\lim_{t \downarrow 0} \nu_t &= \delta \quad \text{weakly},
\end{aligned}
$$

where $\nu_t * \nu_s(A)$ denotes the convolution $\int_{\mathbb{R}^d} \nu_t(A - y)\nu_s(dy)$ and δ is the Dirac measure concentrated at the origin. The celebrated Lévy-Khinchin formula [23] reads as follows

$$
\begin{cases}
\hat{\nu}_t(x)\left(= \int_{\mathbb{R}^d} e^{i(x,y)} \nu_t(dy)\right) = e^{-t\psi(x)}, \\
\psi(x) = \frac{1}{2}(Sx, x) + \int_{\mathbb{R}^d}(1 - \cos(x, y))J(dy),
\end{cases}
\tag{6.4}
$$

where

S is a non-negative definite $d \times d$ symmetric matrix, (6.5)

J is a symmetric measure on $\mathbb{R}^d \setminus \{0\}$ such that $\displaystyle\int_{\mathbb{R}^d \setminus \{0\}} \frac{|x|^2}{1 + |x|^2} J(dx) < \infty.$ (6.6)

Thus a continuous symmetric convolution semigroup $\{\nu_t, t > 0\}$ is characterized by a pair (S, J) satisfying Eqs. 6.5 and 6.6 through the formula Eq. 6.4.

When S vanishes, the Dirichlet form $\mathscr{E}$ on $L^2(\mathbb{R}^d)$ determined by $\{\nu_t, t > 0\}$ is

$$
\mathscr{E}(u, v) = \frac{1}{2} \int_{\mathbb{R}^d \times (\mathbb{R}^d \setminus \{0\})} \big(u(x + y) - u(x)\big)\big(v(x + y) - v(x)\big)J(dy)dx
\tag{6.7}
$$

$$
D(\mathscr{E}) = \Big\{u \in L^2(\mathbb{R}^d): \int_{\mathbb{R}^d \times (\mathbb{R}^d \setminus \{0\})} \big(u(x + y) - u(x)\big)\big(v(x + y) - v(x)\big)J(dy)dx < \infty\Big\}. \tag{6.8}
$$

Note that J can be extended to a symmetric measure on $\mathbb{R}^d$ by setting $J(\{0\}) = 0$. From [15, page:48, Example 1.5.2] we know that, if $\frac{1}{\psi(x)}$ is locally integrable on $\mathbb{R}^d$, then $\mathscr{E}$ is transient.

A typical example (see [15, page:49]) is $\psi(x) = |x|^\alpha$ with $0 < \alpha \le 2$, which corresponds to

$$
S = 0 \quad \text{and} \quad J(dy) = \frac{\alpha 2^{\alpha-1}\Gamma(\frac{\alpha+d}{2})}{\pi^{d/2}\Gamma(\frac{2-\alpha}{2})}|y|^{-d-\alpha}dy,
$$

then the Dirichlet form $\mathscr{E}$ is transient if and only if $\alpha < d$. This example can be also described in the following way. Let "^" resp. "ˇ" denote Fourier transform, i.e.,

$$
\hat{\phi}(x) = (2\pi)^{-\frac{d}{2}} \int \exp[i \langle x, y \rangle_{\mathbb{R}^d}]\phi(y)dy,
$$

resp. its inverse. Define for $\alpha > 0$

$$
(-\Delta)^{\frac{\alpha}{2}} u := \big(|x|^\alpha \hat{u}\big)^{\check{}} \quad (\in L^2(\mathbb{R}^d; dx)), \quad u \in C_0^\infty(\mathbb{R}^d).
\tag{6.9}
$$

Then $(-\Delta)^{\frac{\alpha}{2}}$ is a symmetric linear operator on $L^2(\mathbb{R}^d; dx)$ with dense domain $C_0^\infty(\mathbb{R}^d)$. Then $(\mathscr{E}, D(\mathscr{E}))$ in Eqs. 6.7 and 6.8 is the closure of the form

$$
\mathbb{D}^{(\frac{\alpha}{2})}(u, v) := \frac{1}{2} \int \hat{u}\overline{\hat{v}}|x|^\alpha dx, \quad u, v \in C_0^\infty(\mathbb{R}^d).
$$

For more details we refer to [28, page:43].

Finally, we would like to mention that if in Eq. 6.4 we choose $\psi(x) := f(|x|^2)$, where f is a Bernstein function (cf. [37]) such that $\frac{1}{f(|x|^2)}$ is locally integrable, then in Eq. 6.9 we get $f(-\Delta)$ instead of $(-\Delta)^{\frac{\alpha}{2}}$, i.e., $L = -f(-\Delta)$ in Eq. 1.1.

Appendix

A.1 Auxiliary Results

In this part we aim to prove Eq. A.4, which has been used in the proof of Claims 4.1 and 4.2.

Lemma A.1 *Let $v, \varepsilon, \lambda \in (0, 1]$. For all $x \in F_{1,2}^*$, we have*

$$\big\langle (v - L)((\Psi_\lambda + \lambda I)(J_\varepsilon(x))), x \big\rangle_{F_{1,2,v}^*}$$
$$= \big\langle (\Psi_\lambda + \lambda I)(J_\varepsilon(x)), J_\varepsilon(x) \big\rangle_2 + \varepsilon \|(v - L)((\Psi_\lambda + \lambda I)(J_\varepsilon(x)))\|_{F_{1,2,v}^*}^2. \quad \text{(A.1)}$$

For all $x \in L^2(\mu)$,

$$\big\langle (v - L)(\Psi_\lambda + \lambda I)(J_\varepsilon(x)), x \big\rangle_2$$
$$= \big\langle (v - L)(\Psi_\lambda + \lambda I)(J_\varepsilon(x)), J_\varepsilon(x) \big\rangle_2 + \varepsilon \big| (v - L)(\Psi_\lambda + \lambda I)(J_\varepsilon(x)) \big|_2^2. \quad \text{(A.2)}$$

Proof Recall from Eq. 4.4 that

$$J_\varepsilon(x) + \varepsilon(v - L)\big((\Psi_\lambda + \lambda I)(J_\varepsilon(x))\big) = x, \quad \forall x \in F_{1,2}.$$

For $x \in F_{1,2}^*$, to prove Eq. A.1, we rewrite

$$\big\langle (v - L)((\Psi_\lambda + \lambda I)(J_\varepsilon(x))), x \big\rangle_{F_{1,2,v}^*}$$
$$= \big\langle (v - L)((\Psi_\lambda + \lambda I)(J_\varepsilon(x))), J_\varepsilon(x) \big\rangle_{F_{1,2,v}^*}$$
$$+ \big\langle (v - L)((\Psi_\lambda + \lambda I)(J_\varepsilon(x))), \varepsilon(v - L)\big((\Psi_\lambda + \lambda I)(J_\varepsilon(x))\big) \big\rangle_{F_{1,2,v}^*}$$
$$= \big\langle (\Psi_\lambda + \lambda I)(J_\varepsilon(x)), J_\varepsilon(x) \big\rangle_2 + \varepsilon \|(v - L)((\Psi_\lambda + \lambda I)(J_\varepsilon(x)))\|_{F_{1,2,v}^*}^2.$$

The proof of Eq. A.2 is analogous due to the fact that J_ε is $\frac{1}{\sqrt{v\varepsilon\lambda}}$-Lipschitz in $L^2(\mu)$, so $A_\lambda^{v,\varepsilon} \in L^2(\mu)$ if $x \in L^2(\mu)$.

Lemma A.2 *Let $x \in L^2(\mu)$. Then for $v, \varepsilon, \lambda \in (0, 1]$, $t \in [0, T]$, we have*

$$\mathbb{E}|X_\lambda^{v,\varepsilon}(t)|_2^2 + 2\mathbb{E}\int_0^t \big\langle (v - L)(\Psi_\lambda + \lambda I)(J_\varepsilon(X_\lambda^{v,\varepsilon}(s))), J_\varepsilon(X_\lambda^{v,\varepsilon}(s)) \big\rangle_2 ds \le e^{C_3 T}|x|_2^2. \text{(A.3)}$$

Proof Applying Itô's formula to $|X_\lambda^{v,\varepsilon}|_2^2$, we obtain

$$d|X_\lambda^{v,\varepsilon}(t)|_2^2 + 2\big\langle (v - L)((\Psi_\lambda + \lambda I)(J_\varepsilon(X_\lambda^{v,\varepsilon}(t)))), (X_\lambda^{v,\varepsilon}(t)) \big\rangle_2 dt$$
$$= \|B(t, X_\lambda^{v,\varepsilon}(t))\|_{L_2(L^2(\mu), L^2(\mu))}^2 dt + 2\big\langle X_\lambda^{v,\varepsilon}, B(t, X_\lambda^{v,\varepsilon}(t))dW(t) \big\rangle_2$$

which by Eq. A.2 yields,

$$d|X_\lambda^{v,\varepsilon}(t)|_2^2 + 2\big\langle (v - L)((\Psi_\lambda + \lambda I)(J_\varepsilon(X_\lambda^{v,\varepsilon}(t)))), J_\varepsilon(X_\lambda^{v,\varepsilon}(t)) \big\rangle_2 dt$$
$$+ 2\varepsilon \big| (v - L)((\Psi_\lambda + \lambda I)(J_\varepsilon(X_\lambda^{v,\varepsilon}(t)))) \big|_2^2 dt$$
$$= \|B(t, X_\lambda^{v,\varepsilon}(t))\|_{L_2(L^2(\mu), L^2(\mu))}^2 dt + 2\big\langle X_\lambda^{v,\varepsilon}, B(t, X_\lambda^{v,\varepsilon}(t))dW(t) \big\rangle_2.$$

Taking expectation of both sides, by **(H3)(i)** we get

$$\mathbb{E}|X_\lambda^{\nu,\varepsilon}(t)|_2^2 + 2\mathbb{E}\int_0^t \big\langle(\nu - L)((\Psi_\lambda + \lambda I)(J_\varepsilon(X_\lambda^{\nu,\varepsilon}(s)))), J_\varepsilon(X_\lambda^{\nu,\varepsilon}(s))\big\rangle_2 ds$$

$$+2\varepsilon\mathbb{E}\int_0^t \big|(\nu - L)((\Psi_\lambda + \lambda I)(J_\varepsilon(X_\lambda^{\nu,\varepsilon}(s))))\big|_2^2 ds$$

$$\leq |x|_2^2 + C_3\mathbb{E}\int_0^t |X_\lambda^{\nu,\varepsilon}(s)|_2^2 ds.$$

Then by Eq. 4.16 and Gronwall's lemma we get Eq. A.3 as claimed.

Proposition A.1 *Let $x \in L^2(\mu)$. Then for $\nu, \varepsilon, \lambda \in (0, 1]$, $t \in [0, T]$, we have*

$$\mathbb{E}\int_0^t \big\|(\nu - L)\big((\Psi_\lambda + \lambda I)(J_\varepsilon(X_\lambda^{\nu,\varepsilon}(s)))\big)\big\|_{F_{1,2,\nu}^*}^2 ds \leq \frac{1}{2}\Big(\frac{1}{\lambda} + \lambda + C_5\Big)e^{C_3 T}|x|_2^2. \quad \text{(A.4)}$$

Proof Let $x \in L^2(\mu)$. Then

$$\big\|(\nu - L)\big((\Psi_\lambda + \lambda I)(J_\varepsilon(x))\big)\big\|_{F_{1,2,\nu}^*}^2$$

$$= \|(\Psi_\lambda + \lambda I)(J_\varepsilon(x))\|_{F_{1,2,\nu}}^2$$

$$= \int \frac{1}{2}\Gamma\big((\Psi_\lambda + \lambda I)(J_\varepsilon(x)), (\Psi_\lambda + \lambda I)(J_\varepsilon(x))\big)d\mu$$

$$+\nu\big\langle(\Psi_\lambda + \lambda I)(J_\varepsilon(x)), (\Psi_\lambda + \lambda I)(J_\varepsilon(x))\big\rangle_2$$

$$\leq C_5 \int \frac{1}{2}\Gamma\big(J_\varepsilon(x), (\Psi_\lambda + \lambda I)(J_\varepsilon(x))\big)d\mu + \nu\big(\frac{1}{\lambda} + \lambda\big)\big\langle(\Psi_\lambda + \lambda I)(J_\varepsilon(x)), J_\varepsilon(x)\big\rangle_2$$

$$\leq \big(\frac{1}{\lambda} + \lambda + C_5\big)\langle J_\varepsilon(x), (\Psi_\lambda + \lambda I)(J_\varepsilon(x))\rangle_{F_{1,2,\nu}}$$

$$= \big(\frac{1}{\lambda} + \lambda + C_5\big)\langle(\nu - L)(\Psi_\lambda + \lambda I)(J_\varepsilon(x)), J_\varepsilon(x)\rangle_2,$$

where in the first inequality we used **(H4)**, the fact that $r(\Psi_\lambda(r) + \lambda r) \geq 0$ for all $r \in \mathbb{R}$ and Ψ_λ is $\frac{1}{\lambda}$-Lipschitz ([3, Page:41, Proposition 2.3 (ii)]), the last equality comes from the fact that $(\Psi_\lambda + \lambda I)(J_\varepsilon(x)) \in D(L)$. Now from Eq. A.3, we get the assertion.

A.2 The L^p-Itô Formula in Expectation

The purpose in this section is to prove Theorem A.1 below, which has been used in Lemmas 4.2 and 4.3.

Let ℓ_2 be the space of all square-summable sequences in $\mathbb{R}$ and $p \in [2, \infty)$. In addition, to the real-valued L^p-space, $L^p(\mu) := L^p(E, \mu)$ we consider the ℓ_2-valued L^p-space $L^p(\mu; \ell_2) := L^p(E, \mu; \ell_2)$. We set

$$|g|_p^p := |g|_{L^p(\mu;\ell_2)}^p = \int_E \|g(x)\|_{\ell_2}^p \mu(dx) = \int_E \Big(\sum_{k=1}^\infty |g_k(x)|^2\Big)^{\frac{p}{2}}\mu(dx).$$

Let $\mathscr{P}$ denote the predictable σ-algebra on $[0, T] \times \Omega$ corresponding to $(\Omega, \mathscr{F}, (\mathscr{F}_t)_{t\geq0}, \mathbb{P})$. For $p \in [2, \infty)$ we set

$$\mathbb{L}^p(T) := L^p([0, T] \times \Omega, \mathscr{P}; L^p(\mu))$$

and

$$\mathbb{L}^p(T; \ell_2) := L^p([0, T] \times \Omega, \mathscr{P}; L^p(\mu; \ell_2)),$$

equipped with their standard L^p-norms. Since $(E, \mathcal{B})$ is a standard measurable space, by definition there exists a complete metric d on E, such that (E, d) is separable, i.e., a Polish space, whose Borel σ-algebra coincides with $\mathcal{B}$. Below we fix this metric d and denote the corresponding set of all bounded continuous functions by $C_b(E)$.

Let $\mathcal{E}$ be all $g = (g_k)_{k \in \mathbb{N}} \in L^\infty([0, T] \times \Omega; L^\infty(\mu; \ell_2) \cap L^1(\mu; \ell_2))$ such that there exists $j \in \mathbb{N}$ and bounded stopping times $\tau_0 \leq \tau_1 \leq \cdots \leq \tau_j \leq T$ such that

$$g_k = \begin{cases} \sum_{i=1}^{j} g_k^i 1_{(\tau_{i-1}, \tau_i]}, & \text{if } k \leq j; \\ 0, & \text{if } k > j, \end{cases}$$

where $g_k^i \in C_b(E) \cap L^1(\mu)$, $1 \leq i \leq j$.

Claim A.1 $\mathcal{E}$ is dense in $\mathbb{L}^p(T; \ell_2)$ for all $p \in [2, \infty)$.

Proof Let $f = (f_k)_{k \in \mathbb{N}} \in L^q(T; \ell_2)$, with $q := \frac{p}{p-1}$, be such that

$$_{\mathbb{L}^q(T;\ell_2)}\langle f, g \rangle_{\mathbb{L}^p(T;\ell_2)} = \mathbb{E} \int_0^T \int_E \sum_{k=1}^\infty f_k g_k d\mu ds = 0 \ \forall g \in \mathcal{E}.$$

Now let $\sigma \leq \tau$ be two stopping times and $k \in \mathbb{N}$. Define $g \in \mathbb{L}^p(T; \ell_2)$ by $g = (g_k \delta_{ik})_{i \in \mathbb{N}}$, where

$$g_k := g_k^k I_{(\sigma, \tau]}$$

and $g_k^k \in C_b(E) \cap L^1(\mu)$. Then $g \in \mathcal{E}$, hence

$$0 = {}_{\mathbb{L}^q(T;\ell_2)}\langle f, g \rangle_{\mathbb{L}^p(T;\ell_2)}$$
$$= \mathbb{E} \int_0^T \int_E f_k g_k^k d\mu \ I_{(\sigma, \tau]}(t) dt,$$

which implies that

$$\int_E f_k g_k^k d\mu = 0 \ dt \otimes \mathbb{P} - a.s.,$$

since all sets of the type $(\sigma, \tau]$ generate the σ-algebra $\mathscr{P}$ and since f_k is $\mathscr{P}$-measurable. Therefore, since $C_b(E) \cap L^1(\mu)$ is dense in $L^p(\mu)$,

$$f_k = 0 \ \text{in } L^q(\mu) \ dt \otimes \mathbb{P} - a.s., \ \text{ for all } k \in \mathbb{N}.$$

Now the assertion follows by the Hahn-Banach theorem ([38, page: 61, Corollary 4.23]). $\qquad\blacksquare$

Remark A.1 Let $\mathcal{S}$ be the set of all functions $f \in L^\infty([0, T] \otimes \Omega; L^\infty(\mu) \cap L^1(\mu))$ such that there exist $l \in \mathbb{N}$ and bounded stopping times $\tau_0' \leq \tau_1' \leq \cdots \leq \tau_l' \leq T$ such that $f = \sum_{i=1}^{l} f^i 1_{(\tau_{i-1}', \tau_i']}$, where $f^i \in C_b(E) \cap L^1(\mu)$, $1 \leq i \leq l$. Similarly to Claim A.1, one can prove that $\mathcal{S}$ is dense in $\mathbb{L}^p(T)$ for all $p \in [2, \infty)$.

Define $\mathbf{M} : \mathcal{E} \longmapsto \bigcap_{p \geq 1} L^p(\Omega; C([0, T]; L^p(\mu)))$ as follows:

$$
\begin{aligned}
\mathbf{M}(g)(t) = \int_0^t g \, dW(s) &:= \sum_{k=1}^{\infty} \int_0^t g_k \, dW_k(s) \\
&= \sum_{i,k=1}^{j} g_k^i \big(W_k(t \wedge \tau_i) - W_k(t \wedge \tau_{i-1})\big), \ t \in [0, T], \ g \in \mathcal{E}. \text{(A.5)}
\end{aligned}
$$

Let us note that the right hand-side of Eq. A.5 is $\mathbb{P}$-a.s. for every $t \in [0, T]$ a continuous μ-version of $\mathbf{M}(g)(t) \in L^p(E, \mu)$, which for every $x \in E$ is a continuous real-valued martingale and is equal to

$$
\sum_{k=1}^{\infty} \int_0^t g_k(s, x) \, dW_k(s), \ x \in E, \ t \in [0, T]. \tag{A.6}
$$

Claim A.2 *Let $p \in [2, \infty)$. Then $\mathbf{M}$ extends to a linear continuous map $\overline{\mathbf{M}}$ from $\mathbb{L}^p(T; \ell_2)$ to $L^p(\Omega; C([0, T]; L^p(\mu)))$, such that $\overline{\mathbf{M}}(g)$ is a continuous martingale in $L^p(\mu)$ for all $g \in \mathbb{L}^p(T; \ell_2)$.*

Proof We have

$$
\begin{aligned}
&\mathbb{E}\Big[\sup_{t \in [0,T]} \int_E \Big| \int_0^t g \, dW(s)\Big|^p d\mu\Big] \\
&= \mathbb{E}\Big[\sup_{t \in [0,T]} \int_E \Big| \sum_{k=1}^{\infty} \int_0^t g_k(s, x) \, dW_k(s)\Big|^p d\mu\Big] \\
&\leq \int_E \Big[\mathbb{E} \sup_{t \in [0,T]} \Big| \sum_{k=1}^{\infty} \int_0^t g_k(s, x) \, dW_k(s)\Big|^p\Big] d\mu \\
&\leq c_p \int_E \Big[\mathbb{E} \Big\langle \sum_{k=1}^{\infty} \int_0^{\cdot} g_k(s, x) \, dW_k(s)\Big\rangle_T^{\frac{p}{2}}\Big] d\mu \\
&= c_p \int_E \mathbb{E}\Big(\sum_{k=1}^{\infty} \int_0^T g_k^2(s, x) \, ds\Big)^{\frac{p}{2}} d\mu \\
&= c_p \mathbb{E}\Big[\int_E \Big(\int_0^T |g(s, x)|_{\ell_2}^2 \, ds\Big)^{\frac{p}{2}} d\mu\Big]^{\frac{2}{p} \cdot \frac{p}{2}} \\
&\leq c_p \mathbb{E}\Big[\int_0^T \Big(\int_E |g(s, x)|_{\ell_2}^p \, d\mu\Big)^{\frac{2}{p}} ds\Big]^{\frac{p}{2}} \\
&\leq c_p T^{\frac{p}{2} - 1} \mathbb{E} \int_0^T \big|g(s, \cdot)\big|_{L^p(\mu; \ell_2)}^p \, ds,
\end{aligned}
$$

where we have used the BDG inequality applied to the real-valued martingale in Eq. A.6 in the third step, the assumption that $p \geq 2$ and Minkowski's inequality in the sixth step and Hölder's inequality in the last step. Hence the first part of the assertion follows.

To prove the second let $g \in \mathbb{L}^p(T; \ell_2)$. It suffices to prove that for all $f \in L^q(\mu)$ with $q := \frac{p}{p-1}$,

$$
\int_E f \, \overline{\mathbf{M}}(g)(t) \, d\mu, \ t \in [0, T],
$$

is a real-valued martingale (see e.g. [26, Remark 2.2.5]). But since for some $g_n \in \mathcal{E}$, $n \in \mathbb{N}$, we have $\forall \, t \in [0, T]$ that

$$\mathbf{M}(g_n)(t) \overrightarrow{\; n \to \infty \;} \overline{\mathbf{M}}(g)(t) \ \text{ in } \ L^p(\Omega; L^p(\mu)),$$

it follows that

$$\int_E f \, \mathbf{M}(g_n)(t) d\mu \overrightarrow{\; n \to \infty \;} \int_E f \, \overline{\mathbf{M}}(g)(t) d\mu \ \text{ in } \ L^1(\Omega).$$

So, we may assume that $g \in \mathcal{E}$. But in this case by Eq. A.5 it follows immediately that $\int_E f \, \mathbf{M}(g)(t) d\mu$, $t \in [0, T]$, is a real-valued martingale.

Below we define for $g \in \mathbb{L}^p(T; \ell_2)$, $p \in [2, \infty)$,

$$\int_0^t g(s) dW(s) := \overline{\mathbf{M}}(g)(t), \ \ t \in [0, T],$$

where $\overline{\mathbf{M}}$ is as in Claim A.2.

Now we fix $p \in [2, \infty)$ and consider the following process

$$u : \Omega \times [0, T] \to L^p(\mu),$$

defined by

$$u(t) := u(0) + \int_0^t f(s) ds + \int_0^t g(s) dW(s), \tag{A.7}$$

where $u(0) \in L^p(\Omega, \mathcal{F}_0; L^p(\mu))$, $f \in \mathbb{L}^p(T)$ and $g \in \mathbb{L}^p(T; \ell_2)$.

Theorem A.1 *"Itô-formula in expectation"* Let $p \in [2, \infty)$, $f \in \mathbb{L}^p(T)$, $g \in \mathbb{L}^p(T; \ell_2)$. Let u be as in Eq. A.7. Then for all $t \in [0, T]$,

$$\mathbb{E}|u(t, x)|_p^p = \mathbb{E}|u(0)|^p + \mathbb{E} \int_0^t \int_E p|u(s, x)|^{p-2} u(s, x) f(s, x) \mu(dx) ds$$

$$+ \frac{1}{2} p(p-1) \mathbb{E} \int_0^t \int_E |u(s, x)|^{p-2} |g(s, x)|_{\ell_2}^2 \mu(dx) ds. \tag{A.8}$$

Remark A.2 *In the case $E = \mathbb{R}^d$, $\mu =$ Lebesgue measure, N. Krylov proved Itô's formula for the L^p-norm of a large class of $W^{1,p}$-valued stochastic processes in his fundamental paper [25]. In particular, Lemma 5.1 in that paper gives a pathwise Itô formula for processes u as in Eq. A.7, which immediately implies Eq. A.8. The proof, however, uses a smoothing technique by convoluting the process u in x with Dirac-sequence of smooth functions, which is not available in our more general case, where $(E, \mathcal{B})$ is just a standard measurable space with a σ-finite measure μ, without further structural assumptions that we wanted to avoid to cover applications e.g. to underlying spaces E which are fractals. Fortunately, the above Itô formula in expectation is enough to prove all main results in this paper without any further assumptions. After the preparations above, its proof is quite simple.*

We recall the following well-known result (see e.g. Theorem 21.7 in [10]):

Lemma A.3 *Let $p \in [1, \infty)$, $v_n, v \in L^p(\mu)$ such that $v_n \to v$ in μ-measure as $n \to \infty$ and*

$$\lim_{n \to \infty} |v_n|_p = |v|_p.$$

Then

$$\lim_{n \to \infty} v_n = v \ \text{ in } \ L^p(\mu).$$

Proof of Theorem A.1   By Claim A.1 and Remark A.1, we can find $f_n \in \mathcal{S}$, $n \in \mathbb{N}$, and $g_n \in \mathbb{L}^p(T; \ell_2)$, $n \in \mathbb{N}$, such that as $n \to \infty$

$$f_n \to f \ \text{ in } \ \mathbb{L}^p(T), \tag{A.9}$$

and

$$g_n \to g \ \text{ in } \ \mathbb{L}^p(T; \ell_2). \tag{A.10}$$

For $n \in \mathbb{N}$, define

$$u_n(t) := u(0) + \int_0^t f_n(s)ds + \int_0^t g_n(s)dW(s).$$

By Eqs. A.7, A.9, A.10 and Claim A.2, it follows that as $n \to \infty$,

$$
\begin{aligned}
\int_0^\cdot f_n(s)ds &\to \int_0^\cdot f(s)ds, \\
\int_0^\cdot g_n(s)dW(s) &\to \int_0^\cdot g(s)dW(s), \\
u_n &\to u,
\end{aligned}
\tag{A.11}
$$

in $L^p(\Omega; C([0, T]; L^p(\mu)))$.

Applying the Itô formula to the real-valued semi-martingale $|u_n(t, x)|_p^p$ for each $x \in E$, and integrating w.r.t. $x \in E$ and $\omega \in \Omega$, we obtain

$$
\mathbb{E}\int_E |u_n(t, x)|^p \mu(dx) = \mathbb{E}|u(0)|^p + \mathbb{E}\int_E \int_0^t p|u_n(s, x)|^{p-2}u_n(s, x) \cdot f_n(s, x)ds\mu(dx)
$$
$$
+ \frac{1}{2}p(p-1)\mathbb{E}\int_E \int_0^t |u_n(s, x)|^{p-2} \cdot |g_n(s, x)|_{\ell_2}^2 ds\mu(dx) \tag{A.12}
$$

Note that by Lemma A.3 and Eq. A.11

$$
\begin{aligned}
|u_n(s)|^{p-2}u_n(s) &\to |u(s)|^{p-2}u(s) \ \text{ in } \ L^{\frac{p}{p-1}}(\mu), \\
|u_n(s)|^{p-2} &\to |u(s)|^{p-2} \ \text{ in } \ L^{\frac{p}{p-2}}(\mu),
\end{aligned}
$$

as $n \to \infty$. Hence by Eqs. A.9 and A.10 we may pass to the limit $n \to \infty$ in Eq. A.12 to get Eq. A.8.   □

Data Availability Data sharing not applicable to this article as no datasets were generated or analysed during the current study.

Declarations

Conflict of Interest The authors declare no conflict of interest.

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