Potential Analysis (2024) 60:1369–1382
https://doi.org/10.1007/s11118-023-10091-8

Manifolds with Density and the First Steklov Eigenvalue

Márcio Batista [1] · **José I. Santos** [2]

Received: 20 September 2021 / Accepted: 17 July 2023 / Published online: 31 July 2023

Abstract
There are many interesting eigenvalue problems in a variety of settings; one of them is the well-known Steklov eigenvalue problem. In this work, we are interested in studying some Steklov eigenvalue problems for elliptic operators of second and fourth order using a well-known Reilly formula. Some upper and lower bounds for the first eigenvalue are obtained, and the rigidity case is carefully analyzed.

Keywords Steklov's problem · Eigenvalue estimates · Weighted area · Manifolds

Mathematics Subject Classification (2010) 35P15 · 58C40

1 Introduction

In 1902 Steklov introduced the following eigenvalue problem

$$\begin{cases} \Delta u = 0 & \text{in } \Omega, \\ \frac{\partial u}{\partial \nu} = \sigma u & \text{on } \partial\Omega, \end{cases}$$

steady-state temperature on a domain, and the flux on the boundary is proportional to the temperature; see, for instance, [18] for a nice description of the relationship between mathematics and physics in this problem. Afterward, this problem was studied by Payne in [14] for bounded domains in the plane with non-negative curvature using a new approach.

In the meantime, Kuttler and Sigillito introduced in [11], among other problems, the following problem:

$$\begin{cases} \Delta^2 u = 0 & \text{in } M, \\ u = \Delta u - q\frac{\partial u}{\partial \nu} = 0 & \text{on } \partial M, \end{cases}$$

✉ Márcio Batista
mhbs@mat.ufal.br

José I. Santos
jissivan@gmail.com

1 CPMAT-IM, Universidade Federal de Alagoas, Maceió, AL 57072-970, Brazil

2 Instituto Federal de Alagoas, Palmeiras dos Índios, AL 57608-180, Brazil

since the eigenvalues of this problem are related to optimal constants in a priori inequalities that have applications in bounding solutions of some elliptic equations. Thenceforth, many authors have studied this subject, and they have provided a good understanding of this theme; see, for instance, [3–7, 11, 13, 14, 16, 18–20] and references therein.

One attractive approach to understanding some geometrical properties of the ambient in focus is to understand the behavior of solutions of PDEs such as those introduced above. In many situations, the existence of such a solution implies some rigidity in the ambient studied. Perelman in [15] considered the space of metrics and functions on a manifold as a way to enlarge the space of variables for the Ricci flow, and so he was able to reinterpret the Ricci flow as the L^2-gradient of the Fischer information functional. Since the geometry of spaces and PDEs are related and it is attractive for many mathematicians, we decided to work in the setting of manifolds furnished with a Riemannian metric and a density function, aiming to obtain estimates of the first eigenvalue of weighted Steklov's problems and also make a careful analysis of the rigidity case. Next, we shall introduce the necessary notions.

We recall that a weighted Riemannian manifold is a Riemannian manifold (M, g) endowed with a real-valued smooth function $f : M \to \mathbb{R}$ which is used as a density to measure geometric objects on M, that is, denoting dv the Riemannian measure, we have the new measure $\mu = e^{-f} dv$ which is known as the weighted measure. Associated to this structure, we have an important second-order differential operator defined by

$$\Delta_f u = e^f div(e^{-f} \nabla u),$$

where $u \in C^\infty$. This operator is known as Drift Laplacian. We can iterate the Drift Laplacian and get the biharmonic Drift Laplacian Δ_f^2 which is a fourth-order elliptic operator.

Also, following Lichnerowich [12] and Bakry and Émery [1], the natural generalizations of Ricci curvatures are defined as

$$\text{Ric}_f = \text{Ric} + \text{Hess}\, f$$

and

$$\text{Ric}_f^k = \text{Ric}_f - \frac{df \otimes df}{k - n - 1},$$

where $k > n + 1$ or $k = n + 1$ and f a constant function.

Throughout this paper, we will consider M^{n+1} a compact oriented Riemannian manifold with boundary ∂M. Let $i : \partial M \hookrightarrow M$ be the standard inclusion and ν the outward unit normal on ∂M. We will denote by A its second fundamental form associate to ν, $\langle \nabla_X \nu, Y \rangle = A(X, Y)$, and by H the mean curvature of ∂M, that is, the trace of A over n.

We recall that the weighted mean curvature, introduced by Gromov in [9], of the inclusion i is given by

$$H_f = H - \frac{1}{n} \langle \nu, \nabla f \rangle.$$

Introduced the setting, we will focus on obtaining upper or lower bounds for the first eigenvalue of the weighted Steklov problems listed below. We also characterize the geometry of the Riemannian manifold for specific values of the first eigenvalues in some problems.

The problems we are interested are:

$$\begin{cases} \Delta_f u = 0 & \text{in } M, \\ \frac{\partial u}{\partial \nu} = pu & \text{on } \partial M; \end{cases} \tag{1.1}$$

$$\begin{cases} \Delta_f^2 u = 0 & \text{in } M, \\ u = \Delta_f u - q\frac{\partial u}{\partial \nu} = 0 & \text{on } \partial M; \end{cases} \tag{1.2}$$

$$\begin{cases} \Delta_f^2 u = 0 & \text{in } M, \\ u = \frac{\partial^2 u}{\partial \nu^2} - q\frac{\partial u}{\partial \nu} = 0 & \text{on } \partial M, \end{cases} \tag{1.3}$$

where ν denotes the outward unit normal on ∂M. The first non-zero eigenvalues of the above problems will be denoted by p_1 and q_1, respectively. We will use the same letter for the first non-zero eigenvalues of the last two problems because whenever the weighted mean curvature of ∂M is constant, then the problems are equivalent. Lastly, for the sake of simplicity, we will omit the weighted measure μ in the integrals throughout the text.

Before continuing, we recall that the first non-zero eigenvalues of the problems Eqs. 1.1-1.3 are characterized, respectively, by the following Rayleigh-Ritz quotients:

$$p_1 = \min\left\{\frac{\int_M |\nabla w|^2}{\int_{\partial M} w^2} : \int_{\partial M} w = 0\right\}; \tag{1.4}$$

$$q_1 = \min\left\{\frac{\int_M (\Delta_f w)^2}{\int_{\partial M} (\frac{\partial w}{\partial \nu})^2} : w = 0 \text{ on } \partial M\right\}; \tag{1.5}$$

and

$$q_1 = \min\left\{\frac{\int_M (\Delta_f w)^2 - n\int_{\partial M} H_f (\frac{\partial w}{\partial \nu})^2}{\int_{\partial M} (\frac{\partial w}{\partial \nu})^2} : w = 0 \text{ on } \partial M\right\},$$

where H_f is the f-mean curvature of the boundary; see [11] for more details about the Riemannian case ($f = 0$). Finally, we recall that the first non-zero eigenvalue of the Drift Laplacian on ∂M is given by

$$\lambda_1 = \min\left\{\frac{\int_{\partial M} |\nabla w|^2}{\int_{\partial M} w^2}\right\}.$$

Now, we are able to introduce our results. Our first result reads as follows:

Theorem 1.1 *Let M^{n+1} be a compact weighted Riemannian manifold with boundary ∂M. Assume that $\mathrm{Ric}_f^k \geq 0$ and $H_f \geq \frac{(k-1)c}{n}$, to some positive constant c, and that second fundamental form $A \geq cI$, in the quadratic form sense. Denote by λ_1 the first non-zero eigenvalue of the Drift Laplacian acting on functions on ∂M. Let p_1 be the first eigenvalue of the weighted Steklov eigenvalue problem* Eq. 1.1. *Then,*

$$p_1 \leq \frac{\sqrt{\lambda_1}}{(k-1)c}\left(\sqrt{\lambda_1} + \sqrt{\lambda_1 - (k-1)c^2}\right)$$

with equality occurs if and only if M is isometric to an n-dimensional euclidean ball of radius $\frac{1}{c}$, f is constant and $k = n + 1$.

We also obtain an upper bound for the p_1 under the weaker condition $\mathrm{Ric}_f \geq 0$. The big difference here is what happens in the equality case. The result is the following:

Theorem 1.2 *Let M^{n+1} be a compact weighted Riemannian manifold with boundary ∂M. Assume that* $\mathrm{Ric}_f \geq 0$, $H_f > 0$ *and second fundamental form A strictly convex. Denote by λ_1 the first non-zero eigenvalue of the Drift Laplacian acting on functions on ∂M. Let p_1 be the first eigenvalue of the weighted Steklov eigenvalue problem* Eq. 1.1. *Then,*

$$p_1 \leq \frac{\sqrt{\lambda_1}}{n \inf H_f}(\sqrt{\lambda_1} + \sqrt{\lambda_1 - n \inf H_f \inf A})$$

with equality occurs if and only if $\frac{\partial f}{\partial \nu}$ is constant, $A = (\inf A)I$ and there exists a smooth function u such that $\mathrm{Hess}\, u = 0$, $\frac{\partial u}{\partial \nu} = \frac{\lambda_1}{nH_f}u$, $df(\nabla u) = 0$ *and* $\mathrm{Ric}(\nabla u, \nabla u) = 0$.

The third result is the following:

Theorem 1.3 *Let M^{n+1} be a compact connected weighted Riemannian manifold with boundary ∂M. Assume that* $\mathrm{Ric}_f^k \geq 0$ *and* $H_f \geq \frac{k-1}{k}c$, *to some positive constant c. Let q_1 be the first eigenvalue of the weighted Steklov eigenvalue problem* Eq. 1.2. *Then*

$$q_1 \geq nc.$$

Moreover, equality occurs if and only if M is isometric to a euclidean ball of radius $\frac{1}{c}$ in $\mathbb{R}^{n+1}$, f is constant and $k = n + 1$.

The next results are

Theorem 1.4 *Let M^{n+1} be a compact connected weighted Riemannian manifold with boundary ∂M. Denote by A, V the weighted area of ∂M and the weighted volume of M, respectively. Let q_1 be the first eigenvalue of the weighted Steklov eigenvalue problem* Eq. 1.2. *Then,*

$$q_1 \leq \frac{A}{V}.$$

Moreover, if in addition that the Ric_f^k *of M is non-negative and that there is a point $x_0 \in \partial M$ such that $H_f(x_0) \geq \frac{(k-1)A}{k\,n\,V}$, and $q_1 = \frac{A}{V}$, then M is isometric to an $(n+1)$-dimensional Euclidean ball, f is constant and $k = n + 1$.*

and

Theorem 1.5 *Let M^{n+1} be a compact connected weighted Riemannian manifold with boundary ∂M. Assume that* $\mathrm{Ric}_f^k \geq 0$ *and* $H_f \geq \frac{(k-1)c}{n}$, *for some positive constant c. Let q_1 be the first eigenvalue of the problem* Eq. 1.3. *Then*

$$q_1 \geq c.$$

Moreover, equality occurs if and only if M is isometric to a ball of radius $\frac{1}{c}$ in $\mathbb{R}^{n+1}$, f is constant and $k = n + 1$.

The paper is organized in this way: In section 2, we provide some well-known results and give a sharp proof of a lower bound for the first non-zero eigenvalue of the Drift Laplacian on closed manifolds. In section 3, we provide the proofs of the results.

2 Preliminaries

In this section, we recall some well-known results necessary to prove the theorems announced in the introduction.

The following result is a direct consequence of the Cauchy-Schwarz inequality.

Proposition 2.1 *Assuming either $k > n+1$ or $k = n+1$ and f is a constant. Let u be a smooth function on M^{n+1}, then we have*

$$|\mathrm{Hess}\, u|^2 + \mathrm{Ric}_f(\nabla u, \nabla u) \geq \frac{(\Delta_f u)^2}{k} + \mathrm{Ric}_f^k(\nabla u, \nabla u).$$

Moreover, equality holds if and only if $\mathrm{Hess}\, u = \frac{\Delta u}{n+1}\langle\, ,\rangle$ *and* $\langle \nabla u, \nabla f\rangle = -\frac{k-n-1}{k}\Delta_f u$[1].

In [13], the authors showed that for a smooth function u defined on an n-dimensional compact weighted manifold M with boundary ∂M the following identity holds if $h = \frac{\partial u}{\partial \nu}$, $z = u|_{\partial M}$ and Ric_f denotes the generalized Ricci curvature of M:

$$\int_M [(\Delta_f u)^2 - |\mathrm{Hess}\, u|^2 - \mathrm{Ric}_f(\nabla u, \nabla u)] = \int_{\partial M} \left[n H_f h^2 + 2h\overline{\Delta}_f z + A(\overline{\nabla} z, \overline{\nabla} z)\right],$$

where $\overline{\Delta}$ and $\overline{\nabla}$ represent the Laplacian and the gradient on ∂M with respect to the induced metric on ∂M, respectively.

Using the Proposition 2.1 we get

$$\int_M \frac{k-1}{k}[(\Delta_f u)^2 - \mathrm{Ric}_f^k(\nabla u, \nabla u)] \geq \int_{\partial M} [n H_f h^2 + 2h\overline{\Delta}_f z + A(\overline{\nabla} z, \overline{\nabla} z)] \qquad (2.1)$$

In the next result, we recall a sharp lower bound for the first non-zero eigenvalue of the Drift Laplacian on closed submanifolds. This result is a slight modification of Theorem 1.6 in [10], and we include a short proof for the sake of completeness.

Proposition 2.2 *Let M^{n+1} be a compact weighted Riemannian manifold with nonempty boundary ∂M and $\mathrm{Ric}_f^k \geq 0$. If the second fundamental form of ∂M satisfies $A \geq cI$, in the quadratic form sense, and $H_f \geq \frac{k-1}{n}c$, then*

$$\lambda_1 \geq (k-1)c^2,$$

where λ_1 is the first non-zero eigenvalue of the Drift Laplacian acting on functions on ∂M. The equality holds if and only if M is isometric to an Euclidean ball of radius $\frac{1}{c}$, f is constant and $k = n+1$.

Proof Let z be an eigenfunction corresponding to the first non-zero eigenvalue λ_1 of the Drift Laplacian of ∂M, that is, $\overline{\Delta}_f z + \lambda_1 z = 0$. Let $u \in C^\infty(M)$ be the solution of the Dirichlet problem

$$\begin{cases} \Delta_f u = 0 & \text{in } M, \\ u = z & \text{on } \partial M. \end{cases}$$

Follows from Eq. 2.1 and the non-negativity of Ric_f^k of M that

$$0 \geq \int_{\partial M} [n H_f h^2 + 2h\overline{\Delta}_f z + A(\overline{\nabla} z, \overline{\nabla} z)].$$

[1] This term only appear in the case of a non constant function.

By hypothesis on A and noticing that z is an eigenfunction we get

$$\begin{aligned} 0 &\geq \int_{\partial M} [(k-1)ch^2 - 2\lambda_1 z h + c\lambda_1 z^2] \\ &= \int_{\partial M} \left[(k-1)c\left(h - \frac{\lambda_1 z}{(k-1)c}\right)^2 + \lambda_1 \left(c - \frac{\lambda_1}{(k-1)c}\right) z^2 \right] \\ &\geq \lambda_1 \left(c - \frac{\lambda_1}{(k-1)c}\right) \int_{\partial M} z^2. \end{aligned}$$

Thus,

$$\lambda_1 \geq (k-1)c^2,$$

which proof the first part of theorem. Assuming that the equality holds, from Proposition 2.1 and definition of u we get

$$\operatorname{Hess} u = 0 \text{ in } M \text{ and } \frac{\partial u}{\partial \nu} = cu \text{ on } \partial M.$$

So $|\nabla u|$ is constant. After a straightforward computation we obtain

$$|\nabla u|^2 \mu(\partial M) = kc^2 \int_{\partial M} z^2 \text{ and } c \int_{\partial M} z^2 = |\nabla u|^2 \mu(M).$$

Since $H_f = \frac{k-1}{n} c > 0$ and

$$\mu(M) = \frac{k-1}{k} \int_{\partial M} \frac{1}{nH_f} d\mu,$$

we conclude that M is isometric to a ball, f is constant and $k = n+1$ by Theorem 1.1 in [2] (or see Theorem 1.1 in [10]). The converse is immediate. □

In the following result, our hypothesis is just on the Bakry-Émery-Ricci curvature Ric_f.

Proposition 2.3 *Let M^{n+1} be a compact weighted Riemannian manifold with nonempty boundary ∂M and weighted Ricci curvature is greater or equal to $-c$ and $2c \leq n \inf H_f \inf A$. If the second fundamental form of ∂M is strictly convex and $H_f > 0$, then*

$$\lambda_1 \geq \frac{1}{2}\left(n \inf H_f \inf A - c + \sqrt{(n \inf H_f \inf A)^2 - 2cn \inf H_f \inf A} \right),$$

where λ_1 is the first non-zero eigenvalue of the Drift Laplacian acting on functions on ∂M. The equality holds if and only if $\frac{\partial f}{\partial \nu}$ is constant, $A = (\inf A)I$ and there exists a smooth function u such that $\operatorname{Hess} u = 0$, $\frac{\partial u}{\partial \nu} = \frac{\lambda_1 + c/2}{nH_f} u$, $df(\nabla u) = 0$ and $\operatorname{Ric}(\nabla u, \nabla u) = 0$.

Proof Let z be an eigenfunction corresponding to the first non-zero eigenvalue λ_1 of the Drift Laplacian of ∂M, that is, $\overline{\Delta}_f z + \lambda_1 z = 0$. Let $u \in C^\infty(M)$ be the solution of the Dirichlet problem

$$\begin{cases} \Delta_f u = 0 & \text{in } M, \\ u = z & \text{on } \partial M. \end{cases}$$

Follows from Eq. 2.1 and the non-negativity of Ric_f of M that

$$c \int_{\partial M} hz = c \int_M |\nabla u|^2 \geq \int_{\partial M} [nH_f h^2 + 2h\overline{\Delta}_f z + A(\overline{\nabla} z, \overline{\nabla} z)].$$

By our hypotheses and noticing that z is an eigenfunction we get

$$\begin{aligned}0 &\geq \int_{\partial M} [n \inf H_f h^2 - 2(\lambda_1 + c/2)zh + \inf A \lambda_1 z^2] \\ &= \int_{\partial M} \left[n \inf H_f \left(h - \frac{(\lambda_1 + c/2)z}{n \inf H_f} \right)^2 + \left(\lambda_1 \inf A - \frac{(\lambda_1 + c/2)^2}{n \inf H_f} \right) z^2 \right] \\ &\geq \left(\lambda_1 \inf A - \frac{(\lambda_1 + c/2)^2}{n \inf H_f} \right) \int_{\partial M} z^2.\end{aligned}$$

Thus, after a carefull analyzes, we obtain

$$\lambda_1 \geq \frac{1}{2} \left(n \inf H_f \inf A - c + \sqrt{(n \inf H_f \inf A)^2 - 2cn \inf H_f \inf A} \right),$$

which proves the first part of theorem. Assuming that the equality holds, we deduce that $A = (\inf A)I$, $\operatorname{Hess} u = 0$, $\operatorname{Ric}_f(\nabla u, \nabla u) = -c|\nabla u|^2$, $H_f = \inf H_f$ and $h = \frac{\lambda_1 + c/2}{nH_f} z$. A direct computation show us that $df(\nabla u) = 0$, $\operatorname{Ric}(\nabla u, \nabla u) = -c|\nabla u|^2$ and $\frac{\partial f}{\partial \nu} = cte$. The reciprocal is a straightforward computation. □

Recall the following version of the Hopf boundary point lemma; see its proof in [8], Lemma 3.4.

Proposition 2.4 (Hopf boundary point lemma) *Let (M^n, g) be a complete Riemannian manifold and let $\Omega \subset M$ be a closed domain. If $u : \Omega \to \mathbb{R}$ is a function with $u \in C^2(int(\Omega))$ satisfying*

$$\Delta u + \langle X, \nabla u \rangle \geq 0,$$

where X is a bounded vector field, $x_0 \in \partial\Omega$ is a point where

$$u(x) < u(x_0) \ \ \forall x \in \Omega,$$

u is continuous at x_0, and Ω satisfies the interior sphere condition at x_0, then

$$\frac{\partial u}{\partial \nu}(x_0) > 0,$$

if this outward normal derivative exists.

3 Proof of the Results and Applications

In this section, we will provide the proofs of our results.

Proof of Theorem 1.1 Let u be the solution of the following problem

$$\begin{cases} \Delta_f u = 0 & \text{in } M, \\ u = z & \text{on } \partial M, \end{cases}$$

where z satisfies $\overline{\Delta}_f z + \lambda_1 z = 0$ on ∂M. Set $h = \frac{\partial u}{\partial \nu}\big|_{\partial M}$, then we have from the Rayleigh-Ritz quotient Eq. 1.4 that

$$p_1 \leq \frac{\int_M |\nabla u|^2}{\int_{\partial M} z^2}, \tag{3.1}$$

and from the previous one we obtain

$$p_1 \leq \frac{\int_{\partial M} h^2}{\int_M |\nabla u|^2}. \tag{3.2}$$

Indeed, Eq. 3.1 follows from the variational principle, because $\int_{\partial M} u = 0$. Furthermore, Eq. 3.2 is obtained using integration by part and Cauchy-Schwartz inequality as follows:

$$p_1 \leq \frac{\int_M |\nabla u|^2}{\int_{\partial M} z^2} = \frac{(\int_{\partial M} u\langle \nabla u, \nu\rangle)^2}{\int_{\partial M} z^2 \int_M |\nabla u|^2} \leq \frac{\int_{\partial M} z^2}{\int_{\partial M} z^2} \cdot \frac{\int_{\partial M} \langle \nabla u, \nu\rangle^2}{\int_M |\nabla u|^2} = \frac{\int_{\partial M} h^2}{\int_M |\nabla u|^2}.$$

Plugging u into the equation Eq. 2.1 and using our hypotheses we get

$$0 \geq \int_{\partial M} [nH_f h^2 + 2h\overline{\Delta}_f z + A(\overline{\nabla} z, \overline{\nabla} z)] \geq \int_{\partial M} [(k-1)ch^2 - 2\lambda_1 hz + c|\overline{\nabla} z|^2].$$

Noticing that z is an eigenfuction

$$\begin{aligned} 0 &\geq (k-1)c \int_{\partial M} h^2 - 2\lambda_1 \int_{\partial M} hz + c\lambda_1 \int_{\partial M} z^2 \\ &\geq (k-1)c \int_{\partial M} h^2 - 2\lambda_1 \left(\int_{\partial M} h^2\right)^{\frac{1}{2}} \left(\int_{\partial M} z^2\right)^{\frac{1}{2}} + c\lambda_1 \int_{\partial M} z^2 \\ &= \frac{(k-1)c^2 - \lambda_1}{c} \int_{\partial M} h^2 + \left[\sqrt{\frac{\lambda_1}{c}} \left(\int_{\partial M} h^2\right)^{\frac{1}{2}} - \sqrt{c\lambda_1} \left(\int_{\partial M} z^2\right)^{\frac{1}{2}}\right]^2, \end{aligned}$$

and so

$$\frac{\sqrt{\lambda_1 - (k-1)c^2}}{\sqrt{c}} \left(\int_{\partial M} h^2\right)^{\frac{1}{2}} \geq \sqrt{\frac{\lambda_1}{c}} \left(\int_{\partial M} h^2\right)^{\frac{1}{2}} - \sqrt{c\lambda_1} \left(\int_{\partial M} z^2\right)^{\frac{1}{2}}.$$

Thus,

$$\frac{\sqrt{\lambda_1} - \sqrt{\lambda_1 - (k-1)c^2}}{\sqrt{c}} \left(\int_{\partial M} h^2\right)^{\frac{1}{2}} \leq \sqrt{c\lambda_1} \left(\int_{\partial M} z^2\right)^{\frac{1}{2}},$$

that is,

$$\begin{aligned} \left(\int_{\partial M} h^2\right)^{\frac{1}{2}} &\leq \frac{c\sqrt{\lambda_1}}{\sqrt{\lambda_1} - \sqrt{\lambda_1 - (k-1)c^2}} \left(\int_{\partial M} z^2\right)^{\frac{1}{2}} \\ &= \frac{\sqrt{\lambda_1}}{(k-1)c} (\sqrt{\lambda_1} + \sqrt{\lambda_1 - (k-1)c^2}) \left(\int_{\partial M} z^2\right)^{\frac{1}{2}} \end{aligned}$$

and from Eqs. 3.1 and 3.2 we get the upper bound. Let's assume the equality holds. So

$$\left(\int_{\partial M} h^2\right)^{\frac{1}{2}} = \frac{\sqrt{\lambda_1}}{(k-1)c} \left(\sqrt{\lambda_1} + \sqrt{\lambda_1 - (k-1)c^2}\right) \left(\int_{\partial M} z^2\right)^{\frac{1}{2}}$$

and all inequalities become equalities. Thus $h = \alpha z$ and

$$\alpha = \frac{(\alpha^2 \int_{\partial M} z^2)^{\frac{1}{2}}}{(\int_{\partial M} z^2)^{\frac{1}{2}}} = \frac{\sqrt{\lambda_1}}{(k-1)c} \left(\sqrt{\lambda_1} + \sqrt{\lambda_1 - (k-1)c^2}\right),$$

that is,

$$h = \frac{\sqrt{\lambda_1}}{(k-1)c}\left(\sqrt{\lambda_1} + \sqrt{\lambda_1 - (k-1)c^2}\right)z.$$

Furthermore we infer, by Proposition 2.1, that $\operatorname{Hess} u = 0$. On the other hand, on the boundary ∂M we can write

$$\nabla u = (\nabla u)^{\top} + h\nu,$$

where $(\nabla u)^{\top}$ is tangent to ∂M. Taking a local orthonormal frame $\{e_i\}_{i=1}^{n}$ tangent to ∂M we do the following computations:

$$\begin{aligned}
0 &= \sum_{i=1}^{n} \operatorname{Hess} u(e_i, e_i) = \sum_{i=1}^{n} (\overline{\operatorname{Hess}}\, z + hA)(e_i, e_i) \\
&= \overline{\Delta} z + nHh \\
&= \overline{\Delta}_f z - \frac{\partial f}{\partial \nu} h + nHh \\
&= \overline{\Delta}_f z + nH_f h \\
&= -\lambda_1 z + c(k-1)\frac{\sqrt{\lambda_1}}{(k-1)c}\left(\sqrt{\lambda_1} + \sqrt{\lambda_1 - (k-1)c^2}\right)z,
\end{aligned}$$

and so

$$\lambda_1 = (k-1)c^2.$$

Therefore, it follows from Proposition 2.2 that M is isometric to an $(n+1)$-dimensional Euclidean ball of radius $\frac{1}{c}$, f is constant, and so $k = n+1$. The converse is direct. □

Proof of Theorem 1.2. Mimic the computations in the previous proof and use Proposition 2.3 with $c = 0$ to obtain the desired result. □

Example 1 *Consider* $M = B(0, R)$ *in* $\mathbb{R}^{n+1}$ *with the standard metric and the measure* $e^{-|x|^2/2}dx$. *In such space,* $\operatorname{Ric}_f = \langle\, , \,\rangle$ *and whether* $R < 1$ *we obtain:*

$$\lambda_1(\mathbb{S}^n_R) \geq \frac{1-R^2}{R^2},$$

and

$$p_1 \leq \frac{R}{1-R^2}\lambda_1\left(\sqrt{\lambda_1} + \sqrt{\lambda_1 - \frac{1-R^2}{R^2}}\right).$$

We notice that such a constant can be improved using a constant $c \neq 0$ *in Proposition 2.3 and Theorem 1.2.*

Proof of Theorem 1.3. Let w be an eigenfunction corresponding to the first eigenvalue q_1 of problem Eq. 1.2, that is,

$$\begin{cases} \Delta_f^2 w = 0 & \text{in } M, \\ w = \Delta_f w - q_1 \frac{\partial w}{\partial \nu} = 0 & \text{on } \partial M. \end{cases}$$

Set $h = \frac{\partial w}{\partial \nu}|_{\partial M}$ and using integration by parts twice we get

$$\begin{aligned}\int_M (\Delta_f w)^2 &= -\int_M \langle \nabla(\Delta_f w), \nabla w\rangle + \int_{\partial M} \Delta_f w \,\langle \nabla w, \nu\rangle \\ &= \int_M w\,\Delta_f(\Delta_f w) - \int_{\partial M} w\,\langle \nabla(\Delta_f w), \nu\rangle + \int_{\partial M} \Delta_f w\,\langle \nabla w, \nu\rangle \\ &= q_1 \int_{\partial M} h^2.\end{aligned}$$

Plugging w into Eq. 2.1 and using our hypotheses

$$\begin{aligned}\frac{k-1}{k}\int_M (\Delta_f w)^2 &\geq \int_M \mathrm{Ric}_f^k(\nabla w, \nabla w) + \int_{\partial M} nH_f h^2 \\ &\geq \frac{(k-1)nc}{k}\int_{\partial M} h^2,\end{aligned}$$

and so $q_1 \geq nc$ as we desired.

Assuming $q_1 = nc$ all inequalities become equalities and consequently $H_f = \frac{k-1}{k}c$. Furthermore, by Proposition 2.1, we infer Hess $w = \frac{\Delta w}{n+1}\langle\, ,\rangle$ and $\Delta_f w = \frac{k}{n+1}\Delta w$.

Taking an orthonormal frame $\{e_1, \ldots, e_n\}$ on ∂M, since $w|_{\partial M} = 0$ we get

$$e_i(h) = \langle \nabla_{e_i}\nabla w, \nu\rangle + \langle \nabla w, \nabla_{e_i}\nu\rangle \qquad = \mathrm{Hess}\, w(e_i, \nu) + A((\nabla w)^\top, e_i) = 0,$$

that is, h is constant on the boundary, and so $(\Delta_f w)|_{\partial M} = nch$ is also constant. Using the fact that $\Delta_f w$ is f-harmonic function on M, we conclude by maximum principle that $\Delta_f w$ is constant on M. Since $\Delta_f w = \frac{k}{n+1}\Delta w$, then w satisfies

$$\begin{cases}\mathrm{Hess}\, w = \frac{\Delta_f w}{k}\langle\, ,\rangle & \text{in} \quad M,\\ w = 0 \quad \text{on} \quad \partial M.\end{cases}$$

Thus, from Lema 3 in [17] we conclude that M is isometric to a ball in $\mathbb{R}^{n+1}$ of radius c^{-1}. Hence, using that hessian of w is a multiple of the metric we deduce that $w = \frac{\lambda}{2}r^2 + C$, where $\lambda = \frac{\Delta_f w}{k}$ and r is the distance function from one minimal point q_0, see [17] where we borrowed this idea.

Finally we prove that f is constant and $k = n+1$. Indeed, if $k > n+1$ and f is not constant we know that $\langle \nabla f, \nabla w\rangle$ is constant and integrating along the geodesics starting from q_0 we get $f = -(k-n-1)\ln r + C$, however it is a contradiction because f is a smooth function. □

Example 2 *Set $M = \mathbb{R}^{n+1}$ endowed with the standard metric and the density $f(x) = \frac{1}{2}|x|^2$, after a straightforward computation we deduce that for any $k > n+1$, the balls centered at 0 with radius $R \leq \min\{\sqrt{k-n-1}, \sqrt{\frac{n}{k}}\}$ satisfies the hypothesis of Theorem 1.3 and so $q_1 > nR^{-1}$.*

***Proof of Theorem* 1.4.** Let w be the solution of the following Drift Laplace equation

$$\begin{cases}\Delta_f w = 1 \quad \text{in } M,\\ w = 0 \quad \text{on } \partial M.\end{cases}$$

Follows from Rayleigh-Ritz characterization of q_1, see Eq. 1.5, that

$$q_1 \leq \frac{\int_M (\Delta_f w)^2}{\int_{\partial M} h^2} = \frac{V}{\int_{\partial M} h^2},$$

where $h = \frac{\partial w}{\partial \nu}$. Integrating $\Delta_f w = 1$ on M and using integration by parts we get

$$V = \int_{\partial M} h.$$

Hence, we infer from Cauchy-Schwarz inequality that

$$V^2 \leq A \int_{\partial M} h^2. \tag{3.3}$$

Consequently,

$$q_1 \leq \frac{V}{\int_{\partial M} h^2} \leq \frac{V}{V^2/A} = \frac{A}{V}.$$

Assuming $\mathrm{Ric}_f^k \geq 0$, $H_f(x_0) \geq \frac{(k-1)A}{knV}$ for some $x_0 \in \partial M$ and $q_1 = \frac{A}{V}$ we conclude that Eq. 3.3 become an equality and so $h = \frac{V}{A}$ is constant. Consider the function ϕ on M given by

$$\phi = \frac{1}{2}|\nabla w|^2 - \frac{w}{k}.$$

From Proposition 2.1, the Bochner formula and our hypotheses we get

$$\begin{aligned} \frac{1}{2}\Delta_f \phi &= |\mathrm{Hess}\, w|^2 + \langle \nabla w, \nabla(\Delta_f w)\rangle + \mathrm{Ric}_f(\nabla w, \nabla w) - \frac{1}{k} \\ &\geq \frac{1}{k}(\Delta_f w)^2 - \frac{1}{k} = 0. \end{aligned} \tag{3.4}$$

Thus ϕ is f-subharmonic. We claim that $\phi = \frac{1}{2}\left(\frac{V}{A}\right)^2$ on the boundary. Indeed, as w vanishes on the boundary we can write $\nabla w = h\nu$ on ∂M. On the other hand,

$$1 = \Delta_f w = q_1 h = \frac{A}{V} h \;\text{ which implies }\; h = \frac{V}{A}$$

and we get the claim. From the f-subharmonicity of ϕ and its constancy on the boundary we conclude from Proposition 2.4 that

$$\text{either } \phi = \frac{1}{2}\left(\frac{V}{A}\right)^2 \quad \text{in } M$$

or

$$\frac{\partial \phi}{\partial \nu} > 0, \quad on\ \partial M. \tag{3.5}$$

As w vanishes on ∂M we obtain

$$\begin{aligned} 1 = (\Delta_f w)|_{\partial M} &= nHh + \mathrm{Hess}\, w(\nu,\nu) - \frac{V}{A}\frac{\partial f}{\partial \nu} \\ &= \frac{nV}{A}\left(H_f + \frac{1}{n}\frac{\partial f}{\partial \nu}\right) + \mathrm{Hess}\, w(\nu,\nu) - \frac{V}{A}\frac{\partial f}{\partial \nu} \\ &= \frac{nV}{A} H_f + \mathrm{Hess}\, w(\nu,\nu). \end{aligned}$$

Hence

$$\begin{aligned}\frac{\partial \phi}{\partial \nu} &= \frac{V}{A}\text{Hess}\, w(\nu,\nu) - \frac{V}{k\,A}\\ &= \frac{V}{A}\left(1 - \frac{nV}{A}H_f\right) - \frac{V}{k\,A}\\ &= n\frac{V}{A}\left(\frac{k-1}{k\,n} - H_f\frac{V}{A}\right),\end{aligned}$$

which shows that Eq. 3.5 is not true since $H_f(x_0) \geq \frac{(k-1)A}{k\,n\,V}$ and therefore ϕ is constant on M. Since the Drift Laplacian of ϕ vanishes, we infer that equality must hold in Eq. 3.4 and that this gives us equality in the Proposition 2.1, and consequently $1 = \Delta_f w = \frac{k}{n+1}\Delta w$ and $\text{Hess}\, w = \frac{\Delta w}{n+1}\langle\, ,\, \rangle$. Hereafter, we follow the last two paragraphs in the proof of Theorem 1.3. □

Example 3 *Set $M = \mathbb{R}^{n+1}$ endowed with the standard metric and the density $f(x) = \dfrac{1}{2}|x|^2$, after a straightforward computation we deduce that for any $k > n+1$, the balls centered at* 0 *with radius $R \leq \min\{\sqrt{k-n-1}, \sqrt{\frac{n}{k}}\}$ satisfies the hypothesis of Theorem 1.3 and using Theorem 1.4 we obtain*

$$\frac{n}{R} < q_1 < \frac{e^{-R^2/2}R^n}{\int_0^R r^n e^{-r^2/2}dr}.$$

Proof of Theorem 1.5. Let w be an eigenfunction corresponding to the first eigenvalue q_1 of the problem Eq. 1.3:

$$\begin{cases}\Delta_f^2 u = 0 & \text{in } M,\\ u = \frac{\partial^2 u}{\partial \nu^2} - q\frac{\partial u}{\partial \nu} = 0 & \text{on } \partial M.\end{cases}$$

Observe that w is not constant. Otherwise, we would get from $w|_{\partial M} = 0$ that $w \equiv 0$. Set $h = \frac{\partial w}{\partial \nu}|_{\partial M}$ we claim that $h \neq 0$. Indeed, assuming $h = 0$ we get w, ∇w and $\frac{\partial^2 w}{\partial \nu^2}$ vanish on ∂M. Using an adapted frame on the boundary we conclude that $(\Delta_f w)|_{\partial M} = 0$ and so $\Delta_f w = 0$ on M by the maximum principle, which in turn implies that $w = 0$ where such is not possible.

Using integration by part we have

$$\int_M \langle \nabla w, \nabla(\Delta_f w)\rangle = -\int_M w\Delta_f^2 w = 0,$$

and

$$\int_{\partial M} h\,\Delta_f w = \int_M \langle \nabla(\Delta_f w), \nabla w\rangle + \int_M (\Delta_f w)^2 = \int_M (\Delta_f w)^2. \tag{3.6}$$

Using an adapted orthonormal frame on ∂M we can write $\nabla w = h\nu$ and

$$\begin{aligned}(\Delta_f w)|_{\partial M} &= \frac{\partial^2 w}{\partial \nu^2} + nH\frac{\partial w}{\partial \nu} - \langle \nabla f, \nabla w\rangle\\ &= q_1 h + nH_f h + \frac{\partial f}{\partial \nu}h - \frac{\partial f}{\partial \nu}h\\ &= q_1 h + nH_f h.\end{aligned} \tag{3.7}$$

From Eqs. 3.6 and 3.7 we obtain

$$q_1 = \frac{\int_M (\Delta_f w)^2 - n \int_{\partial M} H_f h^2}{\int_{\partial M} h^2}.$$

On the other hand, substituting w into Eq. 2.1 we get

$$\frac{k-1}{k} \int_M (\Delta_f w)^2 = \int_M \mathrm{Ric}_f^k(\nabla w, \nabla w) + \int_{\partial M} n H_f h^2 \geq \int_{\partial M} n H_f h^2,$$

that is,

$$\int_M (\Delta_f w)^2 - \int_{\partial M} n H_f h^2 \geq \frac{n}{k-1} \int_{\partial M} H_f h^2 \geq c \int_{\partial M} h^2, \tag{3.8}$$

and thus we conclude the desired estimate $q_1 \geq c$.

Assuming $q_1 = c$ all inequalities in Eq. 3.8 become equalities. Thus, from Proposition 2.1 we have

$$\mathrm{Hess}\, w = \frac{\Delta w}{n+1} \langle\, ,\, \rangle \qquad \text{and} \qquad \Delta_f w = -\frac{k}{k-n-1} \langle \nabla f, \nabla w \rangle. \tag{3.9}$$

Picking up an adapted orthonormal frame $\{e_1, \ldots, e_n = \nu\}$ on ∂M and using that w vanishes on ∂M we get for $i = 1, \ldots, n-1$,

$$0 = \mathrm{Hess}\, w(e_i, e_n) = e_i e_n(w) - \nabla_{e_i} e_n(w) \qquad = e_i(h) - \langle \nabla_{e_i} e_n, e_n \rangle h = e_i(h),$$

and so h is constant. From Eq. 3.8 $H_f = \frac{k-1}{n} c$ and as h is constant, we conclude from Eq. 3.7 that $(\Delta_f w)|_{\partial M}$ is constant, and hence $\Delta_f w$ is constant on M by the maximum principle, which implies from Eq. 3.9 that Δw is constant on M. Henceforth, we simply follow the same arguments in the last two paragraphs in the proof of Theorem 1.3. □

Acknowledgements The authors would like to thank the referee for the valuable comments and suggestions, which helped us to improve and clarify the manuscript

Funding The first author was partially supported by Alagoas Research Foundation [Grant: E:60030.0000001758/2022], and by the Brazilian National Council for Scientific and Technological Development, Brazil [Grant: 308440/2021-8 and 405468/2021-0], and both authors were partially supported by Coordination for the Improvement of Higher Education Personnel [Finance code - 001]

Data Availability This manuscript has no associated data.

References

1. Bakry, D., Émery M.: Diffusions hypercontractives. Séminaire de probabilities, XIX, 1983/84, Lecture Notes in Math, **1123**, Springer, Berlin, pp 177–206 (1985)
2. Batista, M., Cavalcante, M.P., Pyo, J.: Some Isoperimetric Inequalities and Eigenvalue Estimates in Weighted Manifolds. J. Math. Anal. Appl. **419**, 617–626 (2014)
3. Binoy, R., Santhanam, G.: Sharp upperbound and a comparison theorem for the first nonzero Steklov eigenvalue. J. Ramanujan Math. Soc. **29**(2), 133–154 (2014)
4. Escobar, J.F.: The geometry of the first non-zero Stekloff eigenvalue. J. Funct. Anal. **150**(2), 544–556 (1997)
5. Escobar, J.F.: An isoperimetric inequality and the first Steklov eigenvalue. J. Funct. Anal. **165**101–116 (1999)
6. Escobar, J.F.: A comparison theorem for the first non-zero Steklov eigenvalue. J. Funct. Anal. **178**(1), 143–155 (2000)
7. Girouard, A., Polterovich, I.: Spectral Geometry of the Steklov problem. J. Spectral Theory **7**321–359 (2017)

8. Gilbarg, D., Trudinger N.S.: Elliptic partial differential equations of second order. Reprint of the 1998 ed. Cassics in Mathematics. Springer-Verlag, Berlin, (2001)
9. Gromov, M.: Isoperimetric of waists and concentration of maps. Geom. Funct. Anal. **13**(1), 178–215 (2003)
10. Huang, Q., Ruan, Q.: Application of Some Elliptic Equations in Riemannian Manifolds. J. Math. Anal. Appl. **409**(1), 189–196 (2014)
11. Kuttler, J.R., Sigillito, V.G.: Inequalities for membrane and Stekloff eigenvalues. J. Math. Anal. Appl. **23**, 148–160 (1968)
12. Lichnerowich, A.: Variétés riemanniennes á tenseur C non négatif. C. R. Acad. Sci. Paris Sér. A-B **271**, A650–A653 (1970)
13. Ma, L., Du, S.H.: Extension of Reilly formula with applications to eigenvalue estimates for drifting Laplacians. C. R. Math. Acad. Sci. Paris **348**(21–22), 1203–1206 (2010)
14. Payne, L.E.: Some isoperimetric inequalities for harmonic functions. SIAM J. Math. Anal. **1**, 354–359 (1970)
15. Perelman, G.: The entropy formula for the Ricci flow and its geometric applications. Preprint available at arXiv:math.DG0211159
16. Raulot, S., Savo, A.: On the first eigenvalue of the Dirichlet-to-Neumann operator on forms. J. Funct. Anal. **262**(3), 889–914 (2012)
17. Reilly, R.: Geometric applications of the solvability of Neumann problems on a Riemannian manifold. Arch. Rational Mech. Anal. **75**(1), 23–29 (1980)
18. Stekloff, M.W.: Sur les problèmes fondamentaux de la physique mathématique. Ann. Sci. École Norm. Sup. **19**, 455–490 (1902)
19. Weinstock, R.: Inequalities for a classical eigenvalue problem. J. Rational Mech. Anal. **3**, 745–753 (1954)
20. Xia, C., Wang, Q.: Sharp bounds for the first non-zero Stekloff eigenvalues. J. Funct. Anal. **257**(8), 2635–2644 (2009)