



The Stochastic Klausmeier System and A Stochastic Schauder-Tychonoff Type Theorem

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Abstract

On the one hand, we investigate the existence and pathwise uniqueness of a nonnegative martingale solution to the stochastic evolution system of nonlinear advection-diffusion equations proposed by Klausmeier with Gaussian multiplicative noise. On the other hand, we present and verify a general stochastic version of the Schauder-Tychonoff fixed point theorem, as its application is an essential step for showing existence of the solution to the stochastic Klausmeier system. The analysis of the system is based both on variational and semigroup techniques. We also discuss additional regularity properties of the solution.

Keywords Stochastic Klausmeier evolution system · Stochastic Schauder-Tychonoff type theorem · Pattern formation in ecology · Nonlinear stochastic partial differential equation · Flows in porous media, pathwise uniqueness

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1 Introduction

Pattern formation at the ecosystem level has recently gained a lot of attention in spatial ecology and its mathematical modeling. Theoretical models are a widely used tool for studying e.g. banded vegetation patterns. One important model is the system of advection-diffusion equations proposed by Klausmeier [41]. This model for vegetation dynamics in semi-deserted areas is based on the “water redistribution hypothesis”, using the idea that rain water in dry regions is eventually infiltrated into the ground. Water falling down onto bare ground mostly runs off downhill toward the next patch of vegetation which provides a better infiltration capacity. The soil in such regions of the world as Australia, Africa, and Southwestern North

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America is prone to nonlocality of water uptake due to the semi-arid environment. Studies of the properties of the system and further developments can be found in e.g. [40, 61–64, 71].

The Klausmeier system is a generalization of the so-called Gray-Scott system [31] (see also [60, 65] for earlier accounts employing similar models) which already exhibits effects similar to Turing patterns [20, 23, 50, 70], see for instance the discussion in [72]. We refer to [45, 52, 53] for further reading on pattern formation in biology. A discussion of reaction-diffusion type equations with motivation from biology can be found in [56].

The underlying mathematics of this model is given by a pair of solutions (u, v) to a partial differential equation system coupled by a nonlinearity. The function u represents the surface water content and v represents the biomass density of the plants. In order to model the spread of water on a terrain without a specific preference for the direction in which the water flows, the original models were extended by replacing the diffusion operator by a nonlinear porous media operator, which represents the situation that the ground is partially filled by interconnected pores conveying fluid under an applied pressure gradient.

To this end, let $\mathcal{O} \subset \mathbb{R}^d$ be a bounded domain, $d = 1, 2, 3$, having C^2 -boundary or $\mathcal{O} = [0, 1]^d$. Consider the following problem

$$\begin{cases} \dot{u}(t) = r_u \Delta u^{[\gamma]}(t) - \chi u(t) v^2(t) + k - f u(t), & t > 0, \quad u(0) = u_0, \\ \dot{v}(t) = r_v \Delta v(t) + u(t) v^2(t) - g v(t), & t > 0, \quad v(0) = v_0, \end{cases} \tag{1.1}$$

with Neumann (or periodic when \mathcal{O} is a rectangular domain) boundary conditions and initial conditions $u(0) = u_0$ and $v(0) = v_0$. Here, $z^{[\gamma]} := |z|^{\gamma-1}z$, $\gamma > 1$, $z \in \mathbb{R}$, and further, r_u, r_v, χ, k, f and g denote positive constants.

The deterministic or macroscopic model is derived from the limiting behavior of interacting diffusions — the so-called microscopic model, see [43]. When applying the strong law of large numbers and passing from the microscopic to the macroscopic equation, one is neglecting the random fluctuations. In order to get a more realistic model, it is necessary to add noise, which represents the randomness of the natural environment or the fluctuation of parameters in the model. The introduction to stochasticity to ecological models is supported by the arguments of [33]. Due to the Wong-Zakai principle, this leads to the representation of the noise as a Gaussian stochastic perturbation with Stratonovich increment, see [27, Section 3.4] and [49, 78]. One important consequence is the preservation of energy in the noisy system.

In practice, we are investigating the system (1.1) driven by a multiplicative infinite dimensional Wiener process. Under suitable regularity assumptions on the initial data (u_0, v_0) , on γ and on the perturbation by noise which we shall specify later, we find that there exists a nonnegative (martingale) solution to the system Eq. 1.2 in dimensions $d = 1, 2, 3$, see our main result Theorem 3.6. More precisely, we prove existence of nonnegative solutions to the noisy systems Eq. 3.6–(3.7) (Itô noise), Eq. 3.1–(3.2) (Stratonovich noise) respectively. For $d = 1$, we show that the solution to Eq. 3.6–(3.7) is pathwise unique and that there exists a strong solution in the stochastic sense. Thus, we are actually considering the following system for $\gamma > 1$

$$\begin{cases} \dot{u}(t) = r_u \Delta u^{[\gamma]}(t) - \chi u(t) v^2(t) + u(t) \Xi_1(t), & t > 0, \quad u(0) = u_0, \\ \dot{v}(t) = r_v \Delta v(t) + u(t) v^2(t) + v(t) \Xi_2(t), & t > 0, \quad v(0) = v_0. \end{cases} \tag{1.2}$$

Here, Ξ_1 and Ξ_2 denote independent random Gaussian noises specified to be certain Banach space valued linear Stratonovich Wiener noises later on. Similar stochastic equations with Lévy noise have been studied e.g. in [7, 18]. We note that due to the coupling and the nonlinearity, the solutions are not expected to be stochastically independent.

Nonlinear diffusion systems perturbed by noise exhibit certain improved well-posedness and regularity properties when compared to their deterministic counterparts, see, among others, [4, 15, 27]. Another feature of linear multiplicative noise is that it preserves the non-negativity of the initial condition [1, 68]. Some applications may demand more complicated noises with nonlinear structure or coupling, however, we choose a linear dependence on the noise as a first step toward more general models.

The challenging problem in the system given by Eq. 1.1, respective the noisy system Eq. 1.2, is the nonlinearity appearing once with a negative sign and once with a positive sign. Choosing different speeds of diffusion seems rather natural, as it well known that the characteristic pattern formation may not take place if $r_u = r_v$, cf. [36]. The stochastic perturbation, however, does not restrict pattern formation [12, 79], noting that our choice of noise exhibits rather small intensity and damped high frequency modes. On the other hand, the nonlinearity is not of variational structure, such that energy methods are not available for the analysis, and neither the maximum principle nor Gronwall type arguments work. More direct deterministic (pathwise) methods may also fail in general as the equation for v is non-monotone.

Another difficulty is posed by the nonlinear porous media diffusion operator in the equation for u . Typically, it is studied with the help of variational monotonicity methods, see e.g. [2, 48]. To the nonlinear diffusion, neither semigroup methods nor Galerkin approximation can be applied directly without greater effort. Other approaches are given for instance in [26, 29, 55, 74], to mention a few.

Our motivation to prove a probabilistic Schauder-Tychonoff type fixed point theorem originates from the aim to show existence of a solution to the stochastic counterpart Eq. 1.2 of the system Eq. 1.1. Our result is for instance also applied in [34, 35]. As standard methods for showing existence and uniqueness of solutions to stochastic partial differential equations cannot be applied directly, we perform a fixed point iteration using a mix of so-called variational and semigroup methods with nonlinear perturbation. Here, a precise analysis of regularity properties of nonnegative solutions to regularized and localized subsystems with truncated nonlinearities needs to be conducted. Together with the continuous dependence of each subsystem on the other one, we obtain the fixed point, locally in time, by weak compactness and an appropriate choice of energy spaces. The result is completed by a stopping time gluing argument.

Apart from the probabilistic structure, the main novelty is that we construct the fixed point in the nonlinearity and not in the noise coefficient. This works well because our system is coupled precisely in the nonlinearity. The main task consists then in the analysis of regularity and invariance properties of a regularized system with “frozen” nonlinearity (not being fully linearized, however, as the porous media operator remains).

It may be possible to apply this method in the future to other nonlinear systems, as for example, systems with nonlinear convection-terms as systems with transport or Navier-Stokes type systems. It is certainly possible to apply the method to linear cross diffusion systems. See [19] for a recent work proving existence of martingale solutions to stochastic cross diffusion systems, however, their approach relies on other methods that do not cover the porous media case. See [51] for a previous work using the classical Schauder theorem for stochastic evolution equations with fractional Gaussian noise.

Given higher regularity of the initial data, we also show the pathwise uniqueness of the solution to Eq. 1.2 for spatial dimension $d = 1$. As a consequence of the celebrated result by Yamada and Watanabe [75, 80], we obtain existence and uniqueness of a strong solution, see Corollary 3.12. We refer to [10, 13, 14, 59] for previous works that employ a similar strategy for proving the existence of a unique strong solution.

Structure of the Paper

The stochastic Schauder-Tychonoff type Theorem 2.1 is presented in Section 2. In the subsequent section, i.e. Section 3, we apply this fixed point theorem to show the existence of a martingale solution to the stochastic counterpart Eq. 1.2 of the system Eq. 1.1. Section 3 contains our main result Theorem 3.6. Section 4 is devoted to the proof of Theorem 2.1. In Section 5, we prove several technical propositions that are need for the main result in Section 3. Section 6 contains the proof of pathwise uniqueness, that is, Theorem 3.10. Some auxiliary results are collected in Appendices A, B and C.

2 The Stochastic Schauder-Tychonoff Type Theorem

Let us fix some notation. Let U be a Banach space. Let $\mathcal{O} \subset \mathbb{R}^d$ be an open domain, $d \geq 1$. Let $\mathbb{X} \subset \{\eta : [0, T] \rightarrow E \subset \mathcal{D}'(\mathcal{O})\}$ be a Banach function space¹, let $\mathbb{X}' \subset \{\eta : [0, T] \rightarrow E \subset \mathcal{D}'(\mathcal{O})\}$ be a reflexive Banach function space embedded compactly and densely into \mathbb{X} . In both cases, the trajectories take values in a Banach function space E over the spatial domain \mathcal{O} , where we assume that E has the UMD property, see [73]. Let $\mathfrak{A} = (\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ be a filtered probability space with filtration $\mathbb{F} = (\mathcal{F}_t)_{t \in [0, T]}$ satisfying the usual conditions. Let H be a separable Hilbert space and $(W(t))_{t \in [0, T]}$ be a Wiener process² in H with a linear, nonnegative definite, symmetric trace class covariance operator $Q : H \rightarrow H$ such that W has the representation

$$W(t) = \sum_{i \in \mathbb{I}} Q^{\frac{1}{2}} \psi_i \beta_i(t), \quad t \in [0, T],$$

where $\{\psi_i : i \in \mathbb{I}\}$ is a complete orthonormal system in H , \mathbb{I} a suitably chosen countable index set, and $\{\beta_i : i \in \mathbb{I}\}$ a family of independent real-valued standard Brownian motions on $[0, T]$ modeled in $\mathfrak{A} = (\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$. Due to [16, Proposition 4.7, p. 85], this representation does not pose a restriction.

For $m \geq 1$, define the collection of processes

$$\mathcal{M}_{\mathfrak{A}}^m(\mathbb{X}) := \left\{ \xi : \Omega \times [0, T] \rightarrow E : \begin{aligned} &\xi \text{ is } \mathbb{F}\text{-progressively measurable and } \mathbb{E}|\xi|_{\mathbb{X}}^m < \infty \end{aligned} \right\} \tag{2.1}$$

equipped with the semi-norm

$$|\xi|_{\mathcal{M}_{\mathfrak{A}}^m(\mathbb{X})} := (\mathbb{E}|\xi|_{\mathbb{X}}^m)^{1/m}, \quad \xi \in \mathcal{M}_{\mathfrak{A}}^m(\mathbb{X}).$$

For fixed $\mathfrak{A}, W, m > 1$, we define the operator $\mathcal{V} = \mathcal{V}_{\mathfrak{A}, W} : \mathcal{M}_{\mathfrak{A}}^m(\mathbb{X}) \times L^m(\Omega, \mathcal{F}_0, \mathbb{P}; E) \rightarrow \mathcal{M}_{\mathfrak{A}}^m(\mathbb{X})$ for $\xi \in \mathcal{M}_{\mathfrak{A}}^m(\mathbb{X})$ via

$$\mathcal{V}(\xi) := \mathcal{V}(\xi, w_0) := w,$$

where w is the solution to the following Itô stochastic partial differential equation (SPDE)

$$dw(t) = (Aw(t) + F(\xi, t)) dt + \Sigma(w(t)) dW(t), \quad w(0) = w_0 \in E. \tag{2.2}$$

For convenience, we drop the dependence on the initial datum w_0 in the notation $\mathcal{V}(\xi)$. Here, we implicitly assume that Eq. 2.2 is well-posed and a unique strong solution (in the

¹ Here, $\mathcal{D}'(\mathcal{O})$ denotes the space of Schwartz distributions on \mathcal{O} , that is, the topological dual space of smooth functions with compact support $\mathcal{D}(\mathcal{O}) = C_0^\infty(\mathcal{O})$.

² That is, a Q -Wiener process, see e.g. [16] for this notion.

stochastic sense) $w \in \mathcal{M}_{\mathfrak{A}}^m(\mathbb{X})$ exists for $\xi \in \mathcal{M}_{\mathfrak{A}}^m(\mathbb{X})$. Here, we shall also assume that $A : D(A) \subset E \rightarrow E$ is a possibly nonlinear and measurable (single-valued) operator and $F : \mathbb{X} \times [0, T] \rightarrow E$ a (strongly) measurable map such that

$$\mathbb{P} \left(\int_0^T (|Aw(s)|_E + |F(\xi, s)|_E) ds < \infty \right) = 1,$$

and assume that $\Sigma : E \rightarrow \gamma(H, E)$ is strongly measurable. Here, $\gamma(H, E)$ denotes the space of γ -radonifying operators from H to E , as defined in the beginning of Section 5, which coincides with the space of Hilbert-Schmidt operators $L_{HS}(H, E)$ if E is a separable Hilbert space.

We are ready to formulate our main tool for proving the existence of nonnegative martingale solutions to (1.2), that is, a stochastic variant of the (deterministic) Schauder-Tychonoff fixed point theorem(s) from [30, § 6–7].

Theorem 2.1 *Let H be a Hilbert space, $Q : H \rightarrow H$ such that Q is linear, symmetric, nonnegative definite and of trace class, let U be a Banach space, and let us assume that we have a compact and dense embedding $\mathbb{X}' \hookrightarrow \mathbb{X}$ as above. Let $m > 1$. Suppose that for any filtered probability space $\mathfrak{A} = (\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ and for any Q -Wiener process W with values in H that is modeled on \mathfrak{A} the following holds.*

Suppose that there exist constants $R_1, \dots, R_K > 0$, $K \in \mathbb{N}$, continuous functions $\Psi_i : \mathbb{X} \rightarrow [0, \infty)$, $1 \leq i \leq K$, measurable functions $\Theta_i : \mathbb{X} \rightarrow [0, \infty)$, $1 \leq i \leq K$ with closed sublevel sets $\Theta_i^{-1}([0, \alpha])$, $\alpha \geq 0$, $1 \leq i \leq K$, and a nonempty, sequentially weak-closed, measurable and bounded subset³ $\mathcal{X}_{R_1, \dots, R_K}(\mathfrak{A})$ of $\mathcal{M}_{\mathfrak{A}}^m(\mathbb{X})$ such that:*

- (a) $\mathbb{E}[\Psi_i(\xi)] \leq R_i$, for every $\xi \in \mathcal{X}_{R_1, \dots, R_K}(\mathfrak{A})$ and every $1 \leq i \leq K$,
- (b) $\mathbb{P}(\{\Theta_i(\xi) < \infty, \xi \in \mathcal{X}_{R_1, \dots, R_K}(\mathfrak{A})\}) = 1$, for every $1 \leq i \leq K$.

Let us assume that the operator $\mathcal{V}_{\mathfrak{A}, W}$, defined by Eq. 2.2, restricted to $\mathcal{X}_{R_1, \dots, R_K}(\mathfrak{A})$ satisfies the following properties:

- (i) *the operator $\mathcal{V}_{\mathfrak{A}, W}$ is well-defined on $\mathcal{X}_{R_1, \dots, R_K}(\mathfrak{A})$ for all choices of $R_i > 0$, $1 \leq i \leq K$,*
- (ii) *there exist constants $R_i^0 > 0$, $1 \leq i \leq K$ such that*

$$\mathcal{V}_{\mathfrak{A}, W}(\mathcal{X}_{R_1, \dots, R_K}(\mathfrak{A})) \subset \mathcal{X}_{R_1, \dots, R_K}(\mathfrak{A})$$

for all $R_i \geq R_i^0$ and all $1 \leq i \leq K$,

- (iii) *for all choices of $R_i > 0$, $1 \leq i \leq K$, the restriction $\mathcal{V}_{\mathfrak{A}, W}|_{\mathcal{X}_{R_1, \dots, R_K}(\mathfrak{A})}$ is uniformly continuous on bounded subsets w.r.t. the strong topology of $\mathcal{M}_{\mathfrak{A}}^m(\mathbb{X})$,*
- (iv) *there exist constants $R > 0$, $m_0 \geq m$ such that⁴*

$$\mathbb{E} [|\mathcal{V}_{\mathfrak{A}, W}(\xi)|_{\mathbb{X}'}^{m_0}] \leq R \text{ for every } \xi \in \mathcal{X}_{R_1, \dots, R_K}(\mathfrak{A}),$$

for all $R_i > 0$ and all $1 \leq i \leq K$.

- (v) *for all $R_i > 0$ and all $1 \leq i \leq K$, $\mathcal{V}_{\mathfrak{A}, W}(\mathcal{X}_{R_1, \dots, R_K}(\mathfrak{A})) \subset \mathbb{D}([0, T]; U)$ \mathbb{P} -a.s.⁵*

³ Here, the notation $\mathcal{X}(\mathfrak{A})$ means that $\text{Law}(\xi) = \text{Law}(\tilde{\xi})$ on \mathbb{X} for $\xi \in \mathcal{X}(\mathfrak{A})$ and $\tilde{\xi} \in \mathcal{M}_{\mathfrak{A}}^m$ implies $\tilde{\xi} \in \mathcal{X}(\mathfrak{A})$.

⁴ Note that we can relax assumption (iv) to $m_0 > 1$ if we assume additionally that there exist constants $R > 0$, $m_1 > m$ such that $\mathbb{E} [|\mathcal{V}_{\mathfrak{A}, W}(\xi)|_{\mathbb{X}'}^{m_1}] \leq R$ for every $\xi \in \mathcal{X}_{R_1, \dots, R_K}(\mathfrak{A})$ for all $R_i > 0$ and all $1 \leq i \leq K$.

⁵ Here, $\mathbb{D}([0, T]; U)$ denotes the Skorokhod space of càdlàg paths in U endowed with the Skorokhod J_1 -topology, see [Appendix A2].

Then, there exists a filtered probability space $\mathfrak{A}^* = (\Omega^*, \mathcal{F}^*, \mathbb{F}^*, \mathbb{P}^*)$ (that satisfies the usual conditions) together with a Q -Wiener process W^* modeled on \mathfrak{A}^* and an element $w^* \in \mathcal{M}_{\mathfrak{A}^*}^m(\mathbb{X})$ such that for all $t \in [0, T]$, \mathbb{P}^* -a.s.

$$\mathcal{V}_{\mathfrak{A}^*, W^*}(w^*, w_0^*)(t) = w^*(t)$$

for any initial datum $w_0 \in L^m(\Omega, \mathcal{F}_0, \mathbb{P}; E)$, where $w_0^* \in L^m(\Omega^*, \mathcal{F}_0^*, \mathbb{P}^*; E)$ satisfies $\text{Law}(w_0^*) = \text{Law}(w_0)$.

The proof of Theorem 2.1 is postponed to Section 4. We note that by construction, we get that w^* solves

$$dw^*(t) = (Aw^*(t) + F(w^*, t)) dt + \Sigma(w^*(t)) dW^*(t), \quad w^*(0) = w_0.$$

on \mathfrak{A}^* .

3 Existence of A Solution To The Stochastic Klausmeier System

In this section, we shall prove the existence of a nonnegative solution to the stochastic Klausmeier system. First, we will introduce some notation and the definition of a (martingale) solution. After fixing the function spaces and the main hypotheses on the parameters of those, we present our main result Theorem 3.6. The following proof consists mainly of a verification of the conditions for Theorem 2.1 and a stopping time localization procedure. As pointed out before, we are using compactness arguments to show the existence, which leads to the loss of the initial stochastic basis.

Let H_1 and H_2 be a pair of separable Hilbert spaces, let $\mathfrak{A} = (\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ be a filtered probability space and let W_1, W_2 be a pair of independent Wiener process modeled on \mathfrak{A} taking values in H_1 and H_2 , respectively, with covariance operators Q_1 and Q_2 , respectively. We are interested in the solution to the following reduced Klausmeier system for $x \in \mathcal{O}$ and $t > 0$,

$$du(t, x) = (r_u \Delta u^{[\gamma]}(t, x) - \chi u(t, x) v^2(t, x)) dt + \sigma_1 u(t, x) \circ dW_1(t, x), \tag{3.1}$$

and,

$$dv(t, x) = (r_v \Delta v(t, x) + u(t, x) v^2(t, x)) dt + \sigma_2 v(t, x) \circ dW_2(t, x), \tag{3.2}$$

with Neumann (or periodic if $\mathcal{O} = [0, 1]^d$) boundary conditions and initial conditions $u(0) = u_0$ and $v(0) = v_0$. Let $r_u, r_v, \chi > 0$ be positive constants. Here, we use the abbreviation $x^{[\gamma]} := |x|^{\gamma-1}x$ for $\gamma > 1$. The hypotheses on the linear noise coefficient maps σ_1, σ_2 are specified below.

Due to the nonlinear porous media term, we do neither use solutions in the strong stochastic sense, nor mild solutions, that is, solutions in the sense of stochastic convolutions. Let us define what we mean with a solution on a fixed stochastic basis. The function spaces $H_2^{-1}(\mathcal{O})$ and $H_2^\rho(\mathcal{O})$ used in the following definition are discussed in Appendix B.

Definition 3.1 A couple (u, v) of stochastic processes on \mathfrak{A} is called *solution to the system* Eq. 3.1–(3.2) for initial data (u_0, v_0) if there exists $\rho \in \mathbb{R}$ such that

$$u \in L^2(\Omega; C([0, T]; H_2^{-1}(\mathcal{O}))) \cap L^{\gamma+1}(\Omega \times (0, T) \times \mathcal{O})$$

and

$$v \in L^2(\Omega; C([0, T]; H_2^\rho(\mathcal{O}))) \cap L^2(\Omega \times (0, T); H_2^{\rho+1}(\mathcal{O}))$$

such that u and v are $\mathbb{F} = (\mathcal{F}_t)_{t \in [0, T]}$ -adapted, and satisfy

$$u(t) = u_0 + \int_0^t (r_u \Delta u^{[\gamma]}(s) - \chi u(s)v^2(s)) ds + \int_0^t \sigma_1 u(s) \circ dW_1(s) \tag{3.3}$$

for every $t \in [0, T]$, \mathbb{P} -a.s. in $H_2^{-1}(\mathcal{O})$, and

$$v(t) = v_0 + \int_0^t (r_v \Delta v(s) + u(s)v^2(s)) ds + \int_0^t \sigma_2 v(s) \circ dW_2(s) \tag{3.4}$$

for every $t \in [0, T]$, \mathbb{P} -a.s. in $H_2^\rho(\mathcal{O})$.

Here, $V := L^{\gamma+1}(\mathcal{O})$ is dualized in a Gelfand triple over $\mathcal{H} := H_2^{-1}(\mathcal{O})$ which, in turn, is identified with its own dual by the Riesz isometry. Thus V^* is not to be mistaken to be equal to be the usual Banach space dual of V , namely $L^{\frac{\gamma+1}{\gamma}}(\mathcal{O})$, see the proof of Theorem 5.4 for details. Note that the above ds -integral for u in Eq. 3.3 is initially a V^* -valued Bochner integral, however seen to be in fact $H_2^{-1}(\mathcal{O})$ -valued, see the discussion in [48, Section 4.2] for further details.

As mentioned before, due to the loss of the original probability space, we are considering solutions in the weak probabilistic sense.

Definition 3.2 A martingale solution to the problem (3.1)–(3.2) for initial data (u_0, v_0) is a system

$$(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P}, (W_1, W_2), (u, v)) \tag{3.5}$$

such that

1. the quadruple $\mathfrak{A} := (\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ is a complete filtered probability space with a filtration $\mathbb{F} = (\mathcal{F}_t)_{t \in [0, T]}$ satisfying the usual conditions,
2. W_1 and W_2 are independent H_1 -valued, respectively, H_2 -valued Wiener processes over the probability space \mathfrak{A} with covariance operators Q_1 and Q_2 , respectively,
3. both $u : [0, T] \times \Omega \rightarrow H_2^{-1}(\mathcal{O})$ and $v : [0, T] \times \Omega \rightarrow H_2^\rho(\mathcal{O})$ are \mathbb{F} -adapted processes such that the couple (u, v) is a solution to the system Eq. 3.1 and Eq. 3.2 over the probability space \mathfrak{A} in the sense of Definition 3.1 for some $\rho \in \mathbb{R}$.

Remark 3.3 For our purposes, instead of the Stratonovich formulation the system Eq. 3.1–(3.2), it is convenient to consider the equations in Itô form:

$$du(t, x) = (r_u \Delta u^{[\gamma]}(t, x) - \chi u(t, x) v^2(t, x)) dt + \sigma_1 u(t, x) dW_1(t, x), \tag{3.6}$$

and,

$$dv(t, x) = (r_v \Delta v(t, x) + u(t, x) v^2(t, x)) dt + \sigma_2 v(t, x) dW_2(t, x), \tag{3.7}$$

with initial data $u(0) = u_0$ and $v(0) = v_0$. One reason for this is that the stochastic integral then becomes a local martingale. In order to show the existence of a solution to the Stratonovich system, one would have to incorporate the Itô-Stratonovich conversion term (cf. [25]), which, due to the linear multiplicative noise, is a linear term being just a scalar multiple of u, v , respectively. If one is interested in the exact form of the correction term, we refer to [21]. We also refer to the discussion in [36], where the constant accounting for the correction term is computed explicitly.

From now on, we shall consider the system Eq. 3.1–(3.2) in Itô form, that is, as in Eq. 3.6–(3.7). Definitions 3.1 and 3.2 are supposed to be adjusted in the obvious way, keeping the statement on the regularity of the solution unchanged. We note that we can solve Eq. 3.1–(3.2) by a straightforward modification of the proof given here.

Before presenting our main result, we will first introduce the hypotheses on d, γ, ρ and the initial conditions u_0 and v_0 , and on the multiplication operators σ_1, σ_2 . Most of the hypotheses are technical in nature, as they lead to several different embeddings for function spaces and interpolation spaces that we need to use in our proofs.

Hypothesis 3.4 [Existence] Let $d \in \{1, 2, 3\}$. Let $\gamma > 1$, $\rho \in \mathbb{R}$, and $m > 2$, $m_0 > \frac{2(\gamma+1)}{\gamma}$ such that

$$\frac{d}{2} - \frac{2}{m} - \frac{d}{m_0} \leq \rho < 1 - \frac{d}{2}$$

and $p^* \geq 2$ and $p_0^* \geq 2$ such that

$$\frac{1}{p^*} + \frac{2}{m} < 1 \quad \text{and} \quad \frac{2}{m_0} + \frac{1}{p_0^*} < \frac{\gamma}{\gamma + 1}.$$

Let

$$l > 1 + \left(1 - \frac{d}{2} - \rho\right)^{-1}$$

Let us assume that the initial conditions u_0, v_0 are \mathcal{F}_0 measurable and satisfy

$$\mathbb{E}|u_0|_{L^2}^l \vee \mathbb{E}|u_0|_{L^{p^*}}^{p_0^*} \vee \mathbb{E}|u_0|_{L^{\gamma+1}}^{\gamma+1} < \infty, \quad \text{and} \quad \mathbb{E}|v_0|_{H_2^0}^{m_0} \vee \mathbb{E}|v_0|_{H_m^{-\delta_0}}^{m_0} < \infty, \quad \delta_0 < \frac{1}{m_0},$$

and that u_0 and v_0 are a.e. nonnegative functions (nonnegative Borel measures that are finite on compact subsets, respectively).

Hypothesis 3.5 [Noise] Let $\{\psi_k : k \in \mathbb{Z}\}$ be the eigenfunctions of $-\Delta$ and $\{v_k : k \in \mathbb{Z}\}$ the corresponding eigenvalues. Let W_1 and W_2 be a pair of Wiener processes given by

$$W_j(t, x) = \sum_{k \in \mathbb{Z}} \lambda_k^{(j)} \psi_k(x) \beta_k^{(j)}(t), \quad t \in [0, T], \quad j = 1, 2, \quad x \in \mathcal{O},$$

where $\{\lambda_k^{(j)} : k \in \mathbb{Z}\}$, $j = 1, 2$, is a pair of nonnegative sequences belonging to $\ell^2(\mathbb{Z})$, $\{\beta_k^{(j)} : [0, T] \times \Omega \rightarrow \mathbb{R} : (k, j) \in \mathbb{Z} \times \{1, 2\}\}$ is a family of independent standard real-valued Brownian motions. We assume that $\lambda_k^{(j)} \leq C_j |v_k|^{-\delta_j}$, $j = 1, 2$, where $\delta_1 > \frac{1}{2} \vee \left(\frac{d}{2} - \frac{1}{4}\right)$, $\delta_2 > \frac{1}{2} \vee \left(\frac{d}{2} - \frac{1}{4}\right)$, and $C_1 > 0$, $C_2 > 0$.

Compare also with Subsection 5.1 for details.

Under these hypotheses, the existence of a martingale solution can be shown.

Theorem 3.6 Assume that Hypotheses 3.4–3.5 hold. Then there exists a martingale solution to system Eq. 3.6–(3.7) satisfying the following properties

- (i) $u(t, x) \geq 0$ and $v(t, x) \geq 0$ \mathbb{P} -a.s., for a.e. $t \in [0, T]$ and a.e. $x \in \mathcal{O}$,
- (ii) Let $p \geq 1$ and $\mathbb{E}|u_0|_{L^{p+1}}^{p+1} < \infty$. Then there exists a constant $C_0(p, T) > 0$ such that

$$\begin{aligned} & \mathbb{E} \left[\sup_{0 \leq s \leq T} |u(s)|_{L^{p+1}}^{p+1} \right] + \gamma p(p+1) r_u \mathbb{E} \int_0^T \int_{\mathcal{O}} |u(s, x)|^{p+\gamma-2} |\nabla u(s, x)|^2 dx ds \\ & + (p+1) \chi \mathbb{E} \int_0^T \int_{\mathcal{O}} |u(s, x)|^{p+1} |v(s, x)|^2 dx ds \leq C_0(p, T) \left(\mathbb{E}|u_0|_{L^{p+1}}^{p+1} + 1 \right). \end{aligned}$$

- (iii) for any choice of parameters ρ, m_0, l as in Hypothesis 3.4, there exists a constant $C_2(T) > 0$ such that

$$\begin{aligned} & \mathbb{E} \left[\sup_{0 \leq s \leq T} |v(s)|_{H_2^0}^{m_0} \right] + \mathbb{E} \left(\int_0^T |v(s)|_{H_2^{p+1}}^2 ds \right)^{m_0/2} \\ & \leq C_2(T) \left(1 + \mathbb{E}|v_0|_{H_2^0}^{m_0} + \mathbb{E}|u_0|_{L^2}^l \right). \end{aligned}$$

We shall also collect a standard notion on uniqueness.

Definition 3.7 *The system Eq. 3.6–(3.7) is called pathwise unique if, whenever $(u_i, v_i), i = 1, 2$ are martingale solutions to the system Eq. 3.6–(3.7) on a common stochastic basis $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P}, (W_1, W_2))$, such that $\mathbb{P}(u_1(0) = u_2(0)) = 1$ and $\mathbb{P}(v_1(0) = v_2(0)) = 1$, then*

$$\mathbb{P}(u_1(t) = u_2(t)) = 1 \quad \text{and} \quad \mathbb{P}(v_1(t) = v_2(t)) = 1, \quad \text{for every } t \in [0, T].$$

The theorem of Yamada and Watanabe [80] asserts that (weak) existence and pathwise uniqueness of the solution to a stochastic equation is equivalent to the existence of a unique strong solution. Therefore, showing pathwise uniqueness is a fundamental step for obtaining the existence of a unique strong solution. Under certain additional conditions, as collected below, we are able to prove pathwise uniqueness.

Hypothesis 3.8 [Uniqueness] Let $\delta_0 \in (0, \frac{1}{\nu})$, and $\rho \geq \frac{d}{2} - \frac{1}{2}$.

Remark 3.9 The condition $\delta_0 > 0$ implies that the Hypothesis 3.8 can only be satisfied if $d = 1$.

Theorem 3.10 *Let $T > 0$ and let $\mathfrak{A} = (\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \in [0, T]}, \mathbb{P})$ a probability space satisfying the usual conditions, let (W_1, W_2) be a Wiener process over \mathfrak{A} on H_1 and H_2 and satisfying Hypothesis 3.5. Let initial data $u_0 \in H_2^{-1}(\mathcal{O})$ and $v_0 \in L^2(\mathcal{O})$ together with the parameters $m, m_0, p^*, p_0^*, l, \delta_0$ and ρ satisfying Hypothesis 3.4 and Hypothesis 3.8, where $\mathcal{O} \subset \mathbb{R}$. Assume further that Hypothesis 3.5 holds. Let $(u_i, v_i), i = 1, 2$, be two solutions to the system Eq. 3.6–(3.7) with initial data (u_0, v_0) and belonging \mathbb{P} -a.s. to $C([0, T]; H_2^{-1}(\mathcal{O})) \times C([0, T]; H_2^{-\delta_0}(\mathcal{O}))$. Then, the processes (u_1, v_1) and (u_2, v_2) are indistinguishable in $H_2^{-1}(\mathcal{O}) \times H_2^{-\delta_0}(\mathcal{O})$.*

Remark 3.11 It can be shown that for dimension $d = 1$ such indices $m^*, m_0, p^*, p_0^*, l, \delta_0$ and ρ satisfying Hypothesis 3.4 and Hypothesis 3.8 can be found. Because of our condition $\rho \geq \frac{d}{2} - \frac{1}{2}$, we cannot handle dimensions $d \in \{2, 3\}$ as Hypothesis 3.4 then implies that $\rho < 0$.

Proof The proof is postponed to Section 6. □

By this Theorem at hand, the following corollary is a consequence of the Theorem of Yamada-Watanabe.

Corollary 3.12 *There exists a unique strong solution (in the stochastic sense) to the system Eq. 3.6–(3.7).*

Proof This follows from Theorem 3.6 in combination with Theorem 3.10 and the Yamada-Watanabe theorem, see Appendix E weiroeckner and [44, 54, 57]. □

The proof of Theorem 3.6 is an application of the Schauder-Tychonoff-type Theorem 2.1 and consists of the following five steps to verify the conditions of Theorem 2.1.

In the first step, we are specifying the underlying Banach spaces. In the second step, we shall construct the operator \mathcal{V} for a truncated system and show that the operator \mathcal{V} satisfies the assumptions of Theorem 2.1. In the third step, we localize via stopping times and glue the fixed point solutions together, which exist by Theorem 2.1. In the fourth step, we prove that the stopping times are uniformly bounded. In the fifth step, we show that we indeed yield a martingale solution satisfying the above properties. However, to keep the proof itself simple, we will postpone several technical a priori estimates and further regularity results which are collected in Section 5.

Proof of Theorem 3.6 Step (I). The underlying space(s). Here we define the spaces on which the operator \mathcal{V} will act. Let the probability space $\mathfrak{A} = (\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ be given and let W_1 and W_2 be two independent H_1 and H_2 -valued Wiener processes defined over \mathfrak{A} with covariances Q_1 and Q_2 . Let $W = (W_1, W_2)$, $H = H_1 \times H_2$, with covariance operator

$$Q = \begin{pmatrix} Q_1 & 0 \\ 0 & Q_2 \end{pmatrix}.$$

Let us define the Banach space

$$\mathbb{Y} = L^{\gamma+1}(0, T; L^{\gamma+1}(\mathcal{O})) \cap L^\infty(0, T; H_2^{-1}(\mathcal{O}))$$

equipped with the norm

$$\|\eta\|_{\mathbb{Y}} := \|\eta\|_{L^{\gamma+1}(0, T; L^{\gamma+1})} + \|\eta\|_{L^\infty(0, T; H_2^{-1})}, \quad \eta \in \mathbb{Y},$$

and the reflexive Banach space

$$\mathbb{Z} := L^{m_0}(0, T; L^m(\mathcal{O})),$$

equipped with the norm

$$\|\xi\|_{\mathbb{Z}} := \|\xi\|_{L^{m_0}(0, T; L^m)}, \quad \xi \in \mathbb{Z}.$$

Finally, let us fix an auxiliary Banach space $\mathbb{H}_\rho := L^2(0, T; H_2^{\rho+1}(\mathcal{O})) \cap L^\infty(0, T; H_2^\rho(\mathcal{O}))$ equipped with the norm

$$\|\xi\|_{\mathbb{H}_\rho} := \|\xi\|_{L^2(0, T; H_2^{\rho+1})} + \|\xi\|_{L^\infty(0, T; H_2^\rho)}, \quad \xi \in \mathbb{H}_\rho. \tag{3.8}$$

Remark 3.13 If

$$\frac{d}{2} - \rho \leq \frac{2}{m} + \frac{d}{m_0},$$

then one can show by Sobolev embedding and interpolation theorems (see Proposition A.6 in the appendix) that there exists a constant $C > 0$ such that

$$\|\xi\|_{\mathbb{Z}} \leq C \|\xi\|_{\mathbb{H}_\rho}, \quad \xi \in \mathbb{H}_\rho.$$

Let us fix the compactly embedded reflexive Banach subspace of \mathbb{Z} by⁶

$$\mathbb{Z}' := L^{m_0}(0, T; H_m^\sigma(\mathcal{O})) \cap \mathbb{W}_{m_0}^\alpha(0, T; H_m^{-\delta}(\mathcal{O}))$$

equipped with the norm

$$\|\xi\|_{\mathbb{Z}'} := \left(\|\xi\|_{L^{m_0}(0, T; H_m^\sigma)} + \|\xi\|_{\mathbb{W}_{m_0}^\alpha(0, T; H_m^{-\delta})} \right)^{1/m_0},$$

compare with Appendix B, where the compact embedding $\mathbb{Z}' \hookrightarrow \mathbb{Z}$ is discussed. The parameters $\sigma > 0$, $\alpha \in (0, 1)$, $\delta > 0$ are specified in Proposition 5.9.

Now, denote the space of progressively measurable (pairs of) processes $\mathcal{M}_{\mathfrak{A}}^{2, m_0}(\mathbb{Y}, \mathbb{Z})$ by

$$\begin{aligned} \mathcal{M}_{\mathfrak{A}}^{2, m_0}(\mathbb{Y}, \mathbb{Z}) := & \left\{ (\eta, \xi) : \eta, \xi : [0, T] \times \Omega \rightarrow \mathcal{D}'(\mathcal{O}) \text{ such that} \right. \\ & \eta \text{ and } \xi \text{ are progressively measurable on } \mathfrak{A} \text{ and} \\ & \left. \mathbb{E}\|\eta\|_{\mathbb{Y}}^2 < \infty \text{ and } \mathbb{E}\|\xi\|_{\mathbb{Z}}^{m_0} < \infty \right\} \end{aligned}$$

⁶ For the definition of $\mathbb{W}_{m_0}^\alpha$, we refer to Appendix B.

equipped with the norm

$$\|(\eta, \xi)\|_{\mathcal{M}_{\mathfrak{A}}^{2,m_0}(\mathbb{Y}, \mathbb{Z})} := \left(\mathbb{E}\|\eta\|_{\mathbb{Y}}^2\right)^{\frac{1}{2}} + \left(\mathbb{E}\|\xi\|_{\mathbb{Z}}^{m_0}\right)^{\frac{1}{m_0}}, \quad (\eta, \xi) \in \mathcal{M}_{\mathfrak{A}}^{2,m_0}(\mathbb{Y}, \mathbb{Z}). \quad (3.9)$$

Note that here, progressive measurability is meant relative to the Borel σ -fields of the target spaces $H_2^{-1}(\mathcal{O})$ and $H_2^p(\mathcal{O})$ respectively.

Finally, for fixed $R_0, R_1, R_2 > 0$, let us define the subspace $\mathcal{X}_{\mathfrak{A}} = \mathcal{X}_{\mathfrak{A}}(R_0, R_1, R_2)$ by

$$\begin{aligned} &\mathcal{X}_{\mathfrak{A}}(R_0, R_1, R_2) \\ &:= \left\{ (\eta, \xi) \in \mathcal{M}_{\mathfrak{A}}^{2,m_0}(\mathbb{Y}, \mathbb{Z}) : \right. \\ &\quad \mathbb{E} \left[\sup_{0 \leq s \leq T} |\eta(s)|_{L^{p^*}}^{p_0^*} \right]^{\frac{1}{p_0^*}} \leq R_0, \quad \mathbb{E}\|\xi\|_{\mathbb{Z}}^{m_0} \leq R_1, \quad \mathbb{E}\|\xi\|_{\mathbb{H}_\rho}^{m_0} \leq R_2, \text{ and} \\ &\quad \left. \eta \text{ and } \xi \text{ are nonnegative } \mathbb{P} \otimes \text{Leb-a.e. in } \mathcal{D}'(\mathcal{O}) \right\}. \end{aligned}$$

It is easy to verify that $\mathcal{X}_{\mathfrak{A}}$ is a sequentially weak*-closed and bounded subset of $\mathcal{M}_{\mathfrak{A}}^{2,m_0}(\mathbb{Y}, \mathbb{Z})$. The continuous functions $\Psi_i, i = 0, 1, 2$, satisfying assumption (a) of Theorem 2.1 can be defined in the obvious way. Also, it is easy to find measurable functions $\Theta_i, i = 0, 1, 2$, with closed sublevel sets such that assumption (b) of Theorem 2.1 is satisfied, and can be used to capture the nonnegativity by setting e.g. $\Theta_i((\eta, \xi)) := \infty$ if η or ξ is negative, $i = 0, 1, 2$.

Every $(\eta, \xi) \in \mathcal{X}_{\mathfrak{A}}$ is a pair of a (equivalence class of a) nonnegative function (or a nonnegative Borel measure that is finite on compact subsets of \mathcal{O}).

Remark 3.14 Note that in order to apply Theorem 2.1 formally, we will assume the obvious modification (or extension) of its statement and proof, such that we can treat pairs of spaces with different exponents like $\mathcal{M}_{\mathfrak{A}}^{2,m_0}(\mathbb{Y}, \mathbb{Z})$.

Step (II). The truncated system.

Let $\phi \in \mathcal{D}(\mathbb{R})$ be a smooth cutoff function that satisfies

$$\phi(x) \begin{cases} = 0, & \text{if } |x| \geq 2, \\ \in [0, 1], & \text{if } 1 < |x| < 2, \\ = 1, & \text{if } |x| \leq 1, \end{cases}$$

and let $\phi_\kappa(x) := \phi(x/\kappa), x \in \mathbb{R}, \kappa \in \mathbb{N}$. In addition, for any progressively measurable pair of processes $(\eta, \xi) \in \mathcal{M}_{\mathfrak{A}}^{2,m_0}(\mathbb{Y}, \mathbb{Z})$ let us define for $t \in [0, T]$

$$h(\eta, \xi, t) := \sup_{0 \leq s \leq t} |\eta(s)|_{H_2^{-1}}^2 + \int_0^t |\eta(s)|_{L^{\gamma+1}}^{\gamma+1} ds + \|\xi\|_{[0,t]} \|\mathbb{H}_\rho\|$$

where $\nu \in (0, 1]$ is chosen such that $\frac{1}{p_0^*} + \frac{\nu m_0}{\gamma+1} \leq 1$ and $\frac{1}{p_0^*} + \frac{\nu}{m_0} \leq 1$.

Let us consider the truncated system given by

$$\begin{cases} du_\kappa(t) = [r_u \Delta(u_\kappa(t))^{[\gamma]} - \chi \phi_\kappa(h(u_\kappa, v_\kappa, t)) u_\kappa(t) v_\kappa^2(t)] dt + \sigma_1 u_\kappa(t) dW_1(t), \\ u_\kappa(0) = u_0, \end{cases} \quad (3.10)$$

and

$$\begin{cases} dv_\kappa(t) = [r_v \Delta v_\kappa(t) + \phi_\kappa(h(u_\kappa, v_\kappa, t))u_\kappa(t)v_\kappa^2(t)] dt + \sigma_2 v_\kappa(t) dW_2(t). \\ v_\kappa(0) = v_0. \end{cases} \tag{3.11}$$

We shall show the existence of a martingale solution to system Eq. 3.10–(3.11).

Proposition 3.15 *For any $\kappa \in \mathbb{N}$, there exists constants $R_0 > 0, R_1 > 0, R_2 > 0$, depending on κ , such that there exists a martingale solution (u_κ, v_κ) to system Eq. 3.10–(3.11) contained in $\mathcal{X}_{\mathfrak{A}}(R_0, R_1, R_2) \subset \mathcal{M}_{\mathfrak{A}}^{2,m_0}(\mathbb{Y}, \mathbb{Z})$.*

Proof of Proposition 3.15 The proof consists of several steps. First, we shall define an operator denoted by \mathcal{V}_κ which satisfies the assumptions of Theorem 2.1, yielding the existence of a martingale solution.

Step (a). Definition of the operator \mathcal{V}_κ .

First, define

$$\mathcal{V}_\kappa := \mathcal{V}_{\kappa, \mathfrak{A}} : \mathcal{X}_{\mathfrak{A}}(R_0, R_1, R_2) \subset \mathcal{M}_{\mathfrak{A}}^{2,m_0}(\mathbb{Y}, \mathbb{Z}) \rightarrow \mathcal{M}_{\mathfrak{A}}^{2,m_0}(\mathbb{Y}, \mathbb{Z})$$

by

$$\mathcal{V}_\kappa(\eta, \xi) := (u_\kappa, v_\kappa), \quad \text{for } (\eta, \xi) \in \mathcal{X}_{\mathfrak{A}}(R_0, R_1, R_2)$$

where u_κ is a solution to

$$\begin{cases} du_\kappa(t) = [r_u \Delta(u_\kappa(t))^{[\gamma]} - \chi \phi_\kappa(h(\eta, \xi, t)) u_\kappa(t) \xi^2(t)] dt + \sigma_1 u_\kappa(t) dW_1(t), \\ u_\kappa(0) = u_0, \end{cases} \tag{3.12}$$

and v_κ is a solution to

$$\begin{cases} dv_\kappa(t) = [r_v \Delta v_\kappa(t) + \phi_\kappa(h(\eta, \xi, t)) \eta(t) \xi^2(t)] dt + \sigma_2 v_\kappa(t) dW_2(t), \\ v_\kappa(0) = v_0. \end{cases} \tag{3.13}$$

The operator \mathcal{V}_κ is well-defined for $(\eta, \xi) \in \mathcal{X}_{\mathfrak{A}}(R_0, R_1, R_2)$. In fact, by Theorem 5.4 and Proposition 5.5, given such a pair of processes (η, ξ) , the existence of a nonnegative unique solution u_κ to Eq. 3.12 for nonnegative initial data u_0 with

$$\mathbb{E} \|u_\kappa\|_{\mathbb{Y}}^2 \leq C_1(\kappa, T) \tag{3.14}$$

follows. By Proposition 5.7 and Proposition 5.8, the existence of a unique solution v_κ to Eq. 3.13 with

$$\mathbb{E} \|v_\kappa\|_{\mathbb{H}^\rho}^{m_0} \leq \mathbb{E} |v_0|_{H_\rho^0}^{m_0} + C(\kappa, T) R_0,$$

follows.

Step (b). The \mathcal{V}_κ -invariant set \mathcal{X} .

Let

$$C_0(T) \left(\mathbb{E} |u_0|_{L^{p^*}}^{p_0^*} + 1 \right) \leq R_0,$$

where the constant $C_0(\gamma, T)$ is as in Eq. 5.6. Then, due to Proposition 5.6, we know that

$$\mathbb{E} \left[\sup_{0 \leq s \leq T} |u_\kappa(s)|_{L^{p^*}}^{p_0^*} \right] \vee \mathbb{E} \left[\sup_{0 \leq s \leq T} |u_\kappa(s)|_{L^{\gamma+1}}^{\gamma+1} \right] \leq R_0.$$

Furthermore, by Proposition 5.7, the existence of a unique solution to Eq. 3.13 such that

$$\mathbb{E}\|v_\kappa\|_{\mathbb{H}_\rho}^{m_0} \leq \mathbb{E}|v_0|_{H_2^\rho}^{m_0} + R_0 C_2(\kappa, T) \tag{3.15}$$

follows.

Let $R_2 := \mathbb{E}|v_0|_{H_2^\rho}^{m_0} + R_0 C_2(\kappa, T)$. Finally, by Proposition 5.9 we can show that for $R_1 \geq C(T) \left\{ \mathbb{E}|v_0|_{H_m^{-\delta_0}}^{m_0} + C(\kappa) \right\}$ (here the constants are given by Proposition 5.9)

$$\mathbb{E}\|v_\kappa\|_{\mathbb{Z}}^{m_0} \leq R_1. \tag{3.16}$$

Then, \mathcal{V}_κ maps $\mathcal{X}_{\mathfrak{A}}(R_0, R_1, R_2)$ into itself. Hence, assumption (i) of Theorem 2.1 is satisfied.

Step (c). Continuity of \mathcal{V}_κ on \mathcal{X} .

Next we show that assumption (ii) of Theorem 2.1 is satisfied. We need to show that the restriction of the operator \mathcal{V}_κ to $\mathcal{X}_{\mathfrak{A}}(R_0, R_1, R_2)$ is continuous in the strong topology of $\mathcal{M}_{\mathfrak{A}}^{2,m_0}(\mathbb{Y}, \mathbb{Z})$. Let $\{(\eta^{(n)}, \xi^{(n)}) : n \in \mathbb{N}\} \subset \mathcal{X}_{\mathfrak{A}}(R_0, R_1, R_2)$ be a sequence converging strongly to (η, ξ) in $\mathcal{M}_{\mathfrak{A}}^{2,m_0}(\mathbb{Y}, \mathbb{Z})$.

Firstly, due to Proposition 5.12 and Remark 3.13, we know that the sequence $\{u_\kappa^{(n)} : n \in \mathbb{N}\}$ where $u_\kappa^{(n)}$ solves

$$\begin{cases} du_\kappa^{(n)}(t) = \left[r_u \Delta(u_\kappa^{(n)}(t))^{[\gamma]'} - \chi \phi_\kappa(h(\eta^{(n)}, \xi^{(n)}, t)) u_\kappa^{(n)}(t) (\xi^{(n)}(t))^2 \right] dt \\ \quad + \sigma_1 u_\kappa^{(n)}(t) dW_1(t), \\ u_\kappa^{(n)}(0) = u_0. \end{cases} \tag{3.17}$$

converges in $\mathcal{M}_{\mathfrak{A}}^{2,m_0}(\mathbb{Y}, \mathbb{Z})$ to u_κ which solves Eq. 3.12.

Secondly, due to Proposition 5.11 and keeping Remark 3.13 in mind, it follows that the sequence $\{v_\kappa^{(n)} : n \in \mathbb{N}\}$, where $v_\kappa^{(n)}$ solves

$$\begin{cases} dv_\kappa^{(n)}(t) = \left[r_v \Delta v_\kappa^{(n)}(t) + \phi_\kappa(h(\eta^{(n)}, \xi^{(n)}, t)) \eta^{(n)}(t) (\xi^{(n)}(t))^2 \right] dt \\ \quad + \sigma_2 v_\kappa^{(n)}(t) dW_2(t), \\ v_\kappa^{(n)}(0) = v_0. \end{cases} \tag{3.18}$$

converges in $\mathcal{M}_{\mathfrak{A}}^{2,m_0}(\mathbb{Y}, \mathbb{Z})$ to v_κ , where v_κ solves Eq. 3.13.

This gives that the operator

$$\mathcal{V}_\kappa|_{\mathcal{X}_{\mathfrak{A}}(R_0, R_1, R_2)} : \mathcal{X}_{\mathfrak{A}}(R_0, R_1, R_2) \subset \mathcal{M}_{\mathfrak{A}}^{2,m_0}(\mathbb{Y}, \mathbb{Z}) \rightarrow \mathcal{M}_{\mathfrak{A}}^{2,m_0}(\mathbb{Y}, \mathbb{Z})$$

is continuous.

Step (d). Tightness.

Next we show that assumption (iii) of Theorem 2.1 is satisfied.

First, note that the embedding $\mathbb{Z}' \hookrightarrow \mathbb{Z}$ is compact by the Aubin-Lions-Simon Lemma B.1.

Second, by Proposition 5.9 it follows that for any $\kappa \in \mathbb{N}$ there exists a constant $C = C(\kappa, T) > 0$ such that for all $(\eta, \xi) \in \mathcal{X}_{\mathfrak{A}}(R_0, R_1, R_2)$

$$\mathbb{E}\|v_\kappa\|_{\mathbb{Z}'}^{m_0} \leq C,$$

where v_κ solves Eq. 3.13. Observe that we have the compact embedding $L^{\gamma+1}(\mathcal{O}) \hookrightarrow H_2^{-1}(\mathcal{O})$. Hence, by Proposition 5.13 and by standard arguments (cf. [24]), we know that

the laws of the set $\{u_\kappa : (\eta, \xi) \in \mathcal{X}_{2\mathfrak{A}}(R_0, R_1, R_2)\}$ are tight on $C([0, T]; H_2^{-1}(\mathcal{O}))$. By Proposition 5.13 and the Aubin-Lions-Simon Lemma B.1, the laws of $\{u_\kappa : (\eta, \xi) \in \mathcal{X}_{2\mathfrak{A}}(R_0, R_1, R_2)\}$ are tight on $L^{\gamma+1}(0, T; L^{\gamma+1}(\mathcal{O}))$. Therefore, it follows that the set $\{\mathcal{V}_\kappa(\eta, \xi) : (\eta, \xi) \in \mathcal{X}_{2\mathfrak{A}}(R_0, R_1, R_2)\}$ is tight on $\mathbb{Y} \times \mathbb{Z}$, which implies assumption (iii) of Theorem 2.1 for

$$\mathbb{X}' := L^{\gamma+1}(0, T; L^{\gamma+1}(\mathcal{O})) \times \mathbb{Z}' \hookrightarrow \mathbb{Y} \times \mathbb{Z} =: \mathbb{X}$$

Step (e). Continuity of paths.

To show assumption (iv) of Theorem 2.1, we notice that by choosing

$$U := H_2^{-1}(\mathcal{O}) \times H_2^\rho(\mathcal{O}),$$

it follows from Proposition 5.13 and standard arguments on continuity of solutions to non-linear SPDEs that

$$\mathcal{V}_\kappa(\eta, \xi) \in C([0, T]; U) \subset \mathbb{D}([0, T]; U) \quad \mathbb{P}\text{-a.s.}$$

for all $(\eta, \xi) \in \mathcal{X}_{2\mathfrak{A}}(R_0, R_1, R_2)$.

Step (f). Existence of a fixed point.

Finally, by Theorem 2.1, for each $\kappa \in \mathbb{N}$, there exists a probability space $\tilde{\mathfrak{A}}_\kappa = (\tilde{\Omega}_\kappa, \tilde{\mathcal{F}}_\kappa, \tilde{\mathbb{F}}_\kappa, \tilde{\mathbb{P}}_\kappa)$, a Wiener process $\tilde{W}^\kappa = (\tilde{W}_1^\kappa, \tilde{W}_2^\kappa)$, modeled on $\tilde{\mathfrak{A}}_\kappa$, and elements

$$(\tilde{u}_\kappa, \tilde{v}_\kappa) \in \mathcal{X}_{2\tilde{\mathfrak{A}}_\kappa}(R_0, R_1, R_2) \cap C([0, T]; U)$$

such that we have a $\tilde{\mathbb{P}}$ -a.s. fixed point

$$\mathcal{V}_{\kappa, \tilde{\mathfrak{A}}_\kappa}(\tilde{u}_\kappa(t), \tilde{v}_\kappa(t)) = (\tilde{u}_\kappa(t), \tilde{v}_\kappa(t))$$

for every $t \in [0, T]$. Due to the construction of $\mathcal{V}_{2\mathfrak{A}}$, the pair $(\tilde{u}_\kappa, \tilde{v}_\kappa)$ solves the system Eq. 3.12–(3.13) over the stochastic basis $\tilde{\mathfrak{A}}_\kappa$ with the Wiener noise \tilde{W}^κ , see Section 4 for details.

Step (III). Here, we will construct a family of solutions $\{(\tilde{u}_\kappa, \tilde{v}_\kappa) : \kappa \in \mathbb{N}\}$ following the solution to the original problem until a stopping time $\tilde{\tau}_\kappa$. In particular, we will introduce for each $\kappa \in \mathbb{N}$ a new pair of processes $(\tilde{u}_\kappa, \tilde{v}_\kappa)$ following the Klausmeier system up to the stopping time $\tilde{\tau}_\kappa$. Besides, we will have $(\tilde{u}_\kappa, \tilde{v}_\kappa)|_{[0, \tilde{\tau}_\kappa)} = (\tilde{u}_{\kappa+1}, \tilde{v}_{\kappa+1})|_{[0, \tilde{\tau}_\kappa)}$.

Let us start with $\kappa = 1$. From Proposition 3.15, we know there exists a martingale solution consisting of a probability space $\mathfrak{A}_1 = (\Omega_1, \mathcal{F}_1, \mathbb{F}_1, \mathbb{P}_1)$, two independent Wiener processes (W_1^1, W_2^1) defined over \mathfrak{A}_1 , and a couple of processes (u_1, v_1) solving \mathbb{P}_1 -a.s. the system

$$\begin{cases} du_1(t) = \left[r_u \Delta u_1^{[\gamma]}(t) - \chi \phi_1(h(u_1, v_1, t)) u_1(t) v_1^2(t) \right] dt + \sigma_1 u_1(t) dW_1^1(t), \\ dv_1(t) = \left[r_v \Delta v_1(t) + \phi_1(h(u_1, v_1, t)) u_1(t) v_1^2(t) \right] dt + \sigma_2 v_1(t) dW_2^1(t), \\ (u_1(0), v_1(0)) = (u_0, v_0). \end{cases}$$

Let us define now the stopping time

$$\tau_1^* := \inf\{s \geq 0 : h(u_1, w_1, s) \geq 1\}.$$

Observe, on the time interval $[0, \tau_1^*)$, the pair (u_1, v_1) solves the system given in (1.2). Now, we define a new pair of processes $(\tilde{u}_1, \tilde{v}_1)$ following (u_1, v_1) on $[0, \tau_1^*)$ and extend this processes to the whole interval $[0, T]$ in the following way.

First, we put $\bar{\mathfrak{A}}_1 := \mathfrak{A}_1$ and $\bar{W}_j^1 := W_j^1, j = 1, 2$, and let us introduce the processes y_1 and y_2 being a strong solution over $\bar{\mathfrak{A}}_1$ to

$$\begin{aligned} dy_1(t, u_1(\tau_1^*), \sigma) &= r_u \Delta y_1^{[y_1]}(t, u_1(\tau_1^*), \sigma) dt + \sigma_1 y_1(t, u_1(\tau_1^*), \sigma) d(\theta_\sigma \bar{W}_1^1)(t) \\ y_1(0, u_1(\tau_1^*)) &= u_1(\tau_1^*), \quad t \geq 0, \\ y_2(t, v_1(\tau_1^*), \sigma) &= e^{r_v \Delta t} v_1(\tau_1^*) \\ &+ \int_0^t e^{(r_v \Delta)(t-s)} \sigma_2 y_2(s, v_1(\tau_1^*), \sigma) d(\theta_\sigma \bar{W}_2^1)(s), \quad t \geq 0. \end{aligned} \tag{3.19}$$

$$\tag{3.20}$$

Here, θ_σ is the shift operator which maps $W_j(t)$ to $W_j(t + \sigma) - W_j(\sigma)$. Since the couple (u_1, v_1) is continuous in $H_2^{-1}(\mathcal{O}) \times H_2^\rho(\mathcal{O})$, we know that $u_1(\tau_1^*)$ and $v_1(\tau_1^*)$ are well-defined random variables belonging \mathbb{P}_1 -a.s. to $L^2(\mathcal{O})$ and $H_2^\rho(\mathcal{O})$, respectively. By [2, Theorem 2.5.1] the existence of a unique solution y_1 over $\bar{\mathfrak{A}}_1$ to Eq. 3.19 in $H_2^{-1}(\mathcal{O})$ is given. Since $(e^{t(r_v \Delta - \tilde{\alpha} Id)})_{t \geq 0}$ is an analytic semigroup on $H_2^\rho(\mathcal{O})$ for $\tilde{\alpha} > 0$, the existence of a unique solution y_2 over $\bar{\mathfrak{A}}_1$ to Eq. 3.20 in $H_2^\rho(\mathcal{O})$ can be shown by standard methods, cf. [16]. Furthermore, it is straightforward to verify that the assumptions on the initial conditions are satisfied. Now, let us define two processes \bar{u}_1 and \bar{v}_1 being identical to u_1 and v_1 , respectively, on the time interval $[0, \tau_1^*)$ and following the porous media, respective, the heat equation (with lower order terms) with noise and without nonlinearity, i.e., $y_1(\cdot, u_1(\tau_1^*), \tau_1^*)$ and $y_2(\cdot, v_1(\tau_1^*), \tau_1^*)$, afterwards. In particular, let

$$\bar{u}_1(t) = \begin{cases} u_1(t) & \text{for } 0 \leq t < \tau_1^*, \\ y_1(t, u_1(\tau_1^*), \tau_1^*) & \text{for } \tau_1^* \leq t \leq T, \end{cases}$$

and

$$\bar{v}_1(t) = \begin{cases} v_1(t) & \text{for } 0 \leq t < \tau_1^*, \\ y_2(t, v_1(\tau_1^*), \tau_1^*) & \text{for } \tau_1^* \leq t \leq T. \end{cases}$$

Let us now construct the probability space and the processes for the next time interval. First, let $(u_1(\tau_1^*), v_1(\tau_1^*))$ have probability law μ_1 on $H_2^{-1}(\mathcal{O}) \times H_2^\rho(\mathcal{O})$. Again, from Proposition 3.15, we know there is a martingale solution consisting of a probability space $\mathfrak{A}_2 = (\Omega_2, \mathcal{F}_2, \mathbb{F}_2, \mathbb{P}_2)$, a pair of independent Wiener processes (W_1^2, W_2^2) such that $(\bar{W}_1^1, \bar{W}_2^1, W_1^2, W_2^2)$ are independent as well, a couple of processes (u_2, v_2) solving \mathbb{P}_2 -a.s. the system

$$\begin{cases} du_2(t) = \left[r_u \Delta u_2^{[y_1]}(t) - \chi \phi_2(h(u_2, v_2, t)) u_2(t) v_2^2(t) \right] dt + \sigma_2 u_2(t) dW_1^2(t), \\ dv_2(t) = \left[r_v \Delta v_2(t) + \phi_2(h(u_2, v_2, t)) u_2(t) v_2^2(t) \right] dt + \sigma_2 v_2(t) dW_2^2(t), \\ (u_2(0), v_2(0)) \sim \mu_1. \end{cases}$$

with initial condition $(u_2(0), v_2(0))$ having law μ_1 . Let us define now the stopping times on \mathfrak{A}_2 ,

$$\tau_2^* := \inf\{s \geq 0 : h(u_2, v_2, s) \geq 2\}.$$

Let $\bar{\mathfrak{A}}_1 := (\bar{\Omega}_1, \bar{\mathcal{F}}_1, \bar{\mathbb{F}}_1, \bar{\mathbb{P}}_1) := \mathfrak{A}_1$, with $\bar{\mathbb{F}}_1 := (\bar{\mathcal{F}}_t)_{t \in [0, T]}$. Let $\bar{\Omega}_2 := \bar{\Omega}_1 \times \Omega_2, \bar{\mathcal{F}}_2 := \bar{\mathcal{F}}_1 \otimes \mathcal{F}_2, \bar{\mathbb{P}}_2 := \bar{\mathbb{P}}_1 \otimes \mathbb{P}_2$ and let $\bar{\mathbb{F}}_2 := (\bar{\mathcal{F}}_t^2)_{t \in [0, T]}$, where

$$\bar{\mathcal{F}}_t^2 := \sigma \left(\left\{ A \cap \{t < \tau_1^*\} : A \in \bar{\mathcal{F}}_t^1 \right\} \cup \bar{\mathcal{F}}_0^1 \cup \left\{ A \cap \{t \geq \tau_1^*\} : A \in \mathcal{F}_{t-\tau_1^*}^2 \right\} \right),$$

where

$$\mathcal{F}_{t-\tau_1^*}^2 := \{A \in \mathcal{F}_\infty^2 : A \cap \{t - \tau_1^* \leq s\} \in \mathcal{F}_s^2 \text{ for all } s \geq 0\},$$

and where $\mathcal{F}_\infty^2 := \sigma(\bigcup_{t \geq 0} \mathcal{F}_t^2)$. Let $\bar{\mathcal{Q}}_2 := (\bar{\Omega}_2, \bar{\mathcal{F}}_2, \bar{\mathbb{P}}_2, \bar{\mathbb{P}}_2)$. Finally, let us set for $j = 1, 2$

$$\bar{W}_j^2(t) := \begin{cases} \bar{W}_j^1(t), & \text{if } t < \tau_1^*, \\ W_j^2(t - \tau_1^*) + \bar{W}_j^1(\tau_1^*), & \text{if } t \geq \tau_1^*. \end{cases}$$

which give independent Wiener processes for $j = 1, 2$, w.r.t. the filtration $\bar{\mathbb{F}}_2$.

Now, let us define two processes \bar{u}_2 and \bar{v}_2 being identical to \bar{u}_1 and \bar{v}_1 , respectively, on the time interval $[0, \tau_1^*]$, being identical to u_2 and v_2 on the time interval $[\tau_1^*, \tau_1^* + \tau_2^*]$ and following the porous media, respective, the heat equation (with lower order terms) with multiplicative noise, afterwards. Let us note, for any initial condition having distribution equal to $u_2(\tau_2^*)$ and $v_2(\tau_2^*)$ that there exists a strong solutions $y_1(\cdot, \cdot, \tau_2^* + \tau_1^*)$ and $y_2(\cdot, \cdot, \tau_2^* + \tau_1^*)$ of the systems Eq. 3.19 and Eq. 3.20, respectively, on $\bar{\mathcal{Q}}_2$. Let for $t \in [0, T]$

$$\bar{u}_2(t) = \begin{cases} \bar{u}_1(t) & \text{for } 0 \leq t < \tau_1^*, \\ u_2(t - \tau_1^*) & \text{for } \tau_1^* \leq t \leq \tau_1^* + \tau_2^*, \\ y_1(t - (\tau_1^* + \tau_2^*), u_2(\tau_2^*), \tau_1^* + \tau_2^*) & \text{for } \tau_2^* + \tau_1^* \leq t \leq T, \end{cases}$$

$$\bar{v}_2(t) = \begin{cases} v_1(t) & \text{for } 0 \leq t < \tau_1^*, \\ v_2(t - \tau_1^*) & \text{for } \tau_1^* \leq t \leq \tau_1^* + \tau_2^*, \\ y_2(t - (\tau_1^* + \tau_2^*), v_2(\tau_2^*), \tau_1^* + \tau_2^*) & \text{for } \tau_1^* + \tau_2^* \leq t \leq T. \end{cases}$$

In the same way we will construct for any $\kappa \in \mathbb{N}$ a probability space $\bar{\mathcal{Q}}_\kappa$, a pair of independent Wiener processes $(\bar{W}_\kappa^1, \bar{W}_\kappa^2)$, over $\bar{\mathcal{Q}}_\kappa$ and a pair of processes $(\bar{u}_\kappa, \bar{v}_\kappa)$ starting at (u_0, v_0) and solving system Eq. 3.10-(3.11) up to time

$$\bar{\tau}_\kappa := \tau_1^* + \dots + \tau_\kappa^* \tag{3.21}$$

and following the porous media, respective, the heat equation afterwards. Besides, we know that $(\bar{u}_\kappa, \bar{v}_\kappa)|_{[0, \bar{\tau}_{\kappa-1}]} = (\bar{u}_{\kappa-1}, \bar{v}_{\kappa-1})|_{[0, \bar{\tau}_{\kappa-1}]}$.

Step (IV). Uniform bounds on the stopping time.

Let us consider the family $\{(\bar{u}_\kappa, \bar{v}_\kappa) : \kappa \in \mathbb{N}\}$. The next aim is to show that there exists R_0, R_1 and $R_2 > 0$ independent of $\kappa \in \mathbb{N}$ such that $\{(\bar{u}_\kappa, \bar{v}_\kappa) : \kappa \in \mathbb{N}\} \subset \mathcal{X}_{\mathcal{Q}}(R_0, R_1, R_2)$.

First, that due to Proposition 5.14 there exists a constant $C_0(1, T) > 0$ such that

$$\mathbb{E} \left[\sup_{0 \leq s \leq \bar{\tau}_\kappa \wedge T} |\bar{u}_\kappa(s)|_{L^2}^2 \right] + 2\gamma r u \mathbb{E} \int_0^{\bar{\tau}_\kappa \wedge T} \int_{\mathcal{O}} |\bar{u}_\kappa(s, x)|^{\gamma-1} |\nabla \bar{u}_\kappa(s, x)|^2 dx ds$$

$$+ 2\chi \mathbb{E} \int_0^{\bar{\tau}_\kappa \wedge T} \int_{\mathcal{O}} |\bar{u}_\kappa(s, x)|^2 |\bar{v}_\kappa(s, x)|^2 dx ds \leq C_0(1, T) (\mathbb{E}|u_0|_{L^2}^2 + 1).$$

From above we can conclude that there exists a constant $C > 0$ such that

$$\mathbb{E} \|\bar{u}_\kappa \mathbb{1}_{[0, \bar{\tau}_\kappa \wedge T]}\|_{\mathbb{Y}}^2 \leq C.$$

Here, it is important that $\bar{u}_\kappa \geq 0$. This estimate can be extended to the time interval $[0, T]$ by standard results (see e.g. [2] and Proposition 5.14). Next, let us assume $R_1 > 0$ is that large that

$$C(T) \left(\mathbb{E}|v_0|_{H_m^{-\delta_0}}^{m_0} + R_0^{\delta_1} R_2^{\delta_1} \right) \leq R_2, \tag{3.23}$$

where the constants $C(T), \delta_0, \delta_1$ are given in Proposition 5.15. Observe, by Proposition 5.15, we have for l as in Hypothesis 3.4

$$\mathbb{E}\|\bar{v}_\kappa\|_{\mathbb{Z}}^{m_0} \leq C(T) \left(\mathbb{E}|v_0|_{H_m^{-\delta_0}}^{m_0} + \left(\mathbb{E} \left(\int_0^{\bar{\tau}_\kappa \wedge T} u_\kappa^2(s) v_\kappa^2(s) ds \right)^l \right)^{\delta_1} R_1^{\delta_1} \right). \tag{3.24}$$

Then, by Proposition 5.14, we know that for $p = 1$, the term

$$\mathbb{E} \left(\int_0^{\bar{\tau}_\kappa \wedge T} u_\kappa^2(s) v_\kappa^2(s) ds \right)^l \leq \mathbb{E}|u_0|_{L^2}^l,$$

which can be uniformly bounded by R_0^l . Hence, we conclude by Eq. 3.23 that $\mathbb{E}\|\bar{v}_\kappa\|_{\mathbb{Z}}^{m_0} \leq R_1$ and the choice of R_2 that $\mathbb{E}\|\bar{v}_\kappa\|_{\mathbb{H}_\rho}^{m_0} \leq R_2$. Finally, by Proposition A.6 we know for $\frac{d}{2} - \rho \leq \frac{2}{m} + \frac{d}{m_0}$ that $\mathbb{H}_\rho \hookrightarrow \mathbb{Z}$. Due to the choice of m and m_0 , we know that the inequality is satisfied. Hence, setting

$$R_1 \geq R_2, \tag{3.25}$$

we know that $\mathbb{E}\|\bar{v}_\kappa\|_{\mathbb{Z}}^{m_0} \leq R_1$. In particular, there exists R_0, R_1 and R_2 such that for all $\kappa \in \mathbb{N}$ we have

$$\mathbb{E} \left[\sup_{0 \leq s \leq T} |\bar{u}_\kappa(s)|_{L^{p^*}}^{p_0^*} \right] \leq R_0^{p_0^*}, \tag{3.26}$$

$\mathbb{E}\|\bar{v}_\kappa\|_{\mathbb{Z}}^{m_0} \leq R_1$, and $\mathbb{E}\|\bar{v}_\kappa\|_{\mathbb{H}_\rho}^{m_0} \leq R_2$.

Step (V). Passing on to the limit.

In the final step, we will show that \mathbb{P} -a.s. a martingale solution to Eq. 3.6-(3.7) exists.

Claim 3.16 *Let*

$$(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P}) := \left(\prod_{\kappa \in \mathbb{N}} \bar{\Omega}_\kappa, \bigotimes_{\kappa \in \mathbb{N}} \bar{\mathcal{F}}_\kappa, \bar{\mathbb{F}}_\infty, \bigotimes_{\kappa \in \mathbb{N}} \bar{\mathbb{P}}_\kappa \right),$$

where $\bar{\mathbb{F}}_\infty := (\bar{\mathcal{F}}_t^\infty)_{t \in [0, T]}$, where, setting $\tau_0^* := 0$,

$$\bar{\mathcal{F}}_t^\infty := \bigcap_{\kappa \in \mathbb{N}} \bar{\mathcal{F}}_t^\kappa.$$

There exists a measurable set $\Omega_0 \subset \Omega$ with $\mathbb{P}(\Omega_0) = 1$ such that a martingale solution (u, v) to the system Eq. 3.6-(3.7) exists on $(\Omega_0, \mathcal{F} \cap \Omega_0, \mathbb{F} \cap \Omega_0, \mathbb{P}|_{\Omega_0}, (\bar{W}_1^\infty|_{\Omega_0}, \bar{W}_2^\infty|_{\Omega_0}))$, where for $j = 1, 2$, setting $\tau_0^* := 0$,

$$\bar{W}_j^\infty(t) := \bar{W}_j^\kappa(t), \text{ if } t \in [\tau_{\kappa-1}^*, \tau_\kappa^*), \quad \kappa \in \mathbb{N}.$$

Proof For any $\kappa \in \mathbb{N}$, let us define the set

$$A_\kappa := \{\omega \in \Omega : \bar{\tau}_\kappa(\omega) \geq T\}.$$

It can be clearly observed that there exists a progressively measurable process (u, v) over \mathfrak{A} such that (u, v) solves \mathbb{P} -a.s. the integral equation given by (3.6)-(3.7) up to time T . In particular, we have for the conditional probability

$$\mathbb{P}(\{\text{there exists a solution } (u, v) \text{ to Eq. 3.6-(3.7)}\} \mid A_\kappa) = 1.$$

Set $\Omega_0 = \bigcup_{\kappa=1}^\infty A_\kappa$. Then, it is elementary to verify that

$$\begin{aligned} & \mathbb{P}(\{\text{there exists a solution } (u, v) \text{ to Eq. 3.6-(3.7)}\} \cap \Omega_0) \\ &= \lim_{\kappa \rightarrow \infty} \mathbb{P}(\{\text{there exists a solution } (u, v) \text{ to Eq. 3.6-(3.7)}\} \cap A_\kappa) \\ &= \lim_{\kappa \rightarrow \infty} [\mathbb{P}(\{\text{there exists a solution } (u, v) \text{ to Eq. 3.6-(3.7)}\} \mid A_\kappa) \mathbb{P}(A_\kappa)]. \end{aligned}$$

Since $\mathbb{P}(\{\text{a solution } (u, v) \text{ to Eq. 3.6-(3.7) exists}\} \mid A_\kappa) = 1$, it remains to show that $\lim_{\kappa \rightarrow \infty} \mathbb{P}(A_\kappa) = 1$. Then, as $A_\kappa \subset A_{\kappa+1}$ for $\kappa \in \mathbb{N}$, it follows that

$$\mathbb{P}(\{\text{there exists a solution } (u, v) \text{ to Eq. 3.6-(3.7)}\} \cap \Omega_0) = 1.$$

However, due to Step IV, there exists a constant $C(T) > 0$ such that

$$\mathbb{E}[h(\bar{u}_\kappa, \bar{v}_\kappa, t)] \leq C(T), \quad t \in [0, T], \quad \kappa \in \mathbb{N},$$

thus by the Markov inequality,

$$\mathbb{P}(\Omega \setminus A_\kappa) \leq \frac{C(T)}{\kappa} \rightarrow 0,$$

as $\kappa \rightarrow \infty$. Thus the solution process is well-defined on $\Omega_0 = \bigcup_{\kappa=1}^\infty A_\kappa$ with $\mathbb{P}(\Omega_0) = 1$. \square

The proof of Theorem 3.6 is complete.

4 Proof of The Stochastic Schauder-Tychonoff Theorem

Proof of Theorem 2.1 Fix \mathfrak{A} and W , and $R_1, \dots, R_K > 0, K \in \mathbb{N}$. In addition, for simplification, we shall omit R_1, \dots, R_K in the notation for $\mathcal{X}_{R_1, \dots, R_K}(\mathfrak{A})$ and write $\mathcal{X}(\mathfrak{A})$ instead of $\mathcal{X}_{R_1, \dots, R_K}(\mathfrak{A})$. Fix an initial datum $w_0 \in L^m(\Omega, \mathcal{F}_0, \mathbb{P}; E)$.

Step (I). In the first step we will approximate the operator $\mathcal{V}_{\mathfrak{A}, W}$. We shall discretize time, as we would like apply the classical Schauder-Tychonoff theorem in a compact subset which is given by a tight collection of laws on a finite time grid. Let us fix a sequence $\{\varepsilon_t : t \in \mathbb{N}\}$ such that $\varepsilon_t \rightarrow 0$.

First, let us introduce a dyadic time grid $\pi_n = \{t_0 = 0 < t_1 < t_2 < \dots < t_{2^n} = T\}$ by $t_k = T \frac{k}{2^n}, k = 0, 1, 2, \dots, 2^n$. The stochastic process will be approximated by an averaging operator over the dyadic time interval. To this end, let us define a step-function $\phi_n : [0, T] \rightarrow [0, T]$ by $\phi_n(s) = T \frac{k}{2^n}$, if $k = 0, 1, 2, \dots, 2^n - 1$ and $T \frac{k}{2^n} \leq s < T \frac{k+1}{2^n}$, i.e., $\phi_n(s) = T 2^{-n} [2^n s], s \geq 0$, where $[t]$ is the largest integer that is less or equal $t \in \mathbb{R}$. Let $\{w_n : n \in \mathbb{N}\} \subset L^m(\Omega, \mathcal{F}_0, \mathbb{P}; E)$ be a sequence, such that $w_n \rightarrow w_0$ in $L^m(\Omega, \mathcal{F}_0, \mathbb{P}; E)$ and

$$\|w_n - w_0\|_{L^m(\Omega, \mathcal{F}_0, \mathbb{P}; E)} \leq \frac{\varepsilon_n}{n}.$$

For a function $\xi \in \mathcal{M}_{\mathfrak{A}}^m(\mathbb{X})$, we define

$$\text{Proj}_n(\xi)(s) := \begin{cases} w_n, & \text{if } s \in [0, T 2^{-n}), \\ \frac{2^n}{T} \int_{\phi_n(s) - T 2^{-n}}^{\phi_n(s)} \xi(r) dr, & \text{if } s \geq T 2^{-n}. \end{cases} \tag{4.1}$$

Note, that $\text{Proj}_n(\xi)$ is a progressively measurable, \mathbb{P} -a.s. piecewise constant, U -valued stochastic process.

Remark 4.1 Observe that the projection operator satisfies

- (i) Proj_n is a linear bounded contraction operator from \mathbb{X} into \mathbb{X} ;
- (ii) If B is a bounded subset of \mathbb{X} , then for all $\varepsilon > 0$ there exists a $n_0 \in \mathbb{N}$ such that

$$\|\text{Proj}_n(\xi) - \xi\|_{\mathbb{X}} < \varepsilon, \quad \xi \in B, \quad n \geq n_0,$$

cf. [11, Appendix B].

Thus, due to Remark 4.1 and the uniform continuity of $\mathcal{V}_{\mathfrak{A},W}$ on the bounded set $\mathcal{X}(\mathfrak{A})$, we know that for any $\varepsilon > 0$, there exists some $n_0 \in \mathbb{N}$ such that for every ξ uniformly in $\mathcal{V}_{\mathfrak{A},W}(\mathcal{X}(\mathfrak{A}))$ such that for any $r \in (1, m]$,

$$(\mathbb{E}\|\text{Proj}_n(\xi) - \xi\|_{\mathbb{X}}^r)^{\frac{1}{r}} \leq \varepsilon, \quad \forall \xi \in \mathcal{V}_{\mathfrak{A},W}(\mathcal{X}(\mathfrak{A})), \quad \forall n \geq n_0.$$

Let $\{n_l : l \in \mathbb{N}\}$ be a sequence such that

$$(\mathbb{E}\|\text{Proj}_{n_l}(\xi) - \xi\|_{\mathbb{X}}^m)^{\frac{1}{m}} \leq \frac{\varepsilon_l}{3}, \quad \forall \xi \in \mathcal{V}_{\mathfrak{A},W}(\mathcal{X}(\mathfrak{A})). \tag{4.2}$$

Finally, let us define the operator

$$\mathcal{V}_{\mathfrak{A},W}^l(\xi) := (\text{Proj}_{n_l} \circ \mathcal{V}_{\mathfrak{A},W})(\xi), \quad \xi \in \mathcal{X}(\mathfrak{A}).$$

Step (II). Denote $\mathbb{U} := \mathbb{D}([0, T]; U)$, the Skorokhod space of càdlàg paths in U endowed with the Skorokhod J_1 -topology, see [39, Appendix A2]. Given the probability space $\mathfrak{A} = (\Omega, \mathcal{F}, \mathbb{P})$, for any $l \in \mathbb{N}$ this operator $\mathcal{V}_{\mathfrak{A},W}^l$ induces an operator \mathcal{V}_l on the set of Borel probability measures on $\mathbb{X} \cap \mathbb{U}$, denoted by $\mathcal{M}_1(\mathbb{X} \cap \mathbb{U})$. The construction of the operator \mathcal{V}_l is done in this step. Define the subset of probability measures \mathcal{X} given by

$$\mathcal{X} := \left\{ \mu \in \mathcal{M}_1(\mathbb{X} \cap \mathbb{U}) : \int_{\mathbb{X}} \Psi_j(\xi) \mu(d\xi) \leq R_j, \right. \\ \left. \text{and } \mu(\{\Theta_j < \infty\}) = 1 \quad \forall j = 1, \dots, K \right\}.$$

Now, let $\mu \in \mathcal{X}$. Then, by the Skorokhod lemma [39, Theorem 4.30], we know that there exists a probability space $\mathfrak{A}_0 = (\Omega_0, \mathcal{F}^0, \mathbb{P}_0)$ and a random variable $\xi : \Omega_0 \rightarrow \mathbb{X} \cap \mathbb{U}$ such that the law of ξ coincides with μ . In particular, the probability measure $\nu_\xi : \mathcal{B}(\mathbb{X} \cap \mathbb{U}) \rightarrow [0, 1]$ induced by ξ and given by

$$\nu_\xi : \mathcal{B}(\mathbb{X} \cap \mathbb{U}) \ni A \mapsto \mathbb{P}_0(\{\omega : \xi(\omega) \in A\})$$

coincides with the probability measure μ .

Due to the definition of \mathbb{U} , we know that ξ is a progressively measurable stochastic process, in particular, $\xi : \Omega_0 \times [0, T] \rightarrow U$ such that $\mathbb{P}_0(\xi \in \mathbb{X} \cap \mathbb{U}) = 1$. Let

$$\mathcal{G}_t^0 := \sigma(\{\xi(s) : 0 \leq s \leq t\} \cup \mathcal{N}_0), \quad t \in [0, T],$$

where \mathcal{N}_0 denotes the collection of zero sets of \mathfrak{A}_0 . Set $\mathfrak{A}_0 := (\Omega_0, \mathcal{F}^0, (\mathcal{G}_t^0)_{t \in [0, T]}, \mathbb{P}_0)$.

Next, we have to construct the Wiener process and extend the probability space. Now, let $\mathfrak{A}_1 = (\Omega_1, \mathcal{F}^1, (\mathcal{F}_t)_{t \in [0, T]}, \mathbb{P}_1)$ be a probability space where a cylindrical Wiener process W on H is defined, and let \mathfrak{A}_μ the product probability space of \mathfrak{A}_0 and \mathfrak{A}_1 . In particular, we set

$$\begin{aligned} \Omega_\mu &= \Omega_0 \times \Omega_1, \\ \mathcal{F}_\mu &= \mathcal{F}^0 \otimes \mathcal{F}^1, \\ \mathcal{G}_t^\mu &= \mathcal{G}_t^0 \otimes \mathcal{G}_t^1, \quad t \in [0, T], \\ \text{and } \mathbb{P}_\mu &= \mathbb{P}_0 \otimes \mathbb{P}_1. \end{aligned}$$

Here, we know that $\xi(t), t \in [0, T]$, is independent of the increments $W(t') - W(t), t' > t$ and $\{\mathcal{G}_t^\mu\}_{t \in [0, t]}$ -progressively measurable. Since $\mu \in \mathcal{X}$, we know $\xi \in \mathcal{X}(\mathfrak{A}_\mu)$.

Next, we have to verify if the family operators

$$\left\{ \mathcal{V}_{\mathfrak{A}_\mu, W}^\iota : \iota \in \mathbb{N} \right\}$$

is well-defined. However, this is follows from assumption (i), and since $\xi \in \mathcal{X}(\mathfrak{A}_\mu)$. In fact, the μ -dependence of the stochastic basis can be removed by lifting to the space of probability measures (path laws). We aim to find a fixed point in the space of probability measures.

Now, for $A \in \mathcal{B}(\mathbb{X} \cap \mathbb{U})$. Then, let us define the mapping \mathcal{V}_ι that maps the probability measure μ , in other words, the probability measure $\nu_\xi : \mathcal{B}(\mathbb{X} \cap \mathbb{U}) \rightarrow [0, 1]$ that is induced by ξ to the probability measure $\nu_{\mathcal{V}_{\mathfrak{A}_\mu, W}^\iota(\xi)} : \mathcal{B}(\mathbb{X} \cap \mathbb{U}) \rightarrow [0, 1]$ given by

$$\nu_{\mathcal{V}_{\mathfrak{A}_\mu, W}^\iota(\xi)}(A) := \mathbb{P}_\mu \left(\left\{ \omega \in \Omega_\mu : \mathcal{V}_{\mathfrak{A}_\mu, W}^\iota(\xi(\omega)) \in A \right\} \right), \quad \mathcal{V}_\iota(\mu) := \nu_{\mathcal{V}_{\mathfrak{A}_\mu, W}^\iota(\xi)}.$$

Note, since $\mathbb{X} \cap \mathbb{U}$ is a complete metric space, the space of probability measures over $\mathbb{X} \cap \mathbb{U}$ equipped with the Prokhorov metric⁷ is complete.

The following points can be easily verified.

- (1) \mathcal{X} is invariant under \mathcal{V}_ι . This follows directly from assumption (ii) and the properties of the projection Proj_n , see also Remark 4.1.
- (2) Due to the fact that $\mathcal{V}_{\mathfrak{A}_\mu, W}^\iota$ restricted to $\mathcal{X}(\mathfrak{A}_\mu)$ is uniformly continuous, \mathcal{V}_ι restricted to \mathcal{X} is continuous on $\mathcal{M}_1(\mathbb{X} \cap \mathbb{U})$ in the Prokhorov metric by [22, Theorem 11.7.1].
- (3) Note that by assumption (v), $\mathcal{V}_{\mathfrak{A}_\mu, W}^\iota(\xi) \in \mathbb{U}$ for $\xi \in \mathcal{X}(\mathfrak{A}_\mu)$. We claim that \mathcal{V}_ι restricted to \mathcal{X} is compact on $\mathcal{M}_1(\mathbb{X} \cap \mathbb{U})$. In particular, it maps bounded sets into compact sets. In fact, we have to show that for all $\iota \in \mathbb{N}$ and $\varepsilon > 0$ there exists a compact subset $K_\varepsilon \subset \mathbb{X} \cap \mathbb{U}$ such that

$$\nu_{\mathcal{V}_\iota(\xi)}((\mathbb{X} \cap \mathbb{U}) \setminus K_\varepsilon) < \varepsilon, \quad \forall \nu_\xi \in \mathcal{X} \quad \text{and} \quad \nu_{\mathcal{V}_\iota(\xi)} := \mathcal{V}_\iota(\nu_\xi).$$

However, by assumption (iv) there exists a constant $R > 0$ with

$$\mathbb{E} \|\mathcal{V}_{\mathfrak{A}_\mu, W}^\iota(\xi)\|_{\mathbb{X}'}^{m_0} \leq R, \quad \xi \in \mathcal{X}(\mathfrak{A}_\mu).$$

Let $\tilde{R} > R^{1/m_0} \varepsilon^{-1/m_0}$ and let $K_\varepsilon := \{x \in \mathbb{X} \cap \mathbb{U} : \|x\|_{\mathbb{X}'} \leq \tilde{R}\}$. Due to the construction of the operator \mathcal{V}_ι , we have that the law is preserved. In particular,

$$\mathbb{P}_\mu \left(\left\{ x \in \mathbb{X} \cap \mathbb{U} \cap \mathcal{X}(\mathfrak{A}_\mu) : \|\mathcal{V}_{\mathfrak{A}_\mu, W}^\iota(x)\|_{\mathbb{X}'} \geq \tilde{R} \right\} \right) = \nu_{\mathcal{V}_\iota(\xi)} \left(\left\{ x \in \mathcal{X} : \|x\|_{\mathbb{X}'} \geq \tilde{R} \right\} \right).$$

⁷ Let $\mathcal{M}_1(X)$ be the set of Borel probability measures on the metric space (X, d) equipped with the weak topology. Let $\nu, \mu \in \mathcal{M}_1(X)$. Then the weak topology can be metrized by the Prokhorov metric, cf. [22],

$$d_\alpha(\mu, \nu) := \inf\{\alpha > 0 : \mu(A) \leq \nu(A_\alpha) + \alpha \text{ and } \nu(A) \leq \mu(A_\alpha) + \alpha \text{ for all } A \in \mathcal{B}(X)\}.$$

Here $A_\alpha := \{x \in X : d(x, A) < \alpha\}$.

Next, by Chebyshev’s inequality, we get that

$$\nu_{\mathcal{V}_i(\xi)}((\mathbb{X} \cap \mathbb{U}) \setminus K_\varepsilon) = \nu_{\mathcal{V}_i(\xi)}\left(\left\{x \in \mathcal{X} : \|x\|_{\mathbb{X}'} \geq \tilde{R}\right\}\right) < \varepsilon.$$

Since $\mathbb{X}' \hookrightarrow \mathbb{X}$ compactly, we have proved the tightness.

(4) \mathcal{X} is a convex subset of $\mathcal{M}_1(\mathbb{X} \cap \mathbb{U})$. Let $\nu, \mu \in \mathcal{X}$, we have to show that for any $\alpha \in (0, 1)$ we have $\alpha\nu + (1 - \alpha)\mu \in \mathcal{X}$. First, analyzing the expectation with respect to Ψ_1, \dots, Ψ_K , this follows by the linearity of the expectation value. Secondly, since $\nu, \mu \in \mathcal{X}$ we know that $\nu(\{\Theta < \infty\}) = 1$ and $\mu(\{\Theta < \infty\}) = 1$, Let $\alpha \in (0, 1)$. Then

$$\begin{aligned} &(\alpha\nu + (1 - \alpha)\mu)(\{\Theta < \infty\}) \\ &= \alpha \underbrace{\nu(\{\Theta < \infty\})}_{=1} + (1 - \alpha) \underbrace{\mu(\{\Theta < \infty\})}_{=1} = 1. \end{aligned}$$

In particular, the mapping \mathcal{V}_i restricted to \mathcal{X} satisfies all assumptions of the classical Schauder-Tychonoff theorem, see [30, § 7, Theorem 1.13., p. 148]:

Lemma 4.2 [Schauder-Tychonoff] *Let C be a nonempty convex subset of a locally convex linear topological space \mathcal{E} , and let $F : C \rightarrow C$ be a compact map, i.e., $F(C)$ is contained in a compact subset of C . Then F has a fixed point.*

Hence, for any $i \in \mathbb{N}$ there exists a probability measure $\nu_i^* \in \mathcal{X}$ such that

$$\mathcal{V}_i(\nu_i^*) = \nu_i^*.$$

Step (III). Note, that the tightness argument in the previous step is independent of i , thus the set

$$\{\nu_i^* : i \in \mathbb{N}\}$$

is tight, therefore there exists a subsequence $\{i_j : j \in \mathbb{N}\}$ and a Borel probability measure ν^* such that $\nu_{i_j}^* \rightarrow \nu^*$, as $j \rightarrow \infty$. In this step, we will construct from the family of probability measures $\{\nu_{i_j}^* : j \in \mathbb{N}\}$ and $\nu^* \in \mathcal{X}$, a filtered probability space \mathfrak{A}^* , a Wiener process W^* , a progressively measurable process w^* , and a family of progressively measurable processes $\{w_{i_j} : j \in \mathbb{N}\}$ that are \mathbb{P}^* -a.s. contained in $\mathbb{X} \cap \mathbb{U}$ over \mathfrak{A}^* such that these objects have probability measures $\{\nu_{i_j}^* : j \in \mathbb{N}\}$ and $\nu_{i_j}^* \in \mathcal{X} \cap \mathbb{U}$.

By the Skorokhod lemma [39, Theorem 4.30], there exists a probability space $\mathfrak{A}_0^* = (\Omega_0^*, \mathcal{F}_0^*, \mathbb{P}_0^*)$ and a sequence of \mathbb{X} -valued random variables $\{w_{i_j}^* : j \in \mathbb{N}\}$ and $w_{i_j}^*$ where the random variable $w_{i_j}^* : \Omega_0^* \rightarrow \mathbb{X} \cap \mathbb{U}$ has the law $\nu_{i_j}^*$ in $\mathbb{X} \cap \mathbb{U}$. In addition, by tightness and the Skorokhod lemma, we have

$$w_{i_j}^* \rightarrow w^* \quad \text{as } j \rightarrow \infty \quad \mathbb{P}_0^*\text{-a.s.} \tag{4.3}$$

on \mathbb{X} .

Moreover, let us introduce the filtration $\mathbb{G}_0^* = (\mathcal{F}_t^{*,0})_{t \in [0, T]}$ given by

$$\mathcal{F}_t^{*,0} := \sigma\left(\left\{(w_{i_j}^*(s), w^*(s)) : 0 \leq s \leq t, j \in \mathbb{N}\right\} \cup \mathcal{N}_0^*\right), \quad t \in [0, T],$$

where \mathcal{N}_0^* denotes the collection of zero sets of \mathfrak{A}_0^* .

Next, similarly as above, let us construct the Wiener process. Let

$$\mathfrak{A}_1^* = \left(\Omega_1^*, \mathcal{F}_1^*, (\mathcal{G}_t^{1,*})_{t \in [0, T]}, \mathbb{P}_1^*\right).$$

be a filtered probability space with a cylindrical Wiener process W^* on H being adapted to the filtration $(\mathcal{G}_t^{1,*})_{t \in [0, T]}$. Let $\mathfrak{A}^* := \mathfrak{A}_0^* \times \mathfrak{A}_1$. In particular, we put

$$\begin{aligned} \Omega^* &= \Omega_0^* \times \Omega_1^*, \\ \mathcal{F}^* &= \mathcal{F}_0^* \otimes \mathcal{F}_1^*, \\ \mathcal{G}_t^* &= \mathcal{G}_t^{0,*} \otimes \mathcal{G}_t^{1,*}, \quad t \in [0, T], \\ \text{and } \mathbb{P}^* &= \mathbb{P}_0^* \otimes \mathbb{P}_1^*. \end{aligned}$$

In addition, $\mathcal{X}(\mathfrak{A}^*)$ can be defined, and also the operators $\mathcal{V}_{\mathfrak{A}^*, W^*}^t$ and $\mathcal{V}_{\mathfrak{A}^*, W^*}$

Step (IV). In this step, we mimic an explicit Euler scheme, to construct a \mathbb{P}^* -a.s. piecewise constant and $\{\mathcal{G}_t^*\}_{t \in [0, T]}$ -progressively measurable process that is a fixed point for the operator $\mathcal{V}_{\mathfrak{A}^*, W^*}^{t_j}$.

Since $v_{t_j}^* \in \mathcal{X}$, the process $w_{t_j}^* \in \mathcal{X}(\mathfrak{A}^*)$ and, hence, $\mathcal{V}_{\mathfrak{A}^*, W^*}^{t_j}(w_{t_j}^*)$ is well-defined. Since $\mathcal{V}_{t_j}(v_{t_j}^*) = v_{t_j}^*$, $\text{Law}(w_{t_j}^*) = v_{t_j}^*$. However, we do not know if the process $w_{t_j}^*$ satisfies

$$\mathbb{P}^* \left(\mathcal{V}_{\mathfrak{A}^*, W^*}^{t_j}(w_{t_j}^*)(s) = w_{t_j}^*(s) \right) = 1 \quad \text{for } 0 \leq s \leq T.$$

In this step, we will construct here a fixed point for the operator $\mathcal{V}_{\mathfrak{A}^*, W^*}^{t_j}$. Let us define a new process by induction. To start with, let

$$w_{t_j, 1}^*(s) := \begin{cases} w_{t_j}^*(s) & \text{if } 0 \leq s < t_1, \\ \left(\mathcal{V}_{\mathfrak{A}^*, W^*}^{t_j}(w_{t_j}^*) \right)(s) & \text{if } t_1 \leq s \leq T, \end{cases} \tag{4.4}$$

and

$$w_{t_j, 2}^*(s) := \begin{cases} w_{t_j, 1}^*(s) & \text{if } 0 \leq s < t_2, \\ \left(\mathcal{V}_{\mathfrak{A}^*, W^*}^{t_j}(w_{t_j, 1}^*) \right)(s) & \text{if } t_2 \leq s \leq T. \end{cases} \tag{4.5}$$

Now, having defined $w_{t_j, k}^*$, let

$$w_{t_j, k+1}^*(s) := \begin{cases} w_{t_j, k}^*(s) & \text{if } 0 \leq s < t_{k+1}, \\ \left(\mathcal{V}_{\mathfrak{A}^*, W^*}^{t_j}(w_{t_j, k}^*) \right)(s) & \text{if } t_{k+1} \leq s \leq T, \end{cases} \tag{4.6}$$

where $t_k \in \pi_n$ are dyadic time points. Let us put $w_{t_j, 0}^*(s) = w_0^*$ for $0 \leq s \leq T$, where w_0^* is a \mathcal{G}_0^* -measurable version of w_0 , and

$$w_{t_j, \infty}^*(s) := w_{t_j, k}^*(s), \quad \text{if } t_{k-1}^{t_j} \leq s < t_k^{t_j}, \quad k = 1, \dots, 2^{t_j}. \tag{4.7}$$

We claim that the process $w_{t_j, \infty}^*$ satisfies

$$\mathbb{P}^* \left(\mathcal{V}_{\mathfrak{A}^*, W^*}^{t_j}(w_{t_j, \infty}^*)(s) = w_{t_j, \infty}^*(s) \right) = 1 \quad \text{for } 0 \leq s \leq T. \tag{4.8}$$

Note, that by the definition of Proj_{t_j} on $[0, t_1^{t_j}]$ the process on $[0, t_1^{t_j}]$ is defined by the initial data. In fact, we have for $0 \leq s < t_1^{t_j}$

$$\mathcal{V}_{\mathfrak{A}^*, W^*}^{t_j}(w_{t_j, \infty}^*)(s) = w_0^*.$$

By equation Eq. 4.7 and equation Eq. 4.4 we have $w_{t_j, \infty}^* = w_{t_j, 1}^* = w_0^*$ for $0 \leq s < t_1^{t_j}$.

In particular, the process on $[0, t_1^{lj})$ is defined by the initial data and we have \mathbb{P}^* -a.s.

$$\mathcal{V}_{\mathfrak{A}^*, W^*}^{lj} \left(w_{i_j, \infty}^* \right) (s) = w_{i_j, \infty}^*(s), \quad \text{for } 0 \leq s < t_1^{lj}.$$

At time t_1^{lj} , we have by equation Eq. 4.7 and equation Eq. 4.4,

$$\mathcal{V}_{\mathfrak{A}^*, W^*}^{lj} \left(w_{i_j, \infty}^* \right) (t_1^{lj}) = \left(\mathcal{V}_{\mathfrak{A}^*, W^*}^{lj} w_{i_j}^* \right) (t_1^{lj}) = w_{i_j, 1}^*(t_1^{lj}).$$

However, we have $\tilde{w}_{i, 1}^*(t_1^{lj}) = w_{i_j, \infty}^*(t_1^{lj})$. Let us analyze what is happening at the next time interval $[t_1^{lj}, t_2^{lj})$. Her the process is constant and equals \mathbb{P}^* -a.s. the value at t_1^{lj} , i.e.

$$\mathcal{V}_{\mathfrak{A}^*, W^*}^{lj} \left(w_{i_j, \infty}^* \right) (s) = w_{i_j, 1}^*(s), \quad \text{for } t_1^{lj} \leq s < t_2^{lj}.$$

Note, also that \mathbb{P}^* -a.s. we have $w_{i_j, 1}^*(s) = w_{i_j, \infty}^*(s)$ for $t_1^{lj} \leq s < t_2^{lj}$, and, hence

$$\mathbb{P}^* \left(w_{i_j, \infty}^*(s) = \mathcal{V}_{i_j} \left(w_{i_j, \infty}^* \right) (s) \right) = 1, \quad \text{for } t_1^{lj} \leq s < t_2^{lj}.$$

Let us analyze what happens in t_2^{lj} . By equation Eq. 4.7, we have

$$\mathcal{V}_{\mathfrak{A}^*, W^*}^{lj} \left(w_{i_j, \infty}^* \right) (t_2^{lj}) = \mathcal{V}_{\mathfrak{A}^*, W^*}^{lj} \left(w_{i_j, 1}^* \right) (t_2^{lj}).$$

By equation of $w_{i_j, 2}^*$, i.e. Eq. 4.5, we have

$$\mathcal{V}_{\mathfrak{A}^*, W^*}^{lj} \left(w_{i_j, \infty}^* \right) (t_2^{lj}) = w_{i_j, 2}^*(t_2^{lj}).$$

Now, we can proceed by induction. Let us assume that in $[0, t_k^{lj})$ we have shown that

$$\mathbb{P}^* \left(\mathcal{V}_{\mathfrak{A}^*, W^*}^{lj} \left(w_{i_j, \infty}^* \right) (s) = w_{i_j, \infty}^*(s) \right) = 1 \quad \text{for } 0 \leq s \leq t_k^{lj}. \tag{4.9}$$

Then, we have by equation Eq. 4.7 and equation Eq. 4.6 we have for $t_k^{lj} \leq s < t_{k+1}^{lj}$

$$\mathcal{V}_{\mathfrak{A}^*, W^*}^{lj} \left(w_{i_j, \infty}^* \right) (s) = \mathcal{V}_{\mathfrak{A}^*, W^*}^{lj} \left(w_{i_j, k-1}^* \right) (s) = w_{i_j, k}^*(s).$$

Step (V). Next, we verify a couple of statements with the goal to pass on to the limit. Here, we point out that the same construction as done for $w_{i_j, \infty}^*$ can be done on the initially given probability space \mathfrak{A} . The resulting process is denoted by $w_{i_j, \infty}$. Due to the construction and by the properties of the projection, it is easy to see that the laws are preserved. In particular, $\text{Law}(w_{i_j, \infty}) = \text{Law}(w_{i_j, \infty}^*)$.

Claim 4.3 *We claim that*

- (1) *there exists a constant $C > 0$ such that $\sup_{j \in \mathbb{N}} \mathbb{E}^* \left[\|w_{i_j, \infty}^*\|_{\mathbb{X}}^{m_0} \right] \leq C$ and*
- (2) *for any $r \in (1, m_0)$ we have*

$$\lim_{j \rightarrow \infty} \mathbb{E}^* \left[\|w_{i_j, \infty}^* - w^*\|_{\mathbb{X}}^r \right] = 0.$$

Proof of Claim 4.3: Clearly, since $\{w_{i_j, \infty}^*\}_{j \in \mathbb{N}} \subset \mathcal{X}(\mathfrak{A}^*)$, and $\mathcal{X}(\mathfrak{A}^*)$ is bounded in \mathbb{X} , we can conclude from the application of the Skorokhod lemma that

$$\mathbb{E} \|w_{i_j, \infty}\|_{\mathbb{X}}^r = \mathbb{E}^* \|w_{i_j, \infty}\|_{\mathbb{X}}^r,$$

for any $r \in [1, m_0]$, so that we get by assumption (iv) that

$$\sup_j \mathbb{E}^* \|w_{t_j, \infty}^*\|_{\mathbb{X}}^{m_0} \leq R\tilde{C} =: C,$$

where $\tilde{C} > 0$ is a constant such that $\|\cdot\|_{\mathbb{X}} \leq \tilde{C}\|\cdot\|_{\mathbb{X}'}$ which exists by the compact and dense embedding $\mathbb{X}' \hookrightarrow \mathbb{X}$.

Hence, we know that $\{\|w_{t_j, \infty}^*\|_{\mathbb{X}}^r\}$ is uniformly integrable for any $r \in (1, m_0]$ w.r.t. the probability measure \mathbb{P}^* . By Eq. 4.3, there exists $w^* \in \mathcal{X}(\mathfrak{Q}^*)$ with $w_{t_j, \infty}^* \rightarrow w^*$ \mathbb{P}^* -a.s., so we get by the Vitali convergence theorem that

$$\lim_{j \rightarrow \infty} \mathbb{E}^* \|w_{t_j, \infty}^* - w^*\|_{\mathbb{X}}^r = 0, \tag{4.10}$$

for any $r \in (1, m_0)$.

Step (VI). In this step we show that w^* over \mathfrak{Q}^* together with the Wiener process W^* is indeed a martingale solution to Eq. 2.2. We shall use an $\varepsilon/3$ -argument and expand

$$\begin{aligned} w^* - \mathcal{V}_{\mathfrak{Q}^*, W^*}(w^*) &= w^* - \underbrace{w_{t_j, \infty}^*}_{=:I} + \underbrace{w_{t_j, \infty}^* - \mathcal{V}_{\mathfrak{Q}^*, W^*}^{t_j}(w_{t_j, \infty}^*)}_{=:II} \\ &\quad + \underbrace{\mathcal{V}_{\mathfrak{Q}^*, W^*}^{t_j}(w_{t_j, \infty}^*) - \mathcal{V}_{\mathfrak{Q}^*, W^*}^{t_j}(w^*)}_{=:III} + \underbrace{\mathcal{V}_{\mathfrak{Q}^*, W^*}^{t_j}(w^*) - \mathcal{V}_{\mathfrak{Q}^*, W^*}(w^*)}_{=:IV}. \end{aligned}$$

Now, we analyze the terms I, II, III , and IV separately.

Due to the convergence, we know that for any $r \in (1, m_0)$, $\varepsilon > 0$, there exists $j_0 \in \mathbb{N}$,

$$\mathbb{E}^* \|w^* - w_{t_j, \infty}^*\|_{\mathbb{X}}^r \leq \frac{\varepsilon}{3} \quad \text{for all } j \geq j_0.$$

Next, to tackle II , we know due to the well-posedness by the existence of fixed point in the step before,

$$\mathcal{V}_{\mathfrak{Q}^*, W^*}^{t_j}(w_{t_j, \infty}^*) = w_{t_j, \infty}^*.$$

To tackle III , due to the uniform continuity of the operator $\mathcal{V}_{\mathfrak{Q}^*, W^*}^{t_j}$, we know that there exists a $\delta = \delta(\varepsilon) > 0$ and $j_0 \in \mathbb{N}$, such that, for any $r \in (1, m_0)$,

$$\mathbb{E}^* \left\| \mathcal{V}_{\mathfrak{Q}^*, W^*}^{t_j}(w_{t_j, \infty}^*) - \mathcal{V}_{\mathfrak{Q}^*, W^*}^{t_j}(w^*) \right\|_{\mathbb{X}}^r \leq \frac{\varepsilon}{3},$$

whenever $j \geq j_0$, and

$$\mathbb{E}^* \|w_{t_j, \infty}^* - w^*\|_{\mathbb{X}}^r < \delta.$$

Finally, since $\mathcal{V}_{\mathfrak{Q}^*, W^*}^{t_j} = \text{Proj}_{n_{t_j}} \circ \mathcal{V}_{\mathfrak{Q}^*, W^*}$, for $r \in (1, m)$, the difference

$$\mathbb{E}^* \left\| \mathcal{V}_{\mathfrak{Q}^*, W^*}^{t_j}(w^*) - \mathcal{V}_{\mathfrak{Q}^*, W^*}(w^*) \right\|_{\mathbb{X}}^r$$

tends to zero by uniform continuity, see Eq. 4.2. However, $m \leq m_0$ by assumption (iv). Thus, IV tends to zero in $L^r(\Omega^*, \mathcal{F}^*, \mathbb{P}^*; \mathbb{X})$ for any $r \in (1, m)$.

As a consequence, we have

$$\mathcal{V}_{\mathfrak{Q}^*, W^*}(w^*) = w^*, \quad \mathbb{P}^*\text{-a.s.}$$

As seen above, $w^* \in \mathcal{X}(\mathfrak{A}^*)$, so that by (v), $\mathcal{V}_{\mathfrak{A}^*, W^*}(w^*) \in \mathbb{D}([0, T]; U)$, and therefore $w^* \in \mathbb{D}([0, T]; U) \mathbb{P}^*$ -a.s. Hence for all $t \in [0, T]$, \mathbb{P}^* -a.s.

$$\mathcal{V}(w^*)(t) = w^*(t)$$

and the proof is complete. By construction, we see that w^* solves

$$dw^*(t) = (Aw^*(t) + F(w^*, t)) dt + \Sigma(w^*(t)) dW^*(t), \quad w^*(0) = w_0^*, \tag{4.11}$$

on \mathfrak{A}^* , where w_0^* is a \mathcal{G}_0^* -measurable version of w_0 .

5 Results On Regularity And Technical Propositions

This section contains the remaining results which are used in the proof of the main result Theorem 3.6.

5.1 Assumptions On The Noise And Consequences

Let us recall, we denoted by $\{\psi_k : k \in \mathbb{N}\}$ the eigenfunctions of the Laplace operator $-\Delta$ in $L^2(\mathcal{O})$ and by $\{\nu_k : k \in \mathbb{N}\}$ the corresponding eigenvalues, where the enumeration is chosen in increasing order counting the multiplicity.

Let us characterize the asymptotic behavior of $\{\nu_k : k \in \mathbb{N}\}$ and $\{\psi_k : k \in \mathbb{N}\}$ for an arbitrary domain \mathcal{O} with C^2 -boundary. Here, we know by Weyl’s law [76, 77] that there exist constants $c, C > 0$ such that,

$$\#\{j : \nu_j \leq \lambda\} \leq C\lambda^{\frac{d}{2}}, \tag{5.1}$$

compare with [37, 46] and [38, Corollary 17.8.5], and there exists a constant $C > 0$ such that,

$$\sup_{x \in \mathcal{O}} |\psi_k(x)| \leq C\nu_k^{\frac{d-1}{2}}, \quad k \in \mathbb{N}, \tag{5.2}$$

compare with [32].

Remark 5.1 If $\mathcal{O} = [0, 1]^d$ is a rectangular domain, then a complete orthonormal system $\{\psi_k : k \in \mathbb{Z}\}$ of the underlying Lebesgue space $L^2(\mathcal{O})$ is given by the trigonometric functions, see [5, p. 352] (now, \mathbb{Z} obviously denotes the integer numbers). Let us define firstly the eigenvalues for the Laplace on $L^2([0, 1])$

$$e_k(x) = \begin{cases} \sqrt{2} \sin(2\pi kx) & \text{if } k \geq 1, x \in \mathcal{O}, \\ 1 & \text{if } k = 0, x \in \mathcal{O}, \\ \sqrt{2} \cos(2\pi kx) & \text{if } k \leq -1, x \in \mathcal{O}. \end{cases} \tag{5.3}$$

For instance, the eigenfunctions for $L^2([0, 1]^2)$ for the multi-index $k = (k_1, k_2) \in \mathbb{Z}^2$ are given by the tensor product

$$\psi_k(x_1, x_2) = e_{k_1}(x_1)e_{k_2}(x_2), \quad x \in [0, 1]^2,$$

the corresponding eigenvalues are given by $\nu_k = \pi^2|k|^2$, where $|k| = k_1 + k_2$. The case $d = 3$ works analogously. In this special case the conditions on δ_1 and δ_2 in Hypothesis 3.5 can be relaxed to

$$\lambda_k^{(1)} \leq C\nu_k^{-\delta_1}, \quad \lambda_k^{(2)} \leq C\nu_k^{-\delta_2}, \quad k = (k_1, k_2)$$

to $\delta_1, \delta_2 > \frac{1}{2}$. See also the discussion in [2, Examples 1.2.1 and 2.1.2].

We begin with a remark on the noise coefficients. We partly work in the Banach spaces $L^m(\mathcal{O})$ and $H_m^\delta(\mathcal{O})$ for δ being arbitrary small but positive.

Given a Wiener process W on $H = L^2(\mathcal{O})$ over $\mathfrak{A} = (\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$, and a progressively measurable process $\xi \in \mathcal{M}_{\mathfrak{A}}^2(L^2(\mathcal{O}))$, $\rho \in [0, \frac{1}{2}]$, let us define $\{Y(t) : t \in [0, T]\}$ by

$$Y(t) := \int_0^t \sigma(\xi(s)) dW(s), \quad t \in [0, T].$$

Let E be a function space, specified later. Here, for each $\xi \in E$, $\sigma(\xi)$ is interpreted as a multiplication operator acting on the elements of H , namely, $\sigma : \xi \mapsto \xi\psi \in S'(\mathcal{O})$, where $S'(\mathcal{O})$ denotes the space of *tempered distributions* on \mathcal{O} . Let E be a Banach space⁸ of martingale type 2. An inequality needed in several places within the proof is the Burkholder-Davis-Gundy inequality given for $p \geq 1$ as follows

$$\mathbb{E} \left[\sup_{t \in [0, T]} |Y(t)|_E^p \right] \leq C(p) \mathbb{E} \left[\int_0^T |\xi(t)|_{\gamma(H, E)}^2 dt \right]^{\frac{p}{2}},$$

where $\gamma(H, E)$ denotes the space of γ -radonifying operators and $|\cdot|_{\gamma(H, E)}$ the corresponding norm, cf. [8, 73]. In case E is a Hilbert space, the γ -norm coincides with the Hilbert-Schmidt norm $|\cdot|_{HS(H, E)}$. See Appendix C for further details.

Since $|\xi\psi_k|_{L^m} \leq |\xi|_{L^m} |\psi_k|_{L^\infty}$ and Eq. 5.2, we know that

$$|\xi|_{\gamma(H, L^m)}^2 \leq \sum_{k \in \mathbb{N}} |\lambda_k^{(j)}|^2 |\xi\psi_k|_{L^m}^2 \leq |\xi|_{L^m}^2 \sum_{k \in \mathbb{N}} \nu_k^{-2\delta_j} |\psi_k|_{L^\infty}^2 \leq C |\xi|_{L^m}^2 \sum_{k \in \mathbb{N}} \nu_k^{(d-1)-2\delta_j},$$

which is finite if $\delta_j > \frac{d}{2} - \frac{1}{4}$.

Next, let $E = H_2^{-1}(\mathcal{O})$. Since

$$|\xi\psi|_{H_2^{-1}} \leq |\xi|_{H_2^{-1}} |\psi|_{H_2^\delta} \leq |\xi|_{H_2^\rho} |\psi|_{L^\infty},$$

we know that $\delta > \frac{1}{2}$, then

$$|\sigma_1(\xi)|_{\gamma(H, H_2^\rho)} \leq C |\xi|_{H_2^\rho}.$$

Furthermore, let $E = H_2^\rho(\mathcal{O})$, where ρ is given in Hypothesis 3.4. Since

$$|\xi\psi|_{H_2^\rho} \leq |\xi|_{H_2^\rho} |\psi|_{H_2^{2|\rho|}} \leq |\xi|_{H_2^\rho} |\psi|_{L^\infty},$$

we know that $\delta > \frac{1}{2}$, then

$$|\sigma_2(\xi)|_{\gamma(H, H_2^\rho)} \leq C |\xi|_{H_2^\rho}.$$

Remark 5.2 From Hypotheses 3.5, one can infer that there exists a constant $C > 0$ such that

$$\sum_{k=1}^\infty [(\sigma_j(\xi) f_k)(x)]^2 \leq C |\xi(x)|_{L^2}^2, \quad \forall \xi \in L^2(\mathcal{O}), \quad x \in \mathcal{O}, \quad \text{and } j = 1, 2.$$

Here $\{f_k\}$ is an orthonormal basis in $H_2^{-1}(\mathcal{O})$, compare with [2, Hypothesis 3, p. 42]. In addition, note that

⁸ For Banach spaces of martingale type 2 we refer to [8, 73]

- Firstly, $\sigma_j : H_2^{-1}(\mathcal{O}) \rightarrow L_{HS}(H_j, H_2^{-1}(\mathcal{O}))$ is of linear growth and Lipschitz continuous. In particular, there exists constants $C_1, L_1 > 0$ such that

$$|\sigma_j(\xi)|_{L_{HS}(H_j, H_2^{-1})} \leq C_1(1 + |\xi|_{H_2^{-1}}), \quad \xi \in H_2^{-1}(\mathcal{O}),$$

$$|\sigma_j(\xi) - \sigma_j(\eta)|_{L_{HS}(H_j, H_2^{-1})} \leq L_1|\xi - \eta|_{H_2^{-1}}, \quad \xi, \eta \in H_2^{-1}(\mathcal{O}).$$

- Secondly, $\sigma_j : L^2(\mathcal{O}) \rightarrow L_{HS}(H_j, L^2(\mathcal{O}))$ is of linear growth and Lipschitz continuous. In particular, there exists constants $C_2, L_2 > 0$ such that

$$|\sigma_j(\xi)|_{L_{HS}(H_j, L^2)} \leq C_2(1 + |\eta|_{L^2}), \quad \eta \in L^2(\mathcal{O}),$$

$$|\sigma_j(\xi) - \sigma_j(\eta)|_{L_{HS}(H_j, L^2)} \leq L_2|\xi - \eta|_{L^2}, \quad \xi, \eta \in L^2(\mathcal{O}).$$

- Similarly, straightforward computations and using the fact that $|f_k|_{L^\infty} \leq \nu_k^{\frac{d-1}{2}}$, see Eq. 5.2 and [2, p. 46], we get that

$$|\sigma_2(\xi)|_{\gamma(H_2, L^m)} \leq C_1(1 + |\xi|_{L^m}), \quad \xi \in L^m(\mathcal{O}),$$

$$|\sigma_2(\xi) - \sigma_2(\eta)|_{\gamma(H_2, L^m)} \leq L_1|\xi - \eta|_{L^m}, \quad \xi, \eta \in L^m(\mathcal{O}),$$

where $\gamma(H_2, L^m)$ denotes the space of γ -radonifying operators, cf. [8, 73].

5.2 Properties Of Equation Eq. 3.12

In this subsection, we are analyzing equation Eq. 3.12, where $\kappa \in \mathbb{N}$ and the couple $(\xi, \eta) \in \mathcal{X}_{\mathfrak{A}}(R_0, R_1, R_2) \subset \mathcal{M}_{\mathfrak{A}}^{2, m_0}(\mathbb{Y}, \mathbb{Z})$ are given. The constants R_0, R_1 and R_2 are given as in Eq. 3.26, Eq. 3.23, and Eq. 3.25, respectively.

First, we will show that a unique solution u_κ to the system Eq. 3.12 exists and is nonnegative. Second, we will show by variational methods that this solution satisfies some integrability properties, given $u_0 \in L^{p+1}(\mathcal{O})$, $p \geq 1$. Let us start with proving an inequality that we are going to use in the sequel.

Lemma 5.3 *Assume that the Hypotheses of Theorem 3.6 hold. For any $\varepsilon_1, \varepsilon_2 > 0$ and any $(\xi, \eta) \in \mathcal{X}_{\mathfrak{A}}(R_0, R_1, R_2) \subset \mathcal{M}_{\mathfrak{A}}^{2, m_0}(\mathbb{Y}, \mathbb{Z})$, $u, w \in L^{\gamma+1}(\mathcal{O})$, $\kappa \in \mathbb{N}$, there exists a constant $C = C(\kappa, \chi, \varepsilon_1, \varepsilon_2) > 0$ such that*

$$\chi \left| \int_{\mathcal{O}} (-\Delta)^{-1}(\phi_\kappa(h(\eta, \xi, t))\xi^2(t, x) u) w \, dx \right| \leq C \left(1 + |u|_{H_2^{-1}}^2 \right) + \varepsilon_1 |u|_{L^{\gamma+1}}^{\gamma+1} + \varepsilon_2 |w|_{L^{\gamma+1}}^{\gamma+1}.$$

Proof By Hölder’s inequality, we have for fixed $\omega \in \Omega$ and fixed $t \in [0, T]$,

$$\begin{aligned} & \left| \int_{\mathcal{O}} (-\Delta)^{-1}(\phi_\kappa(h(\eta, \xi, t))\xi^2(t, x) u) w \, dx \right| \\ & \leq |(-\Delta)^{-1}(\phi_\kappa(h(\eta, \xi, t))\xi^2(t)u)|_{L^{(\gamma+1)/\gamma}} |w|_{L^{\gamma+1}} \\ & \leq |\phi_\kappa(h(\eta, \xi, t))\xi^2(t)u|_{H_{\frac{\gamma+1}{\gamma}}^{-2}} |w|_{L^{\gamma+1}} \end{aligned}$$

Now, pick $0 > s_1 > -\frac{3}{\gamma+1}$ with small modulus and $s_2 = \frac{3}{\gamma+1} > -s_1$ and note that $s_2 \geq s_1$, $s_1 + s_2 \leq \frac{d\gamma}{\gamma+1}$, $s_1 + s_2 - \frac{d\gamma}{\gamma+1} \geq -2$, and thus we can apply Proposition A.3. Then we know

$$|\phi_\kappa(h(\eta, \xi, t))\xi^2(t)u|_{H_{\frac{\gamma+1}{\gamma}}^{-2}} \leq C |\phi_\kappa(h(\eta, \xi, t))\xi^2(t)|_{H_{\frac{\gamma+1}{\gamma}}^{s_2}} |u|_{H_{\frac{\gamma+1}{\gamma}}^{s_1}}.$$

Let $l \geq 2$ such that

$$\frac{d}{l} = \frac{1}{2} \left(\frac{d\gamma}{\gamma + 1} - s_2 \right) = \frac{d\gamma}{2(\gamma + 1)} - \frac{3}{2(\gamma + 1)}.$$

Then, we know that (see [58, p. 364]) there exists $c > 0$ with

$$|\phi_\kappa(h(\eta, \xi, t))\xi^2(t)|_{H^{\frac{s_2}{\gamma+1}}} \leq c|(\phi_\kappa(h(\eta, \xi, t)))^{1/2}\xi(t)|_{H_1^{s_2}}^2.$$

Next, if $\delta \geq \frac{d}{2} - \frac{d}{l} + s_2 = \frac{d}{2} + \frac{d\gamma}{2(\gamma+1)} + \frac{3}{2(\gamma+1)}$ we get

$$|\phi_\kappa(h(\eta, \xi, t))\xi^2(t)|_{H^{\frac{s_2}{\gamma+1}}} \leq c|(\phi_\kappa(h(\eta, \xi, t)))^{1/2}\xi(t)|_{H_2^\delta}^2.$$

Applying the Young inequality we know for any $\varepsilon_2 > 0$ there exists a constant $C(\chi, \varepsilon_2) > 0$ such that

$$\begin{aligned} & \chi \left| \int_{\mathcal{O}} (-\Delta)^{-1} (\phi_\kappa(h(\eta, \xi, t)))\xi^2(t, x) u w \, dx \right| \\ & \leq C(\chi, \varepsilon_2) |(\phi_\kappa(h(\eta, \xi, t)))^{1/2}\xi(t)|_{H_2^\delta}^{2\frac{\gamma+1}{\gamma}} |u|_{H^{\frac{s_1}{\gamma}}}^{\frac{\gamma+1}{\gamma}} + \varepsilon_2 |w|_{L^{\gamma+1}}^{\gamma+1} \end{aligned}$$

Let us fix $r \in (2, \gamma + 1)$, such that

$$\frac{1}{r} = \frac{\theta}{\infty} + \frac{1-\theta}{\gamma+1}, \quad s_1 = \theta(-1) + (1-\theta)0 \Rightarrow \frac{1}{r} = \frac{1+s_1}{\gamma+1}.$$

Thus for any $\varepsilon, \varepsilon_2 > 0$, there exists a constant $C(\chi, \varepsilon, \varepsilon_2) > 0$

$$\begin{aligned} & \chi \left| \int_{\mathcal{O}} (-\Delta)^{-1} (\phi_\kappa(h(\eta, \xi, t)))\xi^2(t, x) u w \, dx \right| \\ & \leq C(\chi, \varepsilon, \varepsilon_2) |(\phi_\kappa(h(\eta, \xi, t)))^{1/2}\xi(t)|_{H_2^\delta}^{\frac{2r(\gamma+1)}{r\gamma-(\gamma+1)}} + \varepsilon |u|_{H_r^{s_1}}^r + \varepsilon_2 |w|_{L^{\gamma+1}}^{\gamma+1} \end{aligned}$$

To apply the the cut-off function $\phi_\kappa^{1/2}$, we need

$$\delta = \rho + (1 - \theta), \quad \frac{r\gamma - (\gamma + 1)}{2r(\gamma + 1)} = \frac{\theta}{\infty} + \frac{1 - \theta}{2}.$$

In fact, if

$$\rho > \frac{15 + 3\gamma - 16\gamma^2 + 8d(1 + \gamma)}{16(1 + \gamma)^2}$$

then,

$$\|\xi\|_{L^{\frac{2r(\gamma+1)}{r\gamma-(\gamma+1)}}(0,T;H_2^\delta)} \leq c\|\xi\|_{\mathbb{H}_\rho},$$

and

$$\|\xi\|_{L^{\frac{2r(\gamma+1)}{r\gamma-(\gamma+1)}}(0,T;H_2^\delta)} \leq C(\kappa).$$

Summarizing the calculations above we know for any $\varepsilon, \varepsilon_2 > 0$ there exists a constant $C(\kappa, \chi, \varepsilon, \varepsilon_2) > 0$ such that

$$\begin{aligned} & \chi \left| \int_{\mathcal{O}} (-\Delta)^{-1} (\phi_\kappa(h(\eta, \xi, t)))\xi^2(t, x) u w \, dx \right| \\ & C(\kappa, \chi, \varepsilon, \varepsilon_2) + \varepsilon |u|_{H_r^{s_1}}^r + \varepsilon_2 |w|_{L^{\gamma+1}}^{\gamma+1}. \end{aligned} \tag{5.4}$$

In addition, by interpolation, see e.g. [3], we get that for any $\varepsilon_1 > 0$, there exists $\varepsilon > 0$ and a constant $C(\varepsilon_1, \varepsilon) > 0$ such that

$$\varepsilon |u|_{H_r^{\varepsilon_1}}^r \leq \varepsilon_1 |u|_{L^{\gamma+1}}^{\gamma+1} + C(\varepsilon_1, \varepsilon) |u|_{H_2^{-1}}^2.$$

Setting $C := C(\varepsilon_1, \varepsilon) \vee C(\kappa, \chi, \varepsilon, \varepsilon_2)$ completes the proof. □

Theorem 5.4 *Assume that the Hypotheses of Theorem 3.6 hold. For any $(\xi, \eta) \in \mathcal{X}_{\mathfrak{A}}(R_0, R_1, R_2) \subset \mathcal{M}_{\mathfrak{A}}^{2,m_0}(\mathbb{Y}, \mathbb{Z})$, and for any $u_0 \in L^2(\Omega; \mathcal{F}_0, \mathbb{P}; H_2^{-1}(\mathcal{O}))$ the system*

$$du_{\kappa}(t) = r_u \Delta u_{\kappa}^{[\gamma]}(t) dt - \chi \phi_{\kappa}(h(\eta, \xi, t)) u_{\kappa} \xi^2(t) dt + \sigma_1 u_{\kappa}(t) dW_1(t), \tag{5.5}$$

with $u_{\kappa}(0) = u_0$ has a unique solution u_{κ} on the time interval $[0, T]$ which is \mathbb{P} -a.s. continuous in H_2^{-1} and satisfies

$$\mathbb{E} \left[\sup_{0 \leq s \leq T} |u_{\kappa}(s)|_{H_2^{-1}}^2 \right] + \mathbb{E} \int_0^T |u_{\kappa}(s)|_{L^{\gamma+1}}^{\gamma+1} ds < \infty.$$

In particular, there exists a constant $C(T, \kappa) > 0$ such that

$$\mathbb{E} \|u_{\kappa}\|_{\mathbb{Y}}^2 \leq C(T, \kappa).$$

Proof Before starting with the proof, we introduce the setting used by the book by Barbu, Da Prato and Röckner [2], and the book by Liu and Röckner [48], respectively. Let $\mathcal{H} := H_2^{-1}(\mathcal{O})$, the dual space of $\mathcal{H}^* = H_2^1(\mathcal{O})$ (corresponding to Neumann boundary conditions). Let $V := L^{\gamma+1}(\mathcal{O})$. By the Sobolev embedding theorem, $\mathcal{H}^* \hookrightarrow L^{(\gamma+1)/\gamma}(\mathcal{O})$ densely and compactly. Thus, upon identifying \mathcal{H} with its own dual space via the Riesz-map $(-\Delta)^{-1}$ of \mathcal{H} , we have a Gelfand triple

$$V \subset \mathcal{H} \equiv \mathcal{H}^* \subset V^*.$$

We set

$$A(t, u, \omega) := r_u \Delta u^{[\gamma]} - \chi \phi_{\kappa}(h(\eta(\omega), \xi(\omega), t)) \xi^2(t, \omega) u.$$

Note, due to the assumption $(\xi, \eta) \in \mathcal{X}_{\mathfrak{A}}(R_0, R_1, R_2) \subset \mathcal{M}_{\mathfrak{A}}^{2,m_0}(\mathbb{Y}, \mathbb{Z})$, ξ is adapted such that for any $q_2 \in [1, m_0]$,

$$\left(\int_0^T |\phi_{\kappa}(h(\eta, \xi, t)) \xi(t)|_{L^m}^{m_0} dt \right) \leq (2\kappa)^{1/\nu}. \tag{5.6}$$

Now, given Eq. 5.6, we shall verify the conditions of [48, Theorem 5.1.3] for $\gamma > 1$. First note that for fixed $t \in [0, T]$ and fixed $\omega \in \Omega$, A maps from V to V^* . In particular, by arguments as in the proof of Lemma 5.3, we have for $\gamma > 1$

$$\begin{aligned} & |_{V^*} \langle A(t, u), w \rangle_V | \tag{5.7} \\ &= \left| \int_{\mathcal{O}} \left[r_u u^{[\gamma]}(x) w(x) + (-\Delta)^{-1} (\chi \phi_{\kappa}(h(\eta, \xi, t)) \xi^2(t, x) u(x)) w(x) \right] dx \right| \\ &\leq \left[r_u |u|_{L^{\gamma+1}}^{\gamma} |w|_{L^{\gamma+1}} + \chi |(-\Delta)^{-1} (\phi_{\kappa}(h(\eta, \xi, t)) \xi^2(t) u)|_{L^{(\gamma+1)/\gamma}} |w|_{L^{\gamma+1}} \right] \\ &\leq \left[r_u |u|_{L^{\gamma+1}}^{\gamma} + C(\kappa, \varepsilon_1, \chi) + \varepsilon_1 |u|_{L^{\gamma+1}}^{\gamma+1} + C(\kappa, \varepsilon_1, \chi) |u|_{H_2^{-1}}^2 \right] |w|_{L^{\gamma+1}}, \end{aligned}$$

where only ξ , and $h(\eta, \xi, \cdot)$ depend on t and ω .

Next, we verify Hypothesis (H1), (H2'), (H3), and (H4') of [48, Theorem 5.1.3].

(H1): For $\lambda \in \mathbb{R}$, $u_1, u_2, w \in L^{\gamma+1}(\mathcal{O})$, $t \in [0, T]$, $\omega \in \Omega$ consider the map

$$\lambda \mapsto \langle A(t, u_1 + \lambda u_2), w \rangle$$

and show its continuity. Note, that we have

$$\begin{aligned} & \langle A(t, u_1 + \lambda u_2), w \rangle \\ &= -r_u \int_{\mathcal{O}} (u_1(x) + \lambda u_2(x))^{\lfloor \gamma \rfloor} w(x) \, dx \\ & \quad - \chi \int_{\mathcal{O}} (-\Delta)^{-1} [\phi_{\kappa}(h(\eta, \xi, t)) \xi^2(t, x) (u_1(x) + \lambda u_2(x))] w(x) \, dx, \end{aligned}$$

where only ξ , and $h(\eta, \xi, \cdot)$ depend on t and ω . For the first integral in the above identity, we prove continuity with the same arguments as in [48, Example 4.1.11, p. 87]. The second integral can be bounded by Lemma 5.3 as follows

$$\begin{aligned} & \left| \chi \int_{\mathcal{O}} (-\Delta)^{-1} [\phi_{\kappa}(h(\eta, \xi, t)) \xi^2(t, x) (u_1(x) + \lambda u_2(x))] w(x) \, dx \right| \\ & \leq C \left(1 + |u_1 + \lambda u_2|_{H_2^{-1}}^2 \right) + \varepsilon_1 |u_1 + \lambda u_2|_{L^{\gamma+1}}^{\gamma+1} + \varepsilon_2 |w|_{L^{\gamma+1}}^{\gamma+1}. \end{aligned}$$

Hence, the assumption (H1) of [48, Theorem 5.1.3] is satisfied.

(H2'): Let $u, w \in L^{\gamma+1}(\mathcal{O})$, $t \in [0, T]$, $\omega \in \Omega$. Take (A.1) into account and apply Lemma 5.3 to get that

$$\begin{aligned} & v^* \langle A(t, u) - A(t, w), u - w \rangle_V + |\sigma_1 u - \sigma_1 w|_{L_{\text{HS}}(H_1, \mathcal{H})}^2 \\ & \leq -r_u \int_{\mathcal{O}} (u^{\lfloor \gamma \rfloor}(x) - w^{\lfloor \gamma \rfloor}(x))(u(x) - w(x)) \, dx + C|u - w|_{\mathcal{H}}^2 \\ & \quad + \varepsilon |u - w|_{L^{\gamma+1}}^{\gamma+1} + C(\varepsilon, \kappa, \chi) \left(1 + |u - w|_{H_2^{-1}}^2 \right) \\ & \leq (\varepsilon - 2^{1-\gamma} r_u) |u - w|_{L^{\gamma+1}}^{\gamma+1} + C(1 + |u - w|_{\mathcal{H}}^2) \end{aligned}$$

and (H2') of [48, Theorem 5.1.3] is satisfied.

(H3): Let $u \in L^{\gamma+1}(\mathcal{O})$, $t \in [0, T]$, $\omega \in \Omega$. Straightforward calculations, estimate Eq. 5.7, and taking into account Hypothesis 3.4 give,

$$\begin{aligned} & v^* \langle A(t, u), u \rangle_V + |\sigma_1 u|_{L_{\text{HS}}(H_1, \mathcal{H})}^2 \\ & \leq - \int_{\mathcal{O}} \left[r_u u^{\lfloor \gamma \rfloor}(x) u(x) + \chi (-\Delta)^{-1} (\phi_{\kappa}(h(t, \eta, \xi)) \xi^2(t, x) u(x)) u(x) \right] dx + C|u|_{\mathcal{H}}^2 \\ & \leq -r_u |u|_{L^{\gamma+1}}^{\gamma+1} + C(\kappa, r_u, \chi) + \frac{r_u}{4} |u|_{L^{\gamma+1}}^{\gamma+1} + C(\kappa, r_u, \chi) |u|_{H_2^{-1}}^2 + \frac{r_u}{4} |u|_{L^{\gamma+1}}^{\gamma+1}. \end{aligned}$$

As a consequence, (H3) of [48, Theorem 5.1.3] holds with $\theta := r_u$.

(H4'): Let $u \in L^{\gamma+1}(\mathcal{O})$, $t \in [0, T]$, $\omega \in \Omega$. Then, (H4') of [48, Theorem 5.1.3] holds by Eq. 5.7 with $\alpha := \gamma + 1$, and $\beta := 0$.

The rest of the proof follows by an application of [48, Theorem 5.1.3].

Proposition 5.5 *Under the conditions of Theorem 5.4 and the additional conditions $u_0(x) \geq 0$ for all $x \in \mathcal{O}$ and $\xi(t, x) \geq 0$, for a.e. $(t, x) \in [0, T] \times \mathcal{O}$, the solution u_{κ} to Eq. 3.12 is a.e. nonnegative.*

Proof For the nonnegativity of the solution to Eq. 3.12, we refer to the proof of positivity of the stochastic porous medium equation, see Section 2.6 in [2] and see also [1]. Mimicking the proof of nonnegativity in [2] and applying a comparison principle [42] the nonnegativity of Eq. 3.12 can be shown. \square

In the next proposition we are using variational methods to verify uniform bounds of u_κ , $\kappa \in \mathbb{N}$.

Proposition 5.6 Fix $p \geq 1$. Then, there exists a constant $C_0(p, T) > 0$ such that for every $u_0 \in L^{p+1}(\Omega, \mathcal{F}_0, \mathbb{P}; L^{p+1}(\mathcal{O}))$, and for every $\kappa \in \mathbb{N}$ and for every $(\xi, \eta) \in \mathcal{X}_\Omega(R_0, R_1, R_2)$ such that u_κ solves Eq. 3.12, the following estimate

$$\begin{aligned} & \mathbb{E} \left[\sup_{0 \leq s \leq T} |u_\kappa(s)|_{L^{p+1}}^{p+1} \right] + \gamma p(p+1)r_u \mathbb{E} \int_0^T \int_{\mathcal{O}} |u_\kappa(s, x)|^{p+\gamma-2} |\nabla u_\kappa(s, x)|^2 dx ds \\ & + (p+1)\chi \mathbb{E} \int_0^T \int_{\mathcal{O}} u_\kappa^{[p]}(s, x) \phi_\kappa(h(\eta, \xi, s)) u_\kappa(s, x) \xi^2(s, x) dx ds \\ & \leq C_0(p, T) \left(\mathbb{E}|u_0|_{L^{p+1}}^{p+1} + 1 \right) \end{aligned}$$

is valid.

Proof By Itô’s formula for the functional $u \mapsto |u|_{L^{p+1}}^{p+1}$, we get that u_κ satisfies for $t \in [0, T]$,

$$\begin{aligned} & |u_\kappa(t)|_{L^{p+1}}^{p+1} - |u_0|_{L^{p+1}}^{p+1} \\ & \leq (p+1)r_u \int_0^t \int_{\mathcal{O}} \Delta(u_\kappa^{[\gamma]}(s, x)) u_\kappa^{[p]}(s, x) dx ds \\ & - (p+1)\chi \int_0^t \int_{\mathcal{O}} \phi_\kappa(h(\eta, \xi, s)) u_\kappa(s, x) \xi^2(s, x) u_\kappa^{[p]}(s, x) dx ds \\ & + M(t) + \frac{p}{2}(p+1) \int_0^t |u_\kappa(s)|_{L^{p+1}}^{p-1} |\sigma_1 u_\kappa(s)|_{\gamma(H_1, L^{p+1})}^2 ds, \end{aligned}$$

where $t \mapsto M(t)$ is a local martingale. Integrating by parts, taking expectation (we may as well apply the Burkholder-Davis-Gundy inequality, see Section) and rearranging yields,

$$\begin{aligned} & \mathbb{E} \left[|u_\kappa(s)|_{L^{p+1}}^{p+1} \right] + (p+1)r_u \mathbb{E} \int_0^t \int_{\mathcal{O}} \nabla(u_\kappa^{[\gamma]}(s, x)) \cdot \nabla(u_\kappa^{[p]}(s, x)) dx ds \\ & + (p+1)\chi \mathbb{E} \int_0^t \int_{\mathcal{O}} \phi_\kappa(h(\eta, \xi, s)) u_\kappa(s, x) \xi^2(s, x) u_\kappa^{[p]}(s, x) dx ds \\ & \leq \mathbb{E}|u_0|_{L^{p+1}}^{p+1} + \frac{p}{2}(p+1)C(p)\mathbb{E} \int_0^t |u_\kappa(s)|_{L^{p+1}}^{p+1} ds + \mathbb{E} \int_0^t |\sigma_1 u_\kappa(s)|_{\gamma(H_1, L^{p+1})}^{(p+1)/(p-1)} ds, \end{aligned}$$

and hence by Remark 5.2 and Gronwall’s lemma,

$$\begin{aligned} & \mathbb{E} \left[\sup_{0 \leq s \leq T} |u_\kappa(s)|_{L^{p+1}}^{p+1} \right] + \gamma p(p+1)r_u \mathbb{E} \int_0^T \int_{\mathcal{O}} |\nabla u_\kappa(s, x)|^2 |u(s, x)|^{\gamma+p-2} dx ds \\ & + (p+1)\chi \mathbb{E} \int_0^T \int_{\mathcal{O}} \phi_\kappa(h(\eta, \xi, s)) u_\kappa(s, x) \xi^2(s, x) u_\kappa^{[p]}(s, x) dx ds \\ & \leq C_0(p, T) \left(\mathbb{E}|u_0|_{L^{p+1}}^{p+1} + 1 \right). \end{aligned}$$

\square

We may remark that the above result now permits an application of Proposition A.4. Note that the inequality becomes particularly useful, when $u_\kappa \geq 0$ and $\eta \geq 0$.

5.3 Properties Of Equation Eq. 3.13

Given the couple $(\eta, \xi) \in \mathcal{M}_{\mathbb{Q}}^{2,m_0}(\mathbb{Y}, \mathbb{X})$, we consider the solution v_κ to the equation Eq. 3.13.

First, we will In the next proposition we investigate existence and uniqueness and the regularity of the process v_κ . The constants R_0, R_1 and R_2 are given as in Eq. 3.26, Eq. 3.23, and Eq. 3.25, respectively.

Proposition 5.7 *Assume that the Hypotheses of Theorem 3.6 hold. Then, for any pair $(\eta, \xi) \in \mathcal{X}_{\mathbb{Q}}(R_0, R_1, R_2) \subset \mathcal{M}_{\mathbb{Q}}^{2,m_0}(\mathbb{Y}, \mathbb{Z})$ and for any initial datum v_0 as in Hypothesis 3.4, there exists a solution v_κ to Eq. 3.13 on the time interval $[0, T]$ such that v_κ is continuous in $H_2^\rho(\mathcal{O})$ and such that*

$$v_\kappa \in L^\infty(0, T; H_2^\rho(\mathcal{O})) \cap L^2(0, T; H_2^{\rho+1}(\mathcal{O})) \quad \mathbb{P}\text{-a.s.}$$

In addition, for all $\kappa \in \mathbb{N}$ there exists a constant $C = C(T, \kappa) > 0$ such that

$$\mathbb{E} \|v_\kappa\|_{\mathbb{H}_\rho}^{m_0} \leq \mathbb{E} |v_0|_{H_2^\rho}^{m_0} + C(\kappa, T) R_0. \tag{5.8}$$

Proof Let us consider the following equation with a locally integrable and progressively measurable $t \mapsto F(t)$

$$dw(t) = r_v \Delta w(t) dt + F(t) dt + \sigma_2 w(t) dW_2(t), \quad w(0) = w_0 \in H_2^\rho(\mathcal{O}), \tag{5.9}$$

where the Laplace operator Δ is equipped with periodic boundary conditions on a box or Neumann boundary conditions and initial condition $w(0) = w_0$. A solution to Eq. 5.9 for $F \equiv 0$ is given by standard methods (see e.g. [16, Chapter 6] or [48, Theorem 4.2.4]) such that for any $q \geq 1$,

$$\mathbb{E} \|w\|_{C([0,T]; H_2^\rho)}^q \leq C_1(T) \quad \text{and} \quad \mathbb{E} \|w\|_{L^2([0,T]; H_2^{\rho+1})}^q \leq C_2(T). \tag{5.10}$$

The solution to Eq. 5.9 is given by

$$\begin{aligned} v_\kappa(t) &= w(t) + \int_0^t e^{r_v \Delta(t-s)} F(s) ds, \quad t \in [0, T]. \\ \left| \int_0^t e^{r_v \Delta(t-s)} F(s) ds \right|_{H_2^\rho} &\leq \int_0^t \left| e^{r_v \Delta(t-s)} F(s) \right|_{H_2^\rho} ds \\ &\leq \int_0^t (t-s)^{-\frac{\delta}{2}} |F(s)|_{H_2^{\rho-\delta}} ds \leq C(T) \left(\int_0^t |F(s)|_{L^1}^\mu ds \right)^{\frac{1}{\mu}}. \end{aligned} \tag{5.11}$$

Setting

$$F = \phi_\kappa(h(\eta, \xi, \cdot)) \eta \xi^2,$$

we obtain by the Hölder inequality for $\delta/2 < 1 - \frac{1}{\mu}$ and $\frac{2\mu}{m_0} < 1$

$$\begin{aligned} &\int_0^T |F(s)|_{L^1}^\mu ds \\ &\leq \sup_{0 \leq s \leq T} |\eta(s)|_{L^{p^*}}^\mu \int_0^T (\phi_\kappa(h(\eta, \xi, t)))^\mu |\xi(s)|_{L^m}^{2\mu} ds \leq (2\kappa)^{\frac{2\mu}{m_0}} \sup_{0 \leq s \leq T} |\eta(s)|_{L^{p^*}}^\mu. \end{aligned} \tag{5.12}$$

Observe, since $\rho < 2 - \frac{4}{m} - \frac{d}{2}$, such a δ and μ exists. Taking expectation, we know due to the assumptions on η , that the term can be bounded. It remains to calculate the norm in $L^2(0, T; H_2^{\rho+1}(\mathcal{O}))$. By standard calculations (i.e. applying the smoothing property and the Young inequality for convolution) we get for $\frac{3}{2} \geq \frac{1}{\mu} + \frac{1}{\kappa}$ and $\delta\kappa/2 < 1$

$$\left\| \int_0^\cdot e^{r_v \Delta(\cdot-s)} F(s) ds \right\|_{L^2(0,T;H_2^{\rho+1})} \leq C \|F\|_{L^\mu(0,T;H_2^{\rho+1-\delta})}.$$

The embedding $L^1(\mathcal{O}) \hookrightarrow H_2^{\rho+1-\delta}(\mathcal{O})$ for $\delta - (\rho + 1) > \frac{d}{2}$ gives

$$\|F\|_{L^\mu(0,T;H_2^{\rho-1})} \leq \|F\|_{L^\mu(0,T;L^1)},$$

and similarly to before

$$\mathbb{E} \left\| \int_0^\cdot e^{r_v \Delta(\cdot-s)} F(s) ds \right\|_{L^2(0,T;H_2^{\rho+1})}^{m_0} \leq C(\kappa) R_0. \square$$

Proposition 5.8 *Under the conditions of Theorem 5.7 and the additional condition $v_0(x) \geq 0$ for all $x \in \mathcal{O}$ and $\eta(t, x) \geq 0, \xi(t, x) \geq 0$ for a.e. $(t, x) \in [0, T] \times \mathcal{O}$, the solution v_κ to Eq. 3.13 is nonnegative a.e.*

Proof The heat semigroup, which is generated by the Laplace, maps nonnegative functions to nonnegative functions. In this way we refer to the proof of nonnegativity by Tessitore and Zabczyk [68]. The perturbation can be incorporated by comparison results, see [42]. \square

Proposition 5.9 *Assume that the Hypotheses of Theorem 3.6 hold. In addition let us assume that we have for the couple $(\eta, \xi) \in \mathcal{X}_{\mathfrak{A}}(R_0, R_1, R_2) \subset \mathcal{M}_{\mathfrak{A}}^{2,m_0}(\mathbb{Y}, \mathbb{Z})$. Let v_κ be a solution to*

$$dv_\kappa(t) = [r_v \Delta v_\kappa(t) + \phi_\kappa(h(\eta, \xi, t)) \eta(t) \xi^2(t)] dt + \sigma_2 v_\kappa(t) dW_2(t), \quad v_\kappa(0) = v_0 \in H_m^{-\delta_1}(\mathcal{O}).$$

Then,

(i) *there exists a number $r_0 > 0$ and a constant $\tilde{C}_2(T) > 0$ such that for any $r \leq r_0$, we have for all $\kappa \in \mathbb{N}$*

$$\mathbb{E} \|v_\kappa\|_{L^{m_0}(0,T;H_m^r)}^2 \leq \tilde{C}_2(T) \left\{ \mathbb{E} |v_0|_{H_m^{-\delta_0}}^2 + 4\kappa \right\}.$$

(ii) *there exists a number $\alpha_0 > 0$ and a constant $C = C(T) > 0$ such that for any $\alpha \leq \alpha_0$ we have for all $\kappa \in \mathbb{N}$*

$$\mathbb{E} \|v_\kappa\|_{\mathbb{W}_{m_0}^\alpha(0,T;H_m^{-\sigma})}^2 \leq C(T) \left\{ \mathbb{E} |v_0|_{H_m^{-\delta_0}}^2 + 4\kappa \right\}.$$

Proof We start to show (i). First, we get by the analyticity of the semigroup for $\delta_0, \delta_1 \geq 0$

$$\begin{aligned} \mathbb{E} \|v_\kappa\|_{L^{m_0}(0,T;H_m^r)}^2 &\leq \left[\int_0^T \left\{ t^{-2m_0\delta_0} \mathbb{E} |v_0|_{H_m^{-\delta_0}}^{m_0} \right. \right. \\ &\quad \left. \left. + \mathbb{E} \left| \int_0^t e^{r_v \Delta(t-s)} \phi_\kappa(h(\eta, \xi, s)) \eta(s) \xi^2(s) ds \right|_{H_m^r}^{m_0} \right. \right. \\ &\quad \left. \left. + \mathbb{E} \left| \int_0^t e^{r_v \Delta(t-s)} \sigma_2 v_\kappa(s) dW_2(s) \right|_{H_m^r}^{m_0} \right\} dt \right] \\ &=: (I + II + III). \end{aligned}$$

Since, by the hypotheses,

$$2m_0\delta_0 < 1,$$

the first term, i.e. I is bounded. In particular, we have

$$I \leq T^{\frac{q}{m_0}(1-2m_0\delta_0)} \mathbb{E}|v_0|_{H_m^{-\delta_0}}^2. \tag{5.13}$$

Let us continue with the second term. The smoothing property of the semigroup gives for any $\delta_1 \geq 0$

$$II \leq \mathbb{E} \left[\int_0^T \left(\int_0^t (t-s)^{-\frac{1}{2}(r+\delta_1)} \phi_\kappa(h(\eta, \xi, s)) \left| \eta(s)\xi^2(s) \right|_{H_m^{-\delta_1}} ds \right)^{m_0} dt \right].$$

Using the Sobolev embedding $L^1(\mathcal{O}) \hookrightarrow H_m^{-\delta_1}(\mathcal{O})$, where $\delta_1 \geq d(1 - \frac{1}{m})$, we get

$$II \leq \mathbb{E} \left[\int_0^T \left(\int_0^t (t-s)^{-\frac{1}{2}(r+\delta_1)} \phi_\kappa(h(\eta, \xi, s)) \left| \eta(s)\xi^2(s) \right|_{L^1} ds \right)^{m_0} dt \right].$$

Supposing $l^{\frac{\delta_1+r_0}{2}} < 1$ the Young inequality for convolutions gives for

$$\frac{1}{m_0} + 1 = \frac{1}{l} + \frac{1}{\mu} \quad \text{and} \quad \beta_0 = \frac{1}{l} - \frac{1}{2}(r + \delta_1) \tag{5.14}$$

and therefore,

$$II \leq C_0 T^{\beta_0 \frac{2}{m_0}} \mathbb{E} \left(\int_0^T \left| \phi_\kappa(h(\eta, \xi, s)) \eta(s)\xi^2(s) \right|_{L^1}^\mu ds \right)^{\frac{2}{\mu}}. \tag{5.15}$$

Which can be bounded by Hölder inequality as follows,

$$\begin{aligned} & \int_0^T \left| \phi_\kappa(h(\eta, \xi, s)) \eta(s)\xi^2(s) \right|_{L^1}^\mu ds \\ & \leq \sup_{0 \leq s \leq T} \left| \eta(s) \right|_{L^{p^*}}^\mu \int_0^T \phi_\kappa(h(\eta, \xi, s)) \left| \xi(s) \right|_{L^m}^{m_0} ds \leq C(\kappa) R_0. \end{aligned}$$

Next, let us investigate III . We treat the stochastic term by applying [9, Corollary 3.5 (ii)], from which it follows for $\tilde{\sigma} + r < 1$ and $\beta > 0$

$$\mathbb{E} \left[\int_0^T \left| \int_0^t e^{r_v \Delta(t-s)} \sigma_2 v_\kappa(s) dW_2(s) \right|_{H_m^r}^2 dt \right]^{\frac{q}{m_0}} \leq T^\beta \mathbb{E} \left[\int_0^T \left| v_\kappa(t) \right|_{H_m^{-\tilde{\sigma}}}^{m_0} dt \right].$$

Due to the Sobolev embedding and interpolation, we know that whenever

$$\tilde{\sigma} > \frac{d}{2} - \frac{d}{m} - \tilde{\rho}, \tag{5.16}$$

then there exists some $\theta \in (0, 1)$ such that

$$\left| v_\kappa(t) \right|_{H_m^{-\tilde{\sigma}}} \leq \left| v_\kappa(t) \right|_{L^m}^\theta \left| v_\kappa(t) \right|_{H_m^{\tilde{\rho}}}^{1-\theta} \leq C \left| v_\kappa(t) \right|_{L^m}^\theta \left| v_\kappa(t) \right|_{H_2^{\tilde{\rho}}}^{1-\theta}.$$

Thus, if $\tilde{\rho}$ satisfies Eq. 5.16, we get that

$$\begin{aligned} \mathbb{E} \left[\int_0^T |v_\kappa(t)|_{H_m^{-\tilde{\sigma}}}^{m_0} dt \right]^{\frac{q}{m_0}} &\leq C \mathbb{E} \left\{ \left[\int_0^T |v_\kappa(t)|_{H_2^{\tilde{\rho}}}^{m_0} dt \right]^{1-\theta} \left[\int_0^T |v_\kappa(t)|_{L^m}^{m_0} dt \right]^\theta \right\}^{\frac{q}{m_0}} \\ &\leq C \mathbb{E} \left[\int_0^T |v_\kappa(t)|_{H_2^{\tilde{\rho}}}^{m_0} dt \right]^{(1-\theta)\frac{q}{m_0}} \left[\mathbb{E} \int_0^T |v_\kappa(t)|_{L^m}^{m_0} dt \right]^\theta \frac{q}{m_0} \\ &\leq C(\varepsilon) \mathbb{E} \left[\int_0^T |v_\kappa(t)|_{H_2^{\tilde{\rho}}}^{m_0} dt \right]^{\frac{q}{m_0}} + \varepsilon \mathbb{E} \|v_\kappa\|_{L^{m_0}(0,T;L^m)}^q. \end{aligned}$$

Collecting everything together, choosing $\varepsilon > 0$ sufficiently small and subtracting $\varepsilon \mathbb{E} \|v_\kappa\|_{L^{m_0}(0,T;L^m)}^q$ on both sides, (i) follows.

The rest of the proof is devoted to item (ii). Note, that for $s < t$

$$\begin{aligned} v_\kappa(t) - v_\kappa(s) &= (e^{r_v \Delta(t-s)} - \text{Id})v_\kappa(s) \\ &+ \int_s^t e^{r_v \Delta(t-\tilde{s})} \phi_\kappa(h(\eta, \xi, s)) \eta(\tilde{s}) \xi^2(\tilde{s}) d\tilde{s} + \int_0^s e^{r_v \Delta(t-\tilde{s})} v_\kappa(\tilde{s}) dW_2(\tilde{s}). \end{aligned}$$

Substituting it in the definition of $\mathbb{W}_{m_0}^\alpha([0, T]; H_m^{-\sigma}(\mathcal{O}))$, see Appendix B, we can write

$$\begin{aligned} &\int_0^T \int_0^T \frac{|v_\kappa(t) - v_\kappa(s)|_{H_m^{-\sigma}}^{m_0}}{|t-s|^{1+\alpha m_0}} ds dt \\ &\leq 2 \int_0^T \int_0^t \frac{|v_\kappa(t) - v_\kappa(s)|_{H_m^{-\sigma}}^{m_0}}{|t-s|^{1+\alpha m_0}} ds dt \\ &\leq 2 \int_0^T \int_0^t \frac{|[e^{r_v \Delta(t-s)} - \text{Id}]v_\kappa(s)|_{H_m^{-\sigma}}^{m_0}}{|t-s|^{1+\alpha m_0}} ds dt \\ &+ 2 \int_0^T \int_0^t \frac{|\int_{\tilde{s}}^t e^{r_v \Delta(t-s)} \phi_\kappa(h(\eta, \xi, s)) \eta(s) \xi^2(s) ds|_{H_m^{-\sigma}}^{m_0}}{|t-\tilde{s}|^{1+\alpha m_0}} d\tilde{s} dt \\ &+ 2 \int_0^T \int_0^t \frac{|\int_{\tilde{s}}^t e^{r_v \Delta(t-\tilde{s})} u(\tilde{s}) dW(\tilde{s})|_{H_m^{-\sigma}}^{m_0}}{|t-s|^{1+\alpha m_0}} ds dt =: J + JJ + JJJ. \end{aligned}$$

First, let us note that by part (i) we know that there exists a number $r > 0$ and a constant $\tilde{C}(T) > 0$ such that

$$\mathbb{E} \left[\int_0^T |v_\kappa(s)|_{H_m}^{m_0} ds \right]^{\frac{q}{m_0}} \leq \tilde{C}(T) \left\{ \mathbb{E} |v_0|_{H_m^{-\delta_0}}^2 + 4\kappa \right\}. \tag{5.17}$$

Let us consider the term J . Since

$$|[e^{r_v \Delta(t-s)} - \text{Id}]x|_{H_m^{-\sigma}} = C \int_0^{t-s} \tilde{s}^{2(r+\sigma)-1} d\tilde{s} |x|_{H_m} \leq C(t-s)^{2(r+\sigma)} |x|_{H_m},$$

we can infer

$$J \leq 2 \int_0^T \int_0^t \frac{(t-s)^{2(r+\sigma)m_0} |v_\kappa(s)|_{H_m}^{m_0}}{|t-s|^{1+\alpha m_0}} ds dt.$$

The Young inequality for convolutions (with all exponents equal to 1) gives

$$J \leq \left(\int_0^T s^{m_0 2(r+\delta)-1-\alpha m_0} ds \right) \cdot \left(\int_0^T |v_\kappa(s)|_{H_m^{r_0}}^{m_0} ds \right).$$

In particular, since $2(r + \delta) > \alpha$, the right hand side is bounded by estimate Eq. 5.17.

Now, we consider the second term JJ .

Jensen’s inequality gives for any $q > 1$, $q' := \frac{q}{q-1}$,

$$\begin{aligned} & \left| \int_{\tilde{s}}^t e^{r_v \Delta(t-s)} \phi_\kappa(h(\eta, \xi, s)) \eta(s) \xi^2(s) ds \right|_{H_m^{-\sigma}}^q \\ & \leq (t - \tilde{s})^{\frac{q}{q'}} \left(\int_{\tilde{s}}^t |\phi_\kappa(h(\eta, \xi, s)) \eta(s) \xi^2(s)|_{H_m^{-\sigma}}^q ds \right). \end{aligned}$$

Hence,

$$\begin{aligned} JJ & \leq \int_0^T \int_0^t \frac{\left| \int_{\tilde{s}}^t e^{r_v \Delta(t-s)} \phi_\kappa(h(\eta, \xi, s)) \eta(s) \xi^2(s) ds \right|_{H_m^{-\sigma}}^{m_0}}{|t - \tilde{s}|^{1+\alpha m_0}} d\tilde{s} dt \\ & \leq \int_0^T \int_0^t \frac{(t - \tilde{s})^{\frac{m_0}{m_0'}} \int_{\tilde{s}}^t \phi_\kappa(h(\eta, \xi, s)) |\eta(s) \xi^2(s)|_{H_m^{-\sigma}}^{m_0} ds}{|t - \tilde{s}|^{1+\alpha m_0}} d\tilde{s} dt \\ & \leq \left[\int_0^T \int_0^t (t - s)^{\frac{m_0}{m_0'}-1-\alpha m_0} ds dt \right] \left[\int_0^T \phi_\kappa(h(\eta, \xi, s)) |\eta(s) \xi^2(s)|_{H_m^{-\sigma}}^{m_0} ds \right]. \end{aligned}$$

Taking into account that $\alpha < \frac{1}{m_0'}$, integration and the Sobolev embedding gives

$$\begin{aligned} JJ & \leq \left[\int_0^T t^{\frac{m_0}{m_0'}-\alpha m_0} dt \right] \int_0^T \phi_\kappa(h(\eta, \xi, s)) |\eta(s) \xi^2(s)|_{H_m^{-\tilde{r}}}^{m_0} ds \\ & \leq T^{\frac{m_0}{m_0'}+1-\alpha m_0} \int_0^T \phi_\kappa(h(\eta, \xi, s)) |\eta(s) \xi^2(s)|_{L^1}^{m_0} ds \leq 4T^{\frac{m_0}{m_0'}+1-\alpha m_0} C(\kappa) R_0. \end{aligned}$$

It remains to give an estimate to the second term. We can show the assertion by the same computations as in the part of the proof for (i), i.e. starting at Eq. 5.15.

Next, let us investigate JJJ . Here, again by [9, Corollary 3.5 (ii)], we get for some $\beta > 0$ that

$$\mathbb{E} \left| \int_s^t e^{r_v \Delta(t-\tilde{s})} \sigma_2 v_\kappa(\tilde{s}) dW_2(\tilde{s}) \right|_{H_m^{-\sigma}}^2 \leq (t - s)^\beta \mathbb{E} \left(\int_s^t |v_\kappa(\tilde{s})|_{H_m^{-\tilde{\sigma}}}^{m_0} d\tilde{s} \right).$$

In this way we get

$$\begin{aligned} \mathbb{E}|JJJ| & \leq 2 \int_0^T \int_0^t \frac{|t - s|^{m_0-2}}{|t - s|^{1+\alpha m_0}} \mathbb{E} \int_s^t |v_\kappa(\tilde{s})|_{H_m^{-\tilde{\sigma}}}^{m_0} d\tilde{s} ds dt \\ & \leq 2 \int_0^T \int_0^t |t - s|^{m_0(1-\alpha)-3} \left(\mathbb{E} \int_s^t |v_\kappa(\tilde{s})|_{H_m^{-\tilde{\sigma}}}^{m_0} d\tilde{s} \right) ds dt. \end{aligned}$$

Applying the Hölder inequality, we show that

$$\mathbb{E}|JJJ| \leq C(T) \mathbb{E} \left(\int_0^T |v_\kappa(s)|_{H_m^{-\tilde{\sigma}}}^{m_0} ds \right).$$

Collecting everything together gives (ii). □

Technical Proposition 5.10 *Let us assume that $(\eta, \xi) \in \mathcal{X}_{\mathfrak{A}}(R_0, R_1, R_2) \subset \mathcal{M}_{\mathfrak{A}}^{2,m_0}(\mathbb{Y}, \mathbb{Z})$. Then, for any $\alpha \in [1, \frac{m_0}{2}]$, $q^* \in (1, m_0]$, and $\kappa \in \mathbb{N}$ there exists some real numbers $\delta_1, \delta_2 \in (0, 1]$ and constants $C_1(\kappa, R_0), C_2(\kappa, R_0) > 0$ such that we have*

$$\begin{aligned} & \mathbb{E} \left\| \eta_1 \xi_1^2 - \eta_2 \xi_2^2 \right\|_{L^\alpha(0,T;L^1)}^{q^*} \\ & \leq C_1(\kappa, R_0) \times \left(\mathbb{E} \|\xi_1 - \xi_2\|_{\mathbb{Z}}^{m_0} \right)^{\delta_1} + \left(\mathbb{E} \|\eta_1 - \eta_2\|_{L^{\gamma+1}(0,T;L^{\gamma+1})}^{\gamma+1} \right)^{\delta_2} \\ & \quad + C_2(\kappa, R_0) \times \left(\mathbb{E} \|\xi_1 - \xi_2\|_{\mathbb{Z}}^{m_0} + \mathbb{E} \|\eta_1 - \eta_2\|_{L^{\gamma+1}(0,T;L^{\gamma+1})}^{\gamma+1} \right). \end{aligned}$$

Proof Firstly, observe that due to the cutoff function, at integrating with respect to the time we have to take into account in which interval s belongs to. Let $\tau_j^u := \inf_{s>0} \{h(\eta_j, \xi_j, s) \geq \kappa\}$ and $\tau_j^o := \inf_{s>0} \{h(\eta_j, \xi_j, s) \geq 2\kappa\}$, $j = 1, 2$. Distinguishing the different cases, and noting that $\text{Lip}(\phi_\kappa) \leq 2\kappa^{-1}$, we obtain

$$\begin{aligned} & \left| \phi_\kappa(h(\eta_1, \xi_1, s)) \eta_1(s) \xi_1^2(s) - \phi_\kappa(h(\eta_2, \xi_2, s)) \eta_2(s) \xi_2^2(s) \right|_{L^1}^\alpha \\ & \leq \begin{cases} |\eta_1(s) \xi_1^2(s) - \eta_2(s) \xi_2^2(s)|_{L^1}^\alpha & \text{if } 0 \leq s \leq T \wedge \underline{\tau}^u, \\ |\eta_1(s) \xi_1^2(s) - \eta_2(s) \xi_2^2(s)|_{L^1}^\alpha \text{Lip}(\phi_\kappa)^\alpha \\ \quad \times \left(\|\eta_1 - \eta_2\|_{L^{\gamma+1}(0,T;L^{\gamma+1})}^\nu + \|\xi_1 - \xi_2\|_{L^{m_0}(0,T;L^{m_0})}^\nu \right) & \text{if } T \wedge \underline{\tau}^u < s \leq T \wedge \bar{\tau}^o, \\ 0 & \text{if } T \geq s > T \wedge \bar{\tau}^o, \end{cases} \end{aligned}$$

where $\underline{\tau}^u := \min(\tau_1^u, \tau_2^u)$ and $\bar{\tau}^o := \max(\tau_1^o, \tau_2^o)$. Therefore, we may integrate over $[0, T]$.

Secondly, observe that for any $n \in \mathbb{N}$ and any $a, b \geq 0$, we have

$$\begin{aligned} |a - b| & \leq \left| \sum_{k=1}^n a^{\frac{k-1}{n}} \left(a^{\frac{1}{n}} - b^{\frac{1}{n}} \right) b^{\frac{n-k}{n}} \right| \\ & \leq |a - b|^{\frac{1}{n}} \sum_{k=1}^n a^{\frac{k-1}{n}} b^{\frac{n-k}{n}} \leq C |a - b|^{\frac{1}{n}} (n - 1) \left(a^{\frac{n-1}{n}} + b^{\frac{n-1}{n}} \right), \end{aligned}$$

where the last inequality follows from an application of Young’s inequality. The difference can be split in the following way for $s \in [0, \bar{\tau}^o \wedge T]$

$$\begin{aligned} & \left| \xi_1^2(s, x) \eta_1(s, x) - \xi_2^2(s, x) \eta_2(s, x) \right| \\ & \leq |\xi_1(s, x) - \xi_2(s, x)| \xi_1(s, x) + \xi_2(s, x) |\eta_1(s, x) - \eta_2(s, x)| + |\xi_1^2(s, x) + \xi_2^2(s, x)| |\eta_1(s, x) - \eta_2(s, x)|. \end{aligned}$$

Let $n \in \mathbb{N}$ be that large such that

$$\frac{1}{n(\gamma + 1)} + \frac{n - 1}{n} \cdot \frac{1}{p^*} + \frac{1}{m} \leq 1, \quad \frac{1}{n(\gamma + 1)} + \frac{1}{m_0} \leq \frac{1}{\alpha}, \quad \text{and} \quad \frac{1}{n(\gamma + 1)} + \frac{1}{p^*} \leq \frac{1}{\alpha}.$$

Therefore, we can write

$$\begin{aligned} & \int_0^{\bar{\tau}^o \wedge T} |\eta_1(s)\xi_1^2(s) - \eta_2(s)\xi_2^2(s)|_{L^1}^\alpha ds \\ & \leq \int_0^{\bar{\tau}^o \wedge T} \left(|\eta_1(s) - \eta_2(s)|_{L^{\gamma+1}}^{\frac{1}{n}} |\eta_1(s) - \eta_2(s)|_{L^{p^*}}^{\frac{n-1}{n}} |\xi_1^2(s)|_{L^m} \right)^\alpha ds \\ & \leq \left(\int_0^{\bar{\tau}^o \wedge T} |\eta_1(s) - \eta_2(s)|_{L^{\gamma+1}}^{\gamma+1} ds \right)^{\frac{\alpha}{(\gamma+1)n}} \\ & \quad \sup_{0 \leq s \leq T} |\eta_1(s) + \eta_2(s)|_{L^{p^*}}^{\alpha \frac{n-1}{n}} \left(\int_0^{\bar{\tau}^o \wedge T} |\xi_1^2(s)|_{L^m}^{m_0} ds \right)^{\frac{\alpha}{m_0}}. \end{aligned}$$

Taking expectation gives

$$\begin{aligned} & \mathbb{E} \left\{ \int_0^{\bar{\tau}^o \wedge T} |\eta_1(s)\xi_1^2(s) - \eta_2(s)\xi_2^2(s)|_{L^1}^\alpha ds \right\}^{q^*} \\ & \leq C(\kappa) \left\{ \mathbb{E} \int_0^{\bar{\tau}^o \wedge T} |\eta_1(s) - \eta_2(s)|_{L^{\gamma+1}}^{\gamma+1} ds \right\}^{\frac{q^* \alpha}{(\gamma+1)n}} \left\{ \mathbb{E} \sup_{0 \leq s \leq T} |\eta_1(s) + \eta_2(s)|_{L^{p^*}}^{p_0^*} \right\}^{\frac{q^* \alpha}{p_0^*} \frac{n-1}{n}}. \end{aligned}$$

Similarly we get by the Hölder inequality

$$|\xi_1^2(s)\eta_2(s) - \xi_2^2(s)\eta_1(s)|_{L^1} \leq |\xi_1(s) - \xi_2(s)|_{L^m} |\xi_1(s) + \xi_2(s)|_{L^m} |\eta_2(s)|_{L^{p^*}}.$$

Integration over time gives

$$\begin{aligned} & \int_0^{\bar{\tau}^o \wedge T} |\xi_1^2(s)\eta_2(s) - \xi_2^2(s)\eta_1(s)|_{L^1}^\alpha ds \\ & \leq \sup_{0 \leq s \leq T} |\eta_2(s)|_{L^{p^*}}^\alpha \left(\int_0^{\bar{\tau}^o \wedge T} |\xi_1(s) - \xi_2(s)|_{L^m}^{m_0} ds \right)^{\frac{\alpha}{m_0}} \left(\int_0^{\bar{\tau}^o \wedge T} |\xi_1(s) + \xi_2(s)|_{L^m}^{m_0} ds \right)^{\frac{\alpha}{m_0}}. \end{aligned}$$

Taking expectation we get

$$\begin{aligned} & \left\{ \mathbb{E} \int_0^{\bar{\tau}^o \wedge T} |\xi_1^2(s)\eta_2(s) - \xi_2^2(s)\eta_1(s)|_{L^1}^\alpha ds \right\}^{q^*} \\ & \leq C(\kappa) \left\{ \mathbb{E} \sup_{0 \leq s \leq T} |\eta_2(s)|_{L^{p^*}}^{p_0^*} \right\}^{\frac{q^* \alpha}{p_0^*}} \left\{ \mathbb{E} \int_0^{\bar{\tau}^o \wedge T} |\xi_1(s) - \xi_2(s)|_{L^m}^{m_0} ds \right\}^{\frac{q^* \alpha}{m_0}}. \end{aligned}$$

Finally, take into account that

$$\begin{aligned} & \mathbb{E} \left[\left(\int_0^{\bar{\tau}^o \wedge T} |\eta_1(s)\xi_1^2(s) - \eta_2(s)\xi_2^2(s)|_{L^1}^\alpha ds \right) \right. \\ & \quad \left. \times (\|\eta_1 - \eta_2\|_{L^{\gamma+1}(0,T;L^{\gamma+1})}^\gamma + \|\xi_1 - \xi_2\|_{L^{m_0}(0,T;L^m)}^\gamma) \right]^{q^*} \end{aligned}$$

$$\begin{aligned} &\leq \mathbb{E} \left[\left(\sum_{j=1,2} \sup_{0 \leq s \leq T} |\eta_j(s)|_{L^{p^*}} \int_0^{\bar{\tau} \wedge T} |\xi_j(s)|_{L^m}^{m_0} ds \right) \right. \\ &\quad \times \left. \left(\|\eta_1 - \eta_2\|_{L^{\gamma+1}(0,T;L^{\gamma+1})}^v + \|\xi_1 - \xi_2\|_{L^{m_0}(0,T;L^m)}^v \right) \right]^q \\ &\leq C(\kappa) \left\{ \mathbb{E} \sum_{j=1,2} \sup_{0 \leq s \leq T} |\eta_j(s)|_{L^{p^*}}^{p_0^*} \right\}^{\frac{q}{p_0^*}} \\ &\quad \times \left[\left(\mathbb{E} \|\eta_1 - \eta_2\|_{L^{\gamma+1}(0,T;L^{\gamma+1})}^{\gamma+1} \right)^{\frac{vq^*}{\gamma+1}} + \left(\mathbb{E} \|\xi_1 - \xi_2\|_{L^{m_0}(0,T;L^m)}^{m_0} \right)^{\frac{vq^*}{m_0}} \right], \end{aligned}$$

the assertion is shown. □

The next proposition gives the continuity property of the operator $\mathcal{V}_{\kappa, \mathfrak{A}}$.

Proposition 5.11 *Let $(\eta_1, \xi_1) \in \mathcal{X}_{\mathfrak{A}}(R_0, R_1, R_2) \subset \mathcal{M}_{\mathfrak{A}}^{2,m_0}(\mathbb{Y}, \mathbb{Z})$ and $(\eta_2, \xi_2) \in \mathcal{X}_{\mathfrak{A}}(R_0, R_1, R_2) \subset \mathcal{M}_{\mathfrak{A}}^{2,m_0}(\mathbb{Y}, \mathbb{Z})$. Let $v_{\kappa}^{(1)}$ and $v_{\kappa}^{(2)}$ be the solutions to*

$$dv_{\kappa}^{(j)}(t) = [r_v \Delta v_{\kappa}^{(j)}(t) + \phi_{\kappa}(h(\eta_j, \xi_j, t) \eta_j(t) \xi_j^2(t))] dt + \sigma_2 v_{\kappa}^{(j)}(t) dW_2(t), \quad t \in (0, T],$$

with $v_{\kappa}^{(j)}(0) = v_0, j = 1, 2$. Then, under the assumptions of Theorem 3.6 there exists a constant $C = C(R_0, \kappa) > 0$ and real numbers $\delta_1, \delta_2 > 0$ such that

$$\begin{aligned} &\mathbb{E} \left\| v_{\kappa}^{(1)} - v_{\kappa}^{(2)} \right\|_{L^{m_0}(0,T;L^m)}^{m_0} \\ &\leq C(R_0, \kappa) \left\{ \left[\mathbb{E} \|\xi_1 - \xi_2\|_{L^{m_0}(0,T;L^m)}^{m_0} \right]^{\delta_1} + \left[\mathbb{E} \|\eta_1 - \eta_2\|_{L^{\gamma+1}(0,T;L^{\gamma+1})}^{\gamma+1} \right]^{\delta_2} \right\}. \end{aligned}$$

Proof First, we get by the analyticity of the semigroup,

$$\begin{aligned} &\mathbb{E} \left(\int_0^T \left| v_{\kappa}^{(1)}(t) - v_{\kappa}^{(2)}(t) \right|_{L^m}^{m_0} dt \right) \tag{5.18} \\ &\leq \mathbb{E} \left(\int_0^T \left| \int_0^t e^{r_v \Delta(t-s)} (\phi_{\kappa}(h(\eta_1, \xi_1, s)) \eta_1^1(s) \xi_1^2(s) - \phi_{\kappa}(h(\eta_2, \xi_2, s)) \eta_2(s) \xi_2^2(s)) ds \right|_{L^m}^{m_0} dt \right) \\ &\quad + \mathbb{E} \left(\int_0^T \left| \int_0^t (\sigma_2 v_{\kappa}^{(1)}(s) - \sigma_2 v_{\kappa}^{(2)}(s)) dW_2(s) \right|_{L^m}^{m_0} dt \right) \\ &=: I + II. \end{aligned}$$

Let us consider I . Here, we use the Sobolev embedding $L^1(\mathcal{O}) \hookrightarrow H_m^{-\delta}(\mathcal{O})$, where $\delta \geq d(1 - \frac{1}{m})$. Next, the Young inequality for convolution gives for $l, \mu \geq 1$, with $\frac{1}{m_0} + 1 = \frac{1}{l} + \frac{1}{\mu}$ and $\frac{\delta}{2} < \frac{1}{l}$

$$I \leq T^{\beta_0} \mathbb{E} \left(\int_0^T \left| \phi_{\kappa}(h(\eta_1, \xi_1, s)) \eta_1^1(s) \xi_1^2(s) - \phi_{\kappa}(h(\eta_2, \xi_2, s)) \eta_2(s) \xi_2^2(s) \right|_{L^1}^{\mu} ds \right).$$

Note, the assumptions on m_0 and m in the Hypothesis 3.4 gives the existence of l and μ .

Now, by the technical Proposition 5.10 we can infer that there exists a constant $C = C(R_0, \kappa) > 0$ and $\delta_1, \delta_2 > 0$ such that

$$I \leq C(R_0, \kappa) \left(\left(\mathbb{E} \|\xi_1 - \xi_2\|_{L^{m_0}(0,T;L^m)}^{m_0} \right)^{\delta_1} + \left(\mathbb{E} \|\eta_1 - \eta_2\|_{L^{\gamma+1}(0,T;L^{\gamma+1})}^{\gamma+1} \right)^{\delta_2} \right).$$

It remains to tackle the second term *II*. This can be done by standard arguments using the Burkholder-Davis-Gundy inequality, see Section . □

Next, we shall tackle the continuity of u_κ with respect to η and ξ .

Proposition 5.12 *Let the couples $(\eta_1, \xi_1) \in \mathcal{X}_{\mathfrak{A}}(R_0, R_1, R_2) \subset \mathcal{M}_{\mathfrak{A}}^{2,m_0}(\mathbb{Y}, \mathbb{Z})$ and $(\eta_2, \xi_2) \in \mathcal{X}_{\mathfrak{A}}(R_0, R_1, R_2) \subset \mathcal{M}_{\mathfrak{A}}^{2,m_0}(\mathbb{Y}, \mathbb{Z})$ be given. Let $u_\kappa^{(1)}$ and $u_\kappa^{(2)}$ be the solutions to*

$$du_\kappa^{(j)}(t) = [r_u \Delta(u_\kappa^{(j)})^{[\gamma]}(t) - \chi \phi_\kappa(h(\eta^{(j)}, \xi^{(j)}, t))u_\kappa^{(j)}(t)(\xi^{(j)})^2(t)] dt + \sigma_1 u_\kappa^{(j)}(t) dW_1(t),$$

$$t \in (0, T],$$

with $u_\kappa^{(j)}(0) = u_0, j = 1, 2$. Then, under the assumptions of Theorem 3.6 there exist constants $C = C(\kappa, R_1, R_2) > 0$ and $c, \delta_1 > 0$ such that

$$\mathbb{E} \left[\sup_{0 \leq s \leq T} \|u_\kappa^{(1)}(s) - u_\kappa^{(2)}(s)\|_{H_2^{-1}}^2 \right] + c \mathbb{E} \|u_\kappa^{(1)} - u_\kappa^{(2)}\|_{L^{\gamma+1}(0,T;L^{\gamma+1})}^{\gamma+1}$$

$$\leq C(R_0, \kappa) \left\{ (\mathbb{E} \|\xi_1 - \xi_2\|_{L^{m_0}(0,T;L^m)}^2)^{\delta_1} + \mathbb{E} \|\xi_1 - \xi_2\|_{L^{m_0}(0,T;L^m)}^2 \right\}.$$

Proof Applying the Itô formula and integration by parts gives

$$|u^{(1)}(t) - u^{(2)}(t)|_{H_2^{-1}}^2$$

$$+ 2r_u \int_0^t \int_{\mathcal{O}} ((u^{(1)})^{[\gamma]}(s, x) - (u^{(2)})^{[\gamma]}(s, x))(u^{(1)}(s, x) - u^{(2)}(s, x)) dx ds$$

$$= -2\chi \int_0^t \int_{\mathcal{O}} (\nabla^{-1}(u^{(1)}(s, x) - u^{(2)}(s, x)))$$

$$\times (\nabla^{-1}(\phi_\kappa(h(\eta_1, \xi_1, s))u_\kappa^{(1)}(s, x)\xi_1^2(s, x) - \phi_\kappa(h(\eta_2, \xi_2, s))u_\kappa^{(2)}(s, x)\xi_2^2(s, x)) dx ds$$

$$+ 2 \int_0^t \left\langle u^{(1)}(s) - u^{(2)}(s), \sigma_1(u^{(1)}(s) - u^{(2)}(s)) dW_1(s) \right\rangle$$

$$+ \int_0^t |\sigma_1(u^{(1)}(s) - u^{(2)}(s))|_{L_{HS}(H_1, H_2^{-1})}^2 ds$$

$$=: I + II + III,$$

where $\nabla^{-1} := -(-\Delta)^{1/2}$.

To deal with the second term and third term, i.e. II and III, we apply first the Burkholder Gundy inequality, resp. calculate the trace. Here, we apply Remark 5.2. Next, we obtain by the Young inequality and Lemma A.1 for $\varepsilon \in \left(0, 2^{\frac{2-\gamma}{\gamma+1}} r_u^{\frac{1}{\gamma+1}}\right]$,

$$II + III \leq \varepsilon^{\gamma+1} \mathbb{E} \int_0^t |u^{(1)}(s) - u^{(2)}(s)|_{L^{\gamma+1}}^{\gamma+1} ds$$

$$+ C \mathbb{E} \int_0^t |u^{(1)}(s) - u^{(2)}(s)|_{H_2^{-1}}^2 ds.$$

Before tackling the first term *I*, let us introduce the following definition. Let us set for $k \in \mathbb{N}$

$$\tau_k(\eta, \xi) := \inf\{t \geq 0 : h(\eta, \xi, t) \geq k\}, \quad \Omega_k := \{\omega \in \Omega : \tau_k(\eta_1, \xi_1) \leq \tau_k(\eta_2, \xi_2)\},$$

and $\Omega_k^c := \Omega \setminus \Omega_k$.

Now, we can decompose the third term into the following summands

$$\begin{aligned}
 & \mathbb{E} \int_0^t \int_{\mathcal{O}} (\nabla^{-1}(u^{(1)}(s, x) - u^{(2)}(s, x))) \\
 & \quad \times (\nabla^{-1}(\phi_\kappa(h(\eta_1, \xi_1, s))u_\kappa^{(1)}(s, x)\xi_1^2(s, x) - \phi_\kappa(h(\eta_2, \xi_2, s))u_\kappa^{(2)}(s, x)\xi_2^2(s, x))) dx ds \\
 & \leq \mathbb{E} \mathbb{1}_{\Omega_{2\kappa}} \int_0^{t \wedge \tau_{2\kappa}(\eta_2, \xi_2)} \int_{\mathcal{O}} (\nabla^{-1}(u^{(1)}(s, x) - u^{(2)}(s, x))) \\
 & \quad \times (\nabla^{-1}(\phi_\kappa(h(\eta_1, \xi_1, s))(u_\kappa^{(1)}(s, x) - u_\kappa^{(2)}(s, x))\xi_1^2(s, x))) dx ds \\
 & + \mathbb{E} \mathbb{1}_{\Omega_{2\kappa}} \int_0^{t \wedge \tau_{2\kappa}(\eta_2, \xi_2)} \int_{\mathcal{O}} (\nabla^{-1}(u^{(1)}(s, x) - u^{(2)}(s, x))) \\
 & \quad \times (\nabla^{-1}(\phi_\kappa(h(\eta_1, \xi_1, s)) - \phi_\kappa(h(\eta_2, \xi_2, s)))u_\kappa^{(2)}(s, x)\xi_1^2(s, x)) dx ds \\
 & + \mathbb{E} \mathbb{1}_{\Omega_{2\kappa}} \int_0^{t \wedge \tau_{2\kappa}(\eta_2, \xi_2)} \int_{\mathcal{O}} (\nabla^{-1}(u^{(1)}(s, x) - u^{(2)}(s, x))) \\
 & \quad \times (\nabla^{-1}((\xi_1^2(s, x) - \xi_2^2(s, x))\phi_\kappa(h(\eta_2, \xi_2, s))u_\kappa^{(2)}(s, x))) dx ds \\
 & + \mathbb{E} \mathbb{1}_{\Omega_{2\kappa}^c} \int_0^{t \wedge \tau_{2\kappa}(\eta_1, \xi_1)} \int_{\mathcal{O}} (\nabla^{-1}(u^{(1)}(s, x) - u^{(2)}(s, x))) \\
 & \quad \times (\nabla^{-1}(\phi_\kappa(h(\eta_1, \xi_1, s))(u_\kappa^{(1)}(s, x) - u_\kappa^{(2)}(s, x))\xi_1^2(s, x))) dx ds \\
 & + \mathbb{E} \mathbb{1}_{\Omega_{2\kappa}^c} \int_0^{t \wedge \tau_{2\kappa}(\eta_1, \xi_1)} \int_{\mathcal{O}} (\nabla^{-1}(u^{(1)}(s, x) - u^{(2)}(s, x))) \\
 & \quad \times (\nabla^{-1}(\phi_\kappa(h(\eta_1, \xi_1, s)) - \phi_\kappa(h(\eta_2, \xi_2, s)))u_\kappa^{(2)}(s, x)\xi_1^2(s, x)) dx ds \\
 & + \mathbb{E} \mathbb{1}_{\Omega_{2\kappa}^c} \int_0^{t \wedge \tau_{2\kappa}(\eta_1, \xi_1)} \int_{\mathcal{O}} (\nabla^{-1}(u^{(1)}(s, x) - u^{(2)}(s, x))) \\
 & \quad \times (\nabla^{-1}((\xi_1^2(s, x) - \xi_2^2(s, x))\phi_\kappa(h(\eta_2, \xi_2, s))u_\kappa^{(2)}(s, x))) dx ds \\
 & =: I_1(t) + I_2(t) + I_3(t) + I_4(t) + I_5(t) + I_6(t).
 \end{aligned}$$

Let us start with I_1 . Let us put for $r, r', \tilde{m}, \tilde{m}', p, p'$, and $s^* = \frac{2}{\gamma+1}$,

$$\begin{aligned}
 r &= 2 \left(1 + \frac{1}{\gamma} \right), & r' &= \frac{2(1 + \gamma)}{2 + \gamma}, \\
 \tilde{m} &= 2(\gamma + 1), & \tilde{m}' &= \frac{2(\gamma + 1)}{2\gamma + 1}, & p &= \gamma, & p' &= \frac{\gamma}{\gamma - 1}.
 \end{aligned} \tag{5.19}$$

Using duality with $\tilde{m} \geq \gamma + 1, 2 \leq r < \gamma + 1, \frac{1}{\tilde{m}} + \frac{1}{\tilde{m}'} \leq 1$ and $\frac{1}{r} + \frac{1}{r'} \leq 1$, we know for the integrand of I_1

$$\begin{aligned}
 & \left| \langle \Delta^{-\frac{1}{2} + \frac{s^*}{2}}(u_\kappa^{(1)}(s) - u_\kappa^{(2)}(s)), \Delta^{-\frac{1}{2} - \frac{s^*}{2}}(\phi_\kappa(h(\eta_1, \xi_1, \cdot))u_\kappa^{(1)}\xi_1^2 - \phi_\kappa(h(\eta_1, \xi_1, \cdot))u_\kappa^{(1)}\xi_1^2) \rangle \right| \\
 & \leq \left| \Delta^{-\frac{1}{2} + \frac{s^*}{2}}(u_\kappa^{(1)}(s) - u_\kappa^{(2)}(s)) \right|_{L^r} \\
 & \quad \times \left| \Delta^{-\frac{1}{2} - \frac{s^*}{2}}(\phi_\kappa(h(\eta_1, \xi_1, s))u_\kappa^{(1)}(s)\xi_1^2(s) - \phi_\kappa(h(\eta_1, \xi_1, \cdot))u_\kappa^{(2)}(s)\xi_1^2(s)) \right|_{L^{r'}}.
 \end{aligned}$$

By integration in time, and applying, firstly, the Hölder and, secondly, Young inequality we know for any $\varepsilon_1 > 0$ there exists a $C(\varepsilon_1) > 0$ such that

$$\begin{aligned} & \int_0^{\tau_{2k}^1(\eta_1, \xi_1) \wedge t} \left| \Delta^{-\frac{1}{2} + \frac{s^*}{2}} (u_k^{(1)}(s) - u_k^{(2)}(s)), \right. \\ & \left. \Delta^{-\frac{1}{2} - \frac{s^*}{2}} (\phi_k(h(\eta_1, \xi_1, \cdot)) (u_k^{(1)}(s) - u_k^{(2)}(s))) \xi_1^2(s) \right| ds \\ & \leq C(\varepsilon_1) \left(\int_0^{\tau_{2k}^1(\eta_1, \xi_1) \wedge t} \left| \Delta^{-\frac{1}{2} + \frac{s^*}{2}} (u_k^{(1)}(s) - u_k^{(2)}(s)) \right|_{L^r}^{\tilde{m}} ds \right)^r \\ & \quad + \varepsilon_1 \left(\int_0^{\tau_{2k}^1(\eta_1, \xi_1) \wedge t} \phi_k^{\tilde{m}'}(h(\eta_1, \xi_1, s)) \left| (u_k^{(1)}(s) - u_k^{(2)}(s)) \xi_1^2(s) \right|_{H_r^{s^*-1}}^{\tilde{m}'} ds \right)^{\frac{r'}{\tilde{m}'}}. \end{aligned} \tag{5.20}$$

Let us analyze the first term on the right hand side.

Take some $\theta \in (0, 2/\tilde{m})$ and let

$$\frac{1}{r''} = \frac{\theta}{2} + \frac{1 - \theta}{\gamma + 1}, \quad \tilde{s} = -\theta + (1 - \theta)(s^* - 1).$$

Then, we know by interpolation that

$$|u|_{H_r^{s^*-1}} \leq |u|_{H_2^{-1}}^\theta |u|_{H_r^{\tilde{s}}}^{1-\theta}$$

This gives

$$\begin{aligned} & \mathbb{E} \left\| \Delta^{-\frac{1}{2} + \frac{s^*}{2}} (u_k^{(1)} - u_k^{(2)}) \right\|_{L^{\tilde{m}}(0, T; L^r)}^r \leq C \mathbb{E} \left\| u_k^{(1)} - u_k^{(2)} \right\|_{L^{\tilde{m}}(0, T; H_r^{s^*-1})}^r \\ & \leq C \mathbb{E} \left(\int_0^t |u_k^{(1)}(s) - u_k^{(2)}(s)|_{H_2^{-1}}^{\theta \tilde{m}} |u_k^{(1)}(s) - u_k^{(2)}(s)|_{H_r^{\tilde{s}}}^{(1-\theta)\tilde{m}} ds \right)^{\frac{r}{\tilde{m}}}. \end{aligned}$$

The Hölder inequality gives

$$\begin{aligned} & \mathbb{E} \left\| \Delta^{-\frac{1}{2} + \frac{s^*}{2}} (u_k^{(1)} - u_k^{(2)}) \right\|_{L^{\tilde{m}}(0, T; L^r)}^r \\ & \leq C \mathbb{E} \left(\int_0^t |u_k^{(1)}(s) - u_k^{(2)}(s)|_{H_2^{-1}}^2 ds \right)^{\frac{\theta \tilde{m}}{2} \frac{r}{\tilde{m}}} \left(\int_0^t |u_k^{(1)}(s) - u_k^{(2)}(s)|_{H_r^{\tilde{s}}}^{(1-\theta)\tilde{m} \frac{2}{2-\theta\tilde{m}}} ds \right)^{\frac{r}{\tilde{m}} \frac{2-\theta\tilde{m}}{2}} \\ & \leq C \mathbb{E} \left(\int_0^t |u_k^{(1)}(s) - u_k^{(2)}(s)|_{H_2^{-1}}^2 ds \right)^{\frac{r\theta}{2}} \|u_k^{(1)} - u_k^{(2)}\|_{L^{(1-\theta)\tilde{m} \frac{2}{2-\theta\tilde{m}}}(0, T; H_r^{\tilde{s}})}^{r(1-\theta)}. \end{aligned}$$

By the Young inequality, we know for any $\varepsilon_2 > 0$ there exists some $C(\varepsilon_2) > 0$ such that

$$\begin{aligned} & \mathbb{E} \left\| \Delta^{-\frac{1}{2} + \frac{s^*}{2}} (u_k^{(1)} - u_k^{(2)}) \right\|_{L^{\tilde{m}}(0, T; L^r)}^r \\ & \leq C(\varepsilon_2) \mathbb{E} \int_0^t |u_k^{(1)}(s) - u_k^{(2)}(s)|_{H_2^{-1}}^2 ds + \varepsilon_2 \mathbb{E} \|u_k^{(1)} - u_k^{(2)}\|_{L^{(1-\theta)\tilde{m} \frac{2}{2-\theta\tilde{m}}}(0, T; H_r^{\tilde{s}})}^{r(1-\theta) \frac{2-r\theta}{2}}. \end{aligned}$$

Observe, for small θ , we know $r'' \geq r$, $\tilde{s} \leq (s^* - 1)$, and $\tilde{m}'' := (1 - \theta)\tilde{m} \frac{2}{2-\theta\tilde{m}} \geq \tilde{m}$. Thus we know, since

$$\frac{1}{r} > \frac{-(s^* - 1)}{2} + \frac{1 - s^*}{\gamma + 1}, \quad \frac{1}{\tilde{m}''} \geq \frac{1 - s^*}{\gamma + 1}.$$

that the same holds for r'' , \tilde{m}'' , and \tilde{s} . By Proposition A.2, we know, if $\gamma > 1$ then there exists some $\theta > 0$ such that

$$\begin{aligned} & \|u_\kappa^{(1)} - u_\kappa^{(2)}\|_{L^{(1-\theta)\tilde{m}} \frac{2-r\theta}{2-\theta\tilde{m}}(0,T;H_r^{\tilde{s}})}^{r(1-\theta)\frac{2-r\theta}{2-\theta\tilde{m}}} \\ & \leq \sup_{0 \leq s \leq T} |u_\kappa^{(1)}(s) - u_\kappa^{(2)}(s)|_{H_2^{-1}}^2 + \int_0^T |u_\kappa^{(1)}(s) - u_\kappa^{(2)}(s)|_{L^{\gamma+1}}^{\gamma+1} ds. \end{aligned}$$

This means, that later on, we can cancel this term by the LHS.

Let us analyze the second term of the RHS in Eq. 5.20, i.e.

$$\varepsilon_1 \left(\int_0^{\tau_{2\kappa}^1(\eta_1, \xi_1) \wedge t} (\phi_\kappa(h(\eta_1, \xi_1, s)))^{\tilde{m}'} \left| (u_\kappa^{(1)}(s) - u_\kappa^{(2)}(s)) \xi_1^2(s) \right|_{H_r^{s_1-1-s^*}}^{\tilde{m}'} ds \right)^{\frac{r'}{\tilde{m}'}}.$$

First, we use [58] p. 171, Theorem 1, where we put $s_1 = -s^{**} < 0$ and $s_2 = s^{**} + \varepsilon$, $s^{**} = s^*/2$, $\varepsilon > 0$ very small. In this way we get

$$\left| (u_\kappa^{(1)}(s) - u_\kappa^{(2)}(s)) \xi_1^2(s) \right|_{H_r^{s_1-1-s^*}} \leq C \left| u_\kappa^{(1)}(s) - u_\kappa^{(2)}(s) \right|_{H_r^{s_1}} \left| \xi_1^2(s) \right|_{H_r^{s_2}}.$$

or, applying the Hölder inequality in time,

$$\begin{aligned} & \left(\int_0^{\tau_{2\kappa}^1(\eta_1, \xi_1) \wedge t} \left| (u_\kappa^{(1)}(s) - u_\kappa^{(2)}(s)) \xi_1^2(s) \right|_{H_r^{s_1-1-s^*}}^{\tilde{m}'} ds \right)^{\frac{r'}{\tilde{m}'}} \\ & \leq C \left(\int_0^{\tau_{2\kappa}^1(\eta_1, \xi_1) \wedge t} \left| u_\kappa^{(1)}(s) - u_\kappa^{(2)}(s) \right|_{H_r^{s_1}}^{p\tilde{m}'} ds \right)^{\frac{r'}{\tilde{m}'p}} \left(\int_0^{\tau_{2\kappa}^1(\eta_1, \xi_1) \wedge t} \left| \xi_1^2(s) \right|_{H_r^{s_2}}^{\tilde{m}'p'} ds \right)^{\frac{r'}{\tilde{m}'p'}}. \end{aligned}$$

Now, since for our choice of $r' > 2$, s_1 , and $m'p$, we know that

$$(i) \quad \frac{1}{r'} \geq \frac{-s_1}{2} + \frac{1+s_1}{\gamma+1} = \frac{s^*}{4} + \frac{1-s^*/2}{\gamma+1}, \quad \frac{1}{m'p} \geq \frac{1+s_1}{\gamma+1} = \frac{1-s^*/2}{\gamma+1},$$

we have

$$\begin{aligned} & \mathbb{E} \|u_\kappa^{(1)} - u_\kappa^{(2)}\|_{L^{p\tilde{m}'}(0,T;H_r^{s_1})}^{r'} \\ & \leq C \mathbb{E} \left(\sup_{0 \leq s \leq t} |u_\kappa^{(1)}(s) - u_\kappa^{(2)}(s)|_{H_2^{-1}}^2 + \int_0^t |u_\kappa^{(1)}(s) - u_\kappa^{(2)}(s)|_{L^{\gamma+1}}^{\gamma+1} ds \right). \end{aligned}$$

Next, we tackle $|\xi_1^2(s)|_{H_r^{s_2}}^{\tilde{m}'p'}$. Here, we have by [58, p. 364] for

$$s_2 + 2 \left(\frac{d}{\tilde{r}} - s_2 \right) = 2 \frac{d}{\tilde{r}} - s_2 = \frac{d}{r'} \quad \Rightarrow \quad \frac{d}{\tilde{r}} = \frac{1}{2} \left(\frac{d}{r'} + s_2 \right) = \frac{1}{2} \left(\frac{c(2+\gamma)}{2+2\gamma} + s_2 \right),$$

the estimate

$$|\xi_1^2(s)|_{H_r^{\tilde{r}}}^{\tilde{m}'p'} \leq C |\xi_1(s)|_{H_r^{\tilde{r}}}^{2\tilde{m}'p'}.$$

Let us put $s_2 = \frac{1}{(1/2+\gamma)}$. Then, $\tilde{r} = \frac{2(1+\gamma)}{2+\gamma}$.

Furthermore, note that $r \geq 2$, $r' \leq 2$ and $\tilde{r} \leq 2$ for $s_2 = \frac{1}{(1/2+\gamma)}$.

Next, the Sobolev embedding $H_2^\delta(\mathcal{O}) \hookrightarrow H_r^{s_2}(\mathcal{O})$ with $\delta \geq s_2 + (\frac{d}{2} - \frac{d}{r}) = \frac{1+d\gamma}{4(1+\gamma)} \leq \frac{d}{4}$. In particular, $\delta = \frac{3+2\gamma}{2+6\gamma+4\gamma^2}$. Now, to estimate the term by \mathbb{H}_ρ we need that

$$(ii) \quad \delta = (1 - \theta)(\rho + 1) = \rho + (1 - \theta), \quad \frac{1}{l} = \frac{1 - \theta}{2}, \quad l \geq \min(2, \tilde{m}' p').$$

Hence, $\rho \geq \delta - \frac{2}{2\tilde{m}' p'} = \frac{1+d\gamma}{4(1+\gamma)} - \frac{(1+\gamma)(1+2\gamma)}{2\gamma(2+\gamma)} < 0$. In this way we get

$$\|\xi_1^2\|_{L^{p'\tilde{m}'}(0,T;H_r^{s_1})} \leq c\|\xi_1\|_{L^{2\tilde{m}'p'}(0,T;H_2^\delta)} \leq c\|\xi_1\|_{\mathbb{H}_\rho}^2.$$

It follows by taking into account the cut-off function

$$\begin{aligned} & \left(\int_0^{\tau_{2\kappa}^1(\eta_1, \xi_1) \wedge t} \phi_\kappa(h(\eta_1, \xi_1, \cdot)) \left| (u_\kappa^{(1)}(s) - u_\kappa^{(2)}(s)) \xi_1^2(s) \right|_{H_r^{s_1}^{m'}} ds \right)^{r'} \\ & \leq C \|\phi_\kappa(h(\eta_1, \xi_1, \cdot))\|_{[0, \tau_{2\kappa}^1(\eta_1, \xi_1) \wedge t]} \|\xi_1\|_{\mathbb{H}_\rho}^{2r'} \\ & \quad \times \mathbb{E} \left(\sup_{0 \leq s \leq t} |u_\kappa^{(1)}(s) - u_\kappa^{(2)}(s)|_{H_2^{-1}}^2 + \int_0^t |u_\kappa^{(1)}(s) - u_\kappa^{(2)}(s)|_{L^{\gamma+1}}^{\gamma+1} ds \right)^{r'} \\ & \leq C(\kappa) \left\{ \mathbb{E} \left(\sup_{0 \leq s \leq t} |u_\kappa^{(1)}(s) - u_\kappa^{(2)}(s)|_{H_2^{-1}}^2 + \int_0^t |u_\kappa^{(1)}(s) - u_\kappa^{(2)}(s)|_{L^{\gamma+1}}^{\gamma+1} ds \right)^{r'} \right\}. \end{aligned}$$

In this way, we get for I_1

$$\begin{aligned} \mathbb{E}|I_1(t)| & \leq C(\varepsilon_1, \varepsilon_2) \mathbb{E} \int_0^t |u_\kappa^{(1)}(s) - u_\kappa^{(2)}(s)|_{H_2^{-1}}^2 ds \\ & \quad + \varepsilon_1 \mathbb{E} \left(\sup_{0 \leq s \leq t} |u_\kappa^{(1)}(s) - u_\kappa^{(2)}(s)|_{H_2^{-1}}^2 + \int_0^t |u_\kappa^{(1)}(s) - u_\kappa^{(2)}(s)|_{L^{\gamma+1}}^{\gamma+1} ds \right) \\ & \quad + \varepsilon_2 C(\kappa) \left\{ \mathbb{E} \left(\sup_{0 \leq s \leq t} |u_\kappa^{(1)}(s) - u_\kappa^{(2)}(s)|_{H_2^{-1}}^2 + \int_0^t |u_\kappa^{(1)}(s) - u_\kappa^{(2)}(s)|_{L^{\gamma+1}}^{\gamma+1} ds \right)^{r'} \right\}. \end{aligned}$$

Next, we have to tackle $I_2(t)$. Again take $r, r', \tilde{m}, \tilde{m}', p$, and p' defined as in Eq. 5.19. Again, using duality we know for the integrand of I_2

$$\begin{aligned} & \left| \left\langle \Delta^{-\frac{1}{2} + \frac{s^*}{2}} (u_\kappa^{(1)}(s) - u_\kappa^{(2)}(s)), \Delta^{-\frac{1}{2} - \frac{s^*}{2}} \left(\phi_\kappa(h(\eta_1, \xi_1, \cdot)) u_\kappa^{(1)} \xi_1^2 - \phi_\kappa(h(\eta_1, \xi_1, \cdot)) u_\kappa^{(1)} \xi_1^2 \right) \right\rangle \right| \\ & \leq \left| \Delta^{-\frac{1}{2} + \frac{s^*}{2}} (u_\kappa^{(1)}(s) - u_\kappa^{(2)}(s)) \right|_{L^r} \\ & \quad \times \left| \Delta^{-\frac{1}{2} - \frac{s^*}{2}} (\phi_\kappa(h(\eta_1, \xi_1, s)) - \phi_\kappa(h(\eta_2, \xi_2, s))) u_\kappa^{(2)}(s) \xi_1^2(s) \right|_{L^{r'}} \end{aligned}$$

By the Young inequality we know for any $\varepsilon_1 > 0$ there exists a constant $C(\varepsilon_1) > 0$ such that

$$\begin{aligned} & \mathbb{E} \mathbb{1}_{\Omega_{2\kappa}} \int_0^{t \wedge \tau_{2\kappa}(\eta_2, \xi_2)} \int_{\mathcal{O}} (\nabla^{-1}(u^{(1)}(s, x) - u^{(2)}(s, x))) \\ & \quad \times (\nabla^{-1}(\phi_\kappa(h(\eta_1 \xi_1, s)) - \phi_\kappa(h(\eta_2, \xi_2, s))) u_\kappa^{(2)}(s, x) \xi_1^2(s, x)) dx ds \\ & \leq \varepsilon \mathbb{E} \|u^{(1)} - u^{(2)}\|_{L^m(0,T;H_r^{s^*-1})}^r + C(\varepsilon) \mathbb{E} \|\xi_1 - \xi_2\|_{\mathbb{H}_\rho}^{r'} \|\mathbb{1}_{\Omega_{2\kappa}} u_\kappa^{(2)} \xi_1^2\|_{L^{m'}(0,t \wedge \tau_{2\kappa}(\eta_2, \xi_2); H_r^{s^*-1})}^{r'} \end{aligned}$$

Next, we follow the calculation as done before. First, we use again [58, p. 171, Theorem 1], where we put $s_1 = -s^{**} < 0$ and $s_2 = s^{**} + \varepsilon$, $s^{**} = s^*/2$, $\varepsilon > 0$ very small. In this way we get

$$\left| u_\kappa^{(2)}(s) \xi_1^2(s) \right|_{H_r^{-1-s^*}} \leq C \left| u_\kappa^{(2)}(s) \right|_{H_r^{s_1}} \left| \xi_1^2(s) \right|_{H_r^{s_2}}.$$

Next, applying the Hölder inequality in time,

$$\begin{aligned} & \left(\int_0^{\tau_{2\kappa}^1(\eta_2, \xi_1) \wedge t} \left| u_\kappa^{(2)}(s) \xi_1^2(s) \right|_{H_r^{-1-s^*}}^{\tilde{m}'} ds \right)^{\frac{r'}{\tilde{m}'}} \\ & \leq C \left(\int_0^{\tau_{2\kappa}^1(\eta_2, \xi_1) \wedge t} \left| u_\kappa^{(2)}(s) \right|_{H_r^{s_1}}^{p\tilde{m}'} ds \right)^{\frac{r'}{\tilde{m}'p}} \left(\int_0^{\tau_{2\kappa}^1(\eta_1, \xi_1) \wedge t} \left| \xi_1^2(s) \right|_{H_r^{s_2}}^{\tilde{m}'p'} ds \right)^{\frac{r'}{\tilde{m}'p'}}. \end{aligned}$$

Again, by the choice of $r, r', \tilde{m}, \tilde{m}', p$, and p' , we know that even for a $r'' > r'$ we have

$$\begin{aligned} & \left(\int_0^{\tau_{2\kappa}^1(\eta_1, \xi_1) \wedge t} \left| u_\kappa^{(2)}(s) \right|_{H_r^{s_1}}^{p\tilde{m}'} ds \right)^{\frac{r'}{\tilde{m}'p}} \\ & \leq C \left(\sup_{0 \leq s \leq \tau_{2\kappa}^1(\eta_1, \xi_1) \wedge t} \left| u_\kappa^{(2)}(s) \right|_{H_2^{-1}}^2 + \int_0^{\tau_{2\kappa}^1(\eta_1, \xi_1) \wedge t} \left| u_\kappa^{(2)}(s) \right|_{L^{\gamma+1}}^{\gamma_1} ds \right). \end{aligned}$$

Again, similarly we know

$$\left(\int_0^{\tau_{2\kappa}^1(\eta_1, \xi_1) \wedge t} \left| \xi_1^2(s) \right|_{H_r^{s_2}}^{\tilde{m}'p'} ds \right)^{\frac{r'}{\tilde{m}'p'}} \leq \|\xi_2\|_{\mathbb{H}_\rho}^{r'}.$$

Now, taking into account the cut off function we know

$$\begin{aligned} & \mathbb{E} \mathbb{1}_{\Omega_{2\kappa}} \|\xi_1 - \xi_2\|_{\mathbb{H}_\rho}^{r'} \wedge \tau_{2\kappa}(\eta_2, \xi_2) \|u_\kappa^{(2)} \xi_1^2\|_{L^{m'}(0, t \wedge \tau_{2\kappa}(\eta_2, \xi_2); H_r^{-s^*-1})}^{r'} \\ & \leq \mathbb{E} \left\{ \mathbb{1}_{\Omega_{2\kappa}} \|\xi_1 - \xi_2\|_{\mathbb{H}_\rho}^{r'} \wedge \tau_{2\kappa}(\eta_2, \xi_2) \|h(\xi_2, \cdot)\xi_2\|_{\mathbb{H}_\rho}^{r'} \right. \\ & \quad \times C \left(\sup_{0 \leq s \leq \tau_{2\kappa}^1(\eta_1, \xi_1) \wedge t} \left| u_\kappa^{(2)}(s) \right|_{H_2^{-1}}^2 + \int_0^{\tau_{2\kappa}^1(\eta_1, \xi_1) \wedge t} \left| u_\kappa^{(2)}(s) \right|_{L^{\gamma+1}}^{\gamma_1} ds \right)^{\frac{r'}{r''}} \left. \right\} \\ & \leq C(\kappa) \left(\mathbb{E} \|\xi_1 - \xi_2\|_{\mathbb{H}_\rho}^2 \right)^{1 - \frac{r'}{r''}} \\ & \quad \times \left\{ \mathbb{E} \left(\sup_{0 \leq s \leq \tau_{2\kappa}^1(\eta_1, \xi_1) \wedge t} \left| u_\kappa^{(2)}(s) \right|_{H_2^{-1}}^2 + \int_0^{\tau_{2\kappa}^1(\eta_1, \xi_1) \wedge t} \left| u_\kappa^{(2)}(s) \right|_{L^{\gamma+1}}^{\gamma_1} ds \right) \right\}^{\frac{r'}{r''}}. \end{aligned}$$

Since the operator is invariant, the second expectation value is smaller than R_1 .

Next, let us treat I_3 .

$$\begin{aligned} & \mathbb{E} \mathbb{1}_{\Omega_{2\kappa}} \int_0^{t \wedge \tau_{2\kappa}(\eta_2, \xi_2)} \int_{\mathcal{O}} (\nabla^{-1}(u^{(1)}(s, x) - u^{(2)}(s, x))) \\ & \quad \times (\nabla^{-1}((\xi_1^2(s, x) - \xi_2^2(s, x)) \phi_\kappa(h(\eta_2, \xi_2, s))) u_\kappa^{(2)}(s, x) dx ds \end{aligned}$$

$$\leq \varepsilon \mathbb{E} \|u^{(1)} - u^{(2)}\|_{L^{\gamma+1}(0,T;L^{\gamma+1})}^{\gamma+1} + C(\varepsilon) \|\mathbb{1}_{\Omega_{2\kappa}} u_{\kappa}^{(2)} (\xi_1^2 - \xi_2^2)\|_{L^{\frac{\gamma+1}{\gamma}}(0,t \wedge \tau_{2\kappa}(\eta_2, \xi_2); H^{\frac{-2}{\gamma+1}})}.$$

By similar calculation as before, we get

$$\begin{aligned} & \mathbb{E} \mathbb{1}_{\Omega_{2\kappa}} \int_0^{t \wedge \tau_{2\kappa}(\eta_2, \xi_2)} \int_{\mathcal{O}} (\nabla^{-1}(u^{(1)}(s, x) - u^{(2)}(s, x))) \\ & \quad \times (\nabla^{-1}((\xi_1^2(s, x) - \xi_2^2(s, x)) \phi_{\kappa}(h(\eta_2, \xi_2, s)) u_{\kappa}^{(2)}(s, x)) dx ds \\ & \leq \varepsilon \mathbb{E} \|u^{(1)} - u^{(2)}\|_{L^{\gamma+1}(0,T;L^{\gamma+1})}^{\gamma+1} \\ & \quad + C(\varepsilon) \mathbb{E} \left\{ \sup_{0 \leq s \leq T} |u_{\kappa}^{(2)}(s)|_{L^p}^{\frac{\gamma+1}{\gamma}} \|\xi_1^2 - \xi_2^2\|_{L^{\frac{\gamma+1}{\gamma}}(0,t \wedge \tau_{2\kappa}(\eta_2, \xi_2); L^{\gamma})}^{\frac{\gamma+1}{\gamma}} \right\} \\ & \leq \varepsilon \mathbb{E} \|u^{(1)} - u^{(2)}\|_{L^{\gamma+1}(0,T;L^{\gamma+1})}^{\gamma+1} \\ & \quad + C(\varepsilon, \kappa) \mathbb{E} \left\{ \sup_{0 \leq s \leq T} |u_{\kappa}^{(2)}(s)|_{L^p}^{\frac{\gamma+1}{\gamma}} \right\} \left\{ \mathbb{E} \|\xi_1 - \xi_2\|_{L^M(0,T;L^{2r})} \right\}. \end{aligned}$$

$I_4, I_5,$ and I_6 can be estimated by similar steps.

Now, we can collect all terms and get

$$\begin{aligned} & \mathbb{E} |u^{(1)}(t) - u^{(2)}(t)|_{H_{-1}^2}^2 \\ & \quad + 2r_u \mathbb{E} \int_0^t \int_{\mathcal{O}} ((u^{(1)})^{[\gamma]}(s, x) - (u^{(2)})^{[\gamma]}(s, x))(u^{(1)}(s, x) - u^{(2)}(s, x)) dx ds \\ & \leq \varepsilon^{\gamma+1} \mathbb{E} \int_0^t |u^{(1)}(s) - u^{(2)}(s)|_{L^{\gamma+1}}^{\gamma+1} ds + C \mathbb{E} \int_0^t |u^{(1)}(s) - u^{(2)}(s)|_{H_{-1}^2}^2 ds \\ & \quad + \varepsilon \mathbb{E} \|u^{(1)} - u^{(2)}\|_{L^{\gamma+1}(0,T;L^{\gamma+1})}^{\gamma+1} \\ & \quad + C(\varepsilon) \left\{ \mathbb{E} \|\xi_1 - \xi_2\|_{\mathbb{H}^p}^2 \right\}^{\delta_1} \left\{ \mathbb{E} \|\mathbb{1}_{\Omega_{2\kappa}} u_{\kappa}^{(2)} \xi_1^2\|_{L^{\frac{\gamma+1}{\gamma}}(0,t \wedge \tau_{2\kappa}(\eta_2, \xi_2); H^{\frac{-2}{\gamma+1}})}^{\frac{\gamma+1}{\gamma}} \right\}^{\delta_2} \\ & \quad + \varepsilon \mathbb{E} \|u^{(1)} - u^{(2)}\|_{L^{\gamma+1}(0,T;L^{\gamma+1})}^{\gamma+1} + C(\varepsilon, \kappa) \mathbb{E} \left\{ \sup_{0 \leq s \leq T} |u_{\kappa}^{(2)}|_{L^p} \right\} \left\{ \mathbb{E} \|\xi_1 - \xi_2\|_{L^M(0,T;L^{2r})} \right\}. \end{aligned}$$

Term I cancels with the corresponding term on the left hand side. We may apply Gronwall’s lemma in order to deal with the term III . □

Proposition 5.13 *For any initial condition (u_0, v_0) satisfying Hypothesis 3.4,*

(i) *there exists $r = r(T, \gamma) > 0$ such that for any $(\eta, \xi) \in \mathcal{X}_{2\lambda}(R_0, R_1, R_2)$, we have*

$$\mathbb{E} \int_0^T |u_{\kappa}(s)|_{H_{\gamma+1}^r}^{\gamma+1} ds \leq CR_0;$$

(ii) *we have that $t \mapsto u_{\kappa}(t)$ is \mathbb{P} -a.s. strongly continuous in $H_2^{-1}(\mathcal{O})$.*

Proof Let us prove (i) first. We know from Proposition 5.6 with $p = 1$, that for any $(\eta, \xi) \in \mathcal{X}_{2\lambda}(R_0, R_1, R_2)$ we have

$$\mathbb{E} \int_0^T \left| u_{\kappa}^{\left[\frac{\gamma-1}{2} \right]}(s) \nabla u_{\kappa}(s) \right|_{L^2}^2 ds \leq CR_0.$$

On the other side, we know by Proposition A.4 that we have for any $\theta < \frac{2}{p+\gamma}$,

$$\|u_\kappa\|_{L^{p+\gamma}(0,T;H^\rho_{p+\gamma})}^{p+\gamma} \leq C \left(\int_0^T \left| u_\kappa^{\left[\frac{p+\gamma-2}{2}\right]}(s) \nabla u_\kappa(s) \right|_{L^2}^2 ds + \int_0^T |u|_{L^2}^{p+\gamma} ds \right),$$

where the LHS is bounded for $p = 1$, noting that Proposition 5.6 with $p = \gamma$ gives

$$\mathbb{E} \left[\sup_{t \in [0,T]} |u_\kappa(t)|_{L^{\gamma+1}}^{\gamma+1} \right] \leq C.$$

The assertion (ii) follows from [48, Theorem 4.2.5]. □

5.4 The System Eq. 3.10–(3.11) and Uniform Bounds on The Stopping Time

Within this section let us assume that $\{(\bar{u}_\kappa, \bar{v}_\kappa) : \kappa \in \mathbb{N}\}$ the family of stochastic processes constructed in Step III of the proof of Theorem 3.6. The constants R_0, R_1 and R_2 are given as in Eq. 3.26, Eq. 3.23, and Eq. 3.25, respectively.

Proposition 5.14 Fix $p \geq 1$ and suppose $u_0 \in L^{p+1}(\mathcal{O})$. Then, there exists a constant $C_0(p, T) > 0$ such that we have for all $\kappa \in \mathbb{N}$

$$\mathbb{E} \left[\sup_{0 \leq s \leq T} |\bar{u}_\kappa(s)|_{L^{p+1}}^{p+1} \right] + \gamma p(p+1)r_u \mathbb{E} \int_0^T \int_{\mathcal{O}} |\bar{u}_\kappa(s, x)|^{p+\gamma-2} |\nabla \bar{u}_\kappa(s, x)|^2 dx ds + (p+1)\chi \mathbb{E} \int_0^T \int_{\mathcal{O}} |\bar{u}_\kappa(s, x)|^{p+1} |\bar{v}_\kappa(s, x)|^2 dx ds \leq C_0(p, T) \left(\mathbb{E}|u_0|_{L^{p+1}}^{p+1} + 1 \right).$$

Proof Follows from an application of Fatou’s lemma and nonnegativity of u_κ and v_κ in the proof of Proposition 5.6.

Proposition 5.15 Let $\rho < 1 - \frac{d}{2}$ and assume that the Hypotheses of Theorem 3.6 hold. There exists a generic constant $C(T) > 0$, a number $l \geq \frac{2}{|\rho|}$ and $\delta_1, \delta_2 \in (0, 1)$ with $\delta_1 + \delta_2 = 1$ such that for all $\kappa \in \mathbb{N}$ we have

$$\mathbb{E} \|\bar{v}_\kappa\|_{\mathbb{H}^\rho}^{m_0} \leq C(T) \left(\mathbb{E}|v_0|_{H_m^{-\delta_0}}^{m_0} + R_0^{\delta_1} R_1^{\delta_1} \right). \tag{5.21}$$

Before proving Proposition 5.15, we consider the following lemma, which will be essential.

Technical Lemma 5.16 For all $1 \leq \alpha < 2, -\rho < \frac{2(2-\alpha)}{\alpha}$, and for any $r^* \geq 1$ there exist generic constants $C(T) > 0$ and $\delta_1, \delta_2 \in (0, 1)$ with $\delta_1 + \delta_2 = 1$, such that for any pair of nonnegative processes $\eta, \xi : [0, T] \times \Omega \times \mathcal{O} \rightarrow [0, \infty)$ we have

$$\mathbb{E} \|\eta \xi^2\|_{L^\alpha(0,T;L^1)}^{r^*} \leq C(T) \left\{ \mathbb{E} \left(\int_0^T |\eta^p(s) \xi^2(s)|_{L^1} ds \right)^{\frac{r^* p}{\alpha}} \right\}^{\delta_1} \left(\mathbb{E} \|\xi\|_{\mathbb{H}^\rho}^{r^*} \right)^{\delta_2}. \tag{5.22}$$

Proof Let us set $\beta = \frac{1}{p^*}, 1 - \beta = \frac{p^*-1}{p^*} = \frac{1}{p'}$. Observe also that

$$\eta(s) \xi^2(s) = \eta(s) \xi^{2\beta}(s) \xi^{2(1-\beta)}(s), \quad s \in [0, T].$$

Assume $\alpha < p$ and fix $\beta = \frac{1}{p}$, $q\alpha/p = 1$ and $q' = \frac{p}{p-\alpha}$

$$\begin{aligned} \|\eta\xi^2\|_{L^\alpha(0,T;L^1)}^2 &\leq \left(\int_0^T \left(\int_{\mathcal{O}} |\eta^p(s,x)\xi^{2\beta p}(s,x)| dx\right)^{\frac{\alpha}{p}} \left(\int_{\mathcal{O}} |\xi^{2(1-\beta)p'}(s,x)| dx\right)^{\frac{\alpha}{p'}} ds\right)^{\frac{2}{\alpha}} \\ &\leq \left(\int_0^T \int_{\mathcal{O}} |\eta^{p^*}(s,x)\xi^{n^*}(s,x)| dx ds\right)^{\frac{2}{q}} \left(\int_0^T |\xi(s)|_{L^{(1-\beta)p'/2}}^{\frac{\alpha q'}{p'}} ds\right)^{\frac{2}{q'}} \\ &\leq \|\eta^{p^*}\xi^{n^*}\|_{L^1(0,T;L^1)}^{\frac{2}{q}} \|\xi\|_{L^{\frac{\alpha q'}{p'}}(0,T;L^{(1-\beta)p'/2})}^{\frac{2\alpha}{p'}} \\ &\leq \|\eta^{p^*}\xi^{n^*}\|_{L^1(0,T;L^1)}^{\frac{2}{q}} \|\xi\|_{L^{\frac{\alpha(p-1)}{p-\alpha}}(0,T;L^2)}^{\frac{2\alpha}{p'}}. \end{aligned}$$

Here, we need $\alpha/p' < 1$ which gives $2 \leq p < \frac{\alpha}{\alpha-1}$. Also, if

$$\frac{d}{2} - \rho < \frac{2(p-\alpha)}{\alpha(p-1)} + \frac{d}{2},$$

then

$$\|\eta\xi^2\|_{L^\alpha(0,T;L^1)}^2 \leq \|\eta^{p^*}\xi^{n^*}\|_{L^1(0,T;L^1)}^{\frac{2}{q}} \|\xi\|_{\mathbb{H}^\rho}^{\frac{2\alpha}{p'}}.$$

Taking the expectation and applying the Hölder inequality where we have to take into account that $\alpha/p' < 1$ we get the assertion.

Proof of Proposition 5.15: Let us consider the following equation for a locally integrable and progressively measurable $t \mapsto F(t)$,

$$dw(t) = r_v \Delta w(t) dt + F(t) dt + \sigma_2 w(t) dW_2(t), \quad w(0) = w_0 \in H_2^\rho(\mathcal{O}). \tag{5.23}$$

Following the proof of Proposition 5.7 verbatim, we find that

$$\left| \int_0^t e^{r_v \Delta(t-s)} F(s) ds \right|_{H_2^\rho} \leq C(T) \left(\int_0^t |F(s)|_{L^1}^\mu ds \right)^{\frac{1}{\mu}}. \tag{5.24}$$

In particular, for α as in the Technical Lemma 5.16, we have $2 - \frac{2}{\alpha} > \frac{d}{2} + \rho$ and $\alpha < 2$. This gives as condition for ρ , $\rho < 1 - \frac{d}{2}$. In addition, setting $p = 2$ we need $-\rho < \frac{2(2-\alpha)}{\alpha}$, which is not a restriction. However, due the hypotheses we find some $\alpha < 2$ such that the first inequality is satisfied. Note, that due to the condition in Hypothesis 3.4, there exists some $\mu = \alpha \geq 1$ such that ρ satisfies the assumption above and those of the Technical Lemma 5.16. Setting

$$F = \phi_\kappa(h(\eta, \xi, \cdot))\eta\xi^2,$$

we obtain by applying Technical Lemma 5.16 that there exists some constant $C(T) > 0$ and $\delta_1, \delta_2 \in (0, 1)$, $\delta_1 + \delta_2 = 1$ such that

$$\mathbb{E} \left(\int_0^T |F(s)|_{L^1}^\mu ds \right)^{\frac{m_0}{\mu}} \leq C(T) R_0^{\delta_1} R_1^{\delta_2}.$$

It remains to calculate the norm in $L^2(0, T; H_2^{\rho+1}(\mathcal{O}))$. By standard calculations (i.e. applying the smoothing property and the Young inequality for convolution), we get for $\frac{3}{2} \geq \frac{1}{\mu} + \frac{1}{\kappa}$ and $\delta\kappa/2 < 1$

$$\left\| \int_0^\cdot e^{r_v \Delta(\cdot-s)} F(s) ds \right\|_{L^2(0,T;H_2^{\rho+1})} \leq C \|F\|_{L^\mu(0,T;H_2^{\rho+1-\delta})}.$$

The embedding $L^1(\mathcal{O}) \hookrightarrow H_2^{\rho+1-\delta}(\mathcal{O})$ for $\delta - (\rho + 1) > \frac{d}{2}$, gives

$$\|F\|_{L^\mu(0,T;H_2^{\rho-1})} \leq \|F\|_{L^\mu(0,T;L^1)},$$

for and similarly to before we know by hypothesis 3.4, that there exists some $\mu = \alpha$ such that $\delta - (\rho + 1) > \frac{d}{2}$, $\frac{3}{2} \geq \frac{1}{\mu} + \frac{1}{\kappa}$, $\delta\kappa/2 < 1$, and $\alpha(p - 1)/(p - \alpha) \leq m_0$. Therefore, we get by the Technical Lemma 5.16

$$\mathbb{E} \left\| \int_0^\cdot e^{r_v \Delta(\cdot-s)} F(s) ds \right\|_{L^2(0,T;H_2^{\rho+1})}^{m_0} \leq C(T) R_0^{\delta_1} R_1^{\delta_2}.$$

6 Pathwise Uniqueness of the Solution

Proof of Theorem 3.10 Let us remind, since (u_1, v_1) and (u_2, v_2) are solutions to the system Eq. 3.6–(3.7) with $\mathbb{P}(u_1(0) = u_2(0)) = 1$ and $\mathbb{P}(v_1(0) = v_2(0)) = 1$, we can write for $i = 1, 2$,

$$du_i(t) = \left(r_u \Delta u_i^{[\gamma]}(t) - \chi u_i(t) v_i^2(t) \right) dt + \sigma_1 u_i(t) dW_1(t) \quad t > 0, \tag{6.1}$$

$$dv_i(t) = \left(r_v \Delta v_i(t) + u_i(t) v_i^2(t) \right) dt + \sigma_2 v_i(t) dW_2(t) \quad t > 0. \tag{6.2}$$

In the first step we will introduce a family of stopping times $\{\tau_N : N \in \mathbb{N}\}$, and show that on the time interval $[0, \tau_N]$ the solutions u_1 and u_2 , respective, v_1 and v_2 , are indistinguishable. Here, in the second step, we will show that $\mathbb{P}(\tau_N < T) \rightarrow 0$ for $N \rightarrow \infty$. From this follows that u_1 and u_2 are indistinguishable on the time interval $[0, T]$.

By Remark 3.9, $d = 1$. Let us remind that $\delta_0 \in (0, \frac{1}{\gamma})$, $\rho \in [0, \frac{1}{2})$, $\frac{1}{2} < \frac{2}{m} + \frac{1}{m_0} + \rho \leq \frac{3\gamma+2}{2\gamma+2} + \rho$.

Step I.

Let us introduce the stopping times $\{\tau_N : N \in \mathbb{N}\}$ as follows:

$$\begin{aligned} \tau_{N,i}^1 &:= \inf\{t \geq 0 : \|\mathbb{1}_{[0,t]} v_i\|_{\mathbb{H}_\rho} \geq N\} \wedge T, \quad i = 1, 2, \\ \tau_{N,i}^2 &:= \inf\{t \geq 0 : \|\mathbb{1}_{[0,t]} u_i\|_{L^{2\gamma}(0,T;H_2^{\delta_0})} \geq N\} \wedge T, \quad i = 1, 2, \end{aligned}$$

and $\tau_N := \min_{i=1,2}(\tau_{N,i}^1, \tau_{N,i}^2)$.

The aim is to show that (u_1, v_1) and (u_2, v_2) are indistinguishable on the time interval $[0, \tau_N]$.

Remark 6.1 Note, due to the assumptions in Theorem 3.10 and by Corollary A.5, and Proposition 5.14 we know that there exists a constant $C > 0$ such that for any solutions (u, v) to Eq. 3.6–(3.7) we have

$$\mathbb{E}\|v\|_{\mathbb{H}_\rho}^{m_0} \leq C \quad \text{and} \quad \mathbb{E}\|u\|_{L^{2\gamma}(0,T;H_2^{\delta_0})}^{2\gamma} \leq C.$$

Fix $N \in \mathbb{N}$.

To get uniqueness on $[0, \tau_N]$ we first stop the original solution processes at time τ_N and extend the processes (u_1, v_1) and (u_2, v_2) by other processes to the whole interval $[0, T]$. For this purpose, let (y_1, z_1) be solutions to

$$\begin{cases} dy_1(t) = r_u \Delta y_1^{[\gamma]}(t) + \sigma_1 y_1(t) d\theta_{\tau_N} \circ W_1(t), & t \in [\tau_N \wedge T, T], \\ dz_1(t) = r_v \Delta z_1(t) - z_1(t) + \sigma_2 z_1(t) d\theta_{\tau_N} \circ W_2(t), & t \in [\tau_N \wedge T, T], \end{cases} \tag{6.3}$$

with initial data $y_1(0) := u_1(\tau_N), z_1(0) := v_1(\tau_N)$, and

let (y_2, z_2) be solutions to

$$\begin{cases} dy_2(t) = r_u \Delta y_2^{[\gamma]}(t) + \sigma_1 y_2(t) d\theta_{\tau_N} \circ W_1(t), & t \in [\tau_N \wedge T, T], \\ dz_2(t) = r_v \Delta z_2(t) - z_2(t) + \sigma_2 z_2(t) d\theta_{\tau_N} \circ W_2(t), & t \in [\tau_N \wedge T, T], \end{cases} \tag{6.4}$$

with initial data $y_2(0) := u_2(\tau_N)$ and $z_2(0) := v_2(\tau_N)$. Here, θ denotes the shift operator, i.e. $\theta_\tau \circ W_i(t) = W_i(t + \tau) - W_i(\tau), i = 1, 2$. Since (u_1, v_1) and (u_2, v_2) are continuous in $H_2^{-1}(\mathcal{O}) \times L^2(\mathcal{O})$, $(u_1(\tau_N), v_1(\tau_N))$ and $(u_2(\tau_N), v_1(\tau_N))$ are well-defined and belong \mathbb{P} -a.s. to $H_2^{-1}(\mathcal{O}) \times H_2^{-\delta_0}(\mathcal{O})$. By [2, Theorem 2.5.1], we know that there exists a unique solutions y_1 and y_2 to Eq. 6.3 belonging \mathbb{P} -a.s. to $C([0, T]; H_2^{-1}(\mathcal{O}))$. Since $(e^{t(r_v \Delta - \text{Id})})_{t \geq 0}$ is an analytic semigroup on $H_2^\rho(\mathcal{O})$, the existence of unique solutions z_1 and z_2 to Eq. 6.3 and Eq. 6.4 belonging \mathbb{P} -a.s. to $C([0, T]; H_2^{-\delta_0}(\mathcal{O}))$ can be shown by standard methods, cf. [16].

Now, let us define two processes (\bar{u}_1, \bar{v}_1) and (\bar{u}_2, \bar{v}_2) which are equal to (u_1, v_1) and (u_2, v_2) on the time interval $[0, \tau_N)$ and follow the processes (y_1, z_1) and (y_2, z_2) afterwards. In particular, let

$$\bar{u}_1(t) = \begin{cases} u_1(t) & \text{for } 0 \leq t < \tau_N, \\ y_1(t) & \text{for } \tau_N \leq t \leq T, \end{cases} \quad \bar{v}_1(t) = \begin{cases} v_1(t) & \text{for } 0 \leq t < \tau_N, \\ z_1(t) & \text{for } \tau_N \leq t \leq T, \end{cases}$$

and

$$\bar{u}_2(t) = \begin{cases} u_2(t) & \text{for } 0 \leq t < \tau_N, \\ y_2(t) & \text{for } \tau_N \leq t \leq T, \end{cases} \quad \bar{v}_2(t) = \begin{cases} v_2(t) & \text{for } 0 \leq t < \tau_N, \\ z_2(t) & \text{for } \tau_N \leq t \leq T. \end{cases}$$

Note, that (u_1, v_1) and (u_2, v_2) solve on $[0, \tau_N)$ the equation corresponding to Eq. 3.6–(3.7), that is, for $i = 1, 2$,

$$du_i(t) = \left(r_u \Delta u_i^\gamma(t) - \chi u_i(t) v_i^2(t) \right) dt + \sigma_1 u_i(t) dW_1(t) \quad t > 0, \tag{6.5}$$

$$dv_i(t) = \left(r_v \Delta v_i(t) + u_i(t) v_i^2(t) \right) dt + \sigma_2 v_i(t) dW_2(t) \quad t > 0. \tag{6.6}$$

Step II.

Let $\alpha = \rho + \frac{1}{2}$ (which implies $\alpha \geq \frac{1}{2}$ due to the assumption on ρ) and let $\frac{1}{2} > \delta > \delta_0$. Our goal is to show that (u_1, v_1) and (u_2, v_2) are indistinguishable on the interval $[0, \tau_N]$. Applying the Itô formula while setting

$$\Phi(t) := \mathbb{E} \left[\sup_{0 \leq s \leq t} |\bar{u}_1(s) - \bar{u}_2(s)|_{H_2^{-1}}^2 \right]$$

over $[0, \tau_N]$ and taking into account that (\bar{u}_1, \bar{v}_1) and (\bar{u}_2, \bar{v}_2) are solutions to system Eq. 3.6–(3.7) we obtain by standard calculations for $0 \leq t \leq \tau_N$

$$\begin{aligned} & \mathbb{E} \left[|\bar{u}_1(t) - \bar{u}_2(t)|_{H_2^{-1}}^2 \right] + r_u \mathbb{E} \left[\int_0^t |\bar{u}_1(s) - \bar{u}_2(s)|_{L^{\gamma+1}}^{\gamma+1} ds \right] \\ & \leq \chi \mathbb{E} \left[\left| \int_0^t \int_{\mathcal{O}} (-\nabla)^{-1} (\bar{u}_1(s, x) \bar{v}_1^2(s, x) - \bar{u}_2(s, x) \bar{v}_2^2(s, x)) (-\nabla)^{-1} (\bar{u}_1(s, x) - \bar{u}_2(s, x)) dx ds \right| \right] \\ & \quad + C \mathbb{E} \int_0^t |\bar{u}_1(s) - \bar{u}_2(s)|_{H_2^{-1}}^2 ds. \end{aligned} \tag{6.7}$$

Furthermore, we obtain

$$\begin{aligned} & \mathbb{E} \left[\left| \int_0^t \int_{\mathcal{O}} (-\nabla)^{-1} (\bar{u}_1(s, x) \bar{v}_1^2(s, x) - \bar{u}_2(s, x) \bar{v}_2^2(s, x)) (-\nabla)^{-1} (\bar{u}_1(s, x) - \bar{u}_2(s, x)) dx ds \right| \right] \\ & \leq \mathbb{E} \left[\left| \int_0^t \int_{\mathcal{O}} (-\nabla)^{-1} (\bar{u}_1(s, x) - \bar{u}_2(s, x)) \bar{v}_1^2(s, x) (-\nabla)^{-1} (\bar{u}_1(s, x) - \bar{u}_2(s, x)) dx ds \right| \right] \\ & \quad + \mathbb{E} \left[\left| \int_0^t \int_{\mathcal{O}} (-\nabla)^{-1} [\bar{u}_2(s, x) \bar{v}_2(s, x) (\bar{v}_1(s, x) - \bar{v}_2(s, x))] (-\nabla)^{-1} (\bar{u}_1(s, x) - \bar{u}_2(s, x)) dx ds \right| \right] \\ & =: I_1(t) + I_2(t). \end{aligned}$$

Now we have by Proposition A.3 with $\alpha \geq \frac{1}{2} = \frac{d}{2}$,

$$|(\bar{u}_1 - \bar{u}_2) \bar{v}_1^2|_{H_2^{-1}} \leq |\bar{u}_1 - \bar{u}_2|_{H_2^{-1}} |\bar{v}_1^2|_{H_2^\alpha}$$

Since $H_2^\alpha(\mathcal{O})$ is a Banach algebra (see [58, Theorem 2-(18), p. 192]), we have

$$|(\bar{u}_1 - \bar{u}_2) \bar{v}_1^2|_{H_2^{-1}} \leq |\bar{u}_1 - \bar{u}_2|_{H_2^{-1}} |\bar{v}_1|_{H_2^2}^2.$$

Hence, we get by applying Young’s inequality for $\varepsilon > 0$

$$\begin{aligned} I_1(t) & \leq \mathbb{E} \int_0^t |\bar{u}_1(s) - \bar{u}_2(s)|_{H_2^{-1}}^2 |\bar{v}_1(s)|_{H_2^2}^2 ds \\ & \leq \varepsilon \mathbb{E} \int_0^t |\bar{u}_1(s) - \bar{u}_2(s)|_{H_2^{-1}}^2 |\bar{v}_1(s)|_{H_2^2}^4 ds + C(\varepsilon) \mathbb{E} \int_0^t |\bar{u}_1(s) - \bar{u}_2(s)|_{H_2^{-1}}^2 ds \\ & \leq \varepsilon \mathbb{E} \left[\sup_{0 \leq s \leq t} |\bar{u}_1(s) - \bar{u}_2(s)|_{H_2^{-1}}^2 \right] \times \mathbb{E} \int_0^t |\bar{v}_1(s)|_{H_2^2}^4 ds \\ & \quad + C(\varepsilon) \mathbb{E} \int_0^t |\bar{u}_1(s) - \bar{u}_2(s)|_{H_2^{-1}}^2 ds. \end{aligned}$$

Since $\alpha = \rho + \frac{1}{2}$, we have $\|\bar{v}_1\|_{L^4(0, T; H_2^\alpha)} \leq \|\bar{v}_1\|_{\mathbb{H}_\rho}$. Next, note that $|\bar{u}_2(\bar{v}_1^2 - \bar{v}_2^2)| \leq |\bar{u}_2 \bar{v}_1(\bar{v}_1 - \bar{v}_2)| + |\bar{u}_2 \bar{v}_2(\bar{v}_1 - \bar{v}_2)|$. Similarly as above we get for $\delta_1 < \delta$ and $\delta_2 = \delta < \frac{1}{\gamma}$ and $\delta_3 > \frac{1}{2} = \frac{d}{2}$ (see Proposition A.3, [58, Theorem 2, p. 200] and the identities on [58, Proposition, p. 14]) for $i = 1, 2$

$$|\bar{u}_2 \bar{v}_i(\bar{v}_1 - \bar{v}_2)|_{H_2^{-1}} \leq C_1 |\bar{v}_1 - \bar{v}_2|_{H_2^{-\delta_1}} |\bar{u}_2 \bar{v}_i|_{H_2^\delta} \leq C_2 |\bar{v}_1 - \bar{v}_2|_{H_2^{-\delta_1}} |\bar{u}_2|_{H_2^{\delta_2}} |\bar{v}_i|_{H_2^{\delta_3}}. \tag{6.8}$$

By Young inequality, we know that for $\delta_3 \leq \frac{\gamma}{\gamma+1} + \rho$ and $-\delta_0 > -\frac{1}{\gamma}$ and for any $\varepsilon > 0$ there exists a constant $C(\varepsilon) > 0$ such that

$$I_2(t) \leq C(\varepsilon) \mathbb{E} \int_0^t |\bar{v}_1(s) - \bar{v}_2(s)|_{H_2^{-\delta_0}}^2 ds + \varepsilon \mathbb{E} \left(\int_0^t |\bar{u}_1(s)|_{H_2^{\delta_2}}^{2\gamma} ds \right)^{\frac{1}{\gamma}}$$

$$\begin{aligned} & \times \mathbb{E} \left(\int_0^t \left(|\bar{v}_1(s)|_{H_2^{\delta_3}}^{2\frac{\gamma+1}{\gamma}} + |\bar{v}_2(s)|_{H_2^{\delta_3}}^{2\frac{\gamma+1}{\gamma}} \right) ds \right)^{\frac{\gamma}{\gamma+1}} \mathbb{E} \left[\sup_{0 \leq s \leq t} |\bar{v}_1(s) - \bar{v}_2(s)|_{H_2^{-\delta_0}}^2 \right] \\ & \leq C(\varepsilon) \mathbb{E} \int_0^t |\bar{v}_1(s) - \bar{v}_2(s)|_{H_2^{-\delta_0}}^2 ds \\ & \quad + \mathbb{E} \|\bar{u}_1\|_{L^{2\gamma}(0,T;H_2^{\delta_2})} \left(\mathbb{E} \|\bar{v}_1\|_{\mathbb{H}_\rho} + \mathbb{E} \|\bar{v}_2\|_{\mathbb{H}_\rho} \right) \mathbb{E} \left[\sup_{0 \leq s \leq t} |\bar{v}_1(s) - \bar{v}_2(s)|_{H_2^{-\delta_0}}^2 \right]. \end{aligned}$$

The third term in Eq. 6.7 can be handled by the Burkholder-Davis-Gundy inequality. Since $t \leq \tau_N$ and $\delta_2 < \frac{1}{\gamma}$ we know that for any $\varepsilon > 0$ there exists a constant $C(\varepsilon) > 0$ such that

$$\begin{aligned} \Phi(t) & \leq \varepsilon_1 N^2 \mathbb{E} \left[\sup_{0 \leq s \leq t} |\bar{v}_1(s) - \bar{v}_2(s)|_{H_2^{-\delta_0}}^2 \right] + \mathbb{E} \int_0^t \sup_{0 \leq r \leq s} |\bar{v}_1(r) - \bar{v}_2(r)|_{H_2^{-\delta_0}}^2 ds \\ & \quad + \varepsilon N \mathbb{E} \left[\sup_{0 \leq s \leq t} |\bar{u}_1(s) - \bar{u}_2(s)|_{H_2^{-1}}^2 \right] + C(\varepsilon) \mathbb{E} \int_0^t |\bar{u}_1(s) - \bar{u}_2(s)|_{H_2^{-1}}^2 ds. \end{aligned}$$

Step III. Let us set

$$\Psi(t) := \mathbb{E} \left[\sup_{0 \leq s \leq t} \|\bar{v}_1(s) - \bar{v}_2(s)\|_{H_2^{-\delta_0}}^2 \right] + \mathbb{E} \|\bar{v}_1 - \bar{v}_2\|_{L^2(0,t;H_2^{-\delta_0+1})}^2.$$

Then we get by standard calculations for $\tilde{\delta} < 1$

$$\begin{aligned} & \mathbb{E} \left[\sup_{0 \leq s \leq t} \|\bar{v}_1(s) - \bar{v}_2(s)\|_{H_2^{-\delta_0}}^2 \right] + \mathbb{E} \|\bar{v}_1 - \bar{v}_2\|_{L^2(0,t;H_2^{-\delta_0+1})}^2 \\ & \leq C \mathbb{E} \|\bar{u}_1 \bar{v}_1^2 - \bar{u}_2 \bar{v}_2^2\|_{L^2(0,t;H_2^{-\delta_0-\tilde{\delta}})}^2 \\ & \quad + \mathbb{E} \left[\sup_{0 \leq s \leq t} \left| \int_0^t \sigma_2(\bar{v}_1(s) - \bar{v}_2(s)) dW_2(s) \right|_{H_2^{-\delta_0}}^2 \right]. \end{aligned}$$

Next, since $\tilde{\delta}$ can be chosen such that $\delta_0 \geq 1 - \tilde{\delta}$ we can use Eq. 6.8 and obtain for $\delta_1 < \delta$ and $\delta_2 = \delta < \frac{1}{\gamma}$ and $\delta_3 > \frac{1}{2} = \frac{d}{2}$

$$|\bar{u}_1 \bar{v}_1^2 - \bar{u}_2 \bar{v}_2^2|_{H_2^{-\delta_0-\tilde{\delta}}} \leq C_1 |\bar{v}_1 - \bar{v}_2|_{H_2^{-\delta_1}} |\bar{u}_2|_{H_2^{\delta_2}} \left(|\bar{v}_1|_{H_2^{\delta_3}} + |\bar{v}_2|_{H_2^{\delta_3}} \right).$$

For $\alpha > \min(1 + -\delta_0 - \delta + \frac{d}{2}, 1, \frac{d}{2})$ and $\tilde{\delta} < 1$ we have by Proposition A.3 and [58, p. 192, Theorem 2-(18)]

$$|(\bar{u}_1 - \bar{u}_2) \bar{v}_1^2|_{H_2^{-\delta_0-\tilde{\delta}}} \leq |\bar{u}_1 - \bar{u}_2|_{H_2^{-1}} |\bar{v}_1^2|_{H_2^\alpha} \leq |\bar{u}_1 - \bar{u}_2|_{H_2^{-1}} |\bar{v}_1|_{H_2^\alpha}^2.$$

The third term can be handled by an application of the Burkholder-Davis-Gundy inequality. The Young inequality implies that for any $\varepsilon_1, \varepsilon_2 > 0$ there exist constants $C(\varepsilon_1), C(\varepsilon_2) > 0$ such that we have

$$\begin{aligned} \Psi(t) & \leq \mathbb{E} |\bar{v}_1(0) - \bar{v}_2(0)|_{H_2^{-\delta_0}}^2 \\ & \quad + \mathbb{E} \left(\int_0^t |\bar{u}_1(s)|_{H_2^{\delta_2}}^2 \left(|\bar{v}_1(s)|_{H_2^{\delta_3}}^2 + |\bar{v}_2(s)|_{H_2^{\delta_3}}^2 \right) |\bar{v}_1(s) - \bar{v}_2(s)|_{H_2^{-\delta_0}}^2 ds \right) \end{aligned}$$

$$\begin{aligned}
 & + \mathbb{E} \left(\int_0^t |\bar{u}_1(s) - \bar{u}_2(s)|_{H_2^{-1}}^2 |\bar{v}_1(s)|_{H_2^\alpha}^4 ds \right) \\
 \leq & \mathbb{E} |v_1(0) - v_2(0)|_{H_2^{-\delta_0}}^2 \\
 & + \varepsilon_1 \mathbb{E} \left(\int_0^t |\bar{u}_1(s)|_{H_2^{\delta_2}}^2, \left(|\bar{v}_1(s)|_{H_2^{\delta_3}}^2 + |\bar{v}_2(s)|_{H_2^{\delta_3}}^2 \right) |\bar{v}_1(s) - \bar{v}_2(s)|_{H_2^{-\delta_0}}^2 ds \right) \\
 & + C(\varepsilon_1) \mathbb{E} \left(\int_0^t |\bar{v}_1(s) - \bar{v}_2(s)|_{L^2}^2 ds \right) \\
 & + \varepsilon_2 \mathbb{E} \left(\int_0^t |\bar{u}_1(s) - \bar{u}_2(s)|_{H_2^{-1}}^2 |\bar{v}_1(s)|_{H_2^\alpha}^4 ds \right) + C(\varepsilon_2) \mathbb{E} \left(\int_0^t |\bar{u}_1(s) - \bar{u}_2(s)|_{H_2^{-1}}^2 ds \right).
 \end{aligned}$$

$$\begin{aligned}
 I_2(t) \leq & \mathbb{E} \left(\int_0^t |\bar{u}_1(s)|_{H_2^{\delta_2}}^{2\gamma} ds \right)^{\frac{1}{\gamma}} \\
 & \times \mathbb{E} \left(\int_0^t \left(|\bar{v}_1(s)|_{H_2^{\delta_3}}^{\frac{2\gamma+1}{\gamma}} + |\bar{v}_2(s)|_{H_2^{\delta_3}}^{\frac{2\gamma+1}{\gamma}} \right) ds \right)^{\frac{\gamma}{\gamma+1}} \mathbb{E} \left[\sup_{0 \leq s \leq t} |\bar{v}_1(s) - \bar{v}_2(s)|_{H_2^{-\delta_0}}^2 \right] \\
 \leq & \mathbb{E} \|\bar{u}_1\|_{L^{2\gamma}(0, T; H_2^{\delta_2})} \left(\mathbb{E} \|\bar{v}_1\|_{\mathbb{H}_\rho} + \mathbb{E} \|\bar{v}_2\|_{\mathbb{H}_\rho} \right) \mathbb{E} \left[\sup_{0 \leq s \leq t} |\bar{v}_1(s) - \bar{v}_2(s)|_{H_2^{-\delta_0}}^2 \right].
 \end{aligned}$$

Since $\delta_0 \in (0, 1)$ and $\alpha = \rho + \frac{1}{2}$,

$$\begin{aligned}
 \Psi(t) \leq & \mathbb{E} |\bar{v}_1(0) - \bar{v}_2(0)|_{H_2^{-\delta_0}}^2 + \varepsilon_1 \mathbb{E} \|\mathbb{1}_{[0, t]} \bar{u}_1\|_{L^{2\gamma}(0, T; H_2^{\delta_2})} \left(\mathbb{E} \|\bar{v}_1\|_{\mathbb{H}_\rho} + \mathbb{E} \|\bar{v}_2\|_{\mathbb{H}_\rho} \right) \\
 & \times \mathbb{E} \left[\sup_{0 \leq s \leq t} |\bar{v}_1(s) - \bar{v}_2(s)|_{H_2^{-\delta_0}}^2 \right] \\
 & + C(\varepsilon_1) \mathbb{E} \left(\int_0^t |\bar{v}_1(s) - \bar{v}_2(s)|_{H_2^{-\delta_0}}^2 ds \right) \\
 & + \varepsilon_2 \mathbb{E} \left[\sup_{0 \leq s \leq t} |\bar{u}_1(s) - \bar{u}_2(s)|_{H_2^{-1}}^2 \right] \mathbb{E} \|\mathbb{1}_{[0, t]} \bar{v}_1\|_{L^4(0, T; H_2^\alpha)}^2 \\
 & + C(\varepsilon_2) \mathbb{E} \int_0^t |\bar{u}_1(s) - \bar{u}_2(s)|_{H_2^{-1}}^2 ds \\
 \leq & \mathbb{E} |v_1(0) - v_2(0)|_{H_2^{-\delta_0}}^2 + \varepsilon_1 \mathbb{E} \|\mathbb{1}_{[0, t]} \bar{u}_1\|_{L^{2\gamma}(0, T; H_2^{\delta_2})} \left(\mathbb{E} \|\bar{v}_1\|_{\mathbb{H}_\rho} + \mathbb{E} \|\bar{v}_2\|_{\mathbb{H}_\rho} \right) \\
 & \times \mathbb{E} \left[\sup_{0 \leq s \leq t} |\bar{v}_1(s) - \bar{v}_2(s)|_{H_2^{-\delta_0}}^2 \right] \\
 & + C(\varepsilon_1) \mathbb{E} \left(\int_0^t |\bar{v}_1(s) - \bar{v}_2(s)|_{H_2^{-\delta_0}}^2 ds \right) \\
 & + \varepsilon_2 \mathbb{E} \left[\sup_{0 \leq s \leq t} |\bar{u}_1(s) - \bar{u}_2(s)|_{H_2^{-1}}^2 \right] \mathbb{E} \|\mathbb{1}_{[0, t]} \bar{v}_1\|_{\mathbb{H}_\rho}^2 \\
 & + C(\varepsilon_2) \mathbb{E} \int_0^t \sup_{0 \leq r \leq s} |\bar{u}_1(r) - \bar{u}_2(r)|_{H_2^{-1}}^2 ds.
 \end{aligned}$$

Taking into account the definition of τ_N we obtain

$$\begin{aligned} \Psi(t) &\leq \mathbb{E}|\bar{v}_1(0) - \bar{v}_2(0)|_{H_2^{-\delta_0}}^2 \\ &+ \varepsilon_1 N^2 \Psi(t) + \varepsilon_2 N \Phi(t) + C(\varepsilon_1) \int_0^t \Phi(s) ds + C(\varepsilon_2) \int_0^t \Psi(s) ds. \end{aligned} \tag{6.9}$$

Step IV. Noting that the estimates can be extended to the whole interval $[0, T]$ by standard calculations. Next, collecting altogether we know, that for any $\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4 > 0$ there exist constants $C(\varepsilon_1), C(\varepsilon_2), C(\varepsilon_3), C(\varepsilon_4) > 0$ such that

$$\begin{aligned} \Phi(t) + \Psi(t) &\leq \mathbb{E}|\bar{u}_1(0) - \bar{u}_2(0)|_{H_2^{-1}}^2 + \mathbb{E}|\bar{v}_1(0) - \bar{v}_2(0)|_{H_2^{-\delta_0}}^2 \\ &+ \varepsilon_1 N^2 \Psi(t) + C(\varepsilon_1) \int_0^t \Psi(s) ds + \varepsilon_0 N \Phi(t) + C(\varepsilon_0) \int_0^t \Phi(s) ds \\ &+ \varepsilon_3 N^2 \Psi(t) + \varepsilon_4 N \Phi(t) + C(\varepsilon_1) \int_0^t \Phi(s) ds + C(\varepsilon_2) \int_0^t \Psi(s) ds. \end{aligned}$$

Taking the $\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4 > 0$ accordingly, we know, that for any $N \in \mathbb{N}$ there exists a constant $C(N) > 0$ such that

$$\begin{aligned} \Phi(t) + \Psi(t) &\leq \mathbb{E}|\bar{u}_1(0) - \bar{u}_2(0)|_{H_2^{-1}}^2 + \mathbb{E}|\bar{v}_1(0) - \bar{v}_2(0)|_{H_2^{-\delta_0}}^2 \\ &+ C(N) \int_0^t (\Phi(s) + \Psi(s)) ds. \end{aligned}$$

An application of the Gronwall lemma and taking into account that $v_1(0) = v_2(0), u_1(0) = u_2(0)$ give that $\Phi(t) \leq 0$ and $\Psi(t) \leq 0$.

Step V. We show that $\mathbb{P}(\tau_N < T) \rightarrow 0$ as $N \rightarrow \infty$.

Therefore,

$$\begin{aligned} &\mathbb{P}(\tau_N < T) \\ &\leq \mathbb{P}\left(\|\bar{u}_1\|_{L^{2\gamma}(0,T;H_2^{\delta_0})} \geq N\right) + \mathbb{P}\left(\|\bar{u}_2\|_{L^{2\gamma}(0,T;H_2^{\delta_0})} \geq N\right) \\ &\quad + \mathbb{P}\left(\|\bar{v}_1\|_{\mathbb{H}_\rho} \geq N\right) + \mathbb{P}\left(\|\bar{v}_2\|_{\mathbb{H}_\rho} \geq N\right). \end{aligned}$$

Due to Remark 6.1, we can apply the Chebyshev inequality and get by the above that

$$\mathbb{P}(\tau_N < T) \leq \frac{C}{N^2},$$

It follows that

$$\mathbb{P}(\tau_N \leq T) \rightarrow 0,$$

as $N \rightarrow \infty$. Hence, both the processes \bar{u}_1 and \bar{u}_2 , and likewise \bar{v}_1 and \bar{v}_2 are indistinguishable on $[0, T]$. Since the processes (\bar{u}_1, \bar{v}_1) and (\bar{u}_2, \bar{v}_2) solves on $[0, T \wedge \tau_N]$ the system corresponding to Eq. 3.6–(3.7), the last arguments completes the proof of Theorem 3.10. \square

A Some Useful Inequalities

Lemma A.1 *Let $\gamma > 1, x, y \in \mathbb{R}$. Then it holds that*

$$(x^{[\gamma]} - y^{[\gamma]})(x - y) \geq 2^{1-\gamma} |x - y|^{\gamma+1}, \tag{A.1}$$

where $z^{[\gamma]} := |z|^{\gamma-1}z$ for $z \in \mathbb{R}$.

Proof See e.g. [47, Lemma 3.1].

Fix a bounded domain $\mathcal{O} \subset \mathbb{R}^d$ with sufficiently smooth boundary.

Proposition A.2 For any for $r \in (2, q + 1)$, $m \in (q + 1, \infty)$, $s \in (0, 1)$, with $\frac{1}{r} \geq \frac{1}{m} - \frac{s}{2}$, and $\frac{1}{m} \geq \frac{1+s}{q+1}$, there exists a constant $C > 0$ such that

$$\|\xi\|_{L^m(0,T;H^s_r)}^r \leq C \left(\|\xi\|_{L^\infty(0,T;H^{-1}_2)}^2 + \|\xi\|_{L^{q+1}(0,T;L^{q+1})}^{q+1} \right). \tag{A.2}$$

Proof In order that $L^m(0, T; H^s_r(\mathcal{O}))$ is an interpolation space between

$$L^\infty(0, T; H^{-1}_2(\mathcal{O})) \text{ and } L^{q+1}(0, T; L^{q+1}(\mathcal{O})),$$

we need that there exists $\theta \in (0, 1)$ such that for the parameters m, r, s the following inequalities are satisfied, see e.g. [3],

$$\begin{aligned} s &\leq -\theta, \\ \frac{1}{m} &\geq \frac{1-\theta}{q+1}, \\ \frac{1}{r} &\geq \frac{\theta}{2} + \frac{1-\theta}{q+1}. \end{aligned}$$

Now, if $\frac{1}{r} \geq \frac{1}{m} - \frac{s}{2}$ and $\frac{1}{m} \geq \frac{1+s}{q+1}$ are satisfied for $r \in (2, q + 1)$, $m \in (q + 1, \infty)$, $s \in (0, 1)$, then the set of inequalities are satisfied and we obtain Eq. A.2. \square

Proposition A.3 Let $s_1, s_2 \in \mathbb{R}$ and $p > 1$. Let $\mathcal{O} \subset \mathbb{R}^d$. Assume that $s_1 \leq s_2$ and that $s_1 + s_2 > d \left(0 \vee \left(\frac{2}{p} - 1\right)\right)$ and $s_2 < \frac{d}{p}$. Then there exists a constant $C > 0$ such that

$$|uv|_{H^r_p} \leq C |u|_{H^{s_1}_p} |v|_{H^{s_2}_p},$$

for any $r \leq s_1 + s_2 - \frac{d}{p}$ and for any $u \in H^{s_1}_p(\mathcal{O})$ and any $v \in H^{s_2}_p(\mathcal{O})$.

Proof See [58, p. 190, Theorem 1 (iii)]. \square

For the definition of the space $F_{p,q}^s$, we refer to [58, 66, 69]. It translates to classical function spaces as in e.g. [66, Remark 2.1.1], in particular, $F_{p,2}^0 = L^p$, $1 < p < \infty$ (Lebesgue spaces), $F_{p,2}^m = W^m_p$, $m \in \mathbb{N}$ (Sobolev spaces), $1 < p < \infty$ and $F_{p,2}^s = H^s_p$, $s \in \mathbb{R}$, $1 < p < \infty$ (fractional Sobolev spaces).

Next, we shall record a variant of the Stroock-Varopoulos inequality together with its proof. See e.g. [17, Lemma 3.6] for another version of this result.

Proposition A.4 For any bounded domain $\mathcal{O} \subset \mathbb{R}^d$ with sufficiently smooth boundary, for any $T > 0$, $\gamma > 1$, $\theta \in (0, \frac{1}{\gamma})$, there exists a constant $C = C(\gamma, T, \mathcal{O}) > 0$ such that

$$\|\eta\|_{L^{2\gamma}(0,T;H^{2\gamma}_2)}^{2\gamma} \leq C \left(\int_0^T |\eta(s)^{[\gamma-1]} \nabla \eta(s)|_{L^2}^2 ds + \int_0^T |\eta(s)|_{L^2}^{2\gamma} ds \right),$$

where $z^{[\gamma-1]} := |z|^{\gamma-2}z$ for $z \in \mathbb{R}$.

Proof By [58, p. 365], we have for any $p \in (1, \infty)$, $s \in (0, 1)$, $\mu \in (0, 1)$ and $\varepsilon \in (0, s\mu)$,

$$\|w\|^\mu_{H^s_{\frac{p}{\mu}}} = \|w\|^\mu_{F^{s\mu-\varepsilon}_{\frac{p}{\mu}, 2}} \leq C\|w\|^\mu_{F^{s\mu}_{\frac{p}{\mu}, 2}} \leq C\|w\|^\mu_{F^{s}_{p, 2}} = C\|w\|^\mu_{H^s_p}, \quad w \in H^s_p(\mathcal{O}). \tag{A.3}$$

From Eq. A.3 we know that for any $\gamma > 1$, $p \in (1, \infty)$, $\theta \in (0, \frac{1}{\gamma})$, $\varepsilon > 0$, there exists a constant $C > 0$ such that

$$\|w\|_{H^{\theta}_{p\gamma}} = \|w^\gamma\|^{\frac{1}{\gamma}}_{H^{\theta}_{p\gamma}} \leq C\|w\|^\gamma_{H^{\theta\gamma+\varepsilon}_p} \cdot \square$$

In particular, for any $0 < \theta < \frac{1}{\gamma}$ and $p = 2$, there exists a constant $C > 0$ such that

$$\|w\|^\gamma_{H^{2\theta}_{2\gamma}} \leq C\|w\|^\gamma_{H^1_2}.$$

Since we know by the chain rule,

$$\int_0^T |\nabla|\eta(s)|^\gamma|_{L^2}^2 ds = \gamma^2 \int_0^T |\eta^{[\gamma-1]}(s)\nabla\eta(s)|_{L^2}^2 ds$$

we know that for any $\theta < \frac{1}{\gamma}$,

$$\int_0^T |\eta(s)|_{H^{2\theta}_{2\gamma}}^{2\gamma} ds \leq C \left(\int_0^T |\eta^{\gamma-1}(s)\nabla\eta(s)|_{L^2}^2 ds + \int_0^T |\eta(s)|_{L^2}^{2\gamma} ds \right).$$

Clearly, we have by the fractional Rellich-Kondrachov theorem that for any $\varepsilon \in (0, \theta)$,

$$\|w\|^\gamma_{H^{2\theta-\varepsilon}_{2\gamma}} \leq C\|w\|^\gamma_{H^{2\theta}_{2\gamma}}.$$

As $\theta \in (0, \gamma^{-1})$ is arbitrary, this yields the following result.

Corollary A.5 *For any bounded domain $\mathcal{O} \subset \mathbb{R}^d$ with sufficiently smooth boundary, for any $T > 0$, $\gamma > 1$, $\theta \in (0, \frac{1}{\gamma})$, there exists a constant $C = C(\gamma, T, \mathcal{O}) > 0$ such that*

$$\|\eta\|_{L^{2\gamma}(0, T; H^{2\theta}_{2\gamma})}^{2\gamma} \leq C \left(\int_0^T |\eta(s)|^{[\gamma-1]}\nabla\eta(s)|_{L^2}^2 ds + \int_0^T |\eta(s)|_{L^2}^{2\gamma} ds \right),$$

where $z^{[\gamma-1]} := |z|^{\gamma-2}z$ for $z \in \mathbb{R}$.

We have the following embedding.

Proposition A.6 *Let $l_1, l_2 \in (2, \infty)$ with $\frac{d}{2} - \rho \leq \frac{2}{l_1} + \frac{d}{l_2}$. Then there exists $C > 0$ such that*

$$\|\xi\|_{L^1(0, T; L^{l_2})} \leq C\|\xi\|_{\mathbb{H}}, \quad \xi \in \mathbb{H}_\rho,$$

where \mathbb{H}_ρ is defined in Eq. 3.8.

Proof First, let us note that for $\delta > 0$ such that

$$\frac{1}{l_2} \geq \frac{1}{2} - \frac{\delta}{d}, \tag{A.4}$$

we have the embedding $H^\delta_2(\mathcal{O}) \hookrightarrow L^2(\mathcal{O})$, and therefore

$$\|\xi\|_{L^1(0, T; L^{l_2})} \leq C\|\xi\|_{L^1(0, T; H^\delta_2)}, \quad \xi \in \mathbb{X}.$$

Due to interpolation, compare with [3, Theorem 5.1.2, p. 107 and Theorem 6.4.5, p. 152], we have

$$\|\xi\|_{L^1(0,T;H_2^\delta)} \leq C \|\xi\|_{L^\infty(0,T;H_2^\rho)}^\theta \|\xi\|_{L^2(0,T;H_2^{\rho+1})}^{1-\theta}, \quad \xi \in \mathbb{H}_\rho,$$

for $\theta \in (0, 1)$ with,

$$\frac{1}{l_1} \leq \frac{1}{2}(1 - \theta), \quad \delta \leq \theta\rho + (1 - \theta)(\rho + 1). \tag{A.5}$$

Taking into account that we have

$$\frac{d}{2} - \rho \leq \frac{2}{l_1} + \frac{d}{l_2},$$

gives that δ and θ satisfying Eq. A.4 and Eq. A.5 exist and thus the Young inequality for products yields the assertion. \square

B Function Spaces And The Aubin-Lions-Simon Compactness Theorem

Let B be a separable Banach space, $0 \leq c < d < \infty$. Let $C_b^{(\beta)}(c, d; B)$ denote a set of all continuous and bounded functions $u : [c, d] \rightarrow B$ such that

$$\|u\|_{C_b^{(\beta)}(c,d;B)} := \sup_{c \leq t \leq d} |u(t)|_B + \sup_{c \leq s, t \leq d, t \neq s} \frac{|u(t) - u(s)|_B}{|t - s|^\beta},$$

is finite. The space $C_b^{(\beta)}(c, d; E)$ endowed with the norm $\|\cdot\|_{C_b^{(\beta)}(c,d;B)}$ is a Banach space. Let

$$L^p(c, d; ; B) = \left\{ u : [c, d] \rightarrow B : u \text{ measurable and } \int_{[c,d]} |u(t)|_B^p dt < \infty \right\}.$$

In addition, for $1 < p < \infty$ let $W_p^1(\mathcal{O})$ be the standard Sobolev space defined by (compare [6, p. 263])

$$W_p^1(\mathcal{O}) := \left\{ u \in L^p(\mathcal{O}) \mid \exists g_1, \dots, g_d \in L^p(\mathcal{O}) \text{ such that } \int_{\mathcal{O}} u(x) \frac{\partial \phi(x)}{\partial x_i} dx = - \int_{\mathcal{O}} g_i(x) \phi(x) dx \quad \forall \phi \in C_0^\infty(\mathcal{O}), \forall i = 1, \dots, d \right\}$$

equipped with norm

$$|u|_{W_p^1} := |u|_{L^p} + \sum_{j=1}^d \left| \frac{\partial u}{\partial x_j} \right|_{L^p}, \quad u \in W_p^1(\mathcal{O}).$$

Given an integer $m \geq 2$ and a real number $1 \leq p < \infty$, we define by induction the space

$$W_p^m(\mathcal{O}) := \left\{ u \in W_p^{m-1}(\mathcal{O}) \mid Du \in W_p^{m-1}(\mathcal{O}) \right\}$$

equipped with norm

$$|u|_{W_p^m} := |u|_{L^p} + \sum_{\alpha=1}^m |D_\alpha u|_{L^p}, \quad u \in W_p^m(\mathcal{O}).$$

Let $H_2^m(\mathcal{O}) := W_2^m(\mathcal{O})$, and for $\rho \in (0, 1)$ let $H_2^\rho(\mathcal{O})$ be the real interpolation space given by $H_2^\rho(\mathcal{O}) := (L^2(\mathcal{O}), H_2^1(\mathcal{O}))_{\rho,2}$. In addition, let $H_2^{-1}(\mathcal{O})$ be the dual space of

$H_2^1(\mathcal{O})$ and for $\rho \in (0, 1)$ let $H_2^{-\rho}(\mathcal{O})$ be the real interpolation space given by $H_2^{-\rho}(\mathcal{O}) := (L^2(\mathcal{O}), H_2^{-1}(\mathcal{O}))_{1-\rho, 2}$.

Note, by Theorem 3.7.1 [3], $H_2^{-\rho}(\mathcal{O})$ is dual to $H_2^\rho(\mathcal{O})$, $\rho \in (0, 1)$. Furthermore, we have $(H_2^{-\rho}(\mathcal{O}), H_2^\rho(\mathcal{O}))_{\frac{1}{2}, 2} = L^2(\mathcal{O})$ and $(H_2^\alpha(\mathcal{O}), H_2^\beta(\mathcal{O}))_{\rho, 2} = H_2^\theta(\mathcal{O})$ for $\theta = \alpha(1 - \rho) + \beta\rho$, $\rho \in (0, 1)$ and $|\alpha|, |\beta| \leq 1$.

Since we need it to tackle the compactness, let us introduce the following space. Given $p \in (1, \infty)$, $\alpha \in (0, 1)$, let $\mathbb{W}_p^\alpha(I; B)$ be the Sobolev space of all $u \in L^p(0, \infty; B)$ such that

$$\int_I \int_{I \cap [t, t+1]} \frac{|u(t) - u(s)|_B^p}{|t - s|^{1+\alpha p}} ds dt < \infty;$$

equipped with the norm

$$\|u\|_{\mathbb{W}_p^\alpha(I; B)} := \left(\int_I \int_{I \cap [t, t+1]} \frac{|u(t) - u(s)|_B^p}{|t - s|^{1+\alpha p}} ds dt \right)^{\frac{1}{p}}.$$

Theorem B.1 *Let $B_0 \subset B \subset B_1$ be Banach spaces, B_0 and B_1 reflexive, with compact embedding of B_0 to B . Let $p \in (1, \infty)$ and $\alpha \in (0, 1)$ be given. Let X be the space*

$$X = L^p(0, T; B_0) \cap \mathbb{W}_p^\alpha(0, T; B_1).$$

Then the embedding of X to $L^p(0, T; B)$ is compact.

Proof See [67, p. 86, Corollary 5] or [28, Theorem 2.1].

C On the Burkholder-Davis-Gundy inequality

We collect the exact form of the Burkholder-Davis-Gundy inequality needed here. Given a cylindrical Wiener process W on $H_2^\delta(\mathcal{O})$, $\delta > 1$, over $\mathfrak{A} = (\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$, and a progressively measurable process $\xi \in \mathcal{M}_{\mathfrak{A}}^2(H_2^\rho(\mathcal{O}))$, $\rho \in [0, \frac{1}{2}]$, let us define $\{Y(t) : t \in [0, T]\}$ by

$$Y(t) := \int_0^t \xi(s) dW(s), \quad t \in [0, T].$$

Here, for each $t \in [0, T]$, $\xi(t)$ is interpreted as a multiplication operator acting on the elements of $H_2^\delta(\mathcal{O})$, namely, $\xi : H_2^\delta(\mathcal{O}) \ni \psi \mapsto \xi \psi \in \mathcal{S}'(\mathcal{O})$. Since for any $\nu > \frac{1}{2}$ and for any $\varphi \in H_2^\nu(\mathcal{O})$ the product $\xi(t)\varphi$ belongs to $H_2^\rho(\mathcal{O})$ by Proposition A.3, we can view $\xi(t)$ as a linear map from $H_2^\nu(\mathcal{O})$ into $H_2^\rho(\mathcal{O})$. It is shown in Proposition A.3 that

$$|\xi(t)\varphi|_{H_2^\rho} \leq C |\xi(t)|_{H_2^\rho} |\varphi|_{H_2^\nu} \tag{A.6}$$

where the constant $C > 0$ is independent of φ . Consequently, for any $p \geq 1$, $\delta > 1$, and any $\rho \in [0, \frac{1}{2}]$

$$\mathbb{E} \left[\sup_{t \in [0, T]} |Y(t)|_{H_2^\rho}^p \right] \leq C \mathbb{E} \left[\int_0^T |\xi(t)|_{L_{\text{HS}}(H_2^\delta, H_2^\rho)}^2 dt \right]^{\frac{p}{2}},$$

where $|\cdot|_{L_{\text{HS}}(H_2^\delta, H_2^\rho)}$ denotes the Hilbert–Schmidt norm from $H_2^\delta(\mathcal{O})$ to $H_2^\rho(\mathcal{O})$. First, let us note that for $\delta > 1$ there exists a $\nu > \frac{1}{2}$ such that the embedding $H_2^\delta(\mathcal{O}) \hookrightarrow H_2^\nu(\mathcal{O})$ is

a Hilbert–Schmidt operator. Using the fact that $\{\psi_k^{(\delta)} : k \in \mathbb{Z}\}$ is an orthonormal basis of $H_2^\delta(\mathcal{O})$, and $\psi_k^{(\delta)} = \lambda_k \psi_k$ we obtain, by using (A.6),

$$\begin{aligned} |\xi(s)|_{LHS(H_2^\delta, H_2^\rho)}^2 &= \sum_{k \in \mathbb{Z}} \left| \xi(s) \psi_k^{(\delta)} \right|_{H_2^\rho}^2 \leq C \sum_{k \in \mathbb{Z}} |\xi(s)|_{H_2^\rho}^2 \left| \psi_k^{(\delta)} \right|_{H_2^\nu}^2 \\ &= C |\xi(s)|_{H_2^\rho}^2 \sum_{k \in \mathbb{Z}} |\psi_k^{(\delta)}|_{H_2^\nu}^2. \end{aligned} \tag{A.7}$$

If the embedding $H_2^\delta(\mathcal{O}) \hookrightarrow H_2^\rho(\mathcal{O})$ is supposed to be a Hilbert–Schmidt, the right hand side of Eq. A.7 is finite and we obtain

$$\mathbb{E} \left[\sup_{t \in [0, T]} |Y(t)|_{H_2^\rho}^p \right] \leq C \mathbb{E} \left[\int_0^T |\xi(t)|_{H_2^\rho}^2 dt \right]^{\frac{p}{2}}. \tag{A.8}$$

In case $\rho \geq \frac{1}{2}$, we use the fact that $H_2^\rho(\mathcal{O})$ is an algebra and obtain

$$|\xi(s)|_{LHS(H_2^\delta, H_2^\rho)}^2 = \sum_{k \in \mathbb{Z}} \left| \xi(s) \psi_k^{(\delta)} \right|_{H_2^\rho}^2 = |\xi(s)|_{H_2^\rho}^2 \sum_{k \in \mathbb{Z}} \left| \psi_k^{(\delta)} \right|_{H_2^\rho}^2. \tag{A.9}$$

If the embedding $H_2^\delta(\mathcal{O}) \hookrightarrow H_2^\rho(\mathcal{O})$ is a Hilbert–Schmidt operator, then the right hand side of Eq. A.9 is finite and we obtain for $\delta > \rho + 1$

$$\mathbb{E} \left[\sup_{t \in [0, T]} |Y(t)|_{H_2^\rho}^p \right] \leq C \mathbb{E} \left[\int_0^T |\xi(t)|_{H_2^\rho}^2 dt \right]^{\frac{p}{2}}. \tag{A.10}$$

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