Potential Analysis (2024) 60:425–443
https://doi.org/10.1007/s11118-022-10055-4

Strong Maximum Principle and Boundary Estimates for Nonhomogeneous Elliptic Equations

Niklas L.P. Lundström[1] · **Marcus Olofsson**[1] · **Olli Toivanen**[2]

Received: 2 November 2021 / Accepted: 19 November 2022 / Published online: 27 December 2022

Abstract

We give a simple proof of the strong maximum principle for viscosity subsolutions of fully nonlinear nonhomogeneous degenerate elliptic equations on the form

$$F(x, u, Du, D^2u) = 0$$

under suitable assumptions allowing for non-Lipschitz growth in the gradient term. In case of smooth boundaries, we also prove a Hopf lemma, a boundary Harnack inequality, and that positive viscosity solutions vanishing on a portion of the boundary are comparable with the distance function near the boundary. Our results apply, e.g., to weak solutions of an eigenvalue problem for the variable exponent p-Laplacian.

Keywords Osgood condition · Non-Lipschitz drift · Non-standard growth · Variable exponent · Fully nonlinear · Sphere condition · Laplace equation · Hopf Lemma · Boundary Harnack inequality

Mathematics Subject Classification (2010) Primary 35B40 · 35B50 · 35J25 · 35J92 · Secondary 35D40 · 35J60 · 35J70

1 Introduction

We consider fully nonlinear nonhomogeneous degenerate elliptic equations in nondivergence form,

$$F(x, u, Du, D^2u) = 0, \tag{PDE}$$

✉ Niklas L.P. Lundström
niklas.lundstrom@umu.se

Marcus Olofsson
marcus.olofsson@umu.se

Olli Toivanen
olli@ollitoivanen.info

[1] Department of Mathematics and Mathematical Statistics, Umeå University, SE-90187, Umeå, Sweden

[2] Department of Applied Physics, University of Eastern Finland, FI-70210, Kuopio, Finland

where Du is the gradient, D^2u the Hessian, $F : \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^n \times \mathbb{S}^n \to \mathbb{R}$, and $\mathbb{S}^n$ is the set of symmetric $n \times n$ matrices equipped with the positive semi-definite ordering, i.e., for $X, Y \in \mathbb{S}^n$, we write $X \leq Y$ if $\langle (X - Y)\xi, \xi \rangle \leq 0$ for all $\xi \in \mathbb{R}^n$.

We assume that F is *proper*, i.e.,

$$F(x, u, p, X) \geq F(x, v, p, Y) \quad \text{whenever} \quad u \geq v \quad \text{and} \quad X \leq Y, \tag{F_1}$$

and that one of the following structural assumptions hold, depending on the result. First, in order to prove the Strong Maximum Principle (SMaP), a Hopf lemma and some boundary growth estimates we assume that

$$-F(x, 0, p, X) \leq \phi(|p|) - \mathcal{P}^-_{\lambda,\Lambda}(X) \quad \text{whenever} \quad x, p \in \mathbb{R}^n, X \in \mathbb{S}^n, \tag{F_2A}$$

where $\mathcal{P}^-_{\lambda,\Lambda}(X) = -\Lambda \mathrm{Tr}(X^+) + \lambda \mathrm{Tr}(X^-)$ is the Pucci minimal operator for $0 < \lambda \leq \Lambda$ (see Section 2 for details) and $\phi : [0, \infty) \to [0, \infty)$ is a strictly increasing continuous function, $\phi(t) \geq t$, satisfying the Osgood condition

$$\int_0^1 \frac{dt}{\phi(t)} = \infty. \tag{ϕ_A}$$

Observe that when $\phi(t) > t$, F is not necessarily Lipschitz continuous in the drift term and the equation can be nonhomogeneous; a prototype example is the variable exponent p-Laplace equation

$$-\nabla \cdot \left(|\nabla u|^{p(x)-2} \nabla u \right) = 0,$$

where $1 < p^- \leq p(x) \leq p^+ < \infty$ is Lipschitz continuous, satisfying our assumptions with $\phi(t) = C\,(|\log t| + 1)\,t$, $\lambda = \min\{1, p^- - 1\}$, and $\Lambda = \max\{1, p^+ - 1\}$ (see Section 6). A counterexample by Julin [25] shows that (ϕ_A) is necessary for the SMaP, the weak maximum principle, and for the comparison principle and uniqueness (see Remark 4.3).

Assumptions (F_1) and (F_2A) are implied by, e.g., the standard ellipticity assumption

$$\lambda \mathrm{Tr}(Y) \leq F(x, u, p, X) - F(x, u, p, X + Y) \leq \Lambda \mathrm{Tr}(Y) \tag{1.1}$$

for some $\lambda, \Lambda > 0$ whenever Y is positive semi-definite, together with monotonicity in u and

$$-F(x, 0, p, 0) \leq \phi(|p|) \quad \text{whenever} \quad x, p \in \mathbb{R}^n. \tag{1.2}$$

Note that assumptions (F_1) and (F_2A) allow nonlinear operators which do not satisfy (1.1), e.g., operators of the form

$$F(X) = -\Lambda \left(\sum_{i=1}^n \Phi(\mu_i^+) \right) + \lambda \left(\sum_{i=1}^n \Psi(\mu_i^-) \right)$$

where $\mu_i^\pm$, $i = 1, \ldots, n$, are the postitive and negative eigenvalues of the matrix $X \in \mathbb{S}^n$, respectively, and $\Phi, \Psi : [0, \infty) \to [0, \infty)$ are continuous and nondecreasing functions such that $\Phi(s) \leq s \leq \Psi(s)$, see Capuzzo-Dolcetta–Vitolo [17].

Assuming (F_1), (F_2A) and (ϕ_A), we prove that if a viscosity subsolution of (PDE) attains a positive maximum in the interior of Ω, then it must be constant (Theorem 4.1). Moreover, if Ω satisfies an interior sphere condition and u is a nonconstant viscosity subsolution to (PDE), then, for any $\omega \in \partial\Omega$ s.t. $u(w) \geq \limsup_{\Omega \ni x \to w} u(x)$ and $u(w) > 0$ we prove that

$$\limsup_{s \to 0} \frac{u(w + s\gamma) - u(w)}{s} < 0$$

for any $\gamma \in \mathbb{R}^n$ pointing strictly into the domain (Theorem 4.2).

The Strong Minimum Principle (SMiP) for viscosity supersolutions (see Corollary 4.4), as well as some of the boundary estimates introduced below require, in place of (F_2A), one of the assumptions

$$F(x, 0, p, X) \leq \phi(|p|) + \mathcal{P}^+_{\lambda,\Lambda}(X) \quad \text{whenever} \quad x, p \in \mathbb{R}^n, X \in \mathbb{S}^n, \qquad (F_2B)$$

$$F(x, r, p, X) \leq \phi(|p|) + \mathcal{P}^+_{\lambda,\Lambda}(X) + \gamma(r) \quad \text{when} \quad r \in [0, \infty), x, p \in \mathbb{R}^n, X \in \mathbb{S}^n. \quad (F_2B^*)$$

Here, $\mathcal{P}^+_{\lambda,\Lambda}(X) = -\lambda \mathrm{Tr}(X^+) + \Lambda \mathrm{Tr}(X^-)$ is the Pucci maximal operator, ϕ is assumed to satisfy (ϕ_A), and γ is a function dominated by ϕ in the sense that $\gamma(r) \leq C^*\phi(r)$ whenever $0 \leq r \leq 1$.

Strong Maximum Principles for solutions of partial differential equations (PDE)s have attracted lots of attention during the last decades. For linear equations the SMaP dates back to Hopf (see Gilbarg–Trudinger [22] for a proof) and Calabi [16] considered generalized solutions of linear equations in nondivergence form. For nonlinear problems, mainly modeled by the p-Laplacian on divergence form, see Pucci–Serrin [39]. Kawohl–Kutev [27] and Bardi–Da Lio [8–10] considered fully nonlinear degenerate elliptic PDEs and Bardi–Goffi [7] nonlinear subelliptic PDEs modeled on Hörmander vector fields. For nonhomogeneous PDEs of variable exponent p-Laplace type, a proof was given by Fan–Zhao–Zhang [20] while Zhang [43] considered a larger class of equations and proved a Hopf lemma for weak C^1-supersolutions. See also Wolanski [41, Theorem 4.1] for a proof of the SMaP for C^1-subsolutions. Capuzzo-Dolcetta–Vitolo [17] investigated the validity of the Alexandrov–Bakelman–Pucci maximum principle and of the Phragmen–Lindelöf principle for fully nonlinear PDEs. Birindelli–Demengel [12] defined the concept of principal eigenvalue for fully nonlinear second order operators that are elliptic, homogenous and with lower order terms. Mikayelyan–Shahgholian [38] proved a Hopf lemma for singular/degenerate PDEs including the p-Laplacian in certain $C^{1,Dini}$-type domains. Sirakov [40] proved the weak Harnack inequality and the SMaP for elliptic equations and Birindelli–Galise–Ishii [13] investigated maximum principles for degenerate elliptic operators whose higher order term is the sum of k eigenvalues of the Hessian. Julin [25] presented a sharp version of the Harnack inequality for operators similar to those considered in this paper but without the dependence on u. In the present paper, we partly generalize some of the above cited works by allowing for general dependence of $|Du|$ in the first order terms.

The proof of the SMaP and the Hopf lemma builds on comparison with certain classical supersolutions of (PDE) constructed with inspiration from Avelin–Julin [5] in Section 3. Using these solutions in a barrier argument inspired by Aikawa–Kilpeläinen–Shanmugalingam–Zhong [3], we also prove that for domains with $C^{1,1}$-boundary, positive viscosity solutions vanishing on a portion of the boundary are comparable with the distance function near the boundary (Corollary 5.5), i.e.,

$$c^{-1}\, d(x, \partial\Omega) \leq u(x) \leq c\, d(x, \partial\Omega) \quad \text{whenever} \quad x \in \Omega \cap B(w, r). \qquad (1.3)$$

As a direct consequence, we obtain a boundary Harnack inequality (Corollary 5.6) for positive viscosity solutions of (PDE).

Our boundary estimate (1.3) consists of a lower estimate (Theorem 5.1) and an upper estimate (Theorem 5.2) generalizing the main results of Adamowicz–Lundström [1] to viscosity solutions of more general PDEs. To prove the lower estimate we need the stronger structural assumption in (F_2B^*) in order to build a positive barrier function consisting of a classical subsolution (Lemma 3.2). To prove the upper estimate we need to deal with large gradients, forcing us to assume that

$$\int_0^1 f(t)dt \to \infty \quad \text{as} \quad \nu \to \infty \tag{ϕ_B}$$

where $f(t)$ solves the differential equation $f'(t) = -\phi(f(t))$ with initial condition $f(0) = \nu$.

Condition (ϕ_B), whose necessity will be shown by a counterexample in Remark 5.3, holds, e.g., if there are s_0 and C such that

$$\phi(s) \leq Cs^2, \quad \text{whenever} \quad s \geq s_0, \tag{1.4}$$

or, if the Keller-Osserman type condition

$$\int_1^\infty \frac{dt}{\phi(t)} = \infty \tag{1.5}$$

holds (see Remark 5.4).

The studies of boundary Harnack type inequalities for solutions of differential equations have a long history. In the setting of harmonic functions on Lipschitz domains, such a result was proposed by Kemper [28] and later studied by Ancona [4], Dahlberg [18] and Wu [42]. Subsequently, Kemper's result was extended by Caffarelli–Fabes–Mortola–Salsa [15] to a class of elliptic equations, and by Jerison–Kenig [24] to the setting of nontangentially accessible domains. Banuelos–Bass–Burdzy [6] and Bass–Burdzy [11] considered Hölder domains while Aikawa [2] studied uniform domains. Recently, Silva–Savin [19] gave a short proof of a boundary Harnack inequality for solutions of uniformly elliptic equations in divergence and nondivergence form. Concerning nonlinear PDEs, Aikawa–Kilpeläinen–Shanmugalingam–Zhong [3] proved similar results in the setting of positive p-harmonic functions while in the same year Lewis–Nyström [29–31] started to develop a theory for proving boundary estimates, including the boundary Harnack inequality, for such functions allowing for more general geometries such as Lipschitz and Reifenberg-flat domains. Moreover, growth estimates near low-dimensional boundaries were considered in Lindqvist [33], Lundström [34, 35] and Lewis–Nyström [32]. For nonhomogeneous equations Adamowicz–Lundström [1] proved similar boundary estimates as we do in Section 5 but for positive $p(x)$-harmonic functions, and Avelin–Julin [5] proved a sharp boundary Harnack inequality for operators similar to those considered in this paper but without dependence on u. Concerning applications of boundary Harnack type inequalities we mention free boundary problems and studies of the Martin boundary, see, e.g., Lewis–Nyström [30, 31]. We end the paper by showing that our results hold for weak solutions of an eigenvalue problem for the variable exponent p-Laplacian, giving boundary estimates which relate to the asymptotic behavior of solutions near C^2-boundaries proven by Zhang in [44].

2 Preliminaries

By Ω we denote a domain, that is, an open connected set, and for a set $E \subset \mathbb{R}^n$ we let $\overline{E}$ denote the closure and ∂E the boundary of E. Furthermore, $d(x, E)$ denotes the Euclidean distance from $x \in \mathbb{R}^n$ to E, and $B(x, r) = \{y : |x - y| < r\}$ denotes the open ball with radius r and center x. By c we denote a constant ≥ 1, not necessarily the same at each occurrence. Moreover, $c(a_1, a_2, \ldots, a_k)$ denotes a constant ≥ 1, not necessarily the same at each occurrence, depending only on $a_1, a_2, \ldots, a_k$.

Let $X \in \mathbb{S}^n$ have eigenvalues $e_1, e_2, \ldots, e_n$. The Pucci extremal operators $\mathcal{P}^+_{\lambda,\Lambda}$ and $\mathcal{P}^-_{\lambda,\Lambda}$ with ellipticity constants $0 < \lambda \le \Lambda$ are defined by

$$\mathcal{P}^+_{\lambda,\Lambda}(X) := -\lambda \sum_{e_i \ge 0} e_i - \Lambda \sum_{e_i < 0} e_i \quad \text{and} \quad \mathcal{P}^-_{\lambda,\Lambda}(X) := -\Lambda \sum_{e_i \ge 0} e_i - \lambda \sum_{e_i < 0} e_i,$$

and with $\mathcal{A}_{\lambda,\Lambda} := \{A \in \mathbb{S}^n : \lambda I \le A \le \Lambda I\}$ we have

$$\begin{aligned}
\mathcal{P}^+_{\lambda,\Lambda}(X) &= \sup_{A \in \mathcal{A}_{\lambda,\Lambda}} -\mathrm{Tr}(AX) = -\lambda \mathrm{Tr}(X^+) + \Lambda \mathrm{Tr}(X^-), \\
\mathcal{P}^-_{\lambda,\Lambda}(X) &= \inf_{A \in \mathcal{A}_{\lambda,\Lambda}} -\mathrm{Tr}(AX) = -\Lambda \mathrm{Tr}(X^+) + \lambda \mathrm{Tr}(X^-),
\end{aligned}$$

where $X = X^+ - X^-$ with $X^+ \ge 0$, $X^- \ge 0$, and $X^+X^- = 0$. For properties of the Pucci operators see, e.g., Caffarelli–Cabre [14] or Capuzzo-Dolcetta–Vitolo [17].

We next recall the standard definition of viscosity solutions. An upper semicontinuous (USC) function $u : \Omega \to \mathbb{R}$ is a viscosity subsolution if for any $\varphi \in C^2(\Omega)$ and any $x_0 \in \Omega$ such that $u - \varphi$ has a local maximum at x_0 it holds that

$$F(x_0, u(x_0), D\varphi(x_0), D^2\varphi(x_0)) \le 0. \tag{2.1}$$

A lower semicontinuous (LSC) function $u : \Omega \to \mathbb{R}$ is a viscosity supersolution if for any $\varphi \in C^2(\Omega)$ and any $x_0 \in \Omega$ such that $u - \varphi$ has a local minimum at x_0 it holds that

$$F(x_0, u(x_0), D\varphi(x_0), D^2\varphi(x_0)) \ge 0. \tag{2.2}$$

A continuous function is a viscosity solution if it is both a viscosity sub- and viscosity supersolution. In the following we sometimes drop the word "viscosity" and simply write *subsolution, supersolution, and solution*. We say that a subsolution (supersolution) is strict in a domain Ω if the equality in (2.1) ((2.2)) is strict, and we say that a subsolution, supersolution, or solution is *classical* if it is twice differentiable in Ω.

Let u be a subsolution and v a supersolution of (PDE) and let a and b be constants. As (PDE) is not homogeneous, $a+bu$ and $a+bv$ are not necessarily sub- and supersolutions. However, (F_1) guarantees that $u - c$ is a subsolution and $v + c$ is a supersolution whenever $c \ge 0$ is a constant.

We will not discuss the validity of a general comparison principle for viscosity solutions of (PDE) as our proofs only require the comparison with *classical strict* sub- and supersolutions and this comparison is always possible. Indeed, let Ω be a bounded domain, u a viscosity subsolution, and v a classical strict supersolution in Ω. Assume that $u \le v$ on $\partial\Omega$ and that $u \ge v$ somewhere in Ω. By USC the function $u - v$ then attains a maximum ≥ 0 at some point $x_0 \in \Omega$. Since $v \in C^2(\Omega)$, $u - v$ has a maximum at x_0, and u is a viscosity subsolution it follows by definition that

$$F(x_0, u(x_0), Dv(x_0), D^2v(x_0)) \le 0. \tag{2.3}$$

But since v is a classical strict supersolution we have $F(x, v(x), Dv(x), D^2v(x)) > 0$ whenever $x \in \Omega$, and as $u(x_0) \ge v(x_0)$ it follows from (F_1) that $F(x_0, u(x_0), Dv(x_0), D^2v(x_0)) \ge F(x_0, v(x_0), Dv(x_0), D^2v(x_0)) > 0$. This contradicts (2.3) and hence we have proved the following simple lemma.

Lemma 2.1 *Assume (F_1) and let Ω be a bounded domain, $u \in USC(\overline{\Omega})$ a viscosity subsolution and $v \in LSC(\overline{\Omega})$ a viscosity supersolution to (PDE), satisfying $u \le v$ on $\partial\Omega$. If either u is a classical strict subsolution, or v is a classical strict supersolution, then $u < v$ in Ω.*

Remark 2.2 We remark that the choice of viscosity solutions is not necessary for our results; other definitions of "weak solutions" can be considered in our arguments as long as such subsolutions (supersolutions) are USC (LSC) and can be compared to classical strict supersolutions (subsolutions) in the sense of Lemma 2.1. For example, it is not necessary for our proofs that F is continuous w.r.t. x, even though this is often assumed in the viscosity solution framework.

When dealing with boundary estimates we make use of the following sphere conditions. A point $w \in \partial\Omega$, where $\Omega \subset \mathbb{R}^n$ is a domain, satisfies the *interior sphere condition* with radius $r_i > 0$ if there exists $\eta^i \in \Omega$ such that $B(\eta^i, r_i) \subset \Omega$ and $\partial B(\eta^i, r_i) \cap \partial\Omega = \{w\}$. Similarly, $w \in \partial\Omega$ satisfies the *exterior sphere condition* with radius $r_e > 0$ if there exists $\eta^e \in \mathbb{R}^n \setminus \Omega$ such that $B(\eta^e, r_e) \subset \mathbb{R}^n \setminus \Omega$ and $\partial B(\eta^e, r_e) \cap \partial\Omega = \{w\}$. A point $w \in \partial\Omega$ satisfies the *sphere condition* with radius r_b if it satisfies both the interior and exterior sphere condition with radius r_b. A domain $\Omega \subset \mathbb{R}^n$ is said to satisfy the (interior/exterior) *sphere condition* if the corresponding condition holds for all $w \in \partial\Omega$. It is well known that $\Omega \subset \mathbb{R}^n$ satisfies the sphere condition if and only if Ω is a $C^{1,1}$-domain, see Aikawa–Kilpeläinen–Shanmugalingam–Zhong [3, Lemma 2.2] for a proof.

3 Construction of Auxiliary Functions

This section is devoted to the construction of some classical strict sub- and supersolutions of (PDE) which will be used in our proofs of the maximum principles and the boundary estimates. The first lemma gives negative subsolutions and positive supersolutions with arbitrary small gradients in an annulus, needed in the proof of Hopf's lemma, the SMaP, and the SMiP. The arguments, which are inspired by Avelin–Julin [5], were recently further advanced by the first author in [36] for proving sharp Phragmen–Lindelöf principles for subsolutions of (PDE) in halfspaces of $\mathbb{R}^n$.

Lemma 3.1 *Assume that (F_1), (F_2B) and (ϕ_A) hold. Suppose that $r \le r^*$, $M > 0$, and $y \in \mathbb{R}^n$ are given and let $A := B(y, 2r) \setminus \overline{B(y, r)}$. Then there exist positive $m(\lambda, \Lambda, n, \phi, r^*) \le M$, $\mu(\lambda, \Lambda, n, \phi, r^*, M)$, and a negative radial classical strict subsolution $\check{u}$ of (PDE) in A such that*

$$\check{u} = 0 \text{ on } \partial B(y, r), \quad \check{u} = -mr \text{ on } \partial B(y, 2r), \quad \text{and} \quad \mu \le |D\check{u}(x)| \le \mu^{-1} \quad \text{in } A.$$

Moreover, if we assume (F_2A) in place of (F_2B), then $\hat{v} = -\check{u}$ is a positive radial classical strict supersolution to (PDE) in A.

In the above lemma, it is crucial that $M > 0$ is arbitrary and that $0 < m \le M$, meaning that we can construct arbitrarily flat strict sub- and supersolutions. Together with (F_1) this allows us to find a classical strict subsolution $\check{u}$ and supersolution $\hat{v}$ in A satisfying

$$\hat{v} = -\check{u} = \tilde{M} - mr > 0 \quad \text{on } \partial B(y, r) \quad \text{and} \quad \hat{v} = -\check{u} = \tilde{M} \quad \text{on } \partial B(y, 2r) \tag{3.1}$$

whenever $\tilde{M} > 0$ is given.

In order to prove the lower boundary growth estimate for positive solutions of (PDE) we need a positive barrier from below in terms of a classical strict subsolution. The following lemma, which is similar to Lemma 3.1, shows that this is possible. Note however that, contrary to Lemma 3.1, we now need to impose some extra control on the growth of $F(x, r, p, X)$ in the parameter r. Namely, we need (F_2B^*) in place of (F_2B). Due to the generality of the function γ in (F_2B^*) we need to bound the maximal size r^* of the annulus in order to handle the lower order terms in $\gamma(u)$.

Lemma 3.2 *Assume that (F_1), (F_2B^*) and (ϕ_A) hold. Suppose that $r \leq r^* \leq 1$, $0 < M$, and $y \in \mathbb{R}^n$ are given and let $A = B(y, 2r) \setminus \overline{B(y, r)}$. Then there exist positive $m(\lambda, \Lambda, n, \phi, r^*, C^*) \leq M$, $\mu(\lambda, \Lambda, n, \phi, r^*, C^*, M)$, and a positive radial classical strict subsolution $\check{\check{u}}$ of (PDE) in A such that*

$$\check{\check{u}} = mr \text{ on } \partial B(y, r), \quad \check{\check{u}} = 0 \text{ on } \partial B(y, 2r) \quad \text{and} \quad \mu \leq |D\check{\check{u}}(x)| \leq \mu^{-1} \quad \text{in } A.$$

The next lemma gives existence of a classical strict supersolution that is zero on the inner ball of the annulus A and as large as needed on the outer ball; the proof therefore needs (ϕ_B), which is an assumption on $\phi(t)$ for large t, to handle the steep gradients.

Lemma 3.3 *Assume that (F_1), (F_2A) and (ϕ_B) hold. Suppose that $r \leq r^*$, $M > 0$, and $y \in \mathbb{R}^n$ are given. Then there exist positive $m(\lambda, \Lambda, n, \phi, r^*) \geq M$, $\nu(\lambda, \Lambda, n, \phi, r^*, M)$, $k(\lambda, \Lambda, n, \phi, r^*, M) > 1$, and a classical strict supersolution $\hat{\hat{v}}$ of (PDE) in $A_k := B(y, kr) \setminus \overline{B(y, r)}$ such that*

$$\hat{\hat{v}} = 0 \text{ on } \partial B(y, r), \quad \hat{\hat{v}} = mr \text{ on } \partial B(y, kr) \quad \text{and} \quad \nu^{-1} \leq |D\hat{\hat{v}}(x)| \leq \nu \quad \text{in } A.$$

Proof of Lemma 3.1 Let $g(t)$ be a solution to $g'(t) = C\phi(g(t))$ with initial condition $g(0) = \mu > 0$, i.e.,

$$t = \int_{\mu}^{g(t)} \frac{ds}{C\phi(s)}, \tag{3.2}$$

where μ and C are constants to be chosen later (see Fig. 1). By assumption (ϕ_A) the increasing solution $g : [0, 1] \to [\mu, \infty)$ is well defined through the implicit function theorem whenever $\mu = \mu(C, \phi)$ is small enough. Define

$$m(\mu, C, \phi) := \int_0^1 g(t)dt \tag{3.3}$$

and

$$\check{u}(x) = r\left(\int_0^{2-|x-y|/r} g(t)dt - m\right), \quad \text{whenever} \quad x \in A.$$

By construction $\check{u}(x) < 0$ whenever $x \in A$, $\check{u} = 0$ on $\partial B(y, r)$, and $\check{u} = -mr$ on $\partial B(y, 2r)$ for m as defined in (3.3). By assumption (ϕ_A) we also have that $\mu \to 0$ implies $g(t) \to 0$, for all $t \in [0, 1]$, and so $m, \check{u} \to 0$ as $\mu \to 0$. Therefore, by decreasing μ if necessary, we have $m \leq M$. Our choice of μ depends only on C, ϕ and M.

To prove that $\check{u}$ is a strict classical subsolution to (PDE) we first calculate the derivatives of $\check{u}$. For notational simplicity we set $\tau := 2 - |x - y|/r$ and obtain

$$\frac{\partial \check{u}}{\partial x_i} = -\frac{x_i - y_i}{|x - y|} g(\tau) \quad \text{giving} \quad |D\check{u}| = g(\tau).$$

The construction of $g(t)$ implies $g'(t) = C\phi(g(t))$ and hence

$$\frac{\partial^2 \check{u}}{\partial x_j \partial x_i} = \frac{1}{r} \frac{(x_i - y_i)(x_j - y_j)}{|x - y|^2} C\phi(g(\tau)) - \left(\frac{\delta_{ij}}{|x - y|} - \frac{(x_i - y_i)(x_j - y_j)}{|x - y|^3}\right) g(\tau),$$

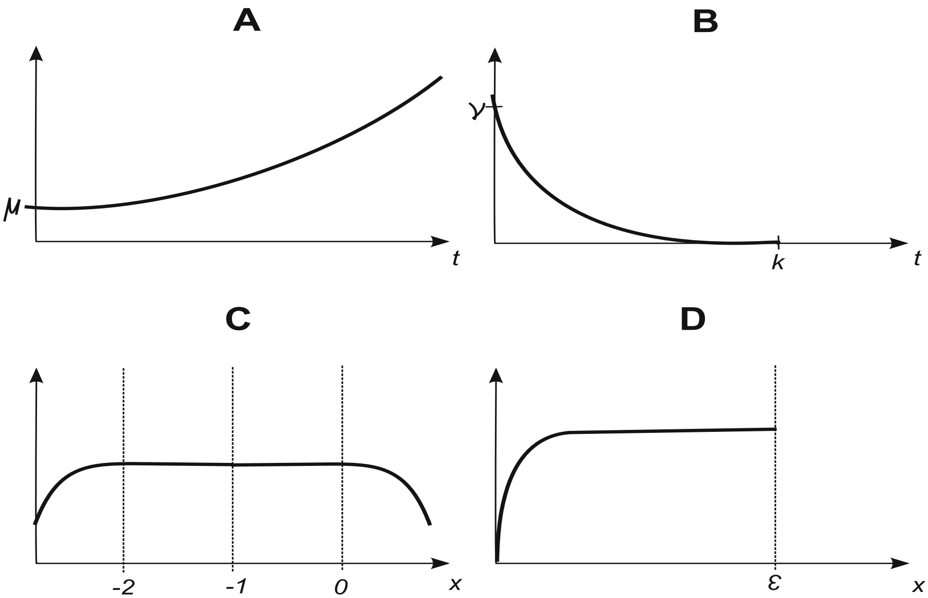

Fig. 1 (A) The function $g(t)$ in the proof of Lemma 3.1, (B) $f(t)$ in the proof of Lemma 3.3, (C) the function $H(x)$ showing the necessity of assumption (ϕ_A) and (D) the function $F(x)$ showing the necessity of assumption (ϕ_B)

implying

$$\mathrm{Tr}(D^2\check{u}) = \frac{1}{r}C\phi\left(g\left(\tau\right)\right) - \frac{n-1}{|x-y|}g\left(\tau\right).$$

We now decompose $D^2\check{u} = D^2\check{u}^+ - D^2\check{u}^-$ so that

$$\mathrm{Tr}\left(D^2\check{u}^+\right) = \frac{1}{r}C\phi\left(g\left(\tau\right)\right) \quad \text{and} \quad \mathrm{Tr}\left(D^2\check{u}^-\right) = \frac{n-1}{|x-y|}g\left(\tau\right).$$

Utilizing assumption (F_2B) gives, since $\check{u} \leq 0$, that

$$F(x, \check{u}, D\check{u}, D^2\check{u}) \leq F(x, 0, D\check{u}, D^2\check{u}) \leq \phi\left(g\left(\tau\right)\right) + \Lambda\tfrac{n-1}{|x-y|}g\left(\tau\right) - \tfrac{\lambda}{r}C\phi\left(g\left(\tau\right)\right).$$

By recalling $t \leq \phi(t)$ and observing that $r \leq |x-y| \leq 2r$ for all $x \in A$ we conclude

$$F(x, \check{u}, D\check{u}, D^2\check{u}) \leq \phi\left(g\left(\tau\right)\right)\left(1 + \Lambda\tfrac{n-1}{r} - C\tfrac{\lambda}{r}\right)$$

and thus by taking C large enough, depending only on λ, Λ, n, and r^*, we obtain

$$F(x, \check{u}, D\check{u}, D^2\check{u}) < 0 \quad \text{in} \quad A.$$

Consequently, $\check{u}$ is a strict classical subsolution to (PDE) in the annulus A. Finally, after decreasing μ if necessary, we have

$$\mu \leq |D\check{u}| \leq \mu^{-1}$$

where μ depends only on $\lambda, \Lambda, n, \phi, r^*$, and M. This completes the proof of the classical strict subsolution $\check{u}$.

By defining $\hat{v} = -\check{u}$ and mimicking the above reasoning, using (F_2A) in place of (F_2B), we conclude that $\hat{v}$ is a classical strict supersolution. The proof of Lemma 3.1 is complete. □

Proof of Lemma 3.2 As the proof largely follows that of Lemma 3.1 we only lay out the main differences. With $g(t)$ as in (3.2) and m as in (3.3), define

$$\check{\check{u}}(x) = r\int_0^{2-|x-y|/r} g(t)dt \quad \text{whenever} \quad x \in A.$$

Then $\check{\check{u}}$ is a positive function satisfying the specified boundary conditions and which has the same derivatives as the subsolution candidate $\check{u}$ in Lemma 3.1. Applying (F_2B^*) we conclude

$$\begin{aligned} F(x, \check{\check{u}}, D\check{\check{u}}, D^2\check{\check{u}}) &\leq \phi\left(g\left(\tau\right)\right) + \Lambda\frac{n-1}{|x-y|}g\left(\tau\right) - \frac{\lambda}{r}C\phi\left(g\left(\tau\right)\right) + \gamma(\check{\check{u}}(x)) \\ &\leq \phi\left(g\left(\tau\right)\right)\left(1 + \Lambda\frac{n-1}{r} - \frac{\lambda C}{2r}\right) + \gamma(\check{\check{u}}(x)) - \frac{\lambda C}{2r}\phi\left(g\left(\tau\right)\right). \end{aligned}$$

Recall C^* given by (F_2B^*) and pick $C = C(\lambda, \Lambda, n, r^*, C^*)$ so large that the first term on the right hand side of the above inequality is negative and such that $\lambda C/(2r^*) \geq C^*$. It then remains to ensure that

$$\gamma(\check{\check{u}}(x)) - C^*\phi\left(g\left(\tau\right)\right) \leq 0 \tag{3.4}$$

whenever $\check{\check{u}}(x)$ has a small gradient, i.e., whenever $g(\tau)$ is small. Since $g(t)$ is a strictly increasing function it follows that $\check{\check{u}}$ satisfies the sub-mean value property and thus

$$\check{\check{u}}(x) \leq r|D\check{\check{u}}(x)| = rg(\tau) \to 0 \quad \text{as} \quad \mu \to 0.$$

By assumption we have $\gamma(r) \leq C^*\phi(r)$ whenever $r \leq 1$ and thus (3.4) holds as long as $r \leq 1$ and $g(\tau) \leq 1$, since then $\check{\check{u}}(x) \leq g(\tau) \leq 1$ and $\gamma(\check{\check{u}}(x)) - C^*\phi\left(g\left(\tau\right)\right) \leq \gamma(\check{\check{u}}(x)) - C^*\phi\left(\check{\check{u}}(x)\right) \leq 0$. The proof of Lemma 3.2 can thus be completed by taking μ small enough. □

Proof of Lemma 3.3 Let $f(t)$ be a solution to $f'(t) = -C\phi(f(t))$ with initial condition $f(0) = \nu > 0$, where ν and C are constants to be chosen later (see Fig. 1). (A positive solution exists in a neighbourhood of $(0, \nu)$ since ϕ is continuous.) For small enough $k = k(C, \phi) > 1$, independent of ν and r, we can define

$$\hat{\hat{v}}(x) = r\int_0^{|x-y|/r-1} f(t)dt, \quad \text{whenever} \quad x \in A_k := B(y, kr) \setminus \overline{B(y, r)}. \tag{3.5}$$

By construction we obtain $\hat{\hat{v}} = 0$ on $\partial B(y, r)$ and $\hat{\hat{v}} = mr$ on $\partial B(y, kr)$ where

$$m = \int_0^{k-1} f(t)dt.$$

Assumption (ϕ_B) implies $m \to \infty$ as $\nu \to \infty$ and therefore we can make $m > M$ by increasing ν, depending only on ϕ, k, C and M.

To prove that $\hat{\hat{v}}$ is a strict classical supersolution to (PDE), put $\theta = |x-y|/r - 1$ and differentiate to obtain

$$\mathrm{Tr}(D^2\hat{\hat{v}}) = -\frac{1}{r}C\phi\left(f\left(\theta\right)\right) + \frac{n-1}{|x-y|}f\left(\theta\right) \quad \text{and} \quad |D\hat{\hat{v}}| = f\left(\theta\right).$$

After decomposing $D^2\hat{\hat{v}}$ as in the proof of Lemma 3.1 and using $\hat{\hat{v}} \geq 0$, (F_1), and (F_2A) we get

$$F(x, \hat{\hat{v}}, D\hat{\hat{v}}, D^2\hat{\hat{v}}) \geq -\phi\left(f\left(\theta\right)\right) - \Lambda\tfrac{n-1}{|x-y|}f\left(\theta\right) + \tfrac{\lambda}{r}C\phi\left(f\left(\theta\right)\right). \tag{3.6}$$

By recalling $t \leq \phi(t)$ and observing that $r \leq |x-y| \leq kr$ we conclude

$$F(x, \hat{\hat{v}}, D\hat{\hat{v}}, D^2\hat{\hat{v}}) \geq \phi\left(f(\theta)\right)\left(-1 - \Lambda \frac{n-1}{r} + C\frac{\lambda}{r}\right) > 0$$

whenever C is large enough, depending only on λ, Λ, n, and r^*. Thus $\hat{\hat{v}}$ is a strict classical supersolution to (PDE) in the annulus A_k. Moreover, $\hat{\hat{v}}$ has bounded gradient since, after increasing ν if necessary,

$$\nu^{-1} \leq |D\hat{\hat{v}}| \leq \nu$$

where $\nu = \nu(\lambda, \Lambda, n, \phi, r^*, M)$. □

4 Strong Maximum Principles

We begin by proving the SMaP using a classical contradiction argument based on comparison with the supersolution in Lemma 3.1.

Theorem 4.1 (Strong Maximum Principle) *Assume that (F_1), (F_2A) and (ϕ_A) hold. Let $\Omega \subset \mathbb{R}^n$ be a domain and suppose that u is a viscosity subsolution of (PDE) in Ω. If u attains a positive maximum in Ω, then u is constant.*

Proof Assume, by contradiction, that a nonconstant USC subsolution u attains its maximum $\tilde{M} > 0$ in Ω and let $K = \{x \in \Omega \,|\, u(x) = \tilde{M}\}$. By assumption $K \neq \Omega$, $K \neq \emptyset$ and therefore, by USC, there exist $\bar{x} \in \Omega \cap \partial K$ and $s_* > 0$ such that $u(\bar{x}) = \tilde{M}$ and for every $s < s_*$ there is $y_s \in \Omega$ such that $B(y_s, s) \in \Omega \setminus K$, $\bar{x} \in \partial B(y_s, s)$ and $u(x) < \tilde{M}$ in $B(y_s, s)$.

Lemma 3.1 yields existence of a classical strict supersolution $\hat{v}$ in the annulus $A = B(y_s, 2r) \setminus \overline{B(y_s, r)}$, and since $\tilde{M} > 0$ we see from (3.1) that we can chose $\hat{v}$ to satisfy

$$\hat{v} = \tilde{M} - mr \text{ on } \partial B(y_s, r), \quad \text{and} \quad \hat{v} = \tilde{M} \text{ on } \partial B(y_s, 2r),$$

with constant m satisfying $\tilde{M} - mr \geq 0$ for all $r \in (0, r_*]$.

Choose r so that $r < s < 2r$ and $B(y_s, 2r) \in \Omega$ (after decreasing s if necessary). It follows that $u \leq \hat{v} = \tilde{M}$ on $\partial B(y_s, 2r)$. Since $u < \tilde{M}$ on $\partial B(y_s, r)$ we can decrease m so that $u \leq \hat{v}$ on $\partial B(y_s, r)$ as well (since $v = \tilde{M} - mr$ there). The weak comparison principle in Lemma 2.1 therefore implies $u < \hat{v}$ in the annulus $A = B(y_s, 2r) \setminus \overline{B(y_s, r)}$. As the gradient of $\hat{v}$ does not vanish in A, $\bar{x} \in A$ and $u(\bar{x}) = \tilde{M}$, we arrive at a contradiction. □

Using a similar argument we next prove the following version of the Hopf lemma.

Theorem 4.2 (Hopf lemma) *Assume that (F_1), (F_2A) and (ϕ_A) hold. Let $\Omega \subset \mathbb{R}^n$ be a domain, $w \in \partial\Omega$, and suppose that Ω satisfies the interior sphere condition with $B(y_w, r_w)$ at w. If u is a nonconstant viscosity subsolution of (PDE) in Ω, upper semicontinuous on $\Omega \cup \{w\}$ so that $u(w) \geq \limsup_{\Omega \ni x \to w} u(x)$ and $u(w) > 0$, then, for any $\gamma \in \mathbb{R}^n$ such that $\gamma \cdot (w - y_w) < 0$ we have*

$$\limsup_{s \to 0} \frac{u(w + s\gamma) - u(w)}{s} < 0.$$

Proof Choose $r < r_w/2$ and let $y_r = w + 2r \frac{(y_w - w)}{|(y_w - w)|}$. Put $\tilde{M} = u(w) > 0$. Lemma 3.1 now yields a classical strict supersolution $\hat{v}$ in the annulus $A = B(y_r, 2r) \setminus \overline{B(y_r, r)}$ satisfying $\hat{v} = \tilde{M}$ on $\partial B(y_r, 2r)$ and $\hat{v} = \tilde{M} - mr$ on $\partial B(y_r, r)$. Choosing r small enough, we can ensure that $\hat{v} > u$ on $B(y_r, r)$ and thus, from the weak comparison principle in Lemma 2.1 we have that $\hat{v} > u$ in A. Hence, since $\hat{v}$ is a radial function with bounded gradient

$$\limsup_{s\to 0} \frac{u(w+s\gamma)-u(w)}{s} \le D\hat{v}(w)\cdot\gamma < 0.$$

□

Remark 4.3 Assumption (ϕ_A) is necessary for Theorems 4.1 and 4.2 to hold.

Indeed, assume that (ϕ_A) does not hold and follow Julin [25] by defining $h : (-1, 1) \to [0, \infty)$ so that $h(x) = 0$ for $(-1, 0)$ and

$$x = \int_0^{h(x)} \frac{dt}{\phi(t)} \quad \text{for} \quad x \in [0, 1).$$

Then $h'(x) = \phi(h(x))$ and the function

$$H(x) = 1 - \int_0^x h(s)ds$$

satisfies $H'' + \phi(|H'|) = 0$ classically on $(-1, 1)$ and violates both the Hopf lemma and the SMaP. (The SMiP is violated by $-H$.) Extending H to $(-3, 1)$ evenly in the line $x = -1$ produces a classical solution contradicting the maximum principle, the comparison principle, and uniqueness since constants solve the equation, see Fig. 1.

We end this section by giving the following analogous results for supersolutions of (PDE):

Corollary 4.4 (I) *Under the assumptions in Theorem 4.2 but with (F_2A) replaced by (F_2B), if v is a viscosity supersolution of (PDE) in Ω, lower semicontinuous on $\Omega \cup \{w\}$ such that $v(w) \le \liminf_{\Omega\ni x\to w} v(x)$ and $v(w) < 0$, then, for any $\gamma \in \mathbb{R}^n$ such that $\gamma\cdot(w-y_w) < 0$ we have*

$$\liminf_{s\to 0} \frac{v(w+s\gamma)-v(w)}{s} > 0.$$

(II) *Under the assumptions in Theorem 4.1 but with (F_2A) replaced by (F_2B) a viscosity supersolution of (PDE) can not attain a negative minimum in Ω unless it is constant.*
(III) *If the stronger assumption (F_2B^*) holds, then (I) holds with $v(\omega) < 0$ replaced by $v(\omega) \le 0$, and assertion (II) holds also when the minimum is nonpositive.*

Proof Statements (I) and (II) follow by using the negative subsolution from Lemma 3.1 in place of the positive supersolution in similar arguments as in the proofs of Theorem 4.1 and Theorem 4.2. Assertion (III) follows using the positive subsolution from Lemma 3.2 in the same arguments. □

5 Boundary Growth Estimates

In this section we prove that a positive solution vanishing on a boundary satisfying the sphere condition vanishes at the rate of the distance to the boundary. We state and prove a lower estimate for a positive supersolution and an upper estimate for a positive subsolution separately and give the full combined result in Corollary 5.5. As an immediate consequence, we also get the boundary Harnack inequality stated in Corollary 5.6.

Theorem 5.1 (Lower estimate) *Assume that (F_1), (F_2B^*) and (ϕ_A) hold. Let $\Omega \subset \mathbb{R}^n$ be a domain satisfying the interior sphere condition with radius r_i and take $w \in \partial\Omega$ and r s.t. $0 < 2r < r_i$. Assume that $v \in LSC(\overline{\Omega} \cap B(w, 6r))$ is a positive viscosity supersolution of (PDE) in $\Omega \cap B(w, 6r)$ satisfying $v = 0$ on $\partial\Omega \cap B(w, 6r)$. Then there exists a constant $c = c(\lambda, \Lambda, n, \phi, r_i, r^{-1}\inf_{\Gamma_{w,r}} v, C^*)$ such that*

$$c\, v(x) \geq d(x, \partial\Omega) \quad whenever \quad x \in \Omega \cap B(w, r),$$

where $\Gamma_{w,r} = \{x \in \Omega | r < d(x, \partial\Omega) < 3r\} \cap B(w, 6r)$.

Proof We follow the lines of Aikawa–Kilpeläinen–Shanmugalingam–Zhong [3, Lemma 3.1]. Take $x \in \Omega \cap B(w, r)$ and let $\eta \in \partial\Omega$ be such that $d(x, \partial\Omega) = |x - \eta|$. By the interior sphere condition at η we can find a point η^i such that $B(\eta^i, r_i) \subset \Omega$ and $\eta \in \partial B(\eta^i, r_i)$. Now, take the point η^i_{2r} which is such that $\eta = \eta^i_{2r} + 2r\frac{\eta - \eta_i}{|(\eta - \eta_i)|}$, i.e., $\eta^i_{2r} \in \Omega$ lies on a straight line ℓ between η and η^i on a distance $2r$ from the boundary $\partial\Omega$. Then $B(\eta^i_{2r}, 2r) \subset \Omega$ so v is a positive viscosity supersolution in $B(\eta^i_{2r}, 2r)$. By construction $|x - \eta| < r$ so $|\eta^i_{2r} - x| > r$ and by the interior sphere condition we get that x lies on the line ℓ and thus $x \in A = B(\eta^i_{2r}, 2r) \setminus \overline{B(\eta^i_{2r}, r)}$.

Next we note that $B(\eta^i_{2r}, r) \subset \Gamma_{w,r}$ and that, for $m > 0$ small enough, we have

$$v(x) \geq \inf_{\Gamma_{w,r}} v \geq mr > 0 \quad \text{whenever} \quad x \in \Gamma_{w,r}.$$

Now apply Lemma 3.2 with $r^* = r_i/2$ and $M = r^{-1}\inf_{\Gamma_{w,r}} v$ to ensure existence of a classical strict subsolution $\check{\check{u}}$ in the annulus $A \subset \Omega \cap B(w, 6r)$. Moreover, this subsolution can be taken to satisfy $\check{\check{u}} = 0$ on $\partial B(\eta^i_{2r}, 2r)$ and $\check{\check{u}} = mr \leq v(x)$ on $\partial B(\eta^i_{2r}, r)$, for some $m = m(\lambda, \Lambda, n, \phi, r_i) \in (0, r^{-1}\inf_{\Gamma_{w,r}} u)$. Since $\check{\check{u}} \leq v$ on the boundaries of the annulus $A = B(\eta^i_{2r}, 2r) \setminus \overline{B(\eta^i_{2r}, r)}$ and $\check{\check{u}}$ is a classical strict subsolution we get from the weak comparison principle in Lemma 2.1 that $\check{\check{u}}(x) \leq v(x)$ for all $x \in A$.

Since $x \in A$ the result will follow after ensuring that $\check{\check{u}}$ does not vanish faster than $d(x, \partial\Omega)$ as $x \to \partial\Omega$. However, this is a consequence of Lemma 3.2 which gives the existence of a constant $c = c(\lambda, \Lambda, n, \phi, r^{-1}\inf_{\Gamma_{w,r}} v, C^*)$, independent of x, such that $c^{-1} \leq |D\check{\check{u}}| \leq c$. The proof is thus complete. □

We next prove an upper estimate of viscosity subsolutions vanishing on a portion of a domain satisfying an exterior sphere condition.

Theorem 5.2 (Upper estimate) *Assume that (F_1), (F_2A) and (ϕ_B) hold. Let $\Omega \subset \mathbb{R}^n$ be a domain satisfying the exterior ball condition with radius r_e and take $w \in \partial\Omega$ and r s.t. $0 < 2r < r_e$. Assume that $u \in USC(\overline{\Omega} \cap B(w, 6r))$ is a positive viscosity subsolution of (PDE) in $\Omega \cap B(w, 6r)$ satisfying $u = 0$ on $\partial\Omega \cap B(w, 6r)$. Then there exists a constant $c = c(\lambda, \Lambda, n, \phi, r_e, r^{-1}\sup_{B(w,6r)\cap\Omega} u)$ such that*

$$u(x) \leq c\, d(x, \partial\Omega) \quad for \quad x \in \Omega \cap B(w, r).$$

Proof We follow the lines of Aikawa–Kilpeläinen–Shanmugalingam–Zhong [3, Lemma 3.3] and take $x \in \Omega \cap B(w, r)$ and let $\eta \in \partial\Omega$ be such that $d(x, \partial\Omega) = |x - \eta|$. By the exterior sphere condition at η we can find a point η^e such that $B(\eta^e, r_e) \subset \mathbb{R}^n \setminus \Omega$ and

$\eta \in \partial B(\eta^e, r_e)$. Now, take the point η_r^e which is such that $\eta = \eta_r^e + r\frac{\eta-\eta^e}{|(\eta-\eta^e)|}$ (i.e., $\eta_r^e \notin \Omega$ lies on the straight line γ between η and η^e on a distance r from the boundary $\partial\Omega$).

Applying Lemma 3.3 we ensure existence of a classical strict supersolution $\hat{\hat{v}}$ in the annulus $A = B(\eta_r^e, 2r) \setminus \overline{B(\eta_r^e, r)}$ satisfying $\hat{v} = 0$ on $\partial B(y, r)$ and $\hat{v} = mr$ on $\partial B(y, 2r)$. Moreover, we are free to choose m such that $m \geq M = r^{-1} \sup_{B(w,6r)\cap\Omega} u$ so that $\hat{\hat{v}} > u$ on the boundary of $B(\eta_r^e, 2r) \cap \Omega$ and since $\hat{\hat{v}} \geq 0$ also dominates u on $\Omega \cap B(w, 6r)$ we are in a position to apply the weak comparison principle in Lemma 2.1. We thus obtain $u \leq \hat{\hat{v}}$ in $B(\eta_r^e, 2r) \cap \Omega$ and since x belongs to this set the result follows by showing that $\hat{\hat{v}}$ vanishes at least as fast as $d(x, \partial\Omega)$ when $x \to \partial\Omega$. However, this is immediate by the gradient bound in Lemma 3.3, with constant depending on λ, Λ, n, ϕ, r_e and $M = r^{-1} \sup_{B(w,6r)\cap\Omega} u$. □

We now comment on the necessity of the assumptions in Theorem 5.1 and Theorem 5.2. Indeed, necessity of (ϕ_A) for the lower bound follows by the counterexample in Remark 4.3 as the derivative of $1 - H$ approaches zero as the function itself vanishes near the origin. To see the necessity of (ϕ_B) for the upper bound, assume that it does not hold. Then for $\epsilon > 0$ there is $C < \infty$ such that

$$\int_0^\epsilon f(t)dt \leq C \quad \text{for all} \quad \nu > 0$$

where $f(t)$ solves the differential equation $f'(t) = -\phi(f(t))$ with initial condition $f(0) = \nu$. Define

$$F(x) = \int_0^x f(t)dt, \quad \text{whenever} \quad x \in (0, \epsilon)$$

and note that then $F'(x) = f(x)$ and $F(x)$ satisfies the differential equation

$$F'' + \phi(|F'|) = 0.$$

Moreover, $F'(0) = f(0) = \nu$ which explodes as $\nu \to \infty$, see Fig. 1. This contradicts the upper bound in Theorem 5.2 and we conclude the following remark:

Remark 5.3 Condition (ϕ_A) is necessary for the lower bound in Theorem 5.1 and condition (ϕ_B) is necessary for the upper bound in Theorem 5.2.

As assumption (ϕ_B) is a bit technical we here comment on some functions ϕ satisfying it. To this end we restrict to $\phi(s) = s^k$, $k \geq 1$, and note that the solution to $f'(t) = -f(t)^k$ with initial condition $f(0) = \nu$ yields

$$f(t) = \begin{cases} \nu e^{-t} & \text{if } k = 1, \\ \left((k-1)t + \nu^{1-k}\right)^{\frac{1}{1-k}} & \text{if } k > 1. \end{cases}$$

Therefore,

$$\int_0^\epsilon f(t)dt = \begin{cases} \nu\left(1 - e^{-\epsilon}\right) & \text{if } k = 1, \\ \log(1 + \epsilon\nu) & \text{if } k = 2, \\ \frac{1}{2-k}\left(\nu^{2-k} - \left((k-1)\epsilon + \nu^{1-k}\right)^{\frac{2-k}{1-k}}\right) & \text{otherwise,} \end{cases}$$

and the above integral diverges for any $\epsilon > 0$ whenever $1 \le k \le 2$ and converges if $k > 2$, as $\nu \to \infty$. Thus assumption (ϕ_B) is satisfied if and only if $k \le 2$ and it follows that condition (1.4) implies (ϕ_B). Moreover, as

$$\int_1^\infty \frac{dt}{\phi(t)} = \infty$$

ensures that (1.4) holds we have proved:

Remark 5.4 The Keller-Osserman type condition (1.5) implies (1.4) which in turn implies (ϕ_B).

Using the lower and upper estimates in Theorems 5.1 and 5.2 we conclude that positive viscosity solutions vanishing on a portion of the boundary are comparable with the distance function near the boundary.

Corollary 5.5 *Let $\Omega \subset \mathbb{R}^n$ be a domain satisfying the sphere condition with radius r_b. Assume that (F_1), (F_2A), (F_2B^*), (ϕ_A), and (ϕ_B) hold. Let $w \in \partial\Omega$ and $0 < r < r_b$. Assume that u is a positive viscosity solution of (PDE) in $\Omega \cap B(w, 6r)$, continuous in $\overline{\Omega} \cap B(w, 6r)$ satisfying $u = 0$ on $\partial\Omega \cap B(w, 6r)$. Then there exists a constant c such that*

$$c^{-1}\, d(x, \partial\Omega) \le u(x) \le c\, d(x, \partial\Omega) \quad \textit{whenever} \quad x \in \Omega \cap B(w, r).$$

The constant c has dependence according to Theorems 5.1 and 5.2.

An immediate consequence of Corollary 5.5 is the boundary Harnack inequality.

Corollary 5.6 (Boundary Harnack inequality) *Let Ω, w, r, and u be as in Corollary 5.5 and suppose that v is another solution satisfying the same properties as u. Then there exists a constant c such that*

$$\frac{1}{c} \le \frac{u(x)}{v(x)} \le c \quad \textit{whenever} \quad x \in \Omega \cap B(w, r).$$

The constant c has dependence according to Theorems 5.1 and 5.2.

6 A Variable Exponent Eigenvalue Problem

In this section we discuss our results in the setting of the eigenvalue problem

$$-\Delta_{p(x)} := -\nabla \cdot \left(|\nabla u|^{p(x)-2}\nabla u\right) = -a|u|^{q(x)-2}u, \tag{6.1}$$

where $1 < p^- \le p(x) \le p^+ < \infty$ is Lipschitz continuous, $a \ge 0$, and where $q(x)$ is to be specified later. To be able to apply our results to (6.1) we define

$$-F(x, u, Du, D^2u) = \mathrm{Tr}(D^2u) + (p(x) - 2)\Delta_\infty u + \log|Du|\langle Dp, Du\rangle - a|u|^{q(x)-2}u,$$

where $\Delta_\infty u$ denotes the normalized infinity Laplace operator $\Delta_\infty u = \langle D^2u\frac{Du}{|Du|}, \frac{Du}{|Du|}\rangle$. Note that the above operator coincides with (6.1) if $|Du| \ne 0$ (which is enough for our needs). Moreover, it satisfies (1.1) and (1.2) and therefore also (F_2A) (without further restrictions on q) and if $q(x) \ge 2$ it satisfies also (F_2B^*) with

$$\lambda = \min\{1, p^- - 1\}, \quad \Lambda = \max\{1, p^+ - 1\}, \quad \phi(t) = C\,(|\log t| + 1)\,t, \quad \text{and} \quad \gamma(t) = at, \tag{6.2}$$

where $\phi(t) = C\,(|\log t| + 1)\,t$ satisfies both the Osgood condition (ϕ_A) and the Keller-Osserman condition (1.5). By these observations, we can conclude that given the validity of a comparison principle, our main results apply to solutions (considered in a suitable weak sense) of (6.1).

To be a bit more precise, but still leaving a general discussion of (6.1) to experts in the field of variable exponent eigenvalue problems, we note that solutions of (6.1) are often considered in the following weak sense: A function $u \in W^{1,p(x)}(\Omega)$ is a weak solution of (6.1) in Ω if

$$\int_\Omega \left(\langle |\nabla u|^{p(x)-2} \nabla u, \nabla \psi \rangle + a|u|^{q(x)-2} u\psi \right) dx = 0, \tag{6.3}$$

whenever $\psi \in W_0^{1,p(x)}(\Omega)$, see, e.g., Harjulehto–Hästö–Lê–Nuortio [23] for details, applications, and a general overview of variable exponent problems. It is a non-trivial task to show that weak solutions in the sense of (6.3) are also viscosity solutions, see, e.g., Juutinen–Lukkari–Parviainen [26] and Julin [25] for the case $a = 0$. However, as the results of Sections 4 and 5 only rely on the classical sub- and supersolutions constructed in Section 3 together with the weak comparison principle in Lemma 2.1, we can, using the corresponding comparison principle in Lemma 6.2 below, conclude that the results of Sections 4 and 5 hold also in the setting of weak solutions of (6.1) (cf. Remark 2.2). In particular:

Corollary 6.1 *If $q(x) = p(x)$ and $q(x) \geq 2$ then all Theorems in* Sections 4 *and* 5 *apply to continuous weak solutions of* (6.1). *The dependence of constants in the Theorems of* Section 5 *can be traced via* (6.2).

As already mentioned in the Introduction, earlier results on maximum principles for variable exponent problems can be found in Fan–Zhao–Zhang [20] and Zhang [43], and we also remark that our boundary estimates in Corollary 6.1 relate to the asymptotic behavior of solutions near C^2-boundaries proven by Zhang in [44].

We proceed by noting that existence of weak solutions of $-\Delta_{p(x)} = -a|u|^{p(x)-2}u$ when

$$a = \frac{\int_\Omega |\nabla u(x)|^{p(x)}\, dx}{\int_\Omega |u(x)|^{p(x)}\, dx}$$

follows by Mendez [37]. Next, we give the following proof of the comparison principle based on Fleckinger-Pellé–Takáč [21, Proposition 4.1]:

Lemma 6.2 *Assume that $u, v \in W^{1,p(x)}(\Omega)$ are weak solutions, respectively, of*

$$-\Delta_{p(x)} u = -a|u|^{p(x)-2}u + f(x), \quad u = g \text{ on } \partial\Omega, \tag{6.4}$$

and

$$-\Delta_{p(x)} v = -a|v|^{p(x)-2}v + f'(x), \quad v = g' \text{ on } \partial\Omega. \tag{6.5}$$

Let p' be such that $1/p + 1/p' = 1$ and assume $a > 0$, $f \leq f'$ in $L^{p'(x)}(\Omega)$, and $g \leq g'$ in $W^{1/p',p}(\partial\Omega)$. Then $u \leq v$ a.e. in Ω.

Proof Set $w = u - v$ and $w^+ = \max\{w, 0\}$, $w^- = \max\{-w, 0\}$. Then $w = w^+ - w^-$ by definition and $w^+ \in W^{1,p(x)}(\Omega)$ with trace $w^+ = 0$ on $\partial\Omega$. By multiplying (6.4) with w^+, integrating over Ω, and applying integration by parts, we get

$$\begin{aligned}\int_\Omega |\nabla u|^{p(x)-2} \langle \nabla u, \nabla w^+ \rangle\, dx &= -\int_\Omega \nabla \cdot (|\nabla u|^{p(x)-2} \nabla u) w^+\, dx \\ &= -a \int_\Omega |u|^{p(x)-2} u w^+\, dx + \int_\Omega f w^+\, dx.\end{aligned}$$

Applying the same procedure to (6.5), subtracting the two equations, and recalling $f \leq f'$, we get

$$\int_\Omega \left(|\nabla u|^{p(x)}\nabla u - |\nabla v|^{p(x)-2}\nabla v\right)\cdot \nabla w^+\,dx + a\int_\Omega \left(|u|^{p(x)-2}u - |v|^{p(x)-2}v\right) w^+\,dx$$
$$= \int_\Omega (f - f')w^+\,dx \leq 0.$$

Next consider the sets

$$\Omega^+ = \{x \in \Omega \mid u(x) > v(x)\} \quad \text{and} \quad \Omega_0^- = \{x \in \Omega \mid u(x) \leq v(x)\}.$$

Note that $\Omega = \Omega^+ \cup \Omega_0^-$ and that in Ω^+ we have $w^+ = u - v > 0$ and $\nabla w^+ = \nabla u - \nabla v$, while in Ω_0^- we have $w^+ = \nabla w^+ = 0$ (see Gilbarg–Trudinger [22, Lemma 7.6]). Using this the previous equation can be written as

$$\begin{aligned} &\int_{\Omega^+} \left(|\nabla u|^{p(x)-2}\nabla u - |\nabla v|^{p(x)-2}\nabla v\right)\cdot \nabla (u-v)\,dx \\ &+a\int_{\Omega^+} \left(|u|^{p(x)}u - |v|^{p(x)-2}v\right)(u-v)\,dx \\ &= \int_{\Omega^+} (f - f')w^+\,dx \leq 0. \end{aligned} \tag{6.6}$$

Next, assume that $u \leq v$ does not hold in a set of positive measure, i.e., assume that $|\Omega^+| > 0$. We will now prove a contradiction by showing that the left-hand side of (6.6) must be positive in this case. First, since $|u|^{p(x)-2}u$ is monotonically increasing for a.e. $x \in \Omega$ and $a > 0$, we have

$$a\int_{\Omega^+} \left(|u|^{p(x)-2}u - |v|^{p(x)-2}v\right)(u-v)\,dx > 0.$$

Second, using Cauchy–Schwarz inequality we observe that

$$\begin{aligned} &\textstyle\int_{\Omega^+} \left(|\nabla u|^{p(x)-2}\nabla u - |\nabla v|^{p(x)-2}\nabla v\right)\cdot \nabla (u-v)\,dx \\ &\geq \textstyle\int_{\Omega^+} \left(|\nabla u|^{p(x)-1} - |\nabla v|^{p(x)-1}\right)(|\nabla u| - |\nabla v|)\,dx \geq 0. \end{aligned}$$

The above two displays together with (6.6) give the contradiction and thus the desired result follows. □

We end the paper by noting that in case of the eigenvalue problem (6.1), our auxiliary functions constructed in Section 3 can be replaced by those constructed in Adamowicz–Lundström [1, Lemma 4.1]. Indeed, restricting our attention to (6.1) we may prove most of our main results using the following lemma, together with a similar result for subsolutions:

Lemma 6.3 *Let $p(x) \in (p^-, p^+)$ belong to $\mathcal{C}^1(\bar{\Omega})$, $q(x) \in (q^-, q^+)$, $M > 0$ and define*

$$\hat{u}(x) = \frac{M}{e^{-\mu} - e^{-4\mu}}\left(e^{-\mu} - e^{-\mu\frac{|s-y|^2}{r^2}}\right) \quad \textit{whenever} \quad x \in B(y, 2r)\setminus B(y, r).$$

Then there are constants $r_* = r_*(p^-, \|p\|_{L^\infty})$ and $\mu_* = \mu_*(p^+, p^-, q^-, q^+, n, \|p\|_{L^\infty}, M)$ such that for $r \leq r_*$ and $\mu \geq \mu_*$ it holds that $\hat{u}(x) = M$ on $\partial B(y, 2r)$, $\hat{u}(x) = 0$ on $\partial B(y, r)$ and $\hat{u}$ is a classical supersolution of (6.1).

Proof The proof in Adamowicz–Lundström [1, Lemma 4.1] hinges on calculating $\Delta_{p(x)}u$ for $u = Ae^{-\mu\frac{|s-y|^2}{r^2}} + B$ and then choosing A and B appropriately. The estimates (4.1), (4.6), and (4.8) in [1] carry forward the need for $\Delta_{p(x)}u \leq 0$ by choosing μ and r so that

$$\begin{aligned}
&\mu(8r\|\nabla p\|_{L^\infty} - 2(p^- - 1)) \\
&+2r\|\nabla p\|_{L^\infty}\left(\log\left(\frac{4}{1-e^{-3\mu}}\right) + |\log M| + |\log r|\right) \\
&+n + p^+ - 2 \leq 0.
\end{aligned} \tag{6.7}$$

The first term is negative if $r \leq \frac{p^- - 1}{4\|\nabla p\|_{L^\infty}}$ and with a small upper bound on r, $r|\log r|$ is increasing, meaning the second term does not blow up for smaller r. The inequality then holds for a big enough positive μ.

In the present case we have $\Delta_{p(x)}u \leq a|u|^{q(x)-2}u$ (in place of $\Delta_{p(x)}u \leq 0$), which enters on the R.H.S. of (6.7). Sticking to the same choice of A and B as in [1] we have $0 \leq u \leq M$ and the inequality still holds trivially if $a \geq 0$. If $a < 0$, we can move the extra term to the L.H.S. of (6.7) and note that

$$-a|u|^{q(x)-2}u \leq -a\left|\frac{M}{e^{-\mu} - e^{-4\mu}}(e^{-\mu} - e^{-\frac{|x-y|^2}{r^2}})\right|^{q(x)-1} \leq -a\max\{M^{q^+-1}, M^{q^--1}\}$$

since

$$0 < \frac{e^{-\mu} - e^{-\frac{|x-y|^2}{r^2}}}{e^{-\mu} - e^{-4\mu}} < 1.$$

This means the additional term is bounded so that, with μ large enough, the L.H.S. of (6.7) will stay non-positive as in [1]. (The case $a < 0$ is not very useful as the comparison principle is not valid.) This completes the proof of the lemma. □

Acknowledgements We thank Tomasz Adamowicz for useful discussions, ideas, and comments. Niklas Lundström and Marcus Olofsson were partially funded by the Swedish research council grant 2018-03743; this support is hereby gratefully acknowledged.

Funding Open access funding provided by Umea University.

References

1. Adamowicz, T., Lundström, N.L.P.: The boundary Harnack inequality for variable exonent p-Lalacian, Carleson estimates, barrier functions and $p(\cdot)$-harmonic measures. Ann. Mat. Pura Appl. **195.2**, 623–658 (2016)
2. Aikawa, H.: Boundary Harnack principle and Martin boundary for a uniform domain. J. Math. Soc. Jpn. **53.1**, 119–145 (2001)
3. Aikawa, H., Kilpeläinen, T., Shanmugalingam, N., Zhong, X.: Boundary Harnack principle for p-harmonic functions in smooth euclidean domains. Potential Analysis **26.3**, 281–301 (2007)
4. Ancona, A.: Principe de Harnackà la frontière et théorème de Fatou pour un opérateur elliptique dans un domaine Lipschitzien. Ann. Inst. Fourier (Grenoble) **28.4**, 169–213 (1978)

5. Avelin, B., Julin, V.: A Carleson type inequality for fully nonlinear elliptic equations with non-Lipschitz drift term. J. Funct. Anal. **272.8**, 3176–3215 (2017)
6. Banuelos, R., Bass, R., Burdzy, K.: Hölder domains and the boundary Harnack principle. Duke Math. J. **64.1**, 195–200 (1991)
7. Bardi, M., Goffi, A.: New strong maximum and comparison principles for fully nonlinear degenerate elliptic PDEs, Calculus of Variations and Partial Differential Equations 58.6, Paper no.184 (2019)
8. Bardi, M., Da Lio, F.: On the strong maximum principle for fully nonlinear degenerate elliptic equations. Arch. Math. **73.4**, 276–285 (1999)
9. Bardi, M., Da Lio, F.: Propagation of maxima and strong maximum principle for viscosity solutions of degenerate elliptic equations. I: Convex operators. Nonlinear Anal. Theory Methods Appl. **44.8**, 991–1006 (2001)
10. Bardi, M., Da Lio, F.: Propagation of maxima and strong maximum principle for viscosity solutions of degenerate elliptic equations. II. Concave operators. Indiana Univ. Math. J. **52.3**, 607–627 (2003)
11. Bass, R., Burdzy, K.: A boundary Harnack principle in twisted hölder domains. Ann. Math. **134.2**(2), 253–276 (1991)
12. Birindelli, I., Demengel, F.: Eigenvalue, maximum principle and regularity for fully non linear homogeneous operators. Commun. Pure Appl. Anal. **6**(2), 335–366 (2007)
13. Birindelli, I., Galise, G., Ishii, H.: 2018, A family of degenerate elliptic operators: maximum principle and its consequences. Annales de l'Institut Henri Poincare C Analyse non lineaire **35**(2), 417–441 (2018)
14. Caffarelli, L.A., Cabre, X.: Fully Nonlinear Elliptic Equations American Mathematical Society Colloquium Publications 43. American Mathematical Society, Providence (1995)
15. Caffarelli, L.A., Fabes, E., Mortola, S., Salsa, S.: Boundary behavior of nonnegative solutions of elliptic operators in divergence form. Indiana Univ. Math. J. **30.4**, 621–640 (1981)
16. Calabi, E.: An extension of E, Hopf's maximum principle with an application to Riemannian geometry. Duke Math. J. **25.1**, 45–56 (1958)
17. Capuzzo-Dolcetta, I., Vitolo, A.: On the maximum principle for viscosity solutions of fully nonlinear elliptic equations in general domain. Matematiche (Catania) **62.2**, 69–91 (2007)
18. Dahlberg, B.: On estimates of harmonic measure. Arch. Ration. Mech. Anal. **65.3**, 275–288 (1977)
19. De Silva, D., Savin, O.: A short proof of Boundary Harnack Principle. J. Differ. Equ. **269.3**, 2419–2429 (2020)
20. Fan, X., Zhao, Y., Zhang, Q.: A strong maximum principle for p(x)-Laplace equations. Chin. J. Contemp. Math. **24.3**, 277–282 (2003)
21. Fleckinger-Pellé, J., Takáč, P.: Uniqueness of positive solutions for nonlinear cooperative systems with the p-Laplacian. Indiana Univ. Math. J. **43.4**, 1227–1253 (1994)
22. Gilbarg, D., Trudinger, N.S.: Elliptic Partial Differential Equations of Second Order. Springer, Berlin, Heidelberg (2015)
23. Harjulehto, P., Hästö, P., LêÚt, V., Nuortio, M.: Overview of differential equations with non-standard growth. Nonlinear Anal. Theory Methods Appl. **72.12**, 4551–4574 (2010)
24. Jerison, D., Kenig, C.: Boundary behavior of harmonic functions in nontangentially accessible domains. Adv. Math. **46.1**, 80–147 (1982)
25. Julin, V.: Generalized Harnack inequality for nonhomogeneous elliptic equations. Arch. Ration. Mech. Anal. **2.216**, 673–702 (2015)
26. Juutinen, P., Lukkari, T., Parviainen, M.: Equivalence of viscosity and weak solutions for the p(x)-Laplacian. Annales de l'Institut Henri Poincaré, Analyse Non Lineaire **27.6**, 1471–1487 (2010)
27. Kawohl, B., Kutev, N.: Strong maximum principle for semicontinuous viscosity solutions of nonlinear partial differential equations. Arch. Math. **70.6**, 470–478 (1998)
28. Kemper, J.T.: A boundary Harnack principle for Lipschitz domains and the principle of positive singularities. Commu. on Pure and Appl. Maths. **25**, 247–255 (1972)
29. Lewis, J.L., Nyström, K.: Boundary behaviour for p-harmonic functions in Lipschitz and starlike Lipschitz ring domains. Annales scientifiques de l'Ecole normale supé,rieure **40.5**, 765–813 (2007)
30. Lewis, J.L., Nyström, K.: Boundary behavior and the Martin boundary problem for p-harmonic functions in Lipschitz domains. Ann. Math. **172.3**(2), 1907–1948 (2010)
31. Lewis, J.L., Nyström, K.: Regularity and free boundary regularity for the -Laplace operator in Reifenberg flat and Ahlfors regular domains. J. Am. Math. Soc. **25**, 827–862 (2012)
32. Lewis, J.L., Nyström, K.: Quasi-linear PDEs and low-dimensional sets. J. Eur. Math. Soc. **20.7**, 1689–1746 (2018)
33. Lindqvist, P.: On the growth of the solutions of the differential equation $\nabla \cdot (|\nabla u|^{p-2}\nabla u) = 0$ in n-dimensional space. J. Differ. Equ. **58**, 307–317 (1985)
34. Lundström, N.L.P.: Estimates for p-harmonic functions vanishing on a flat. Nonlinear Anal. **74.18**, 6852–6860 (2011)

35. Lundström, N.L.P.: Phragmén-lindelöf theorems and p-harmonic measures for sets near low-dimensional hyperplanes. Potential Analysis **44.2**, 313–330 (2016)
36. Lundström, N.L.P.: Growth of subsolutions to fully nonlinear equations in halfspaces. J. Differ. Equ. **320**, 143–173 (2022)
37. Méndez, O.: Eigenvalues and Eigenfunctions of the $p(\cdot)$-Laplacian. A Convergence Analysis, in Jarosz, K. Contemporary Mathematics 645 Functions spaces in analytics. Trans. Am. Math. Soc., Providence (2015)
38. Mikayelyan, H., Shahgholian, H.: Hopf's lemma for a class of singular/degenerate PDE's. Ann. Acad. Sci. Fenn. Math. **40**, 475–484 (2015)
39. Pucci, P., Serrin, J.: The Maximum Principle. Basel, Birkhäuser Basel (2007)
40. Sirakov, B.: Boundary Harnack estimates and quantitative strong maximum principles for uniformly elliptic PDE. Int. Math. Res. Not. **24**, 7457–7482 (2018)
41. Wolanski, N.: Local bounds, Harnack's inequality and hölder continuity for divergence type elliptic equations with nonstandard growth. Revista de la Unió,n Matemática Argentina **56.1**, 73–105 (2015)
42. Wu, J.-N.: Comparisons of kernel functions, boundary Harnack principle and relative Fatou theorem on Lipschitz domains. Ann. Inst. Fourier. (Grenoble) **28.4**, 147–167 (1978)
43. Zhang, Q.: A strong maximum principle for differential equations with nonstandard $p(x)$-growth conditions. J. Math. Anal. Appl. **312.1**, 24–32 (2005)
44. Zhang, Q.: Existence and asymptotic behavior of positive solutions to $p(x)$-Laplacian equations with singular nonlinearities. J. Inequalities Appl. **2007**, 1–9 (2007)